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# Topological preons from algebraic spinors

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**Abstract** It is demonstrated that many of the assumed rules that govern the structure of a previously proposed topological preon model, in which simple non-trivial braids consisting of three twisted ribbons are mapped to the first generation of leptons and quarks, are automatically adhered to when the algebraic spinors of two complex Clifford algebras are identified with braids via a suitable map. Much of the assumed topological architecture of this model can therefore be interpreted as a direct consequence of the deeper algebraic structures upon which the minimal ideals of these Clifford algebras are constructed. This result deepens the understanding of how these two complementary descriptions, one topological and one algebraic, of Standard Model symmetries are intimately connected despite originating from very different perspectives.

## 1 Introduction

Preon models were first developed in the 1970s with the hope of deriving the properties and quantum numbers of the SM particles from a smaller set of constituent particles. The most famous of these is the Harari-Shupe preon model, based on just two fundamental particles [1,2]. Using the Harari-Shupe model as inspiration, it was shown in [3] that one generation of SM fermions can be represented topologically in terms of triplets of ribbons, possibly carrying twists of $\pm 2\pi$, bounded together at the top and bottom by a parallel disk. The three ribbons are allowed to braid each other which allows the ribbons to be distinguished by their relative crossings. With the ribbons distinguished in this way, the *twist structure* of the ribbons accounts for the electrocolor symmetries. The *braid structure* on the other hand encodes the weak symmetry and chirality.

Although this topological preon model provides an economical representation of leptons and quarks, the class of braids considered to correspond to physical states is heavily restricted by a set of rules (axioms), which although simple, are arbitrary and lack any theoretical motivation. There is no intrinsic reason to bind ribbons in triplets in particular (as opposed to any other arbitrary number of ribbons), nor why the twisting on each ribbons should be restricted to only $\pm 2\pi$. Most importantly, there is no a priori reason for excluding braids composed of ribbons that carry opposite twists, yet this assumption is crucial to the model, and without it one ends up with 27 electrocolor states instead of 16.[1] The braiding between ribbons is unrestricted in the model. Without a suitable mechanism to prevent ever more complex braiding, one ends up with an unbounded number of generations [4].

The minimal left ideals of $\mathbb{C}\ell(6)$ were previously shown to contain one generation of leptons and quarks transforming correctly under the unbroken $SU(3)_c \times U(1)_{em}$ gauge symmetry [5]. This paper demonstrates that the rules that govern the permitted topological architecture of triplets of twisted ribbons in [3] can at a deeper level be understood as a reflection of the algebraic structure of the minimal one-sided ideals of $\mathbb{C}\ell(6)$. The construction of ideals relies fundamentally on a Witt basis composed of nilpotent anti-commuting ladder operators. Via a simple map from this Witt basis to elements of a suitable braid group with half the number of generators, every basis state of the minimal ideals of $\mathbb{C}\ell(6)$, each representing a distinct fermion in [5], is associated with a unique braid that is isotopic to the triplets of twisted ribbons that represent the electrocolor symmetries of one generation of leptons and quarks via the twist structure in [3]. That is, starting with the algebraic states together with a simple map, the resulting topological structures automatically adhere to the rules imposed by hand in [3]. The finite dimensionality of the exterior algebra generated from the $\mathbb{C}\ell(6)$ Witt basis

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[1] It is worth noting that a similar assumption is made in other preon models, including the Harari-Shupe model [1,2].

means that the twisting on each ribbon never exceeds $\pm 2\pi$ and that combinations of ribbons that carry opposite twist are naturally excluded.

Likewise, the minimal right ideals of $\mathbb{C}\ell(4)$ were shown previously to contain one generation of chiral weak states transforming correctly under $SU(2)_L$ [6]. Therefore, by supplementing the minimal left ideals of $\mathbb{C}\ell(6)$ with the minimal right ideals of $\mathbb{C}\ell(4)$, every lepton and quark can be algebraically represented as simultaneously belonging to a minimal left ideals of $\mathbb{C}\ell(6)$ and minimal right ideal of $\mathbb{C}\ell(4)$ [7]. We demonstrate that the braiding of (twisted) ribbons in [4] can be generated from these minimal right ideals of $\mathbb{C}\ell(4)$. The finite dimensionality of the exterior algebra generated from the $\mathbb{C}\ell(4)$ Witt basis in this case restricts the possible complexity of braiding, thereby avoiding the problem of an unbounded number of generations. We are therefore able to reproduce (with some minor differences) the topological model in [3] starting with the minimal ideals of $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$, without having to introduce additional rules about the permitted topological structures. These permitted states emerge automatically as a consequence of the underlying algebraic structure of the minimal ideals.

This paper builds on earlier works where a structural correspondence between the algebraic characterization of leptons and quarks as minimal one-sided ideals of Clifford algebras, and their topological representation as triplets of braided ribbons was established. It was shown in [8] that mapping a Witt basis of $\mathbb{C}\ell(6)$ to specific braids in the circular Artin braid group $B_3^c$[2] makes it possible to replicate the twist structure describing electrocolor symmetries in this preon model. This result makes use of the important fact that braiding and twisting are interchangeable [4,9]. This result was subsequently extended to include the $SU(2)_L$ chiral weak symmetry in terms of the braid structure. This is achieved by mapping a Witt basis of an additional $\mathbb{C}\ell(4)$ algebra to braids in $B_3$, taken to be a subgroup of $B_3^c$ [7,10].

Whereas the purpose of these earlier works was to establish a structural correspondence between the algebraic characterization of leptons and quarks as minimal one-sided ideals of Clifford algebras, and their topological representation as braids, the focus here is to demonstrate that the rules imposed to restrict the permitted topological structures in [3], are at a deeper level actually a reflection of the algebraic structure of the minimal one-sided ideals of $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$. Our starting point is the algebras $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$. We do not focus on justifying the choice of these two algebras here, other than to say that both can be generated from tensor products of normed division algebras acting on themselves. The reader is directed to [5,11–15] for more information.

## 2 A topological model of composite preons

We begin by providing a brief overview of the building blocks and rules of the braid model [3]. The fundamental preonic object is a ribbon which may be twisted by $\pm\pi$. The permitted topological structure of these ribbons are governed by the following three rules:

**(i) Unordered pairing** *Twists combine in pairs, called helons, so that their total twist is 0 modulo $2\pi$, and the ordering of twists within a pair is unimportant*. This results in each ribbon carrying a twist of $0, +2\pi$, or $-2\pi$. We may simply write this as $0, +1, -1$ respectively.
**(ii) Helons bind into triplets** *Helons are bound into triplets by a mechanism represented as the tops of each strand being connected to each other, and likewise for the bottoms of each strand*. Such a triplet of helons may be written as a vector $[a, b, c]$ where by the first assumption $a, b, c \in \{0, +1, -1\}$.
**(iii) No charge mixing** *When constructing braided triplets, combinations of helons with twists in opposite direction in the same triplet are not allowed.* A permissible triplet of helons can then be written as $[a, b, c]$ where now $a, b, c \in \{0, +1\}$ or $a, b, c \in \{0, -1\}$.

With these assumptions, supplemented by a simple choice for how the ribbons are braided, the braids in Fig. 1, represent the first generation of SM leptons and quarks.

In this representation, the *twist structure* of the ribbons accounts for the electrocolor symmetries, with charges of $\pm e/3$ represented by $\pm 2\pi$ twists, and the permutations of twisted ribbons representing color. The *braid structure* of ribbons on the other hand encodes the weak symmetry and chirality, with a top to bottom reflection corresponding to mapping between particles and anti-particles, and a left to right reflection corresponding to a parity transformation. Weak interaction are represented via braid composition, which corresponds to joining the bottom of the ribbons of the first braid to the tops of the ribbons of the second braid, and then sliding (isotop) the twists from each component braid upward. No explicit assumption is made about the permitted braiding of ribbons, and the model in Fig. 1 chooses just one simple possibility. More complex braiding is assumed to correspond to additional generations. However, without any restriction on the complexity of braiding, this leads to an unbounded number of generation [4].

The joining of three helons at the top and bottom is equivalent to two parallel disks connected by a triplet of ribbons. Without any braiding of these ribbons, the order of $a, b, c$ in

[2] The Artin braid group on $n$ strands is denoted by $B_n$ and is generated by elementary braids $\{\sigma_1, ..., \sigma_{n-1}\}$ satisfying $\sigma_i\sigma_j = \sigma_j\sigma_i$, whenever $|i-j| > 1$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, ...., n-2$. The circular braid group $B_n^c$ has $n$ strings attached to the outer edges of two disks, and has one additional generator $\sigma_n$ which crosses the $n$-th string over the first string.

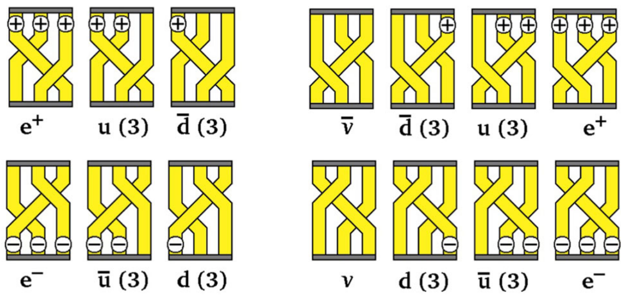


**Fig. 1** In the braid model, leptons and quarks are represented as braids of three twisted ribbons, restricted by certain rules. Source, [3]

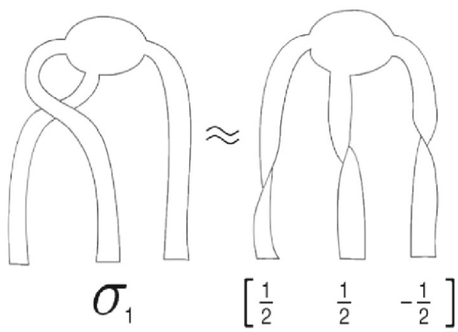


**Fig. 2** One can go from the framed braid on the left (with braiding but no twisting) to the one on the right (with twisting but no braiding) by turning over the disk at the top by $\pi$ around the axis that passes through its center and between the first and second ribbons. Source, [17]

the twist vector $[a, b, c]$ is of no importance, since a simple overall rotation will permute $a, b, c$, making it impossible to distinguish the ribbons. This means one would no longer be able to assign color to quarks. When the braid structure is included, $[1, 0, 0]\sigma_1\sigma_2^{-1}$ is not isotopic to $[0, 1, 0]\sigma_1\sigma_2^{-1}$, and so all three colored quarks are topologically distinct. The union of the ribbons and disks form a closed surface which may or may not be orientable. The resulting topological objects, corresponding to capped framed braids in the circular Artin braid group $B_3^c$, are called 3-belts [9,16].

Crucial to our construction later on is the observation that the twist and braid structure are not individually conserved, but are interchangeable [4,9]. An example for the braid generator $\sigma_1$ is given in Fig. 2.

As a 3-belt, $\sigma_1$ is isotopic to $[1/2, 1/2, -1/2]$. For the generators of $B_3^c$ we write:

$$\sigma_1 \approx [1/2, 1/2, -1/2]\,, \;\; \sigma_2 \approx [-1/2, 1/2, 1/2]\,,$$
$$\sigma_3 \approx [1/2, -1/2, 1/2]\,. \tag{1}$$

The twist vectors of $\sigma_i^{-1}$ correspond to the negatives of $\sigma_i$, so that, for example, $\sigma_1^{-1} \approx [-1/2, -1/2, 1/2]$. Care must be taken when consider longer braid words since each braid generator induces a permutation. Exchanging between twisting and braiding is possible for $n$-belts in general. However, (orientable) 3-belts are unique in that they can always be written in a form in which all the twisting or all the braiding has been eliminated [9].

## 3 Construction of $\mathbb{C}\ell(2n)$ spinors

The algebras $\mathbb{C}\ell(2n)$ each have only one irreducible representations, the algebraic spinors, of dimension $2^n$. These algebraic spinors correspond to minimal one-sided ideals. In this section we review the well-known construction of these minimal ideals [18], and subsequently how the minimal ideals of $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$ can be used to represent the unbroken electrocolor, and chiral weak symmetry respectively, of a single generation of leptons and quarks [5].

The construction of algebraic spinors requires three key ingredients. These are a Witt basis of nilpotent ladder operators, the exterior algebra generated from these ladder operators, and finally a primitive idempotent. Let $V$ be the generating space of $\mathbb{C}\ell(2n)$ spanned by $\{e_i\}$, $i = 1, ..2n$ over $\mathbb{C}$. These basis vectors $e_i$ are anticommuting. Via a Witt decomposition of this basis, we can define a set of $n$ creation operators, and an adjoint set of annihilation operators:

$$\alpha_j^\dagger := \frac{1}{2}(e_j + ie_{j+n}), \quad \alpha_j := \frac{1}{2}(-e_j + ie_{j+n}),$$
$$j = 1, .., n \tag{2}$$

These ladder operators are nilpotent ($\alpha_i^2 = 0$), and satisfy the fermionic anti-commutation relations

$$\{\alpha_i^\dagger, \alpha_j^\dagger\} = 0, \quad \{\alpha_i, \alpha_j\} = 0, \quad \{\alpha_i^\dagger, \alpha_j\} = \delta_{ij}. \tag{3}$$

Here, $\dagger$ takes $i \mapsto -i$, $e_j \mapsto -e_j$, and reverses the order of multiplications. The ladder operators, $\{\alpha_i\}$ and $\{\alpha_i^\dagger\}$ each form the basis of an $n$ complex dimensional space, $\chi_n^\dagger \cong \mathbb{C}^n$ and $\chi_n \cong \mathbb{C}^{*n}$ respectively, corresponding to maximal totally isotropic subspaces.

Via the Clifford product,[3] $\{\alpha_i^\dagger\}$ and $\{\alpha_i\}$ generate the $2^n$-dimensional exterior algebras $\bigwedge \mathbb{C}^n$ and $\bigwedge \mathbb{C}^{*n}$ respectively, including the nilpotents $w = \bigwedge^n \mathbb{C}^n$, and $w^\dagger = \bigwedge^n \mathbb{C}^{*n}$, from which the primitive idempotents $ww^\dagger$ and $w^\dagger w$ can be constructed. Subsequently $\bigwedge \mathbb{C}^n ww^\dagger$ and $\bigwedge \mathbb{C}^{*n} w^\dagger w$ define two minimal left ideals of $\mathbb{C}\ell(2n)$. We can write

$$\mathbb{C}\ell(2n)ww^\dagger \equiv \bigwedge \mathbb{C}^n ww^\dagger,$$
$$\mathbb{C}\ell(2n)w^\dagger w \equiv \bigwedge \mathbb{C}^{*n} w^\dagger w. \tag{4}$$

The unitary symmetry that preserves these one-sided ideals is $SU(n) \times U(1)$, with the generators of this symmetry constructed from the bivectors of the algebra, expressible in terms of the Witt basis.

[3] Because the inner product vanishes on $\chi$ and $\chi^\dagger$, the Clifford product on the subalgebra coincides with the exterior product.

*Electrocolor symmetries from $\mathbb{C}\ell(6)$*

Using this construction, the first minimal left ideal of $\mathbb{C}\ell(6)$, $S^u \equiv \mathbb{C}\ell(6)\omega\omega^\dagger = \bigwedge \chi^\dagger\omega\omega^\dagger$, containing the isospin-up fermions is explicitly given

$$\begin{aligned} S^u \equiv\ & \nu\omega\omega^\dagger + \bar{d}^r\alpha_1^\dagger\omega\omega^\dagger + \bar{d}^g\alpha_2^\dagger\omega\omega^\dagger + \bar{d}^b\alpha_3^\dagger\omega\omega^\dagger u^r\alpha_3^\dagger\alpha_2^\dagger\omega\omega^\dagger \\ & + u^g\alpha_1^\dagger\alpha_3^\dagger\omega\omega^\dagger + u^b\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger \\ & + e^+\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger, \end{aligned} \tag{5}$$

where $\omega\omega^\dagger = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$, and $\nu$, $\bar{d}^r$ etc. are complex coefficients denoting the isospin-up fermions. The conjugate system gives a second, linearly independent, minimal left ideal of isospin-down fermions $S^d \equiv \mathbb{C}\ell(6)\omega^\dagger\omega = \bigwedge \chi\omega^\dagger\omega$ spanned by the states

$$\{1, \alpha_1, \alpha_2, \alpha_3, \alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_1\alpha_2, \alpha_1\alpha_2\alpha_3\}\,\omega^\dagger\omega. \tag{6}$$

In [5] it was shown that $S^u$ and $S^d$, are preserved by the electrocolor symmetry $SU(3)_c \times U(1)_{em}$, with each basis state, via the commutator, transforming as a specific lepton or quark as indicated by their suggestively labeled complex coefficients. The reader is directed to [5] for the explicit representation of the $SU(3)_c \times U(1)_{em}$ generators, which are of no importance to us in what follows.

*Chiral weak symmetries from $\mathbb{C}\ell(4)$*

Including the weak symmetry requires an additional Clifford algebra, linearly independent from $\mathbb{C}\ell(6)$, whose minimal one-sided ideals are preserved by $SU(2)_L$. This algebra is $\mathbb{C}\ell(4)$ with Witt basis $\{\beta_1, \beta_2, \beta_1^\dagger, \beta_2^\dagger\}$, satisfying the same anticommutation relations (3). Two minimal right ideals are given by $\Omega\Omega^\dagger\mathbb{C}\ell(4)$ and $\Omega^\dagger\Omega\mathbb{C}\ell(4)$, with explicit bases

$$\Omega^\dagger\Omega\{1, \beta_1^\dagger, \beta_2^\dagger, \beta_1^\dagger\beta_2^\dagger\}, \quad \Omega\Omega^\dagger\{1, \beta_1, \beta_2, \beta_2\beta_1\}, \tag{7}$$

where $\Omega = \beta_2\beta_1$ and $\Omega^\dagger = \beta_1^\dagger\beta_2^\dagger$. These ideals each contain an $SU(2)$ doublet and two $SU(2)$ singlets [6], allowing us to assign chirality to the previously obtained $\mathbb{C}\ell(6)$ states, so that now

$$\nu_R = \omega\omega^\dagger\Omega^\dagger\Omega, \quad \nu_L = \omega\omega^\dagger\Omega^\dagger\Omega\beta_1^\dagger, \tag{8}$$

$$\begin{aligned} e_L^- &= \alpha_1\alpha_2\alpha_3\omega^\dagger\omega\Omega^\dagger\Omega\beta_2^\dagger, \\ e_R^- &= \alpha_1\alpha_2\alpha_3\omega^\dagger\omega\Omega^\dagger\Omega\beta_1^\dagger\beta_2^\dagger. \end{aligned} \tag{9}$$

It is apparent that the neutrino and electron live in different $\mathbb{C}\ell(6)$ ideals, but in the same $\mathbb{C}\ell(4)$ ideal. One can write down the quark states in a similar manner (see [7] for details). The eight weak-doublets are then identified as

$$\begin{aligned} \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix} &= \begin{pmatrix} \omega\omega^\dagger\Omega^\dagger\Omega\beta_1^\dagger \\ \alpha_1\alpha_2\alpha_3\omega^\dagger\omega\Omega^\dagger\Omega\beta_2^\dagger \end{pmatrix}, \\ \begin{pmatrix} u_L^{(3)} \\ d_L^{(3)} \end{pmatrix} &= \begin{pmatrix} \alpha_j^\dagger\alpha_i^\dagger\omega\omega^\dagger\Omega^\dagger\Omega\beta_1^\dagger \\ \epsilon_{ijk}\alpha_k\omega^\dagger\omega\Omega^\dagger\Omega\beta_2^\dagger \end{pmatrix}, \end{aligned} \tag{10}$$

$$\begin{aligned} \begin{pmatrix} e_R^+ \\ \bar{\nu}_R \end{pmatrix} &= \begin{pmatrix} \alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger\Omega\Omega^\dagger\beta_2 \\ \omega^\dagger\omega\Omega\Omega^\dagger\beta_1 \end{pmatrix}, \\ \begin{pmatrix} \bar{d}_R^{(3)} \\ \bar{u}_R^{(3)} \end{pmatrix} &= \begin{pmatrix} \alpha_i^\dagger\omega\omega^\dagger\Omega\Omega^\dagger\beta_2 \\ \epsilon_{ijk}\alpha_j\alpha_k\omega^\dagger\omega\Omega\Omega^\dagger\beta_1 \end{pmatrix} \end{aligned} \tag{11}$$

All of the other physical states, such as $(\bar{u}_L^{(3)}) = (\epsilon_{ijk}\alpha_j\alpha_k\omega^\dagger\omega\Omega\Omega^\dagger\beta_2\beta_1)$, are weak singlets.

All of these states transform correctly under the unbroken electrocolor symmetry, and chiral weak symmetry, with the symmetry generators explicitly given in [7]. The combined ideals can be written as minimal left ideals of $\mathbb{C}\ell(6) \otimes \mathbb{C}\ell(4) \cong \mathbb{C}\ell(10)$ in a way that preserves individually the $\mathbb{C}\ell(6)$ structure and $\mathbb{C}\ell(4)$ structure of physical states. One advantage of this model is that it captures many of the attractive features of the Georgi and Glashow $SU(5)$ Grand Unified Theory without introducing proton decay or other unobserved processes. Such processes are naturally excluded because they do not preserve the underlying algebraic structure.

## 4 From $\mathbb{C}\ell(6)$ to triplets of twisted ribbons

This section contains the first main result of this paper. At the core of our construction is a map $\mathbb{C}^3 \mapsto B_3^{c+}$, together with the dual map $\mathbb{C}^{*3} \mapsto B_3^{c-}$. Here $B_3^c$ denotes the circular braid group of three strands, and the $+$ and $-$ indicate positive braids (composed of $\sigma_i$) and negative braids (composed of $\sigma_i^{-1}$) respectively. These maps extend to the exterior algebras $\bigwedge\mathbb{C}^3$ etc in such a way that the algebraic product in $\bigwedge\mathbb{C}^3$ corresponds to braid composition in $B_3^c$. A particularly important feature for 3-belts is that twisting and braiding are interchangeable. Because this is a unique feature of 3-belts, our construction does not work for general maps $\mathbb{C}^n \mapsto B_n^c$. We demonstrate that via these maps, every basis state of the $\mathbb{C}\ell(6)$ ideals $S^u$ and $S^d$ is associated with a unique braid that is isotopic to a triplets of twisted ribbons that respects the rules (i)–(iii) in Sect. 2, and coincide with the twist structures in Fig. 1. We establish this result in three steps: (1) we interpret the twist structures found in the topological model as a binary code; (2) we find a suitable map from the Witt basis of $\mathbb{C}\ell(6)$ to the circular braid group $B_3^c$; and (3) eliminate the resulting braiding by exchanging it for twisting, using Eq. (1).

*Twist structure as a 3-bit binary code*

By assumption (iii) in Sect. 2, oppositely charged ribbons cannot be part of the same triplet. An admissible triplet of ribbons can then be written as $[a, b, c]$ where $a, b, c \in \{0, +1\}$ or $a, b, c \in \{0, -1\}$. These twist vectors may be interpreted as a 3-bit binary code, precisely because ribbons with opposite twists do not bind together. For three ribbons, this generates $2^3 = 8$ possible twist states for a given twist direction. This matches the dimensionality of $\bigwedge \mathbb{C}^3$ and $\bigwedge \mathbb{C}^{*3}$ and hence the minimal left ideals of $\mathbb{C}\ell(6)$.[4] This is essentially the same how in the $SU(5)$ and $SO(10)$ grand unified theories each particle and antiparticle can be represented via a 5-bit binary code [19]. Each bit, representing a basic property of a particle such as isospin up or red, can be treated as a basis vector of $\mathbb{C}^5$. Subsequently, the 32-complex dimensional algebra $\bigwedge \mathbb{C}^5$ has a basis given by the wedge product of these five basis vectors, and the representation of $SU(5)$ is then used to describe fermions.

*From algebraic spinors to triplets of twisted ribbons*

Since the number of basis vectors of the minimal left ideal $S^u$ of $\mathbb{C}\ell(6)$ matches the number of distinct twist vectors, one might look for a one-one map from the former to the latter. This is possible by first mapping the creation operators $\alpha_i^\dagger$ to specific braids in $B_3^c$, and subsequently exchanging the resulting braiding for twisting. It is easy to check, using Eq. (1), that

$$\begin{aligned}&[1, 0, 0] = \sigma_3\sigma_2[0, 0, 0], \quad [0, 1, 0] = \sigma_1\sigma_3[0, 0, 0],\\&[0, 0, 1] = \sigma_2\sigma_1[0, 0, 0].\end{aligned} \tag{12}$$

One might therefore consider the mapping $\alpha_i^\dagger \mapsto \sigma_{i+2}\sigma_{i+1}$ where $i+1, i+2$ are modulo 3.[5] Explicitly

$$\alpha_1^\dagger \mapsto (\sigma_3\sigma_2), \quad \alpha_2^\dagger \mapsto (\sigma_1\sigma_3), \quad \alpha_3^\dagger \mapsto (\sigma_2\sigma_1). \tag{13}$$

With this choice, each of the eight basis states in $\bigwedge \chi^\dagger$ maps to a distinct twist vector that happens to coincide precisely with the twist vectors of the leptons and quarks in Fig. 1 in a one-to-one manner. Crucially, this construction avoids triplets containing ribbons with twists in opposite directions, thereby automatically satisfying both rules (i) and (iii). The finite dimensionality of the exterior algebra also means that the twisting on each ribbon never exceeds $\pm 2\pi$.

As an explicit example, consider the twist structure of a green up quark $u^g$:

$$u^g : \alpha_1^\dagger\alpha_3^\dagger\omega\omega^\dagger \mapsto (\sigma_3\sigma_2)(\sigma_2\sigma_1)[0, 0, 0] = [1, 1, 0]. \tag{14}$$

We can do the same for the second ideal, $S^d$, via the maps (see [10] for more details)

$$\alpha_1 \mapsto (\sigma_2^{-1}\sigma_3^{-1}), \quad \alpha_2 \mapsto (\sigma_3^{-1}\sigma_1^{-1}), \quad \alpha_3 \mapsto (\sigma_1^{-1}\sigma_2^{-1}). \tag{15}$$

Note that whereas $\alpha_i$ and $\alpha_i^\dagger$ are dual, the braids they map to are inverses.[6] However, their actions are the same. At the algebra level we have, for example $(\alpha_1^\dagger)(\alpha_1\alpha_2\omega^\dagger\omega) = (1 - \alpha_1\alpha_1^\dagger)\alpha_2\omega^\dagger\omega = \alpha_2\omega^\dagger\omega$. The action of multiplying by $\alpha_1^\dagger$ cancels the action of $\alpha_1$ (in the presence of the primitive idempotent $\omega^\dagger\omega$). At the braid group level this is replicated when $\alpha_1^\dagger$ and $\alpha_1$ are mapped to braid inverses. The primitive idempotents $\omega\omega^\dagger$ and $\omega^\dagger\omega$ both lie in the kernal of our combined map (13) and (15), which is therefore not one-to-one.

We finish this section by discussing the insights and advantages offered by this algebraic construction. Even though from the braid group point of view there are infinitely many 3-belts, the nilpotent nature of the algebraic ladder operators, together with their anti-commuting properties means that the maps (13) and (15) each select a finite set of $2^3 = 8$ 3-belts, one for each algebraically nonzero product of ladder operators. The other 3-belts do not correspond to basis states of minimal ideals, cannot be represented as a product of ladder operators, and therefore do not correspond to a physical state. Furthermore, triplets composed of oppositely twisted ribbons do not arise from this algebraic construction, and the twisting on each ribbon never exceeds $\pm 2\pi$.

## 5 From $\mathbb{C}\ell(4)$ to braided ribbons

Without braid structure, the order of $a, b, c$ in the vector $[a, b, c]$ representing the twist structure is of no importance, making it impossible to distinguish the ribbons, and subsequently to be able to assign color to quarks if the braid structure is ignored. In this section we supplement the minimal left ideals of $\mathbb{C}\ell(6)$ with the minimal right ideals of $\mathbb{C}\ell(4)$, and subsequently generate suitable braid structure from these ideals of $\mathbb{C}\ell(4)$. We again construct a map, $\mathbb{C}^2 \mapsto B_3 \subset B_3^c$ together with its dual. We would like $\beta_i$ and $\beta_i^\dagger$ to again map to inverse braids. One choice would be to map $\beta_1^\dagger \mapsto \sigma_1\sigma_2$, and $\beta_2^\dagger \mapsto \sigma_2\sigma_1$. However in that case both $\alpha_3^\dagger$ and $\beta_2^\dagger$ map to the same braid, making $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$ not independent

[4] More generally, for $n$ ribbons there are $2^n$ states, representable via an $n$-bit binary code. This dimensionality matches that of the exterior algebra $\bigwedge \mathbb{C}^n$, which we saw plays a central role in the construction of the algebraic spinors of $\mathbb{C}\ell(2n)$.

[5] One may wonder why the simpler map $\alpha_i^\dagger \mapsto \sigma_i$ (along with $\alpha_i \mapsto \sigma_i^{-1}$) is not chosen instead. In that case, when the braiding is exchanged for twisting, it gives a twist of only $\pm\pi$ on each of the ribbons, and furthermore will lead to some ribbons being twisted in opposite directions, as is clear from Eq. (1). This violates rules (i) and (iii).

[6] This is a slight but important deviation from the original construction in [8], where $\alpha_i^\dagger$ and $\alpha_i$ were not mapped to inverse braids of each other.

from each other. Taking inspiration from Fig. 1 we instead consider the maps

$$\beta_1^\dagger \mapsto \sigma_1\sigma_2^{-1}, \qquad \beta_2^\dagger \mapsto \sigma_1^{-1}\sigma_2, \tag{16}$$

$$\beta_1 \mapsto \sigma_2\sigma_1^{-1}, \qquad \beta_2 \mapsto \sigma_2^{-1}\sigma_1. \tag{17}$$

Again, both the primitive idempotents $\Omega\Omega^\dagger$ and $\Omega^\dagger\Omega$ map to the unbraid $\mathbb{I}$.

Finally, every physical state in $\mathbb{C}\ell(6)\otimes\mathbb{C}\ell(4)$ can now be mapped to a unique braid with both twist structure and braid structure. For example

$$\begin{aligned}\begin{pmatrix} u_L^r \\ d_L^r \end{pmatrix} &= \begin{pmatrix} \alpha_3^\dagger\alpha_2^\dagger\omega\omega^\dagger\Omega^\dagger\Omega\beta_1^\dagger \\ \alpha_1\omega^\dagger\omega\Omega^\dagger\Omega\beta_2^\dagger \end{pmatrix} \mapsto \begin{pmatrix} [1,0,1]\sigma_1\sigma_2^{-1} \\ [0,-1,0]\sigma_1^{-1}\sigma_2 \end{pmatrix} \\ &= \begin{pmatrix} [1,0,1]\sigma_1\sigma_2^{-1} \\ \sigma_1^{-1}\sigma_2[-1,0,0] \end{pmatrix}. \end{aligned} \tag{18}$$

It is here that it is important to remember that the twist structure is permuted by the braid structure, so that $[0,-1,0]\sigma_1^{-1}\sigma_2 = \sigma_1^{-1}\sigma_2[-1,0,0]$. Similarly, for the associated weak singlets we find

$$\left(u_R^r\right) \mapsto \left([1,0,1]\mathbb{I}\right), \quad \left(d_R^r\right) \mapsto \left(\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2[0,0,-1]\right). \tag{19}$$

The full list of particle states are listed in the Appendix of [10]. It is apparent that it is no longer true that the braid structures of all particles are the same length. However it remains true that all weakly interacting particle states have the same length, and are equivalent to their representations in [3].

Importantly, the complexity of braid structure is naturally limited by the algebraic structure of the ideals, and only $2^2 = 4$ different braids (together with their inverses) appear in this construction. That is, the underlying algebraic structure of the minimal right ideals of $\mathbb{C}\ell(4)$ provides the required mechanism that limits the complexity of braiding, and hence the introduction of an unbounded number of generations.

## 6 Discussion

The purpose of this paper has been to demonstrate that the arbitrary rules that restrict the permitted topological structures in [3], are at a deeper level actually a reflection of the underlying algebraic structure of the minimal one-sided ideals of $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$. Despite the topological and algebraic models originating from very different perspectives, remarkably they share a common underlying mathematical structure. This connection should be investigated further.

A Witt-decomposition of $\mathbb{C}\ell(2n)$ splits the algebra into $n$ raising and $n$ lowering operators, all nilpotent. These ladder operators are subsequently mapped to braids in an appropriate braid group. The choice of braid group is determined by the number of raising operators. For $\mathbb{C}\ell(6)$ the possible braid groups are $B_3^c$ or $B_4$, each of which has three generators, whereas for $\mathbb{C}\ell(4)$ the braid group $B_3$ with two generators is the unique choice. To represent every lepton and quark as a single topological object in a way that reflects the algebraic representation of leptons and quarks as simultaneously belonging to a minimal left ideal of $\mathbb{C}\ell(6)$ and minimal right ideal of $\mathbb{C}\ell(4)$ forces us to map the $\mathbb{C}\ell(6)$ ideals to braids in $B_3^c$. Every lepton and quark is then represented topologically as a 3-belt, corresponding to a triplet of ribbons connected to each other at the top and bottom. This is assumption (ii) in Sect. 2.

Within $B_3^c$, the braiding of ribbons is exchangeable for twisting. Using the maps (13) and (15), the basis states of the $\mathbb{C}\ell(6)$ ideals replicate the twist structure of [3] with the twist on each ribbon equal to 0, $+2\pi$, or $-2\pi$. This is assumption (i) in Sect. 2. Furthermore, each minimal ideal is constructed entirely from either raising or lowering operators acting on a primitive idempotent, but not both. As a direct consequence of this algebraic structure, combinations of ribbons with twists in opposite direction in the same triplet are excluded. This is assumption (iii) in Sect. 2.

Finally, the nilpotent nature of the ladder operators means the dimensionality of the minimal ideals correspond to the dimensionality of the exterior algebra of the ladder operators. Since each basis state of a minimal ideal is mapped to a braid, this gives a finite spectrum of braids. Therefore the algebraic structure of the minimal ideals, and the nilpotent nature of the ladder operators naturally restricts the number of distinct physical states. This avoids the problem of generating an unbounded number of generations via ever more complex braiding.

A more general construction which maps the minimal ideals of $\mathbb{C}\ell(2n)$ and $\mathbb{C}\ell(2n-2)$ to $n$-belts is unlikely to work. This is because in general the twisting or braiding cannot be made trivial in an $n$-belt. The connection between $\mathbb{C}\ell(6)$ and $\mathbb{C}\ell(4)$, with a topological representation of leptons and quarks in terms of 3-belts is therefore essentially unique. This is interesting because these Clifford algebras (and others) can be generated from the left and right actions of the octonions and quaternions on themselves respectively (or tensor products of division algebras acting on themselves) [5,20–22].

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