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Diphoton decay of the higgs from the Epstein–Glaser viewpoint

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Abstract We revisit a nearly 10-year old controversy on the diphoton decay of the Higgs particle. To a large extent, the controversy turned around the respective merits of the regularization techniques employed. The novel aspect of our approach is that *no* regularization techniques are brought to bear: we work within the Bogoliubov–Epstein–Glaser scheme of renormalization by extension of distributions. Solving the problem actually required an expansion of this method's toolkit, furnished in the paper.

Die Eule der Minerva beginnt erst mit der einbrechenden Dämmerung ihren Flug

– Georg Wilhelm Friedrich Hegel

1 Introduction: the controversy

Due to its cleanness, it is hard to overstate the experimental importance of the decay of the Higgs particle into two photons. It goes mainly via virtual W-bosons, the heavier charged particles of flavourdynamics. The amplitude of this contribution was calculated to the first non-vanishing order (one-loop, cubic in the couplings) long ago in the light-higgs limit [1] – and then "exactly" in [2]. The accepted result was confirmed many times – see [3] for a particularly clever calculation. It does *not* vanish in the heavy-higgs limit – which seems to fly in the face of the "decoupling theorem" (DT) in [4], as often understood.

Much more recently, those calculations were questioned in [5,6]. The ensuing debate highlights the *theoretical* relevance of this decay. The authors of these papers made the point that, since the higgs cannot couple directly to the photons, the one-loop contribution must be finite: there are no couplings requiring "renormalization". The roundabout procedures through "renormalizable gauges", they concluded, were unnecessary. Eschewing dimensional regularization, they recomputed the amplitude in the unitary gauge of electroweak (EW) theory. They did obtain a result differing from the standard one by an additive constant, which shows up for instance in the heavy-higgs limit – whereby their result is equal to zero.

There was no shortage of rejoinders [7–14] to [5,6]. The authors of [9] are the ones of the original calculation [2]. Those papers made several points, some rather implausibly arguing that at a given point in the calculation in [6] electromagnetic gauge invariance is lost, and criticizing the interpretation of the DT made in [5,6]. There was in some of the the rejoinders an explanatory reliance on the heuristics of the Brout–Englert–Higgs mechanism, throwing back the so-called "equivalence theorem" (GBET).

The criticisms received a rejoinder in turn in [15]. This later paper argues by the example that two computations of the same process in different gauges (R_ξ versus unitary gauge) may yield different results. This goes against the grain, although of course no theorem contradicts such an assertion. Meanwhile, a dispersion relation calculation carried out in [16] appeared to support the contentions of [5,6], and got in turn a – quite thoughtful – rejoinder in [17]. More recent papers dealing with the same or related issues are [18,19].

By and large, the majority's opinion and the experimental results [20] support the first tally. On the other hand, from the theoretical point of view the situation is still obscure: it had to be so, since both parties draw strength from different casuistics of the calculations in perturbative quantum field theory.

The debate about the uses and abuses of the unitary gauge and the role of the decoupling and equivalence "theorems"

To the memory of Günter Scharf and Raymond Stora.

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is to be saluted as salutary. And it is safe to admit that up to now we lack a full conceptual understanding of the problem. The cleanest way to address this lack is surely to renounce *all* the heuristics of mathematically ill-defined quantities, in favour of a method in which there can be no argument on the meaning of infinite terms. Such is the truly (perturbatively) stringent scheme by Bogoliubov, Epstein and Glaser (BEG) of "renormalization" without regularization, by extension of distributions.

In the BEG construction, governed by causality, there is no such thing as a "divergent diagram": one never encounters infinities. There may, however, remain in the extension procedures some additive *ambiguity*, that can be restricted (but not always completely removed) by physical principles. This is rather to be regarded as a strength of the BEG paradigm, because those ambiguities express precisely how, and to what extent, the theory is determined by the fundamental principles of perturbative QFT.

A particular advantage of the inductive BEG construction [21] of the (functional) $\mathbb{S}$-matrix is that in principle one is allowed to stay on configuration space, which makes more transparent the physics under examination. For examples of calculations within the BEG scheme explicitly carried out in configuration space, see [22] or [23, Sect. 3.5]. It is only for computational convenience that we switch at some moment to momentum space.

Since we do not deal in infinities, we refer as *normalization* to the processes taking the place of regularization and renormalization in the BEG framework. For its relative paucity of diagrams, in our context the underlying argument is made clearer by working mostly in the unitary gauge – whereupon only the physical particles' data are brought to bear.[1]

To summarize, so far: we were motivated to tackle this subject by wondering why most knowledgeable people, borrowing different (but all apparently sound) methods to work on such a basic process, were divided on the outcome. It all turns around a subtlety uncovered by use of the BEG normalization. That condenses the purpose of the present paper.

1.1 Main results and plan of the article

In Appendix A we introduce our conventions and notations, recalling a few well-known formulae of QFT needed in the body of the paper, in particular the propagators for the EW theory in the unitary gauge. Let m_h denote the mass of the higgs h. The amplitude coming from the one-loop calculations may be quoted as [25–27]:

$$\mathcal{A} = \frac{g\alpha}{2\pi M} F_1(\rho) P_{\mu\nu},$$

with α the fine structure constant, g the EW coupling constant, M the mass of the intermediate W-boson and $\rho := m_h^2/4M^2$. The polarization factor $P_{\mu\nu}$, reflecting electromagnetic gauge invariance (EGI) of $\mathcal{A}$,[2] is written in this paper as

$$P_{\mu\nu} := (k_1k_2)g_{\mu\nu} - k_{1\nu}k_{2\mu}; \quad (P_{\bullet\nu}k_1) = (P_{\mu\bullet}k_2) = 0, \tag{1.1}$$

with k_1, k_2 the outgoing photons' momenta. Finally, for the dimensionless factor:

$$F_1(\rho) := 2 + \frac{3}{\rho} + \frac{3}{\rho}\left(2 - \frac{1}{\rho}\right) f(\rho). \tag{1.2}$$

Now that we are at that, we quote as well the comparable result for a charged scalar particle of mass M at the place of the W-boson:

$$F_0(\rho) = \frac{1}{\rho}\left(1 - \frac{f(\rho)}{\rho}\right); \quad \text{so that}$$
$$F_1(\rho) = 3F_0(\rho) + \frac{6f(\rho)}{\rho} + 2. \tag{1.3}$$

For the benefit of the reader coming to the subject of this paper for the first time, Appendix B introduces the distribution $f(\rho)$ appearing in both F_1 (1.2) and F_0 (1.3) – as well as in the amplitude of diphoton decay of h via virtual fermions.

The bone of contention is that the first summand 2 in (1.2) should not be there, according to [5,6,16]. Relations (B.3) and (B.6) tell us that, as $\rho \downarrow 0$:

$$F_1 = 2 + \frac{3}{\rho} + \left(\frac{6}{\rho} - \frac{3}{\rho^2}\right)\left(\rho + \frac{\rho^2}{3} + \frac{8\rho^3}{45} + \cdots\right)$$
$$= 7 + \frac{22}{15}\rho + O(\rho^2);$$

so $F_1(0) = 7$ and $F_1(\infty) = 2$ from (1.2). Precisely the former figure is what was calculated in the paper [1]. The result argued by the "heretics" in the controversy is $F_1 - 2$, so their respective assertions are instead $F_1(0) = 5$ and $F_1(\infty) = 0$. Also, from (1.3): $F_0(0) = -1/3$ and $F_0(\infty) = 0$.

Appendices A and B of this paper deal with conventions and mathematical prerequisites. The basics of the BEG scheme are recalled in Appendix C. Understanding of the BEG method is indispensable in what follows, and even readers familiar with it are advised not to miss our review. The relation between the normalization problem by extension of distributions (or by "distribution splitting") and *dispersion integrals* is treated in its Sect. 1. New results in this respect are required, announced in the short Sect. 2 and proved in Sects. 3.2 and 3.3 of this paper. So for *aficionados* of BEG

[1] The paper [24] dwells usefully on the subject of the R_ξ-versus-unitary gauges, leaning to demonstrate the validity of the latter at the quantum level.

[2] That is, transversality of the outgoing photons.

normalization there is novelty here – whose interest goes beyond the particular problem that motivated it.

Sections 3 and 4 constitute the heart of the paper. The scalar model leading to F_0 is worked out in Sect. 3. One is able to perform the "adiabatic limit" of Epstein and Glaser at an intermediate step, which simplifies computations – this is rigorously justified. This "toy model" allows the reader to familiarize with the BEG construction of time-ordered products in a relatively simple case. For it, the ambiguity in the Epstein–Glaser result can be disposed of, and the unique outcome happens to coincide with the result of a "naive" on-shell calculation, of the kind performed in [16].

Finally, in Sect. 4, we compute the EW amplitude, working first in the unitary gauge. We start in earnest by illustrating in this relevant instance the machinery of the BEG formalism in constructing time-ordered products, at the lowest non-trivial order: from cubic interaction vertices, identified to time-ordered products at first order in the couplings, we *derive* the quartic, second-order $AAWW^\dagger$-vertex.

It is time to aver why the "no-renormalization" argument in [6] is not watertight. A direct $h\gamma\gamma$ coupling in flavourdynamics is forbidden also because of EGI. Thus to obtain the general amplitude, which lives off-shell, one must add to the naive calculations a polynomial in the external momenta, of degree given by the singular order of that amplitude. Computing the 1-loop contribution in the unitary gauge by the Epstein–Glaser method, we ratify this fact. To find the coefficients of that polynomial, beyond EGI here we call upon gauge-fixing independence of the on-shell amplitude. This locks in the indetermination; and in the end we do obtain $F_1(\rho)$. Within the unitary gauge, a different argument to the same purpose is discussed at the end of this Sect. 4. Section 5 is the conclusion.

2 The obstruction to distribution splitting for null momenta

Formula (C.16) in Appendix C is our main workhorse: in momentum space the Epstein–Glaser distribution splitting amounts to a dispersion integral. But it pertains to remark that, by construction, prescriptions (C.14) and (C.16) are in principle valid *only for timelike* k. Thus, in order to solve the problem in this paper, one has to run an extra mile. The explicit splitting procedure introduced here exhibits relevant novel features: we have to compute the central solution $a^c(k_1,k_2)$ for *null momenta*. Hence, one cannot immediately use the dispersion integrals (C.14) or (C.16). On trying to work instead with the convolution integral (C.13), there appears the problem that, in spite of $k_j^2=0$, it generally holds that $(k_j-v_j)^2\neq 0$ because $v_j\in V_+$; it *does not suffice* to know the causal distribution $d(k_1,k_2)$ only for $k_1^2=0=k_2^2$.

The next section solves this problem for models such that $0<(k_1+k_2)^2<4M^2$ and $k_1^0k_2^0>0$. The proof's strategy is as follows: starting from the dispersion integral (C.14) for $k_1^2>0$, $k_2^2>0$ and $k_1^0k_2^0>0$, we intend to show that $d(k_1,k_2)$ is regular enough that this integral commutes with the limit $(k_1^2\downarrow 0\wedge k_2^2\downarrow 0)$. Therefore the dispersion integrals (C.14) and (C.16) keep their usefulness for $k_1^2=0=k_2^2$: indeed, for computing $a^c(k_1,k_2)|_{k_1^2=0=k_2^2}$ it suffices to know $d(k_1,k_2)$ only for $k_1^2=k_2^2=0$, because $k_1^2=k_2^2=0$ implies $(tk_1)^2=(tk_2)^2=0$ for all t.

Crucially, in the resulting dispersion integrals (C.14) and (C.16) for $k_1^2=0=k_2^2$, the parameter ω is the singular order of the *off-shell* $d(k_1,k_2)$. As a consequence, the general solution (prior to imposition of other invariance rules) of the distribution splitting is obtained by adding to $a^c(k_1,k_2)|_{k_1^2=0=k_2^2}$ a polynomial in k_1,k_2, in principle arbitrary, whose degree is given by the singular order of the off-shell amplitude $d(k_1,k_2)$. Now, it frequently happens that the singular order of $d(k_1,k_2)|_{k_1^2=0=k_2^2}$ has a *smaller* value. Consequently, it may happen that the required dispersion integral appears to be "oversubtracted" – i.e., it would be convergent also for a smaller value of ω. Examples for this are the "toy model" in the next section and the EW diphoton decay of the higgs in the unitary gauge (Sects. 3.3 and 4.3, respectively).

These issues were realized by Raymond Stora, who, referring to the very subject process of this paper, pointed out to one of us that the good behaviour of the absorptive part of the form factor involving Compton scattering of the W-bosons should not make one forget that BEG-generated dispersion integrals, just as perturbative renormalization theory in general, applies off-shell. [3]

3 Higgs to diphoton decay via a charged scalar field

The scalar electrodynamics computation leading to F_0 works like a kind of toy model, allowing the reader to familiarize with our methods in a less complicated, although non-trivial case. We develop it in the present section. Notice the following: in the Epstein–Glaser scheme the "seagull" $e^2AA\varphi\varphi^\dagger$-vertex *is derived* by implementing EGI within the construction rules of the method – as any other part of T_2 [28]. We give full details on how this comes about for the quartic vertex in the EW theory in Sect. 4.1. The game here would be similar, only simpler. The reader is advised to keep in mind the methods and standard notations recalled in Sect. C.2.

[3] Private communication, early 2013.

3.1 A causal distribution on-shell

The starting point is given by the lower order time-ordered products (TOPs):

$$\begin{aligned}
&T_1(x_3) = gM\, h(x_3)\, \varphi(x_3)\, \varphi^\dagger(x_3);\\
&T_1(x_j) = -ieA^\lambda(x_j)\, \varphi^\dagger(x_j)\, \overleftrightarrow{\partial_\lambda}\, \varphi(x_j), \quad j = 1, 2;\\
&T_2(x_1, x_2)\\
&\quad = -e^2 A^\mu(x_1) A^\nu(x_2) \big[\varphi^\dagger(x_1)\, \partial_\mu \Delta^F(x_1 - x_2)\, \partial_\nu \varphi(x_2)\\
&\qquad - \partial_\mu \varphi^\dagger(x_1) \Delta^F(x_1 - x_2)\, \partial_\nu \varphi(x_2)\\
&\qquad + \varphi^\dagger(x_1) \big(\partial_\nu \partial_\mu \Delta^F(x_1 - x_2) + ig_{\mu\nu}\, \delta(x_1 - x_2)\big) \varphi(x_2)\\
&\qquad - \partial_\mu \varphi^\dagger(x_1)\, \partial_\nu \Delta^F(x_1 - x_2)\, \varphi(x_2)\big]\\
&\qquad + (x_1 \leftrightarrow x_2) + T_1(x_1)\, T_1(x_2)\\
&\qquad + [\text{irrelevant loop diagram terms}],
\end{aligned}$$

where Δ^F denotes the Feynman propagator (A.3).

From our formulas (C.4) and (C.5):[4]

$$\begin{aligned}
&D_3(x_1, x_2, x_3) = -[\overline{T}_1(x_1), T_2(x_2, x_3)]\\
&\quad -[\overline{T}_1(x_2), T_2(x_1, x_3)] + [\overline{T}_2(x_1, x_2), T_1(x_3)]. \qquad (3.1)
\end{aligned}$$

Because the photons emitted at x_1, x_2 are on-shell, only the third commutator is relevant here – in the language of Cutkosky rules, one needs only the triangle cut separating the higgs vertex from the propagator connecting the photons. We give the explanation further on. From the general formula for the antichronological product (C.3), we particularly know that

$$\begin{aligned}
&\overline{T}_1(x_1) = T_1(x_1); \quad \overline{T}_2(x_1, x_2) = -T_2(x_1, x_2)\\
&\quad + T_1(x_1) T_1(x_2) + T_1(x_2) T_1(x_1). \qquad (3.2)
\end{aligned}$$

For the same reasons just argued, only the connected tree diagram part of the $T_2(x_1, x_2)$ summand in $\overline{T}_2(x_1, x_2)$ contributes.

A most convenient parallel for the coming calculation is the treatment of the vertex function in QED in the first edition of the finite QED book by Scharf [29, Sect. 3.8]. Going to the contractions, bringing in the vertices and the propagators (A.2), (A.4), apart from a factor $4ge^2M$ we obtain:

$$\begin{aligned}
&A_\mu(x_1) A_\nu(x_2) h(x_3) \big[\Delta^-(1)\, \partial^\mu \Delta^F(1-2)\, \partial^\nu \Delta^-(2)\\
&\quad - \partial^\mu \Delta^-(1)\, \partial^\nu \Delta^F(1-2)\, \Delta^-(2)\\
&\quad - \partial^\mu \Delta^-(1)\, \Delta^F(1-2)\, \partial^\nu \Delta^-(2) + \Delta^-(1) \big(\partial^\mu \partial^\nu \Delta^F(1-2)\\
&\quad + ig^{\mu\nu}\, \delta(1-2)\big) \Delta^-(2)\\
&\quad - [\text{the same four terms with } \Delta^- \text{ replaced by } \Delta^+] + \cdots \big]\\
&\quad =: A_\mu(x_1) A_\nu(x_2) h(x_3)\, d^{\mu\nu}(1, 2),
\end{aligned}$$

where $1 \equiv y_1 := x_1 - x_3$, $2 \equiv y_2 := x_2 - x_3$. Here and further down, the dots stand for the terms coming from the other two cuts and further terms not contributing to the on-shell amplitude. Note the advertised additional $+ig^{\mu\nu}\,\delta$ to $\partial^\mu \partial^\nu \Delta^F$, corresponding to the "closed seagull" or fish-like diagram contribution to the $h \to 2\gamma$ decay in this model.

We now proceed to momentum space, where computations are carried out more simply. For Fourier transformations, consult the convention (C.7). In this section and the next, in keeping with physicists' notation, we indicate the transforms by just exhibiting the variables, namely: $d^\mu(k_1, k_2) \equiv \hat{d}^\mu(k_1, k_2)$. We obtain

$$\begin{aligned}
d^{\mu\nu}(k_1, k_2) = &\frac{1}{(2\pi)^2} \Big[4(I_+^{\mu\nu} - I_-^{\mu\nu}) + 2k_2^\nu (I_+^\mu - I_-^\mu)\\
&- 2k_1^\mu (I_+^\nu - I_-^\nu) - k_1^\mu k_2^\nu (I_+ - I_-)\\
&- \frac{i}{(2\pi)^2}\, g^{\mu\nu}\, (J_+ - J_-)\Big] + \cdots \qquad (3.3)
\end{aligned}$$

with the integrals

$$\begin{aligned}
I_\pm^{\{\cdot|\mu|\mu\nu\}}(k_1, k_2) &:= \int d^4k\, \{1|k^\mu|k^\mu k^\nu\}\, \Delta^\pm(k_1 - k)\\
&\quad \Delta^F(k)\, \Delta^\pm(k + k_2),\\
J_\pm(k_1, k_2) &:= \int d^4k\, \Delta^\pm(k_1 - k)\, \Delta^\pm(k + k_2), \qquad (3.4)
\end{aligned}$$

where the $J_\pm$-term is the contribution of the fish-like diagram. Keep in mind that the terms belonging to $A_3' := A_3 - T_3$ are those coming from the integrals $I_-^{\cdot|\mu|\mu\nu}$ and J_-, whereas the contribution of $R_3' := R_3 - T_3$ is given by the integrals $I_+^{\cdot|\mu|\mu\nu}$ and J_+.

For our purposes one may perform the adiabatic limit already at this stage. Since all internal lines of the diagrams correspond to massive fields, this limit can be done here in the naive way by just setting the switching function $g(x)$ in (C.1) to 1:

$$\begin{aligned}
&\int dx_1\, dx_2\, dx_3\, A^\mu(x_1) A^\nu(x_2) h(x_3)\, d^{\mu\nu}(x_1 - x_3, x_2 - x_3)\\
&= (2\pi)^2 \int dk_1\, dk_2\, h(k_1 + k_2) A^\mu(-k_1) A^\nu(-k_2)\, d^{\mu\nu}(k_1, k_2). \qquad (3.5)
\end{aligned}$$

In this limit the momenta k_1 and k_2 become the momenta of the external photons: $k_1^2 = k_2^2 = 0$.

From now on, we compute $d^{\mu\nu}(k_1, k_2)|_{k_1^2 = 0 = k_2^2}$. Were we to have included the other cuts in (3.1) or $T_1 \overline{T}_1 T_1$-terms, there would appear $\Delta^\pm$-type propagators at the place of the Feynman propagators above. The former are $\sim \delta(k^2 - M^2)$, with k denoting the internal momentum variable in the loop: so to speak, in contrast with the Feynman propagators, the $\Delta^\pm$ are

[4] The D_n are always linear combinations of commutators.

"always on-shell", even within loops.[5] Thus no further internal momenta can be on-shell: assuming $k^2 = M^2$ one obtains $(k_1 - k)^2 = M^2 - 2(k_1 k) \neq M^2$; similarly for $(k + k_2)$.[6]

Scalar integrals $I_\pm$.

We have to compute

$$I_\mp(k_1, k_2) := \frac{i}{(2\pi)^4} \int d^4k\, \theta(\mp(k_1^0 - k^0))\, \delta((k_1 - k)^2 - M^2) \times \frac{1}{k^2 - M^2 + i0}\, \theta(\mp(k^0 + k_2^0))\, \delta((k + k_2)^2 - M^2).$$

Let us make a change of variable $q := k + k_2$, and introduce $P := k_1 + k_2$, noting for later purposes that $P^2 = 2(k_1 k_2)$. One obtains the integral:

$$\int d^4q\, \theta(\mp(P^0 - q^0))\, \delta((P - q)^2 - M^2) \frac{1}{(q - k_2)^2 - M^2 + i0}\, \theta(\mp q^0)\, \delta(q^2 - M^2). \tag{3.6}$$

It follows that $I_\mp(k_1, k_2) \propto \theta(\mp P^0)\, \theta(P^2 - 4M^2)$, and that $\operatorname{sgn} k_1^0 = \operatorname{sgn} k_2^0$ for $P^2 \geq 4M^2$.

Performing the q^0-integration and using the notation $E_q := \sqrt{|\boldsymbol{q}|^2 + M^2}$, we extract

$$I_\mp(k_1, k_2) = \frac{i}{(2\pi)^4}\, \theta(\mp P^0)\, \theta(P^2 - 4M^2) \times \int \frac{d^3q}{2E_q}\, \theta(\mp(P^0 - q^0))\, \delta((P - q)^2 - M^2) \left.\frac{1}{(q - k_2)^2 - M^2 + i0}\right|_{q^0 = \mp E_q}.$$

Since $P^2 > 0$, one may choose a particular Lorentz frame such that

$$P = (P^0, \boldsymbol{0}); \quad \text{hence}$$
$$\boldsymbol{k}_1 = -\boldsymbol{k}_2, \quad k_1^0 = \mp|\boldsymbol{k}_1| = \mp|\boldsymbol{k}_2| = k_2^0 = \frac{1}{2} P^0. \tag{3.7}$$

Taking into account $q^2 = M^2$, we observe that $(P - q)^2 - M^2 = 2P^0(\frac{1}{2} P^0 - q^0)$, which yields

$$\delta((P - q)^2 - M^2) = \frac{\delta(q^0 - \frac{1}{2} P^0)}{2|P^0|} = \frac{\delta(E_q - \frac{1}{2}|P^0|)}{2|P^0|},$$

by using $q^0 = \mp E_q$. For later aims, we point out that in the chosen frame this distribution implies $q^0 = k_2^0$; hence

$$kP = (q - k_2)P = (q^0 - k_2^0)P^0 = 0. \tag{3.8}$$

[5] This point is made in [30, Sect. 6.4].

[6] Compare the discussion after [29, Eq. (3.8.24)].

From $\mp q^0 = \mp\frac{1}{2} P^0$ comes $\mp(P^0 - q^0) = \mp\frac{1}{2} P^0 > 0$. Therefore the factor $\theta(\mp(P^0 - q^0))$ is redundant. Changing the integration variables,

$$\int d^3q \cdots = \int_M^\infty dE_q\, E_q \sqrt{E_q^2 - M^2} \int d\Omega_q \cdots,$$

the E_q-integration can trivially be done, and we are left with:

$$I_\mp(k_1, k_2) = i\, \theta(\mp P^0)\, \theta(P^2 - 4M^2) \frac{\sqrt{(P^0)^2 - 4M^2}}{(2\pi)^4\, 8|P^0|} \int \left.\frac{d\Omega_q}{(q - k_2)^2 - M^2 + i0}\right|_{q^0 = P^0/2}. \tag{3.9}$$

Let α be the angle between $\boldsymbol{k}_2$ and $\boldsymbol{q}$, and let $z := \cos\alpha$. Due to $q^2 = M^2$, $k_2^2 = 0$, $|\boldsymbol{q}| = \sqrt{E_q^2 - M^2} = \frac{1}{2}\sqrt{P_0^2 - 4M^2}$ and relations (3.7) and (3.8), we obtain

$$(q - k_2)^2 - M^2 = -2(k_2 q) = -2(k_2^0 q^0 - |\boldsymbol{q}| \cdot |\boldsymbol{k}_2|\, z) = \tfrac{a}{2}(-a + bz), \tag{3.10}$$

where

$$a := |P^0| > 0, \quad 0 \leq b := \sqrt{(P^0)^2 - 4M^2} < a.$$

We point out that $(-a + bz) < 0$ for all $z \in [-1, 1]$: there is no infrared problem in our triangle graph. The remaining Ω_q-integral can be easily computed:

$$\frac{4\pi}{a} \int_{-1}^1 \frac{dz}{-a + bz} = \frac{4\pi}{|P^0| \sqrt{(P^0)^2 - 4M^2}} \log \frac{(P^0)^2 - |P^0| \sqrt{(P^0)^2 - 4M^2} - 2M^2}{2M^2}. \tag{3.11}$$

To obtain the result in a generic Lorentz frame, replace $(P^0)^2$ by $s := P^2 = 2(k_1 k_2)$, so

$$I_\mp(k_1, k_2) = \frac{i\theta(\mp P^0)\, \theta(s - 4M^2)}{4(2\pi)^3\, s} \log\left[\frac{s - \sqrt{s(s - 4M^2)}}{2M^2} - 1\right] =: \theta(\mp P^0)\, \theta(s - 4M^2) F(s). \tag{3.12}$$

The result for $J_\pm(k_1, k_2)$ can be read off from (3.9) by omitting the Feynman propagator $i(2\pi)^{-2}((q - k_2)^2 - M^2 + i0)^{-1}$. One obtains for the contribution of the J-integrals:

$$J_\pm(k_1, k_2) = \frac{1}{8\pi}\, \theta(\pm P^0)\, \theta(s - 4M^2) \sqrt{1 - 4M^2/s}\,.$$

Vector integrals $I_\mp^\mu$.

For the same reasons as for the scalar integral, it must hold that $I_\mp^\mu(k_1, k_2) \propto \theta(\mp P^0)\, \theta(s - 4M^2)$. From Lorentz covariance and $I_\pm^\mu(k_1, k_2) = -I_\pm^\mu(k_2, k_1)$ it follows

$$I_\mp^\mu(k_1, k_2) = \theta(\mp P^0)\, \theta(s - 4M^2)\, (k_1^\mu - k_2^\mu)\, G(s)$$

for appropriate $G(s)$. An immediate consequence is $I^\mu P_\mu = 0$. To procure $G(s)$, compute

$$k_{2,\mu} I^\mu_\mp(k_1,k_2) = \frac{1}{2}\theta(\mp P^0)\,\theta(s-4M^2)\,s\,G(s) \\ = \bigl(-i/8\,(2\pi)^3\bigr)\,\theta(\mp P^0)\,\theta(s-4M^2)\,\sqrt{1-4M^2/s}$$

The second equality is obtained by comparing with the scalar integral: there is an extra factor $(k_2k) = (k_2q) = -a(-a+bz)/4$, where (3.10) is used. Then the Ω_q-integral becomes trivial. Thus we glean

$$G(s) = \frac{-i}{32\,\pi^3\,s}\sqrt{1-4M^2/s}\,. \tag{3.13}$$

Tensor integrals $I^{\mu\nu}_\mp$.

Proceeding analogously to the vector integrals, one argues that

$$I^{\mu\nu}_\mp(k_1,k_2) = \theta(\mp P^0)\,\theta(s-4M^2)\bigl[(k_1^\mu k_1^\nu + k_2^\mu k_2^\nu)\,A(s) \\ +(k_1^\mu k_2^\nu + k_2^\mu k_1^\nu)\,B(s) + g^{\mu\nu}\,C(s)\bigr].$$

We need three independent identities to compute $A(s)$, $B(s)$ and $C(s)$. A first one is:

$$I^{\mu\nu}_\mp k_{2\mu}k_{2\nu} = \theta(\mp P^0)\,\theta(s-4M^2)\,A(s)\,s^2/4 \\ = \theta(\mp P^0)\,\theta(s-4M^2)\,\frac{-i}{2^5\,(2\pi)^3}\,s\,\sqrt{1-4M^2/s}\,. \tag{3.14}$$

The second equality is obtained by a modification of the computation of the scalar integral: there is the extra factor $(k_2k)^2 = a^2(-a+bz)^2/16$. This yields $A(s) = G(s)/2$. A second identity is given by the *trace*. The result is again obtained by comparing with the computation of the scalar integral: there is an additional factor $k^2 = (q-k_2)^2 = M^2 - 2(k_2q) = M^2 - 2(kk_2)$, hence

$$I^\mu_{\mp,\mu} = \theta(\mp P^0)\,\theta(s-4M^2)\,(sB+4C) = M^2 I_\mp - 2k_{2,\mu}I^\mu_\mp.$$

A third identity following from (3.8) reads:

$$I^{\mu\nu}_\mp P_\nu = \theta(\mp P^0)\,\theta(s-4M^2)\,P^\mu\bigl((A+B)s/2 + C\bigr) = 0.$$

Pulling together these results, one arrives at

$$B(s) = -M^2F(s)/s \quad\text{and}\quad C(s) = M^2F(s)/2 - s\,G(s)/4.$$

•At this point we are able to show that the triangle plus fish-like parts constitute a gauge-invariant quantity. For that, insert the results already known for the integrals into (3.3), obtaining:

$$d^{\mu\nu}(k_1,k_2)\Big|_{k_1^2=0=k_2^2} = \frac{\mathrm{sgn}(P^0)\,\theta(s-4M^2)}{(2\pi)^2}\bigl[k_1^\mu k_2^\nu[4G(s) \\ -(1+4M^2/s)F(s)] \\ +2M^2g^{\mu\nu}F(s) - k_1^\nu k_2^\mu\frac{4M^2}{s}\,F(s)\bigr] \\ = \mathrm{sgn}(P^0)\,\theta(s-4M^2)\frac{4M^2}{(2\pi)^2}\,P^{\mu\nu}\,\frac{F(s)}{s}. \tag{3.15}$$

The $k_1^\mu k_2^\nu$-terms have been dropped in the last identity, due to $k_\mu A^\mu(-k) = 0$. The remainder is electromagnetically gauge-invariant. Introducing the dimensionless variable

$$\tilde\rho := \frac{s}{4M^2} = \frac{P^2}{4M^2},$$

keeping in mind formula (3.12), and on use of (B.5), equation (3.15) can be rewritten as

$$d^{\mu\nu}_{\mathrm{gi}}(k_1,k_2)\Big|_{k_1^2=0=k_2^2} := \frac{i\ \mathrm{sgn}(P^0)\,\theta(\tilde\rho-1)}{(2\pi)^5}\,P^{\mu\nu}\,b(\tilde\rho) \tag{3.16}$$

with

$$b(\tilde\rho) := \frac{1}{16\,M^2\,\tilde\rho^2}\,\log\bigl(2\tilde\rho - 2\sqrt{\tilde\rho(\tilde\rho-1)} - 1\bigr) \\ = -\frac{1}{16\,M^2\,\tilde\rho^2}\,\log\frac{1+\sqrt{1-\tilde\rho^{-1}}}{1-\sqrt{1-\tilde\rho^{-1}}},$$

where 'gi' stands for the gauge invariant part. The singular order of $d^{\mu\nu}_{\mathrm{gi}}\big|_{k_1^2=0=k_2^2}$ is $\omega = -2$ by power counting; whereas for the off-shell $d^{\mu\nu}(k_1,k_2)$ the value is $\omega = 0$.

3.2 Regularity of absorptive parts in momentum space

This subsection is devoted to prove essential regularity properties of the *off-shell* d-distribution, more precisely of $d^{\mu\nu}(k_1,k_2)$, for $(k_1,k_2) \in \mathcal{V} := \bigl(\overline{V}_+\setminus\{0\}\bigr)^{\times 2} \cup \bigl(\overline{V}_-\setminus\{0\}\bigr)^{\times 2}$. We look at the terms coming from (3.3) by means of (3.4). Introducing the new integration variable $q := -k + \frac{1}{2}(k_1 - k_2)$, the internal lines' momenta are

$$q_1 = q + \frac{1}{2}P, \quad q_2 = q - \frac{1}{2}P, \quad q_3 = q - \frac{1}{2}(k_1 - k_2), \tag{3.17}$$

and one sees that the considered terms are all of the type

$$H^{\mu\nu}(k_1,k_2) \\ := \int d^4q\,\bigl(\theta(q_1^0)\,\theta(-q_2^0) \\ -\theta(-q_1^0)\,\theta(q_2^0)\bigr)\,\delta(q_1^2-M^2)\,\delta(q_2^2-M^2)\,\frac{h^{\mu\nu}(k_1,k_2,q)}{M^2-q_3^2} \tag{3.18}$$

for $(k_1,k_2)\in\mathcal{V}_1 := \big(\overline{V}\setminus\{0\}\big)^{\times 2}$ with $\overline{V} := \overline{V}_+\cup\overline{V}_-$, and where $h^{\mu\nu}\colon\mathbb{R}^{4\times 3}\to\mathbb{C}$ is a polynomial of degree 2. We have used that for $(k_1,k_2)\in\mathcal{V}_1$ it holds true that

$$\int d^4q\,\big(\theta(q_1^0)\,\theta(-q_2^0)\\ -\theta(-q_1^0)\,\theta(q_2^0)\big)\,\delta(q_1^2-M^2)\,\delta(q_2^2-M^2)\,\delta(q_3^2-M^2)=0. \tag{3.19}$$

This last relation can be argued as follows:[7] the various θ- and δ-distributions yield the restrictions $(q_1,q_2)\in(H_M^+\times H_M^-)\cup(H_M^-\times H_M^+)$ and $q_3\in H_M^+\cup H_M^-$; taking moreover into account that $q_3=q_2+k_2$ and $q_3=q_1-k_1$, it ensues that the various restrictions on q_3 are not compatible.

The same identity implies that terms of the kind $T_1(x_{\pi 1})\,T_1(x_{\pi 2})\,T_1(x_{\pi 3})$ do not contribute to the third commutator in formula (3.1) for D_3 when $(k_1,k_2)\in\mathcal{V}_1$, for all permutations π: the *whole* contribution to $d^{\mu\nu}(k_1,k_2)|_{(k_1,k_2)\in\mathcal{V}_1}$ coming from this commutator is of the kind (3.18).

The contributions to $d^{\mu\nu}(k_1,k_2)|_{(k_1,k_2)\in\mathcal{V}}$ coming from the other two commutators in (3.1) are of the same form up to cyclic permutations $k_1\mapsto k_2\mapsto -(k_1+k_2)\mapsto k_1$ of the external momenta. Here we use that $(k_1,k_2)\in\mathcal{V}$ implies $(k_2,-k_1-k_2)\in\mathcal{V}_1$ and $(-k_1-k_2,k_1)\in\mathcal{V}_1$, hence we may apply the identity (3.19) also for the permuted momenta. However, note that the polynomials $h_j^{\mu\nu}$, $j=2,3$, belonging to these other two cuts are not obtained by cyclic permutations of the external momenta in the original polynomial $h_1^{\mu\nu}$, meant in (3.18). This is due to the difference between the higgs vertex and the photon vertices; in particular, these other two cuts contain no term giving rise to a fish-like diagram. Summing up, it holds that

$$d^{\mu\nu}(k_1,k_2)\big|_{(k_1,k_2)\in\mathcal{V}}\\ =H_1^{\mu\nu}(k_1,k_2)+H_2^{\mu\nu}(k_2,-k_1-k_2)+H_3^{\mu\nu}(-k_1-k_2,k_1), \tag{3.20}$$

for some $H_j^{\mu\nu}$ $(j=1,2,3)$ of the form (3.18), the pertinent polynomials $h_j^{\mu\nu}$ being of degree 2.

Lemma 1 *Let q_1,q_2,q_3 and $\mathcal{V}_1$ be defined as above in (3.17) and after (3.18), and let $H^{\mu\nu}:\mathbb{R}^{4\times 2}\to\mathbb{C}$ be given in terms of a generic polynomial $h^{\mu\nu}:\mathbb{R}^{4\times 3}\to\mathbb{C}$ of degree $\zeta\in\mathbb{N}_0$, as in (3.18). Then for all $(k_1,k_2)\in\mathcal{V}_1$ and for some $C>0$ the function $H^{\mu\nu}$ is* ***continuous*** *in the region $\mathcal{V}_1$, and can be bounded as follows:*

$$|H^{\mu\nu}(k_1,k_2)|\\ \le C\,\frac{(1+|(k_1,k_2)|)^\zeta}{|(k_1k_2)|}\,\theta((k_1+k_2)^2-4M^2)\,\log((k_1+k_2)^2/M^2). \tag{3.21}$$

Note that $|(k_1k_2)|>0$ if $(k_1,k_2)\in\mathcal{V}_1$ and $(k_1+k_2)^2\ge 4M^2$.

Proof Let $P:=k_1+k_2$ and $k:=k_1-k_2$. We first observe, on the strength of

$$q_1^2-q_2^2=2(Pq),\quad q_1^2+q_2^2-2M^2=2(q^2+\tfrac14P^2-M^2)$$
and of $M^2-q^3=(M^2-q^2-\tfrac14P^2)+\tfrac14P^2-\tfrac14k^2+(kq)$ that
$$H^{\mu\nu}(k_1,k_2)\sim\operatorname{sgn}(P^0)\int d^4q\,\delta(q^2+\tfrac14P^2-M^2)\\ \delta((Pq))\,\frac{h^{\mu\nu}(k_1,k_2,q)}{P^2/4-k^2/4+(kq)},$$

omitting irrelevant prefactors. Since $q_1-q_2=P$ and $(q_1,q_2)\in(H_M^+\times H_M^-)\cup(H_M^-\times H_M^+)$, we know that $H^{\mu\nu}(k_1,k_2)$ vanishes for $P^2<4M^2$. Hence, to perform the integrals in q^0 and $|\boldsymbol{q}|$ using the Dirac deltas, we may work in the frame in which $\boldsymbol{P}=0$. There the δ-distributions yield $q^0=0$ and $|\boldsymbol{q}|=\sqrt{P_0^2/4-M^2}$.

With the notation $\hat p:=\boldsymbol{p}/|\boldsymbol{p}|$ for $p\in\{q,k\}$, it follows that

$$\tfrac14P^2-\tfrac14k^2+(kq)=(k_1k_2)\Big(1-(\hat q\,\hat k)\sqrt{1-4M^2/P^2}\,\frac{|P^0|\,|\boldsymbol{k}|}{2(k_1k_2)}\Big),$$

and one verifies that

$$0\le P_0^2\,|\boldsymbol{k}/2|^2=(k_1k_2)^2-k_1^2k_2^2, \tag{3.22}$$

with $k_1=(k_1^0,\boldsymbol{k}_1)$ and $k_2=(P^0-k_1^0,-\boldsymbol{k}_1)$. With the help of these results we obtain

$$(k_1k_2)\,H^{\mu\nu}(k_1,k_2)\sim\operatorname{sgn}(P^0)\,\theta(P^2-4M^2)\sqrt{1-4M^2/P^2}\\ \times\int_{\mathbb{S}^2}d\Omega(\hat q)\,\frac{h^{\mu\nu}\big(k_1,k_2,(0,\sqrt{P^2/4-M^2}\,\hat q)\big)}{1-(\hat q\,\hat k)\sqrt{(1-4M^2/P^2)\,(1-k_1^2k_2^2/(k_1k_2)^2)}}, \tag{3.23}$$

valid in the frame in which $\boldsymbol{P}=\boldsymbol{0}$. Let moreover $\mathcal{V}_1^M:=\{(k_1,k_2)\in\mathcal{V}_1:(k_1+k_2)^2\ge 4M^2\}$. We know that

$$4M^2/P^2\in(0,1]\quad\text{and}\quad k_1^2k_2^2/(k_1k_2)^2\in[0,1]\quad\text{for}\quad(k_1,k_2)\in\mathcal{V}_1^M;$$
$$\text{hence}\quad a:=\sqrt{(1-4M^2/P^2)\,(1-k_1^2k_2^2/(k_1k_2)^2)}\in[0,1).$$

In particular, the denominator in the integrand of (3.23) does not vanish for $(k_1,k_2)\in\mathcal{V}_1^M$. Since $\theta(P^2-4M^2)\sqrt{1-4M^2/P^2}$ is continuous, $H^{\mu\nu}$ *is continuous* on $\mathcal{V}_1$.

Observe now that for all $\hat q\in\mathbb{S}^2$ the inequality

$$\big|h^{\mu\nu}\big(k_1,k_2,(0,\sqrt{P^2/4-M^2}\hat q)\big)\big|\le\text{const}(1+|(k_1,k_2)|)^\zeta$$

holds, with $|(k_1,k_2)|^2:=\sum_{j=0}^3(k_{1j}^2+k_{2j}^2)$.

Setting $z:=(\hat q\hat k)$, the remaining integral is of the type

$$\int_{-1}^1\frac{dz}{1-az}=\frac1a\log\Big(\frac{1+a}{1-a}\Big)\le 2(1-\log(1-a)),$$

[7] We borrow the standard notation for the mass shell: $H_M^\pm:=\{p\in\mathbb{R}^4:p^2=M^2,\ \pm p_0>0\}$.

valid for $a \in [0,1)$. Using that $a \leq \sqrt{1-4M^2/P^2} \leq (1-2M^2/P^2)$ and monotonicity of the logarithm, we see that

$$-\log(1-a) = \log\frac{1}{1-a} \leq \log\frac{P^2}{2M^2}\,.$$

Putting together the estimates, we end up with

$$\begin{aligned}&|(k_1k_2)\,H^{\mu\nu}(k_1,k_2)|\\&\quad\leq \text{const}\cdot\theta(P^2-4M^2)\,(1+|(k_1,k_2)|)^\zeta\left(1+\log(P^2/2M^2)\right),\end{aligned}\tag{3.24}$$

implying (3.21), since $1+\log(P^2/2M^2) < 2\,\log(P^2/M^2)$ for $P^2 \geq 4M^2$. □

The reader should keep in mind that $d^{\mu\nu}(k_1,k_2)$ is supported outside a certain neighbourhood of the origin on momentum space – have a look back at Eq. (3.23).

Corollary 2 *The off-shell d-distribution $d^{\mu\nu}(k_1,k_2)$ given in (3.20) is continuous on $\mathcal{V}$ and fulfills the bound:*

$$\begin{aligned}&|d^{\mu\nu}(k_1,k_2)| \leq \text{const}\,\frac{(1+|(k_1,k_2)|)^{\omega+2}}{|(k_1k_2)|}\\&\quad\log(2+|(k_1,k_2)|/M)\ \textit{for all}\ (k_1,k_2)\in\mathcal{V}.\end{aligned}\tag{3.25}$$

Proof Continuity follows immediately from Lemma 1. For the bound (3.25) we have substituted $\omega+2 \equiv \omega(d)+2$ for ζ of the Lemma, since the singular order of $H_j^{\mu\nu}$ $(j=1,2,3)$ is $\zeta-2$ by power counting in (3.18). In addition, for $H_1^{\mu\nu}(k_1,k_2)$ we have used that $(k_1+k_2)^2 \leq 4\,|(k_1,k_2)|^2$, and in order to omit the θ-distribution we have replaced $\log(2\,|(k_1,k_2)|/M)$ by $2\log(2+|(k_1,k_2)|/M)$. One deals analogously with $H_2^{\mu\nu}(k_2,-k_1-k_2)$ and $H_3^{\mu\nu}(-k_1-k_2,k_1)$. □

3.3 Distribution splitting by the dispersion integral for null momenta

Recall that for $(k_1,k_2) \in V_\eta \times V_\eta$ the advanced part $a^{\mu\nu}$ of $d^{\mu\nu}$ can be computed by the dispersion integral (C.14). Using the regularity properties of $d^{\mu\nu}$ given in Corollary 2, we finally aim to show that the limit $k_1^2 \downarrow 0, k_2^2 \downarrow 0$ in (C.14) commutes with integration; that is, the dispersion integral is also valid for $k_1^2 = 0 = k_2^2$. To formulate the assertion, let

$$\begin{aligned}\mathcal{K} := \{\,&(k_1,k_2)\in(\mathbb{R}^4)^{\times 2} : k_1^2, k_2^2 < 4M^2,\\&(k_1+k_2)^2 < 4M^2,\ (k_1k_2)\neq 0\,\}.\end{aligned}\tag{3.26}$$

Bearing in mind the factors $\theta(q^2-4M^2)$ for $q \in \{k_1,k_2,k_1+k_2\}$ appearing in each term of $d^{\mu\nu}(k_1,k_2)$, we see that for $(k_1,k_2)\in(V_\eta\times V_\eta)\cap\mathcal{K}$, formula (C.14) can be rewritten as:

$$a^{\mu\nu}(k_1,k_2) = \frac{i\eta}{2\pi}\int_{|t|\geq t_{\min}} dt\,\frac{d^{\mu\nu}(tk_1,tk_2)}{t^{\omega+1}(1-t)},\tag{3.27}$$

for some $t_{\min} > 1$ depending on k_1, k_2. Now, as discussed in Sect. 1, one knows $a^{\mu\nu}(k_1,k_2)$ to be analytic on the region $\mathcal{K}$. The Lebesgue dominated convergence theorem [31, Th. 4.6.3] with the bound (3.25) allows us conclude that (3.27) is a valid identity for $(k_1,k_2)\in\mathcal{V}\cap\mathcal{K}$. Indeed, introducing the set of limit points

$$\begin{aligned}\mathcal{M} &:= \mathcal{V}\cap\mathcal{K}\cap\{(k_1,k_2)\in\mathbb{R}^8 : k_i^2 = 0\}\\&= \{(k_1,k_2)\in\mathbb{R}^8 : k_i^2 = 0,\ 0 < (k_1+k_2)^2 < 4M^2\},\end{aligned}$$

it is enough to observe that for any $(\tilde{k}_1,\tilde{k}_2)\in\mathcal{M}$ – implying $(\tilde{k}_1\,\tilde{k}_2) > 0$ and $\tilde{k}_1^0\tilde{k}_2^0 > 0$ – there is a neighbourhood $\mathcal{U}_{(\tilde{k}_1,\tilde{k}_2)}$ such that

$$\begin{aligned}&\left|\theta(|t|-t_{\min})\,\frac{d(tk_1,tk_2)}{t^{\omega+1}(1-t)}\right|\\&\quad\leq \text{const}\cdot\frac{\theta(|t|-t_{\min})}{|t(1-t)|}\,\frac{\left(1+|(k_1,k_2)|\right)^{\omega+2}}{|(k_1\,k_2)|}\\&\quad\log(2+|t|\,|(k_1,k_2)|/M)\\&\quad\leq C\,\frac{\theta(|t|-t_1)}{|t(1-t)|}\,\log(2+C_1\,|t|),\end{aligned}$$

for all $(k_1,k_2)\in(V_\eta\times V_\eta)\cap\mathcal{K}\cap\mathcal{U}_{(\tilde{k}_1,\tilde{k}_2)}$, for some $C, C_1 > 0$ and some $t_1 > 1$ independent of (k_1,k_2). The function on the right hand side is absolutely integrable in t – here we see the reason for the condition $(k_1k_2)\neq 0$ in (3.26).

3.4 Normalization of the scalar model by distribution splitting

We must finally compute the gauge invariant part $t_{\text{gi}}^{\mu\nu}(k_1,k_2)$ for momenta lying on the set $\mathcal{M}$. Considering the formula $T_3 = A_3 - A_3'$ and reckoning that $a'^{\mu\nu}(k_1,k_2)|_{k_1^2=0=k_2^2}$ contains the factor $\theta(P^2-4M^2)$ where $P := k_1+k_2$, we see that on $\mathcal{M}$ its contribution vanishes, that is $t^{\mu\nu} = a^{\mu\nu}$ there.

The upshot of the preceding two subsections is that we may compute valid terms of the central solution $a^{\mu\nu}|_{\mathcal{M}} \equiv a^{c\,\mu\nu}|_{\mathcal{M}}$ by inserting the on-shell amplitude (3.15) into the dispersion integral, with ω the singular order *of the off-shell* $d^{\mu\nu}$, equal to 0 in the present case.

Looking at (3.15), observe that a $k_r^\mu k_s^\nu$- or $g^{\mu\nu}$-term of $d^{\mu\nu}$ goes over to a $k_r^\mu k_s^\nu$- or $g^{\mu\nu}$-term of $a^{\mu\nu}$, respectively. Therefore, such factors may be taken out of the dispersion integral. Since moreover $P^{\mu\nu}(tk) = t^2\,P^{\mu\nu}(k)$, we see that the gauge invariant part $a_{\text{gi}}^{\mu\nu}$ can be obtained by inserting just the gauge invariant part $d_{\text{gi}}^{\mu\nu}$ in (3.16) into the dispersion integral. The latter is of the form (C.15). So we may use the version (C.16) of the dispersion integral.

Lastly, $t_{\text{gi}}^{\mu\nu}|_{\mathcal{M}} = a_{\text{gi}}^{\mu\nu}|_{\mathcal{M}}$ is obtained from (C.16) by setting $\omega = 0$ and substituting there $b(u\tilde{\rho})$ as given in (3.16) for $f\left(u(k_1^2,k_2^2,(k_1+k_2)^2)\right)$ – in our case only $(k_1+k_2)^2 = s$ is present. Allowing for the dilation factor in $P^{\mu\nu}$ this leads, for $(k_1, k_2) \in \mathcal{M}$, to

$$
\begin{aligned}
& t^{\mu\nu}_{\mathrm{gi}}(k_1, k_2) \\
&\quad = -\frac{P^{\mu\nu}}{(2\pi)^6} \int_{\tilde\rho^{-1}}^{\infty} du\, \frac{u\, b(u\tilde\rho)}{u(1-u)} \\
&\quad = \frac{P^{\mu\nu}}{16M^2\,(2\pi)^6} \int_{\tilde\rho^{-1}}^{\infty} \frac{du}{\tilde\rho^2 u^2 (1-u)} \log \frac{1+\sqrt{1-u^{-1}\tilde\rho^{-1}}}{1-\sqrt{1-u^{-1}\tilde\rho^{-1}}} \\
&\quad = -\frac{P^{\mu\nu}}{8\,(2\pi)^6} \frac{J_2(\tilde\rho)}{M^2}, \qquad (3.28)
\end{aligned}
$$

where

$$
2\, J_2(\tilde\rho) := \int_1^\infty dv\, \frac{1}{(v-\tilde\rho)v^2} \log \frac{1+\sqrt{1-v^{-1}}}{1-\sqrt{1-v^{-1}}},
$$

after the change of integration variable $v := u\tilde\rho$. Integrals like J_2 have been computed in [19]. From Appendix C of that reference:

$$
\begin{aligned}
J_1(\tilde\rho, a) &:= \frac{1}{2} \int_1^\infty dv\, \frac{1}{(v-\tilde\rho)(v-a)} \log \frac{1+\sqrt{1-v^{-1}}}{1-\sqrt{1-v^{-1}}} \\
&= \frac{f(\tilde\rho) - f(a)}{\tilde\rho - a} \qquad (3.29)
\end{aligned}
$$

for $0 \le \tilde\rho \le 1$, $0 \le a \le 1$, where f is the distribution (B.3). We infer that

$$
\begin{aligned}
J_2(\tilde\rho) &= \frac{\partial}{\partial a}\bigg|_{a=0} J_1(\tilde\rho, a) \\
&= \frac{f(\tilde\rho)}{\tilde\rho^2} - \frac{1}{\tilde\rho} \quad \text{for} \quad 0 \le \tilde\rho \le 1, \qquad (3.30)
\end{aligned}
$$

by bringing in the values $f(0) = 0$ and $f'(0) = 1$, which can be read off from (B.6).

Summing up, the final result reads, as expected:

$$
\begin{aligned}
t^{\mu\nu}_{\mathrm{gi}}(k_1, k_2) &= \frac{P^{\mu\nu}}{8\,(2\pi)^6} \frac{1}{M^2} \left(\frac{1}{\tilde\rho} - \frac{f(\tilde\rho)}{\tilde\rho^2} \right) \\
&= \frac{P^{\mu\nu}}{8\,(2\pi)^6} \frac{F_0(\tilde\rho)}{M^2} \quad \text{for } (k_1, k_2) \in \mathcal{M}, \qquad (3.31)
\end{aligned}
$$

where F_0 was given in (1.3). We conjecture that this formula holds true for all (k_1, k_2) satisfying $k_1^2 = 0 = k_2^2$ and $(k_1 + k_2)^2 > 0$.

The reader should remember that (3.31) stands in principle for just a member of a solution set. Since $\omega = 0$, the general Lorentz-invariant Epstein–Glaser solution is obtained by adding to expression (3.31) a term of the type $Cg^{\mu\nu}$ with $C \in \mathbb{C}$ arbitrary. But such a term with $C \neq 0$ would violate EGI. Therefore we regard the above result as unique.

Recovering formula (3.5) and the factor $4ge^2M$, one ends up with

$$
\begin{aligned}
& \int dx_1\, dx_2\, dx_3\, T_3(x_1, x_2, x_3) \\
&\quad = \frac{g\alpha}{(2\pi)^3 M} \int dk_1\, dk_2\, h(k_1 + k_2) A^\mu(-k_1) A^\rho(-k_2)\, P_{\mu\nu}\, F_0(\tilde\rho),
\end{aligned}
$$

which, on substituting ρ for $\tilde\rho$, that is, m_h^2 for $s \equiv (k_1 + k_2)^2$, agrees with the literature [25].

Remark 1 In the occasion an (unsubtracted) dispersion integral applied to $b(u)$, performed in [16, Eq. (3.2)], leads to the *same* integral (3.28) and so the same correct result. As the next section shows, this does not hold for the higgs to diphoton decay via EW vector bosons.

4 Higgs to diphoton decay via EW vector bosons

4.1 Derivation of the quartic $AAWW^\dagger$-vertex in the unitary gauge

The amplitude in question in this paper describes an EW decay process at third order in the coupling constant. Its structure is given by the *cubic* vertices in the first TOP T_1 – that is the *sole* "empirical" input. Here in going from T_1 to T_2 we *derive* the $AAWW^\dagger$-vertex which contributes by a "fish-like" diagram to the amplitude to be computed, see Fig. 1.

The general idea is to examine the propagator which is to become the internal line linking the di-photon in the one-loop, three-vertex graph, and to obtain the one-loop, two-vertex graph from a modification of that propagator, demanded by EGI – by which here we precisely understand invariance of the $\mathbb{S}$-matrix under the variations $A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x)$: interaction dictates symmetry. The method is similar to the derivation of the $AA\varphi\varphi^\dagger$ "seagull" vertex from the cubic coupling in scalar QED, first performed in this way in [28].

The concept works on configuration space, as follows. Recall the pertinent Hermitian vertex – see for instance [32, Sect. 7.2.2], explicitly referring to the unitary gauge. With $G_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu$, one has:

$$
T_1(x_1) = ie[(W^\mu G^\dagger_{\mu\nu} - W^{\dagger\mu} G_{\mu\nu}) A^\nu - W^\mu W^\dagger_\nu F_\mu{}^\nu](x_1). \qquad (4.1)
$$

All indicated operator products are Wick products. We copy a second vertex similar to (4.1):

$$
T_1(x_2) = ie[(W^\rho G^\dagger_{\rho\lambda} - W^{\dagger\rho} G_{\rho\lambda}) A^\lambda - W^\rho W^\dagger_\lambda F_\rho{}^\lambda](x_2), \qquad (4.2)
$$

and proceed to make the contractions, according to the BEG method to construct (the relevant sector of) the second-order $T_2(x_1,x_2)$ out of $T_1(x_1), T_1(x_2)$. Now $T_2(x_1,x_2) = T_1(x_1)T_1(x_2)$ for x_1 not to the past of x_2, and $T_2(x_1,x_2) = T_1(x_2)T_1(x_1)$ for x_2 not to the past of x_1.

In view of the triangle diagram given in Fig. 1, we are only interested in contractions yielding a connected tree diagram with the photons uncontracted. Once they are made, we analyze the terms for which the resulting distributions are not uniquely defined on the diagonal $x_1 = x_2$. The ambiguities in this extension are eliminated by EGI as sole selection criterion.

From the last pair of terms in (4.1) and (4.2), and with $\Delta^{\mu\beta}$ standing for the Feynman–Proca propagator (A.8) for the W-bosons, there comes:

$$\begin{aligned}
&-e^2\,\big(F_{\mu\nu}(x_1)W^{\dagger\nu}(x_1)\langle\!\langle \mathrm{T}\, W^{\mu}(x_1)\, W^{\dagger\beta}(x_2)\rangle\!\rangle W^{\alpha}(x_2)F_{\alpha\beta}(x_2)\\
&\quad + F_{\mu\nu}(x_1)W^{\mu}(x_1)\langle\!\langle \mathrm{T}\, W^{\dagger\nu}(x_1)\, W^{\alpha}(x_2)\rangle\!\rangle W^{\dagger\beta}(x_2)F_{\alpha\beta}(x_2)\big)\\
&\quad = -e^2\, F_{\mu\nu}(x_1)F_{\alpha\beta}(x_2)W^{\dagger\nu}(x_1)\,\Delta^{\mu\beta}(x_1-x_2)W^{\alpha}(x_2)\\
&\quad + (x_1 \leftrightarrow x_2),\ \text{ denoted } T_2^D(x_1,x_2) \text{ for later use;}
\end{aligned}\tag{4.3}$$

where explicitly $\Delta^{\mu}_{\beta} = -\big(g^{\mu}_{\beta} + \partial^{\mu}\partial_{\beta}/M^2\big)\Delta^F$ with Δ^F the scalar Feynman propagator (A.2). Electromagnetic gauge variations obviously do not affect (4.3).

By pairing the last term in (4.2) with A-type terms in (4.1) and the last term in (4.1) with A-type terms in (4.2) we obtain eight terms, that we organize as follows:

$$\begin{aligned}
&e^2\, A^{\mu}(x_1)F_{\rho\beta}(x_2)\big[G^{\dagger}_{\alpha\mu}(x_1)\,\Delta^{\alpha\beta}(x_1-x_2)\,W^{\rho}(x_2)\\
&\quad - G_{\alpha\mu}(x_1)\,\Delta^{\alpha\rho}(x_1-x_2)\,W^{\beta\dagger}(x_2) +\\
&W^{\alpha}(x_1)(-g^{\rho}_{\mu}\partial_{\alpha} + g^{\rho}_{\alpha}\partial_{\mu})\Delta^F(x_1-x_2)W^{\beta\dagger}(x_2)\\
&\quad - W^{\alpha\dagger}(x_1)(-g^{\beta}_{\mu}\partial_{\alpha} + g^{\beta}_{\alpha}\partial_{\mu})\Delta^F(x_1-x_2)W^{\rho}(x_2)\big]\\
&\quad + (x_1 \leftrightarrow x_2) =: T_2^B(x_1,x_2) + T_2^C(x_1,x_2),
\end{aligned}\tag{4.4}$$

where T_2^C denotes the $(x_1 \leftrightarrow x_2)$-term. As noted in Appendix A, third-order derivatives of Δ^F cancel here of their own accord. In order to verify EGI in the above expression, note first that it can only be violated on the diagonal $x_1 = x_2$, that is by contact terms – see the discussion on this in Sect. C.2. Therefore in computing the divergence of expressions like the above one selects only such terms that under ∂_1^{μ} produce local expressions. For instance, the third term in (4.4) brings forth:

$$\begin{aligned}
&W^{\alpha}(x_1)\, g^{\rho}_{\alpha}\,\partial_1^{\mu}\partial_{\mu}\,\Delta^F(x_1-x_2)\,W^{\beta\,\dagger}(x_2)\\
&\quad = -i\,W^{\rho}(x_1)W^{\dagger\beta}(x_1)\,\delta(x_1-x_2) + \cdots
\end{aligned}$$

where the dots stand for a term $\sim M^2 W\Delta^F W^{\dagger}$; but it is not hard to see that it is cancelled by a similar one coming from the next term. In conclusion: T_2^B is individually electromagnetically gauge-invariant, and T_2^D, T_2^B, T_2^C calculated up to now have no bearing on the generation of the $AAWW^{\dagger}$-vertex in constructing T_2.

Still, we are left with the most interesting contractions to calculate. By pairing the two first terms in (4.1) with the two first in (4.2) we get:

$$\begin{aligned}
&-e^2\, A^{\mu}(x_1)A^{\rho}(x_2)\big[-W^{\alpha}(x_1)\widetilde{D}_{\alpha\mu\;\beta\rho}(x_1-x_2)W^{\beta\dagger}(x_2)\\
&\quad - G^{\dagger}_{\alpha\mu}(x_1)\,\Delta^{\alpha\beta}(x_1-x_2)\,G_{\beta\rho}(x_2) +\\
&G^{\dagger}_{\alpha\mu}(x_1)(g^{\alpha}_{\rho}\partial_{\beta} - g^{\alpha}_{\beta}\partial_{\rho})\Delta^F(x_1-x_2)W^{\beta}(x_2)\\
&\quad + W^{\alpha\dagger}(x_1)(-g^{\beta}_{\mu}\partial_{\alpha} + g^{\beta}_{\alpha}\partial_{\mu})\Delta^F(x_1-x_2)G_{\beta\rho}(x_2)\big]\\
&\quad + (x_1 \leftrightarrow x_2) =: T_2^A(x_1,x_2).
\end{aligned}\tag{4.5}$$

Outside the diagonal the distribution $\widetilde{D}_{\alpha\mu\;\beta\rho}$ necessarily coincides with $D_{\alpha\mu\;\beta\rho}$, defined in Eq. (A.9) as the propagator for the G-fields. Following the Epstein–Glaser method, we now look for the most general Lorentz-covariant extension of this distribution having the same scaling degree. The solution reads:

$$\widetilde{D}_{\alpha\mu,\beta\rho}(y) = D_{\alpha\mu,\beta\rho}(y) + i(a\, g_{\alpha\beta}g_{\mu\rho} - b\, g_{\alpha\rho}g_{\mu\beta})\,\delta(y)\tag{4.6}$$

with as yet unknown numbers $a, b \in \mathbb{C}$. Note that the second term in the above propagator generates a contact term eventually yielding the $AAWW^{\dagger}$-vertex

$$\begin{aligned}
T_F(x_1,x_2) &:= e^2\, A^{\mu}(x_1)A^{\rho}(x_2)W^{\alpha}(x_1)W^{\dagger\beta}(x_2)\, i(a\, g_{\alpha\beta}g_{\mu\rho}\\
&\quad - b\, g_{\alpha\rho}g_{\mu\beta})\,\delta(x_1-x_2).
\end{aligned}\tag{4.7}$$

A third Lorentz tensor might appear in the Ansatz (4.6), namely $g_{\alpha\mu}g_{\rho\beta}\,\delta(y)$. However, on insertion into (4.5), one obtains the same contribution as $g_{\alpha\rho}g_{\mu\beta}\,\delta(y)$, namely

$$e^2\, A^{\mu}(x_1)A^{\rho}(x_1)W_{\mu}(x_1)W^{\dagger}_{\rho}(x_1)\,\delta(x_1-x_2).$$

Since EGI of T_2 can be violated only by local terms, it will suffice to select the terms which are $\sim (\partial)\delta(x_1-x_2)$ after taking the divergence ∂_1^{μ}. Those can be of two types:

(a) Either the contact term already contains a $\delta(x_1-x_2)$; or
(b) Such terms are generated when the $\partial^{\mu}_{x_1}$ acts on $\partial_{\mu}\Delta^F(x_1-x_2)$ or on $\partial\partial_{\mu}\Delta^F(x_1-x_2)$, due to $(\Box + m^2)\Delta^F = -i\delta$.

From the $W(x_1)\,W^{\dagger}(x_2)$ part in (4.5), we find:

- The type (a) contributions:

$$
\begin{aligned}
& i\,\partial^\mu W^\alpha(x_1)\,(-a\,g_{\alpha\beta}g_{\mu\rho} + b\,g_{\alpha\rho}g_{\mu\beta})\,\delta(x_1-x_2)\,W^{\beta\,\dagger}(x_2) \\
&\quad + i\,W^\alpha(x_1)\,\big((-a\,g_{\alpha\beta}\partial_\rho + b\,g_{\alpha\rho}\partial_\beta)\delta\big)(x_1-x_2)\,W^{\beta\,\dagger}(x_2) \\
&= i\big(-a\,\partial_\rho W^\alpha(x_1) W^\dagger_\alpha(x_1) \\
&\quad + b\,\partial_\beta W_\rho(x_1) W^{\beta\dagger}(x_1)\big)\delta(x_1-x_2) \\
&\quad - ia\,W^\alpha(x_1)W^\dagger_\alpha(x_2)\,\partial_\rho\delta(x_1-x_2) \\
&\quad + ib\,W_\rho(x_1)W^{\beta\dagger}(x_2)\,\partial_\beta\delta(x_1-x_2),
\end{aligned} \tag{4.8}
$$

and the type (b) contribution:

$$
\begin{aligned}
& i\,W^\alpha(x_1)\,\big((g_{\alpha\beta}\partial_\rho - g_{\alpha\rho}\partial_\beta)\delta\big)(x_1-x_2)\,W^{\beta\,\dagger}(x_2) \\
&= i\,W^\alpha(x_1)W^\dagger_\alpha(x_2)\,\partial_\rho\delta(x_1-x_2) \\
&\quad - i\,W_\rho(x_1)W^{\beta\dagger}(x_2)\,\partial_\beta\delta(x_1-x_2).
\end{aligned} \tag{4.9}
$$

- From the $W^\dagger(x_1)\,W(x_2)$-part of the exchange term, we obtain terms (4.8) and (4.9) with $W \leftrightarrow W^\dagger$ interchanged.
- From the $W^\dagger(x_1)\,G(x_2)$-part of the original term, we get the type (b) contribution:

$$
\begin{aligned}
& i\,W^{\alpha\dagger}(x_1)\,g^\beta_\alpha\,\delta(x_1-x_2)\,\big(-\partial_\beta W_\rho(x_2) + \partial_\rho W_\beta(x_2)\big) \\
&= i\big(-W^{\alpha\dagger}(x_1)\,\partial_\alpha W_\rho(x_1) + W^{\alpha\dagger}(x_1)\,\partial_\rho W_\alpha(x_1)\big) \\
&\quad \times \delta(x_1-x_2).
\end{aligned} \tag{4.10}
$$

- From the $W(x_1)\,G^\dagger(x_2)$-part of the exchange term, we obtain the term (4.10) with $W \leftrightarrow W^\dagger$ interchanged.

Summing up, the requirement is:

$$
\begin{aligned}
0 \overset{!}{=}\ & W^\alpha(x_1)W^\dagger_\alpha(x_2)\,\partial_\rho\delta(x_1-x_2)\,(1-a) \\
& + W_\rho(x_1)W^{\beta\dagger}(x_2)\,\partial_\beta\delta(x_1-x_2)\,(b-1) \\
& + \partial_\beta W_\rho(x_1)W^{\beta\dagger}(x_1)\,\delta(x_1-x_2)\,(b-1) \\
& + \partial_\rho W^\alpha(x_1)W^\dagger_\alpha(x_1)\,\delta(x_1-x_2)\,(1-a) \\
& + [W \leftrightarrow W^\dagger].
\end{aligned} \tag{4.11}
$$

Generally, for two free fields $B(x)$, $C(x)$ it holds that the three terms

$$
\partial B(x_1)\,C(x_1)\,\delta(x_1-x_2), \quad B(x_1)\,\partial C(x_1)\,\delta(x_1-x_2) \quad \text{and} \quad B(x_1)\,C(x_2)\,\partial\delta(x_1-x_2)
$$

are linearly independent. We conclude that condition (4.11) has a unique solution, namely $a = 1 = b$. *In fine*, the resulting contact term may be written as

$$
\widetilde{D}_{\alpha\mu,\beta\rho}(y) - D_{\alpha\mu,\beta\rho}(y) = \tfrac{i}{2}(2g_{\alpha\beta}g_{\mu\rho} - g_{\alpha\rho}g_{\mu\beta} - g_{\mu\alpha}g_{\rho\beta})\,\delta(y), \tag{4.12}
$$

which reproduces the seagull $e^2AAWW^\dagger$-vertex in the Feynman rules for the EW interaction.

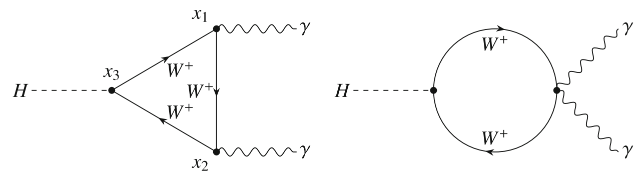

Fig. 1 Diagrams contributing to the amplitude to be computed

•*Recapitulation*: assembling T_3 from T_1 and T_2 by the Epstein–Glaser method, the above derived $AAWW^\dagger$-vertex (which is part of T_2) generates the fish-like diagram in Fig. 1. In the Feynman gauge, the $AAWW^\dagger$-vertex was already derived in [33] by the same method. The fact that we had to modify only $D^{\bullet\bullet}_{\bullet\bullet}$ (i.e., the G-propagator), and not $\Delta^{\bullet}_{\bullet}$ (i.e., the W-propagator), although here both are of the same singular order $\omega = 0$, is in accordance with the corresponding modification in the Feynman gauge or, more generally, in a R_ξ gauge: in such gauges the $AAWW^\dagger$ term *can be derived in the same way*, but only the G-propagator may be modified, because only this propagator has $\omega = 0$. The WG and GW-propagators have singular order $\omega = -1$ (see Eqs. (A.11) and (A.12)) and are not to be modified.

•*In conclusion*: almost the first thing to learn when working in the BEG scheme is that what is usually taken from a top-down, mysterious prevalence in particle physics of classical non-Abelian gauge theories, with their quartic, second-order couplings, is here *derived* by pure quantum field theory operations. To wit, the inductive construction of the time-ordered products respecting "gauge invariance", in the sense of [28,33–36], order by order in the couplings. In fact, **all** of the reductive Lie algebra structure of the Standard Model interactions, up to including at least one scalar particle, comes *automatically* in the BEG formalism from respecting first principles of quantum field theory in building the $\mathbb{S}$-matrix[8] – without invoking unobservable mechanisms. To go further into this subject here would take us too far afield. We have merely dealt with the example necessary for our purposes.

4.2 Computation of the causal distribution at third order

We wish to mention here that triangular graphs in Epstein–Glaser theory have been examined, for instance, in [39, Chap. 3.8] – the vertex correction in QED; in [40] – anoma-

[8] The derivation of the reductive Lie algebra structure was annunciated by Stora [37]. His kind of principled bottom-up approach has long been neglected in textbooks. A refreshing exception is [38, Problem 9.3].

lies; as well as in [41] and [23, Sect. 3.2].[9] Of course, here we shall have more terms and a forest of indices.

Proceeding similarly to Sect. 3, we perform the adiabatic limit already at this stage. Since all the propagators in the loop are massive, we may use (3.5); hence, we only have to compute the restricted distribution $d^{\mu\nu}(k_1,k_2)|_{k_1^2=0=k_2^2}$. Again, because of the kinematic constraints, only the first cut in formula (3.1) for D_3 counts (including the "fish graph" as derived in the previous subsection). Moreover, the discussions and general conclusions of Sects. 2, 3.2 and 3.3 hold here, and will be assumed without further ado.

The two WWA-vertices are given in (4.1) and (4.2), and the WWh-vertex reads

$$T_1(x_3) = gM\,h(x_3)W^\nu(x_3)W_\nu^\dagger(x_3).$$

The first step is to compute $R_3^{\prime A|B|C|D|F}(x_1,x_2;x_3) := T_2^{A|B|C|D|F}(x_1,x_2)\,T_1(x_3)\big|_\triangle$, where the lower-order TOPs $T_2^\bullet$ have been respectively given by lexicographical order in (4.5), (4.4), (4.3), and finally (4.7) together with (4.12). As it happens, their sum is symmetric in (x_1,x_2), so we just introduce a general factor 2 and need not mention exchange anymore.

We start with

$$\begin{aligned}
&R_3^{\prime D}(x_1,x_2;x_3)\\
&\quad= -2e^2 gM\,h(x_3)F^{\mu\nu}(x_1)F^{\alpha\beta}(x_2)\,r'^{D}_{\mu\nu,\alpha\beta}(y_1,y_2), \quad \text{where}\\
&r'^{D}_{\mu\nu,\alpha\beta}(y_1,y_2)\\
&\quad:= \Delta_\nu^{\gamma\,+}(y_1)\,\Delta_{\mu\beta}(y_1-y_2)\,\Delta_{\alpha\gamma}^{+}(y_2), \quad \text{and}\\
&y_k := x_k - x_3.
\end{aligned} \tag{4.13}$$

The "fish" diagram contribution reads:

$$\begin{aligned}
&R_3^{\prime F}(x_1,x_2;x_3) = -2e^2 gM\,h(x_3)A^\mu(x_1)A^\nu(x_2)\,r'^{F}_{\mu\nu}(y_1,y_2) \quad \text{with}\\
&\quad r'^{F}_{\mu\nu}(y_1,y_2) := i\,\delta(y_1-y_2)\big[-g_{\mu\nu}\,\Delta_\beta^{\gamma\,+}(y_1)\,\Delta_\gamma^{\beta\,+}(y_1)\\
&\qquad\qquad + \Delta_\nu^{\gamma\,+}(y_1)\,\Delta_{\mu\gamma}^{+}(y_1)\big].
\end{aligned} \tag{4.14}$$

Next we compute $R_3^{\prime B}$:

$$\begin{aligned}
&R_3^{\prime B}(x_1,x_2;x_3)\\
&\quad= -2e^2 gM\,h(x_3)A^\mu(x_1)F_{\nu\beta}(x_2)\,r'^{B\,\nu\beta}_{\ \mu}(y_1,y_2)\\
&\text{with}\quad r'^{B\,\nu\beta}_{\ \mu}(y_1,y_2)\\
&\quad:= (g_\mu^\gamma\partial_\alpha - g_\alpha^\gamma\partial_\mu)\Delta^+(y_1)\,\Delta^{\alpha\beta}(y_1-y_2)\,\Delta_\gamma^{\nu\,+}(y_2)\\
&\quad+ \Delta^{\alpha\gamma\,+}(y_1)\,(g_\mu^\nu\partial_\alpha - g_\alpha^\nu\partial_\mu)\Delta^F(y_1-y_2)\,\Delta_\gamma^{\beta\,+}(y_2).
\end{aligned} \tag{4.15}$$

The relation $R_3^{\prime C}(x_1,x_2;x_3) := R_3^{\prime B}(x_2,x_1;x_3)$ obviously holds. Finally, for $R_3^{\prime A}$, we collect

$$\begin{aligned}
&R_3^{\prime A}(x_1,x_2;x_3)\\
&\quad= -2e^2 gM\,h(x_3)A^\mu(x_1)A^\nu(x_2)\,r'^{A}_{\mu\nu}(y_1,y_2), \quad \text{with}\\
&r'^{A}_{\mu\nu}(y_1,y_2)\\
&\quad:= -(g_\mu^\gamma\partial_\alpha - g_\alpha^\gamma\partial_\mu)\Delta^+(y_1)\,\Delta^{\alpha\beta}(y_1-y_2)\,(g_{\nu\gamma}\partial_\beta\\
&\qquad - g_{\beta\gamma}\partial_\nu)\Delta^+(y_2)\\
&\qquad - \Delta^{\alpha\gamma\,+}(y_1)\,D_{\alpha\mu,\beta\nu}(y_1-y_2)\,\Delta_\gamma^{\beta\,+}(y_2)\\
&\qquad - (g_\mu^\gamma\partial_\alpha - g_\alpha^\gamma\partial_\mu)\Delta^+(y_1)\,(g_\nu^\alpha\partial_\beta\\
&\qquad - g_\beta^\alpha\partial_\nu)\Delta^F(y_1-y_2)\,\Delta_\gamma^{\beta\,+}(y_2)\\
&\qquad + \Delta^{\alpha\gamma\,+}(y_1)\,(g_{\mu\beta}\partial_\alpha - g_{\alpha\beta}\partial_\mu)\Delta^F(y_1-y_2)\\
&\qquad (g_{\gamma\nu}\partial^\beta - g_\gamma^\beta\partial_\nu)\Delta^+(y_2).
\end{aligned} \tag{4.16}$$

Next we express the resulting integrals by momentum space integrals. By using $F^{\mu\nu}(k) = -i\big(k^\mu A^\nu(k) - k^\nu A^\mu(k)\big)$ and $R_3' := R_3^{\prime A} + R_3^{\prime B} + R_3^{\prime C} + R_3^{\prime D} + R_3^{\prime F}$ we gather

$$\begin{aligned}
&\int dx_1\,dx_2\,dx_3\,R_3'(x_1,x_2;x_3)\\
&\quad= -2e^2 gM(2\pi)^2\int dk_1\,dk_2\,h(k_1+k_2)\\
&\quad A^\mu(-k_1)A^\nu(-k_2)\,r'_{\mu\nu}(k_1,k_2),
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&\text{where}\quad r'_{\mu\nu}(k_1,k_2) := r'^{A}_{\mu\nu}(k_1,k_2) + r'^{F}_{\mu\nu}(k_1,k_2)\\
&\quad+ \big[ik_2^\beta(r'^{B}_{\mu,\beta\nu}(k_1,k_2) - r'^{B}_{\mu,\nu\beta}(k_1,k_2)) + (k_1,\mu)\leftrightarrow(k_2,\nu)\big]\\
&\quad- k_1^\beta k_2^\alpha\big[r'^{D}_{\beta\mu,\alpha\nu}(k_1,k_2) - r'^{D}_{\mu\beta,\alpha\nu}(k_1,k_2)\\
&\quad- r'^{D}_{\beta\mu,\nu\alpha}(k_1,k_2) + r'^{D}_{\mu\beta,\nu\alpha}(k_1,k_2)\big].
\end{aligned} \tag{4.18}$$

To compute the (combinations of) Fourier transformed $r'^{\cdots}_{\cdots}$-distributions appearing on the right hand side of (4.18), we bring in $k_1^2 = 0 = k_2^2$, and omit all terms containing a factor $k_{1\mu}$ or $k_{2\nu}$: this is allowed in view of $k^\lambda A_\lambda(-k) = 0$. Similarly to the analogous computation for the scalar model in the previous section, let us work with the integration variable $q := k + k_2$, and introduce $P := k_1 + k_2$, hence $k_1 - k = P - q$ and $s := P^2 = 2k_1k_2$ and $Pk_1 = k_1k_2 = Pk_2$. Due to the factors $\Delta^+(q)$ and $\Delta^+(P-q)$, we may use the relations

$$\begin{aligned}
&q^2 = M^2, \quad (P-q)^2 = M^2, \quad \text{hence}\\
&\quad 2Pq = s = 2(k_1k_2), \quad \text{implying}\\
&(P-q)q = (k_1k_2) - M^2, \quad 0 = (P(q-k_2)) = (Pk),\\
&\quad (q-k_2)^2 = M^2 - 2(qk_2),
\end{aligned}$$

and $((P-q)\,k_1) = (qk_2)$, that is, $(qk_1) + (qk_2) = (k_1k_2)$. Remember also that we may replace $P_\mu \to k_{2\mu}$ and $P_\nu \to k_{1\nu}$.

[9] Some of these works are very instructive, in that they show that cherished invariance properties cannot always be preserved under distribution splitting.

To tally the fish diagram contribution, we introduce the integrals

$$J^{\{\cdot|\mu|\mu\nu\}}(P) := \int d^4q\,\{1|q^\mu|q^\mu q^\nu\}\,\Delta^+(P-q)\,\Delta^+(q). \tag{4.19}$$

These symbols $J^\bullet(P)$ generalize that of the scalar $J(P)$, defined in (3.4). We obtain:

$$\begin{aligned} r'^F_{\mu\nu}(k_1,k_2) = \frac{i}{(2\pi)^4}\biggl(& g_{\mu\nu}\Bigl(-2+\frac{s}{M^2}-\frac{s^2}{4M^4}\Bigr)J(P) \\ &-\frac{k_{2\mu}k_{1\nu}}{M^2}J(P) \\ &+\frac{1}{M^2}\Bigl(k_{2\mu}J_\nu(P)+k_{1\nu}J_\mu(P)\,\frac{s}{2M^2}\Bigr) \\ &-J_{\mu\nu}(P)\Bigl(\frac{1}{M^2}+\frac{s}{2M^4}\Bigr)\biggr). \end{aligned} \tag{4.20}$$

To figure out J^μ and $J^{\mu\nu}$ one first argues that

$$\begin{aligned} &J^\mu(P) = \theta(P_0)\,\theta(s-4M^2)\,P^\mu\,g(s), \\ &J^{\mu\nu}(P) = \theta(P_0)\,\theta(s-4M^2)\,(P^\mu P^\nu\,a(s)+g^{\mu\nu}c(s)), \end{aligned} \tag{4.21}$$

for appropriate $g(s)$, $a(s)$ and $c(s)$. The latter can be obtained from the identites

$$J^\mu(P)P_\mu = \frac{1}{2}sJ(P), \quad J^\mu_\mu(P) = M^2J(P), \quad J^{\mu\nu}(P)P_\mu P_\nu = \tfrac{1}{4}s^2J(P).$$

and from $J(P) = \theta(P^0)\,\theta(s-4M^2)\,4i\pi^2\,s\,G(s)$ – see (3.13). So we arrive at

$$g(s) = 2i\pi^2 s\,G(s); \quad a(s) = \frac{4i\pi^2}{3}(s-M^2)G(s); \quad c(s) = \frac{4i\pi^2}{3}(M^2-s/4)s\,G(s),$$

yielding

$$\begin{aligned} r'^F_{\mu\nu}(k_1,k_2) = \frac{1}{4\pi^2}\,\theta(P^0)\,\theta(s-4M^2)\,G(s)\biggl[\Bigl(&\frac{14}{3} -\frac{11}{6}\,\frac{s}{M^2}+\frac{5}{12}\,\frac{s^2}{M^4}\Bigr)g_{\mu\nu}(k_1k_2) \\ &+k_{1\nu}k_{2\mu}\Bigl(-\frac{1}{3}+\frac{2}{3}\,\frac{s}{M^2}-\frac{1}{12}\,\frac{s^2}{M^4}\Bigr)\biggr]. \end{aligned} \tag{4.22}$$

Other new integrals. We introduce already all the new integrals required in this section. By means of $(q-k_2)^2-M^2 = -2(qk_2)$, implying $(qk_2)\,\Delta^F(q-k_2) = -i/8\pi^2$, some of the integrals to appear are calculated:

$$\begin{aligned} K^{\{\cdot|\mu|\mu\nu\}}(P) &:= \int d^4q\,\{1|q^\mu|q^\mu q^\nu\}\,(qk_2)\,\Delta^+(P-q) \\ &\quad \Delta^F(q-k_2)\,\Delta^+(q) = -\frac{i}{8\pi^2}\,J^{\{\cdot|\mu|\mu\nu\}}(P), \\ L(P) &:= \int d^4q\,(qk_2)^2\,\Delta^+(P-q) \\ &\quad \Delta^F(q-k_2)\,\Delta^+(q) = k_{2\mu}\,K^\mu. \end{aligned}$$

The following integrals will also be needed:

$$N^{\{\cdot|\mu|\mu\nu\}}(P) := \int d^4q\,\{1|q^\mu|q^\mu q^\nu\}\,\Delta^+(P-q)\Delta^F(q-k_2)\,\Delta^+(q).$$

By using $k_2^\nu = 0$, they are easily be expressed in terms of the integrals computed in Sect. 3.1:

$$N(P) = I(P), \qquad N^\mu(P) = I^\mu(P) + k_2^\mu I(P), \qquad N^{\mu\nu}(P) = I^{\mu\nu}(P) + k_2^\mu I^\nu(P),$$

Electromagnetic gauge invariance. The computation of the individual terms in (4.18) is lengthy, but straightforward. For the r'^B-terms we reap

$$\begin{aligned} &ik_2^\beta\bigl(r'^B_{\mu,\beta\nu}(k_1,k_2) - r'^B_{\mu,\nu\beta}(k_1,k_2)\bigr) \\ &= \frac{1}{(2\pi)^2}\biggl\{-2N(P)(g_{\mu\nu}(k_1k_2)-k_{1\nu}k_{2\mu}) \\ &\quad + g_{\mu\nu}(k_1k_2)\Bigl(2\frac{K(P)}{M^2}-\frac{L(P)}{M^4}\Bigr) - k_{1\nu}k_{2\mu}(k_1k_2)\,\frac{K(P)}{M^4} \\ &\quad + N_{\mu\nu}(P)\Bigl(2\frac{(k_1k_2)}{M^2}-\frac{(k_1k_2)^2}{M^4}\Bigr) \\ &\quad + k_{2\mu}\Bigl(N_\nu(P)\Bigl(-2\frac{(k_1k_2)}{M^2}+\frac{(k_1k_2)^2}{M^4}\Bigr)+K_\nu(P)\,\frac{(k_1k_2)}{M^4}\Bigr) \\ &\quad + k_{1\nu}\,K_\mu(P)\Bigl(-\frac{2}{M^2}+\frac{k_1k_2}{M^4}\Bigr)\biggr\}. \end{aligned}$$

Inserting the integrals calculated above, one reaches for $ik_2^\beta\bigl(r'^B_{\mu,\beta\nu}(k_1,k_2) - r'^B_{\mu,\nu\beta}(k_1,k_2)\bigr) + (k_1,\mu) \leftrightarrow (k_2,\nu)$ the following result:

$$\begin{aligned} \frac{\theta(P^0)\,\theta(s-4M^2)}{(2\pi)^2}\,(g_{\mu\nu}(k_1k_2)-k_{1\nu}k_{2\mu})\,\Bigl(&F(s)\Bigl(-2-\frac{s}{2M^2}\Bigr) \\ &+G(s)\,\frac{s}{M^2}\Bigr). \end{aligned} \tag{4.23}$$

Note that $ik_2^\beta\bigl(r'^B_{\mu,\beta\nu}(k_1,k_2) - r'^B_{\mu,\nu\beta}(k_1,k_2)\bigr)$ is individually gauge invariant. This reflects the known fact that T_2^B given by (4.4) is individually gauge invariant.

Analogously, since T_2^D (4.3) is trivially gauge invariant, we expect the combination

$$-k_1^\beta k_2^\alpha\big(r'^D_{\beta\mu,\alpha\nu}(k_1,k_2)-r'^D_{\mu\beta,\alpha\nu}(k_1,k_2) \\ -r'^D_{\beta\mu,\nu\alpha}(k_1,k_2)+r'^D_{\mu\beta,\nu\alpha}(k_1,k_2)\big),$$

to contain the factor $g_{\mu\nu}(k_1k_2)-k_{1\nu}k_{2\mu}$. Indeed, we obtain

$$\begin{aligned}&\frac{1}{(2\pi)^2}\bigg\{-2(g_{\mu\nu}(k_1k_2)-k_{1\nu}k_{2\mu})\,N(P)\\
&\quad+g_{\mu\nu}\bigg(-2\frac{L(P)}{M^2}+(k_1k_2)\Big(2\frac{K(P)}{M^2}-\frac{L(P)}{M^4}\Big)\bigg)\\
&\quad+k_{1\nu}k_{2\mu}\,(k_1k_2)\,\frac{K(P)}{M^4}+N_{\mu\nu}\bigg(2\frac{(k_1k_2)}{M^2}+\frac{(k_1k_2)^2}{M^4}\bigg)\\
&\quad+k_{1\nu}\bigg(-2\frac{K_{1\mu}(P)}{M^2}-(k_1k_2)\,\frac{K_\mu(P)}{M^4}\bigg)\\
&\quad+k_{2\mu}\bigg(2\frac{K_\nu(P)}{M^2}-2\frac{(k_1k_2)}{M^2}\,N_\nu(P)\\
&\quad-\frac{(k_1k_2)^2}{M^4}\,N_\nu(P)+(k_1k_2)\,\frac{K_{1\nu}(P)}{M^4}\bigg)\bigg\}\\
&=\frac{\theta(P^0)\,\theta(s-4M^2)}{(2\pi)^2}\,(g_{\mu\nu}(k_1k_2)\\
&\quad-k_{1\nu}k_{2\mu})\bigg[\Big(-1+\frac{s}{4M^2}\Big)F(s)-\frac{s^2}{4M^4}\,G(s)\bigg].\end{aligned}\tag{4.24}$$

Finally, for r'^A we get

$$\begin{aligned}&r'^A_{\mu\nu}(k_1,k_2)\\
&=\frac{1}{(2\pi)^2}\bigg\{-2(g_{\mu\nu}(k_1k_2)-k_{1\nu}k_{2\mu})\,N(P)\\
&\quad+g_{\mu\nu}\bigg(\Big(2+\frac{2(k_1k_2)}{M^2}\Big)K(P)-\Big(\frac{2}{M^2}+\frac{(k_1k_2)}{M^4}\Big)L(P)\bigg)\\
&\quad+k_{1\nu}k_{2\mu}\Big(-\frac{(k_1k_2)}{M^4}\Big)K(P)+k_{1\nu}\Big(-\frac{2}{M^2}+\frac{(k_1k_2)}{M^4}\Big)K_\mu(P)\\
&\quad+k_{2\mu}\bigg(\Big(-12+2\frac{(k_1k_2)}{M^2}-\frac{(k_1k_2)^2}{M^4}\Big)N_\nu(P)+\Big(\frac{4}{M^2}\\
&\quad+\frac{(k_1k_2)}{M^4}\Big)K_\nu(P)\bigg)\\
&\quad+\Big(12-2\frac{k_1k_2}{M^2}+\frac{(k_1k_2)^2}{M^4}\Big)N_{\mu\nu}(P)\\
&\quad-\Big(\frac{2}{M^2}+2\frac{(k_1k_2)}{M^4}\Big)K_{\mu\nu}(P)\bigg\}.\end{aligned}$$

After insertion of the integrals, this is equal to

$$\begin{aligned}&\frac{\theta(P^0)\,\theta(s-4M^2)}{(2\pi)^2}\bigg\{(g_{\mu\nu}(k_1k_2)\\
&\quad-k_{1\nu}k_{2\mu})\,F(s)\Big(-3+12\,\frac{M^2}{s}+\frac{s}{4M^2}\Big)\\
&\quad+G(s)\bigg[g_{\mu\nu}(k_1k_2)\Big(-\frac{14}{3}+\frac{5}{6}\,\frac{s}{M^2}-\frac{1}{6}\,\frac{s^2}{M^4}\Big)\\
&\quad+k_{1\nu}k_{2\mu}\Big(\frac{1}{3}+\frac{s}{3M^2}-\frac{s^2}{6M^4}\Big)\bigg]\bigg\}.\end{aligned}$$

The sum $r'^A+r'^F$ is indeed gauge invariant:

$$\begin{aligned}&r'^A_{\mu\nu}(k_1,k_2)+r'^F_{\mu\nu}(k_1,k_2)\\
&=\frac{\theta(P^0)\,\theta(s-4M^2)}{(2\pi)^2}\,(g_{\mu\nu}(k_1k_2)-k_{1\nu}k_{2\mu})\\
&\quad\times\bigg(F(s)\Big(-3+12\,\frac{M^2}{s}+\frac{s}{4M^2}\Big)\\
&\quad+G(s)\Big(-\frac{s}{M^2}+\frac{s^2}{4M^4}\Big)\bigg),\end{aligned}$$

as expected from the outcome of the discussion in Sect. 4.1.

The $G(s)$-terms are seen to cancel out, and for the total r' we obtain the following gauge-invariant result:

$$\begin{aligned}r'_{\mu\nu}(k_1,k_2)&=\frac{\theta(P^0)\,\theta(s-4M^2)}{(2\pi)^2}\,(g_{\mu\nu}(k_1k_2)\\
&\quad-k_{1\nu}k_{2\mu})\,6F(s)\,(-1+2M^2/s).\end{aligned}\tag{4.25}$$

Like for the scalar model, $a'_{\mu\nu}(k_1,k_2)$ is obtained from $r'_{\mu\nu}(k_1,k_2)$ in (4.25) by replacing $\theta(P^0)$ by $\theta(-P^0)$, and hence $d_{\mu\nu}=a'_{\mu\nu}-r'_{\mu\nu}$ by replacing $\theta(P^0)$ by $-\operatorname{sgn}(P^0)$.

Hence $d^{\mu\nu}_{\text{gi}}(k_1,k_2)|_{k_1^2=0=k_2^2}$ is of the form

$$\begin{aligned}&d^{\mu\nu}_{\text{gi}}(k_1,k_2)\big|_{k_1^2=0=k_2^2}=P^{\mu\nu}(k_1,k_2)\,d_0(P),\quad\text{where}\\
&d_0(P):=\frac{i\,\operatorname{sgn}(P^0)\,\theta(\tilde\rho-1)}{(2\pi)^5}\,b_1(\tilde\rho),\\
&b_1(\tilde\rho):=-\frac{3}{16M^2\,\tilde\rho}\,\Big(\frac{1}{\tilde\rho}-2\Big)\,\log\frac{1+\sqrt{1-\tilde\rho^{-1}}}{1-\sqrt{1-\tilde\rho^{-1}}}\,.\end{aligned}\tag{4.26}$$

The factor 3 above was to be expected, since the Proca field has three components. The singular order of the on-shell distribution $d^{\mu\nu}_{\text{gi}}(k_1,k_2)|_{k_1^2=0=k_2^2}$ is clearly equal to zero. It is also easy to show that power counting rules imply that the singular order of the *off-shell* d-distribution [10] $d^{\mu\nu}$ satisfies the bounds $6\ge\omega:=\omega(d^{\mu\nu})\ge 2$.

[10] This notation is badly abused in this paper. But that is hardly avoidable.

4.3 Distribution splitting in EW theory

The off-shell $d^{\mu\nu}$-distribution for the diphoton decay of the higgs via EW vector bosons is also of the genre (3.20), the difference being that the pertinent polynomials $h_j^{\mu\nu}$ are of a higher degree, i.e., $\omega := \omega(d^{\mu\nu})$ has a greater value. Therefore, the distribution splitting method for null momenta developed in Sect. 2 does apply: $t_{\rm gi}^{\mu\nu}|_{\mathcal{M}} = a_{\rm gi}^{c\,\mu\nu}|_{\mathcal{M}}$ can be computed by inserting the light-cone restriction of $d_{\rm gi}^{\mu\nu}(k_1,k_2)$ into the dispersion integral (3.27). Again, this restricted d-distribution is of the genre (3.16) for the given b_1. Hence, the *central solution* compatible with EGI is obtained by:

$$
\begin{aligned}
t_{\rm gi}^{c\,\mu\nu}(k_1,k_2) &= P^{\mu\nu}(k_1,k_2)\, t_{\rm gi}^{c}(P), \quad \text{where} \\
t_{\rm gi}^{c}(P) &= \frac{i\eta}{2\pi}\int_{|t|\geq\sqrt{1/\tilde\rho}} \frac{t^2\, d_0(tP)\,dt}{(1-t)\,t^{\max\{\omega+1,\,0\}}} \\
&= -\frac{1}{(2\pi)^6}\int_{1/\tilde\rho}^{\infty} \frac{u\, b_1(u\tilde\rho)\,du}{u^{\max\{\lfloor\omega/2\rfloor+1,\,0\}}(1-u)},
\end{aligned}
\tag{4.27}
$$

for $(k_1,k_2)\in\mathcal{M}$. Let us first assume that $\omega=2$. Before computing, we recall that the *ambiguity* of the result for $t_{\rm gi}^{\mu\nu}$ will be given in that case by a polynomial of *second* degree in (k_1,k_2) containing the factor $P^{\mu\nu}$; that is, in the occasion, a constant multiple of $P^{\mu\nu}$ itself.

Besides a global factor $-3(16M^2(2\pi)^6)^{-1}$, in view of (4.26) and making the customary change of variable $v=u\tilde\rho$, to obtain $t_{\rm gi}^{c}(\tilde\rho)$ we ought to compute:

$$
\begin{aligned}
&\tilde\rho\int_1^\infty dv\,\bigg[\frac{\log\big(1+\sqrt{1-v^{-1}}\big)/\big(1-\sqrt{1-v^{-1}}\big)}{v^3(\tilde\rho-v)} \\
&\quad - \frac{2\log\big(1+\sqrt{1-v^{-1}}\big)/\big(1-\sqrt{1-v^{-1}}\big)}{v^2(\tilde\rho-v)}\bigg] \\
&= \int_1^\infty dv\,\bigg[\frac{\log\big(1+\sqrt{1-v^{-1}}\big)/\big(1-\sqrt{1-v^{-1}}\big)}{v^2(\tilde\rho-v)} \\
&\quad - \frac{2\log\big(1+\sqrt{1-v^{-1}}\big)/\big(1-\sqrt{1-v^{-1}}\big)}{v(\tilde\rho-v)}\bigg] \\
&\quad + \int_1^\infty dv\,\bigg[\frac{\log\big(1+\sqrt{1-v^{-1}}\big)/\big(1-\sqrt{1-v^{-1}}\big)}{v^3} \\
&\quad - \frac{2\log\big(1+\sqrt{1-v^{-1}}\big)/\big(1-\sqrt{1-v^{-1}}\big)}{v^2}\bigg].
\end{aligned}
$$

Notice that the last two integral terms just yield a number, equal to minus the sum of the values of the two previous ones at $\tilde\rho=0$. The first integral term is already known from the discussion in Sect. 3.4, and in all yields

$$
\frac{3}{8M^2(2\pi)^6}J_2(\tilde\rho) = \frac{-3}{8M^2(2\pi)^6}\left(\frac{1}{\tilde\rho}-\frac{f(\tilde\rho)}{\tilde\rho^2}\right).
$$

This is essentially $3F_0(\tilde\rho)$, that clearly contributes to the expected final result for $F_1(\tilde\rho)$ – recall (1.3). According to Eq. (3.29) – also in Sect. 3.4 – in the case $a=0$, the second integral term yields

$$
-\frac{2\cdot 3}{8M^2(2\pi)^6}J_1(\tilde\rho,0) = -\frac{6}{8M^2(2\pi)^6}\,\frac{f(\tilde\rho)}{\tilde\rho}.
$$

Including the last two integrals, we recover $t_{\rm gi}^{c}(\tilde\rho)\sim F_1(\tilde\rho)-7$. Taking into account the Epstein–Glaser ambiguity as limited by EGI, and bringing in constant prefactor appearing in (4.17), we end up with the amplitude

$$
-2e^2 gM(2\pi)^2\, t_{\rm gi}^{\mu\nu}(k_1,k_2) = \frac{e^2 g}{4(2\pi)^4 M}\,P^{\mu\nu}(k_1,k_2)\big(F_1(\tilde\rho)+C\big),
\tag{4.28}
$$

with C an arbitrary constant.

If $\omega>2$, the central solution within the unitary gauge is obtained by adding the following expression to the above obtained result, more precisely, to $-(8M^2(2\pi)^6)^{-1}\big(F_1(\tilde\rho)-7\big)$:

$$
\begin{aligned}
&-\frac{1}{(2\pi)^6}\int_{\tilde\rho^{-1}}^\infty du\,\frac{b_1(u\tilde\rho)}{1-u}\left(\frac{1}{u^{\lfloor\omega/2\rfloor}}-\frac{1}{u}\right) \\
&= -\frac{1}{(2\pi)^6}\int_{\tilde\rho^{-1}}^\infty du\, b_1(u\tilde\rho)\sum_{k=0}^{\lfloor\omega/2\rfloor-2}\frac{1}{u^{\lfloor\omega/2\rfloor-k}} \\
&= \sum_{k=0}^{\lfloor\omega/2\rfloor-2}\tilde c_k\,\tilde\rho^{\lfloor\omega/2\rfloor-1-k}; \quad \text{where the} \\
&\tilde c_k := -\frac{1}{2\pi}\int_1^\infty dv\,\frac{b_1(v)}{v^{\lfloor\omega/2\rfloor-k}}
\end{aligned}
\tag{4.29}
$$

do not depend on $\tilde\rho$; in the first step we have used the relation

$$
\frac{1}{u^{\lfloor\omega/2\rfloor}}-\frac{1}{u} = (1-u)\sum_{k=0}^{\lfloor\omega/2\rfloor-2}\frac{1}{u^{\lfloor\omega/2\rfloor-k}},
$$

and in the last step we have made again the substitution $v:=u\tilde\rho$. The point is that (4.29) is a polynomial in $\tilde\rho$ allowed by the surviving Epstein–Glaser ambiguity. Therefore, without knowing the precise value of ω, the general Epstein–Glaser solution respecting EGI can be written as

$$
\begin{aligned}
&-2e^2 gM(2\pi)^2\, t_{\rm gi}^{\mu\nu}(k_1,k_2) \\
&= \frac{e^2 g}{4(2\pi)^4 M}\,P^{\mu\nu}(k_1,k_2)\left(F_1(\tilde\rho)+\sum_{k=0}^{\lfloor\omega/2\rfloor-1}C_k\tilde\rho^k\right),
\end{aligned}
\tag{4.30}
$$

where the constants C_k are all arbitrary. However, terms corresponding to $\omega\geq 4$ can be discarded on grounds of perturbative unitarity.

Let us now to come back to reference [16]. It is argued there that the convergent integral

$$t_0(\tilde{\rho}) := \int_{\tilde{\rho}^{-1}}^{\infty} du\, \frac{b_1(u\tilde{\rho})}{1-u}$$

leads to the correct result. From the standpoint of this reference, formula (4.27) is "oversubtracted". One obtains there, yet again

$$\begin{aligned} P^{\mu\nu} t_0(\tilde{\rho}) &= P^{\mu\nu} \int_{\tilde{\rho}^{-1}}^{\infty} du\, \frac{b_1(u\tilde{\rho})}{1-u} \\ &= -\frac{3\, P^{\mu\nu}}{8M^2(2\pi)^6}\, (2J_1(\tilde{\rho},0) - J_2(\tilde{\rho})) \\ &= -\frac{P^{\mu\nu}}{8M^2(2\pi)^6} \left[\frac{3}{\tilde{\rho}} + \frac{6f(\tilde{\rho})}{\tilde{\rho}} - \frac{3f(\tilde{\rho})}{\tilde{\rho}^2}\right] \\ &= -\frac{P^{\mu\nu}}{8M^2(2\pi)^6}\, (F_1(\tilde{\rho}) - 2). \end{aligned} \tag{4.31}$$

So the naive on-shell computation yields a *particular* Epstein–Glaser solution. In the present case, however, equation (4.30) tells us that we are *forced* to add (at least) a polynomial of degree two respecting EGI, that is, a term $C\,P^{\mu\nu}$ for C an indeterminate constant – with which our result for the amplitude is compatible with the generally accepted one.

Remark 2 From our viewpoint, the expression in (4.31) is the unique Epstein–Glaser solution respecting EGI, corresponding to the following causal d-distribution: let the result (4.26) for $d^{\mu\nu}_{\text{gi}}(k_1, k_2)$ (obtained by light-cone restriction of the photon momenta) be *interpreted as an unrestricted element of* $\mathcal{S}'(\mathbb{R}^8)$, that is, all values $(k_1, k_2) \in \mathbb{R}^8$ are admitted. One easily verifies that this d-distribution has causal support, so the splitting problem is well defined, and since its singular order is zero, the EGI requirement selects a unique splitting solution. Writing the latter suitably as a dispersion integral in momentum space, one verifies the claim. This procedure strongly simplifies explicit computations, but it is not conceptually correct.[11]

4.4 Fixing the normalization polynomial by agreement with the Feynman gauge

In order to determine the normalization polynomial we may as well invoke the computation of the $h \to \gamma\gamma$ decay in the Feynman gauge and *gauge-fixing independence*, namely, the requirement that observable quantities should not depend on the choice of gauge.[12] Motivated by results of [43],[13] we contend that the "entirely on-shell" amplitude coming out of our previous computation should coincide with that of an Epstein–Glaser computation in the Feynman gauge. By "entirely on-shell" we mean that not only the photons, but *also the higgs* is on-shell, that is, $\tilde{\rho} = \rho := m_h^2/4M^2$.

Denote the Epstein–Glaser result for the d-distribution in the Feynman gauge by $d^1_{\mu\nu}$. In contrast with the unitary gauge, there additionally contribute diagrams with Stückelberg fields and Faddeev–Popov ghosts (as inner lines) to $d^1_{\mu\nu}$, see e.g. [15]. We spare the reader the details of the construction of the TOPs, and in particular the derivation of the $AAWW^\dagger$-vertex in this context. For photons on-shell with physical polarizations (setting $k_1^2 = 0 = k_2^2$ and omitting pure gauge terms $\sim k_{1\mu}$ or $\sim k_{2\nu}$), our result reads:

$$\begin{aligned} d^1_{\mu\nu}(k_1, k_2) &= -\frac{1}{2^3\,(2\pi)^6\, M^2} \\ &\times \left[(k_1k_2)g_{\mu\nu}\left(-\frac{3}{\tilde{\rho}^2} + \frac{7}{\tilde{\rho}} - \frac{\rho}{\tilde{\rho}^2}\right)\right. \\ &\left. - k_{2\mu}k_{1\nu}\left(-\frac{3}{\tilde{\rho}^2} + \frac{8}{\tilde{\rho}} - \frac{2\rho}{\tilde{\rho}^2}\right)\right]\Im f(\tilde{\rho}). \end{aligned} \tag{4.32}$$

The tedious computation of the above absorptive part was done with the aid of the Mathematica package FeynCalc [44]. The computation proceeds along the lines of the computations of related absorptive parts in scalar electrodynamics and electroweak theory in the unitary gauge presented in full detail in Sects. 3.1 and 4.2 , respectively. As before, all terms contributing to the distribution $d^1_{\mu\nu}$ can be represented by Feynman diagrams with cuts – for the complete list see e.g. [15]. Just like in Sects. 3.1 and 4.2 , because of the kinematic constraints one needs to consider only the cut separating the higgs vertex from the photon vertices. All the appearing expressions have a very similar structure to those that have been already considered in the above-mentioned parts. Thanks to the presence of the cut, each integral over the four-momentum flowing in the loop can be converted into an integral over a sphere, which can be evaluated explicitly. We stress the fact that, due to compactness of the region of integration, the computation of the absorptive part does not involve any regularization.

[11] Actually, in the first edition of the book by Scharf on quantum electrodynamics (i.e., [29] rather than [39]), the vertex function in QED at third order was computed by such a method.

[12] The equivalence or inequivalence of calculations performed in different gauges was a nagging worry of Raymond Stora in his last years. The classic paper [42] illustrates the difficulties lurking here.

[13] This reference works with a formulation of gauge invariance suitable for the BEG scheme. In that framework it was shown for the various R_ξ-gauges that the T-products can be normalized in such a way that the physical S-matrix (i.e., for in- and out-states being on-shell) does not depend on the gauge-fixing parameter ξ in the formal adiabatic limit; and that this normalization is compatible with gauge invariance in the mentioned sense.

An important feature of electroweak theory in the R_ξ-gauges is the fact that all interaction vertices have dimensions lower or equal to four (because dim $W^\mu = 1$, in contrast to the value dim $W^\mu = 2$ for the unitary gauge). In particular, a straightforward power counting argument gives the upper bound $\omega(d^1_{\mu\nu}) \leq 0$ for the singular order of the off-shell distribution $d^1_{\mu\nu}$. Noting that $\tilde{\rho} = (k_1k_2)/2M^2$ and $f(\tilde{\rho}) = O(\log\tilde{\rho})$ we see that the on-shell restriction of $d^1_{\mu\nu}(k_1, k_2)$ in Eq. (4.32) grows logarithmically for big values of $\tilde{\rho}$. For the off-shell $d^1_{\mu\nu}$, this implies the equality $\omega(d^1_{\mu\nu}) = 0$. This should be contrasted with the bounds $6 \geq \omega(d_{\mu\nu}) \geq 2$ in the case of the absorptive part computed in the unitary gauge.

The off-shell distribution $d^1_{\mu\nu}$ is again of the type considered in Sects. 3.2. In particular, the method of distribution splitting developed in Sect. 3.3 is applicable. For photons on-shell with physical polarizations, the central solution reads

$$\begin{aligned} t^{1\,c}_{\mu\nu}(k_1, k_2) &= -\frac{1}{2^3\,(2\pi)^6\,M^2} \\ &\times \left\{ g_{\mu\nu}(k_1k_2)\left[\left(-\frac{3}{\tilde{\rho}^2} + \frac{7}{\tilde{\rho}} - \frac{\rho}{\tilde{\rho}^2}\right) f(\tilde{\rho}) + \frac{3}{\tilde{\rho}} + \frac{2\rho}{\tilde{\rho}}\right] \right. \\ &\left. - k_{1\nu}k_{2\mu}\left[\left(-\frac{3}{\tilde{\rho}^2} + \frac{8}{\tilde{\rho}} - \frac{2\rho}{\tilde{\rho}^2}\right) f(\tilde{\rho}) + \frac{3}{\tilde{\rho}} + \frac{2\rho}{\tilde{\rho}}\right]\right\}. \end{aligned} \tag{4.33}$$

According to the postulate 'Divergence degree' (in Sect. C.1), we have to demand for the off-shell $t^1_{\mu\nu}$ that

$$\omega(t^1_{\mu\nu}) = \omega(d^1_{\mu\nu}) = 0.$$

This implies that the pertaining normalization freedom consists of a constant term which is a tensor with two indices. By the Lorentz invariance such a term has to be proportional to the metric tensor. Consequently, the general off-shell solution of the splitting problem is of the form

$$t^1_{\mu\nu}(k_1, k_2) = t^{1\,c}_{\mu\nu}(k_1, k_2) + g_{\mu\nu}D, \tag{4.34}$$

where D is an arbitrary constant; note that this relation holds also after restriction to on-shell photons with physical polarizations.

Observe that, in contrast to the unitary gauge, as long as the higgs is off-shell, the distributions (4.32) and (4.33) are *not* electromagnetically gauge-invariant. This was to be expected and is related to the presence of unphysical degrees of freedom in electroweak theory in the R_ξ-gauges. However, entirely on-shell EGI can be satisfied: setting $\tilde{\rho} := \rho$ in (4.33), we plainly get

$$\begin{aligned} &t^1_{\mu\nu}(k_1, k_2)\big|_{\tilde{\rho}=\rho} \\ &\quad = -\frac{1}{2^3\,(2\pi)^6\,M^2}\left(P_{\mu\nu}(k_1, k_2)\,F_1(\rho) + D\,g_{\mu\nu}\right), \end{aligned} \tag{4.35}$$

and one sees that D must be put equal to zero. This fixes completely the normalization freedom in the construction of $t^1_{\mu\nu}$ in the Feynman gauge. At this level there is of course coincidence with the result in [45], despite different game rules.

Recall that in the unitary gauge, for on-shell photons with physical polarizations, the general normalization freedom fulfilling electromagnetic gauge invariance and Lorentz covariance is given by the last term in (4.30), where $\omega \equiv \omega(d_{\mu\nu})$. We stress that the constants C_k appearing in that term cannot be fixed without imposing some further normalization conditions. To address this problem, observe that it is possible to adjust the coefficients C_k of the polynomial in the expression (4.30) for $t_{\mathrm{gi},\mu\nu}$ in the unitary gauge in such a way that the following equality

$$t_{\mathrm{gi},\mu\nu}(k_1, k_2)\big|_{\tilde{\rho}=\rho} = t^1_{\mu\nu}(k_1, k_2)\big|_{\tilde{\rho}=\rho}. \tag{4.36}$$

holds entirely on-shell, i.e. for $\tilde{\rho} = \rho$. In fact, we must set $C_0 := 2$ and $C_k := 0$ for all $k \geq 1$ in (4.30), which fixes completely the normalization freedom of $t_{\mathrm{gi},\mu\nu}$. Eq. (4.36) expresses the independence of the physical amplitude of the diphoton decay of the higgs of the choice of the gauge. We regard (4.36) as a normalization condition of time-ordered products. We have shown that this condition can be satisfied in the case at hand and determines uniquely the indeterminate normalization polynomial of $t_{\mathrm{gi},\mu\nu}$ in the expression (4.30).

In summary, our final result for the entirely on-shell EW $h \to \gamma\gamma$ decay reads:

$$t_{\mu\nu}(k_1, k_2)\big|_{\tilde{\rho}=\rho} = -\frac{1}{2^3\,(2\pi)^6\,M^2}\,P_{\mu\nu}(k_1, k_2)\,F_1(\rho),$$

in agreement with the majority of the literature.

4.5 On settling the controversy

Should one infer that by computing in the "physical" unitary gauge there is no way to entirely settle the controversy that motivates this work, by removing the remaining ambiguity in determining the amplitude in question? Not without at least pondering credible "heavy-higgs" (or $M \to 0$) and "light-higgs" (or $M \to \infty$) arguments to bolster the case of $F_1(\rho)$ versus $F_1(\rho) - 2$, that have been made in the literature.

Now, for the present authors the question is not whether either class of arguments is compelling enough. Instead, the question is whether they can be made within the BEG prescriptions, and at the level of rigour of this paper. The arguments in the first-named class involve plays with field transformations, power counting rules and the adiabatic limit that we find hard to countenance in the BEG formalism.

However, those of the second class are persuasive within our purview. Note that $F(0)$, for both scalar and vector boson charged fields, as well as for Dirac fermions, must coincide with (the first coefficient of) the β-function series associated to *electric charge renormalization*.[14] It was a fortunate historical fact that a calculation of the effective Lagrangian for charged Proca particles [46] was already available when the first "exact" computation of the higgs to digamma process that we are aware of was performed [2] – thus making possible a dependable "light-higgs" argument. A computation of the renormalization of the electric charge of massive vector bosons in the unitary gauge by means of BEG technology is in principle feasible – cf. in this respect [41, Sect. 7] and [47] – and expected to yield the required value $F_1(0) = 7$. That would complete the analysis of this paper, without going beyond the unitary gauge framework.

5 Conclusion

Contrary to custom, we begin this section by declaring what we have *not* done in the paper. Finite QFT *à la* Bogoliubov–Epstein–Glaser is mathematically a rigorous method. So, referring to what is found in the literature – like that cited in the Introduction – we have not employed dimensional regularization, deemed an "artifact" by some. Nor do we borrow Pauli–Villars', nor cutoff regularizations, for that matter. We did not have to practice "judicious routings of the external momenta" [6], nor adopt the "loop regularization method" [10], or any of the techniques to handle divergent integrals, resulting from the blind application of Feynman graph technology on momentum space. We do not pore over divergent integrals, at all. Each and every one of the integrals appearing in this paper produces an unambiguous result; each amplitude is finite.

We expected the BEG procedure to yield a conceptually clear understanding of the EW $h \to \gamma\gamma$ decay in the unitary gauge. We have succeeded in this – at a price. According to Epstein and Glaser, the adiabatic limit is to be performed *after* distribution splitting. Such an off-shell procedure for the $h \to \gamma\gamma$ decay in the unitary gauge demands computations more than one order of magnitude greater than the ones performed in this paper – compare the computation of the QED vertex function in [39, Chap. 3.8] and in [48].

We were not disposed to inflict this on ourselves, nor our surviving readers. Thus we were forced to innovate on the method, generalizing the splitting dispersion integral to production of massless particles, and showing that in the present situation the adiabatic limit may be performed before distribution splitting. Only, then one may have to add to the result so obtained an a priori indeterminate polynomial in the external momenta, of a degree given by the singular order of the amplitude off-shell. It is precisely the addition of this polynomial that is missing in references [5,6] and [16]. We have resolved the ambiguity by recourse to gauge-fixing independence of the entirely on-shell amplitude. Alternatively, the ambiguity could be resolved within the unitary gauge in the BEG scheme, by invoking the low-energy argument.[15] We have not attempted here a rigorous proof of this argument, nor computed the relevant coefficient of the beta function, leaving the task for a separate analysis in future work.

[14] $F_0(0) = -1/3$, which has been calculated in this paper, means precisely this.

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Appendix A: Notations and prerequisites

Our Minkowski metric is mostly-negative. The Minkowski inner product of two vectors $x \equiv x^\mu$, $p \equiv p^\nu$ is denoted with parentheses: $(xp) = x^\mu p_\mu$. When (we hope) it does not cause confusion, we often denote $p^2 = (pp)$.

We signal the standard formula for time-ordered 2-point function:

$$\langle\!\langle \mathrm{T}\, \varphi(x)\, \chi(x') \rangle\!\rangle := \frac{i}{(2\pi)^4} \int d^4p\, \frac{e^{-i(p(x-x'))}}{p^2 - M^2 + i0}\, M^{\varphi\chi}(p), \tag{A.1}$$

[15] Variants of the "light-higgs" or "low energy" argument besides [2, 9,17] are found for instance in [27, Ch. 24.8], in [49] and in [50].

where $M^{\varphi\chi}$ is the multiplier appearing in the corresponding 2-point function for the fields φ, χ with the same mass M.

Propagators for a (complex) scalar field. Clearly, for (say, complex) *scalar* fields the Feynman propagator

$$\Delta^F(x-x') := \langle\!\langle \mathrm{T}\,\varphi(x)\,\varphi^\dagger(x')\rangle\!\rangle \tag{A.2}$$

fulfils

$$\Box\Delta^F(x-x') = -M^2\Delta^F(x-x') - i\delta(x-x'), \tag{A.3}$$

where M is the mass of the φ-field. Also, with θ denoting the Heaviside function, the Wightman functions

$$\begin{aligned}&\Delta^+(x-x') := \langle\!\langle \varphi(x)\varphi^\dagger(x')\rangle\!\rangle\\ &= \langle\!\langle \varphi^\dagger(x)\varphi(x')\rangle\!\rangle = \frac{1}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\delta(p^2-M^2)e^{-i(p(x-x'))}\\ &\quad\text{so that}\quad (\Box+M^2)\Delta^+(x)=0, \qquad \Delta^-(x) := -\Delta^+(-x),\end{aligned} \tag{A.4}$$

are used in our calculations.

Massive vector fields. A dreibein $e_r(p)$ on Minkowski momentum space, with the properties:

$$\big(e_r(p)\,e_s(p)\big) = -\delta_{rs} \quad \text{for} \quad r,s=1,2,3; \qquad \big(p\,e_r(p)\big)=0,$$

describes polarization states for particles with squared mass $M^2 = p^2 > 0$ and spin $j=1$. From the above identities, one derives the projector formula:

$$\sum_{r=1}^{3} e_r^\mu(p)e_r^\nu(p) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}. \tag{A.5}$$

The set e is regarded as an intertwiner matrix mapping the natural representation space of the Lorentz group onto the representation space $\mathbb{C}^3$ for spin 1 objects. Let $a_r^\dagger(p)$ and $a_r(p)$ be respectively the creation and annihilation operators on the boson Fock space for such particles – whose 1-particle subspace is the corresponding Wigner unirrep space; and $b_r^\dagger(p)$ and $b_r(p)$ for their antiparticles.

There is a quantum vector field acting on that space given by the formula

$$\begin{aligned}W^\mu(x) := \sum_r \int d\mu(p)\,\big[&e^{i(px)}e_r^\mu(p)\,b_r^\dagger(p)\\ &+e^{-i(px)}e_r^{\mu *}(p)\,a_r(p)\big];\end{aligned} \tag{A.6}$$

In (A.6) and in other formulas $d\mu(p)$ denotes the usual invariant measure $d^3\boldsymbol{p}/2E(p) = d^3\boldsymbol{p}/\sqrt{m^2+|\boldsymbol{p}|^2}$ over the mass hyperboloid $H_M^\pm := \{p\in\mathbb{M}\mid p^2=M^2\wedge \pm p^n>0\}$. By its definition, the charged *Proca* field W is divergenceless: $(\partial W)=0$. Its equations of motion can be variously written as

$$\begin{aligned}(\Box+M^2)W^\mu &= (\Box+M^2)W^\mu - \partial^\mu(\partial W)\\ &= \partial_\nu G^{\nu\mu}(x) + M^2W^\mu = 0,\end{aligned} \tag{A.7}$$

where $G^{\mu\nu} := \partial^\mu W^\nu - \partial^\nu W^\mu$.

The theory of massive vector fields is a gauge theory [51,52], its Proca version being a "unitary gauge" for it. It has been analyzed, in terms parallel to Maxwell field theory, in [53]; wherein the associated BRST machinery is "deconstructed" in terms of Koszul cohomology.

The high-energy limit of $(-g^{\mu\nu}+p^\mu p^\nu/M^2)/(p^2-M^2)$ apparently signals quadratic divergences and trouble with unitarity of the scattering matrix: cross-sections would appear to grow without bound due to the longitudinal momentum states. The difficulty lies with the closure relation (A.5) of the intertwiners e_r, whose dimension does not allow the standard sufficiency criterion for renormalizability. This is usually "cured" nowadays by the cohomological extension of the Wigner representation space for massive spin-1 particles into spaces populated by Faddeev-Popov ghosts and anti-ghosts and Stückelberg fields.

In this paper we work mainly with the Proca field (i.e., we use the unitary gauge), where these additional unphysical fields do not appear; the apparently bad UV-behaviour of the propagators is under control, as we verify, thanks to amazing cancellations in the amplitudes.

Propagators for the EW theory in the unitary gauge. We will make frequent use of

$$\begin{aligned}\Delta^\alpha_\beta(x-x') := \langle\!\langle \mathrm{T}\,W^\alpha(x)\,W^\dagger_\beta(x')\rangle\!\rangle = -(g^\alpha_\beta + \partial^\alpha\partial_\beta/M^2)\\ \times\Delta^F(x-x'),\end{aligned} \tag{A.8}$$

where M is the mass of the W-field, and its properties:

$$\Box\Delta^\alpha_\beta = -M^2\Delta^\alpha_\beta + i(g^\alpha_\beta+\partial^\alpha\partial_\beta/M^2)\delta; \quad \partial_\mu\Delta^\mu_\nu = i\partial_\nu\delta/M^2.$$

The corresponding formulas for the Wightman functions respectively read:

$$\begin{aligned}&\Delta^{\alpha\,+}_\beta(x-x') := \langle\!\langle W^\alpha(x)\,W^\dagger_\beta(x')\rangle\!\rangle = \langle\!\langle W^{\alpha\,\dagger}(x)\,W_\beta(x')\rangle\!\rangle\\ &= -(g^\alpha_\beta+\partial^\alpha\partial_\beta/M^2)\Delta^+(x-x')\\ &\quad\text{and}\quad \Box\Delta^{\alpha\,+}_\beta = -M^2\Delta^{\alpha\,+}_\beta, \quad \partial_\mu\Delta^{\mu\,+}_\nu = 0.\end{aligned}$$

We will invoke also the Maxwell-like fields, where $\# = \dagger$ or naught,

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu; \qquad G^\#_{\mu\nu} := \partial_\mu W^\#_\nu - \partial_\nu W^\#_\mu,$$

and introduce the propagator

$$\begin{aligned}D^{\alpha\mu}_{\beta\rho}(x-x') &:= \langle\!\langle \mathrm{T}\,G^{\alpha\mu}(x)\,G^\dagger_{\beta\rho}(x')\rangle\!\rangle = \langle\!\langle \mathrm{T}\,G^{\alpha\mu\,\dagger}(x)\,G_{\beta\rho}(x')\rangle\!\rangle\\ &= \langle\!\langle \mathrm{T}(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x))(\partial_\beta W^\dagger_\rho(x') - \partial_\rho W^\dagger_\beta(x'))\rangle\!\rangle\\ &= -\partial^\mu\big(\partial_\rho\Delta^\alpha_\beta(x-x') - \partial_\beta\Delta^\alpha_\rho(x-x')\big)\\ &\quad+\partial^\alpha\big(\partial_\rho\Delta^\mu_\beta(x-x') - \partial_\beta\Delta^\mu_\rho(x-x')\big)\\ &= (g^\alpha_\beta\partial^\mu_\rho - g^\alpha_\rho\partial^\mu_\beta - g^\mu_\beta\partial^\alpha_\rho + g^\mu_\rho\partial^\alpha_\beta)\Delta^F(x-x'),\\ \partial^\alpha_\rho &:= \partial^\alpha\partial_\rho,\end{aligned} \tag{A.9}$$

since in

$$\partial_\rho \Delta^\alpha_\beta - \partial_\beta \Delta^\alpha_\rho = (-g^\alpha_\beta \partial_\rho + g^\alpha_\rho \partial_\beta)\Delta^F \tag{A.10}$$

the terms with three derivatives *cancel out*, due to the antisymmetry of $G^\#_{\mu\nu}$. That fact is relevant in this paper. Analogously we obtain

$$\begin{aligned} D^{\alpha\mu\,+}_{\beta\rho}(x-x') &:= \langle\!\langle G^{\alpha\mu}(x)\, G^\dagger_{\beta\rho}(x')\rangle\!\rangle = \langle\!\langle G^{\alpha\mu\,\dagger}(x)\, G_{\beta\rho}(x')\rangle\!\rangle \\ &= (g^\alpha_\beta \partial^\mu_\rho - g^\alpha_\rho \partial^\mu_\beta - g^\mu_\beta \partial^\alpha_\rho + g^\mu_\rho \partial^\alpha_\beta)\Delta^+(x-x'), \end{aligned}$$

without third-order derivatives. We also note that

$$\partial_\mu D^{\alpha\mu}_{\beta\rho} = (g^\alpha_\beta \partial_\rho - g^\alpha_\rho \partial_\beta)\Box\Delta^F = (-g^\alpha_\beta \partial_\rho + g^\alpha_\rho \partial_\beta)(M^2\Delta^F + i\delta),$$

since third-order derivatives appear only in the form $\partial\Box\Delta^F$, removable with the help of (A.3).

For the 2-point functions with one $G^\#$ plus one $W^\#$, we use (A.10) to get rid of the terms with three derivatives. For time-ordered ones we obtain

$$\begin{aligned} &\langle\!\langle \mathrm{T}\, W^\mu(x)\, G^\dagger_{\alpha\nu}(x')\rangle\!\rangle = \langle\!\langle \mathrm{T}\, W^{\mu\,\dagger}(x)\, G_{\alpha\nu}(x')\rangle\!\rangle \\ &\quad = -(\partial_\alpha \Delta^\mu_\nu(x-x') - \partial_\nu \Delta^\mu_\alpha(x-x')) = (g^\mu_\nu \partial_\alpha - g^\mu_\alpha \partial_\nu) \\ &\quad\ \times \Delta^F(x-x'), \\ &\langle\!\langle \mathrm{T}\, G^\dagger_{\alpha\nu}(x)\, W^\mu(x')\rangle\!\rangle \\ &\quad = \langle\!\langle \mathrm{T}\, G_{\alpha\nu}(x)\, W^{\mu\,\dagger}(x')\rangle\!\rangle \\ &\quad = (-g^\mu_\nu \partial_\alpha + g^\mu_\alpha \partial_\nu)\Delta^F(x-x'). \end{aligned} \tag{A.11}$$

With the parallel Wightman functions we proceed similarly:

$$\begin{aligned} \langle\!\langle W^\mu(x)\, G^\dagger_{\alpha\nu}(x')\rangle\!\rangle &= \langle\!\langle W^{\mu\,\dagger}(x)\, G_{\alpha\nu}(x')\rangle\!\rangle \\ &= (g^\mu_\nu \partial_\alpha - g^\mu_\alpha \partial_\nu)\Delta^+(x-x'), \\ \langle\!\langle G^\dagger_{\alpha\nu}(x)\, W^\mu(x')\rangle\!\rangle &= \langle\!\langle G_{\alpha\nu}(x)\, W^{\mu\,\dagger}(x')\rangle\!\rangle \\ &= (-g^\mu_\nu \partial_\alpha + g^\mu_\alpha \partial_\nu)\Delta^+(x-x'). \end{aligned} \tag{A.12}$$

Comparing with the Feynman gauge, in which the $W^\#$ two-point functions Δ^α_β and $\Delta^{\alpha\,+}_\beta$ are replaced by $-g^\alpha_\beta\Delta^F$ (A.13) and $-g^\alpha_\beta\Delta^+$ (A.14), respectively, we find the $W^\# G^\#$, $G^\# W^\#$ and $G^\# G^\#$ two-point functions to be the same, thanks to the cancellations in (A.10).

Propagators for the EW theory in the Feynman gauge. The Feynman propagator and the Wightman two-point function for the W-field in the Feynman gauge read

$$\langle\!\langle \mathrm{T}\, W^\alpha(x)\, W^\dagger_\beta(x')\rangle\!\rangle = -g^\alpha_\beta \Delta^F(x-x'), \tag{A.13}$$

$$\langle\!\langle W^\alpha(x)\, W^\dagger_\beta(x')\rangle\!\rangle = \langle\!\langle W^{\alpha\,\dagger}(x)\, W_\beta(x')\rangle\!\rangle = -g^\alpha_\beta \Delta^+(x-x'). \tag{A.14}$$

Besides the W-field the computation from Sects. 4.4 involves the Stückelberg fields $\varphi^\pm$ and the ghost and anti-ghost fields $C^\pm, \bar C^\pm$ – where $\phi^\pm := \frac{1}{\sqrt{2}}\big(\phi_1 \pm i\phi_2\big)$ for $\phi = \varphi, C, \bar C$. Below we list the non-vanishing Feynman propagators and two-point functions for these fields:

$$\langle\!\langle \mathrm{T}\, \varphi^+(x)\, \varphi^-(x')\rangle\!\rangle = \Delta^F(x-x'), \tag{A.15}$$

$$\langle\!\langle \varphi^+(x)\, \varphi^-(x')\rangle\!\rangle = \langle\!\langle \varphi^-(x)\, \varphi^+(x')\rangle\!\rangle = \Delta^+(x-x'), \tag{A.16}$$

$$\langle\!\langle \mathrm{T}\, C^+(x)\, \bar C^-(x')\rangle\!\rangle = \langle\!\langle \mathrm{T}\, C^-(x)\, \bar C^+(x')\rangle\!\rangle = \Delta^F(x-x'), \tag{A.17}$$

$$\langle\!\langle C^+(x)\, \bar C^-(x')\rangle\!\rangle = -\langle\!\langle \bar C^-(x)\, C^+(x')\rangle\!\rangle = \Delta^+(x-x'), \tag{A.18}$$

$$\langle\!\langle C^-(x)\, \bar C^+(x')\rangle\!\rangle = -\langle\!\langle \bar C^+(x)\, C^-(x')\rangle\!\rangle = \Delta^+(x-x'). \tag{A.19}$$

Appendix B: An interesting distribution

In this appendix we study the distribution $f(\rho)$ appearing in the amplitude of the $h \to \gamma\gamma$ decay via both scalar QED and flavourdynamics.

To define $\sqrt{\cdot}: \mathbb{C} \to \mathbb{C}$ and $\log: \mathbb{C} \to \mathbb{C}$ one uses a cut on the negative real axis:

$$\sqrt{r\, e^{i\varphi}} = \sqrt{r}\, e^{i\varphi/2}, \quad \log r e^{i\varphi} = \log r + i\varphi,$$

both with $\varphi \in (-\pi, \pi]$.

The complex function

$$\tilde f : \begin{cases} \mathbb{C} \setminus \big((-\infty, 0) \cup (1, \infty)\big) \longrightarrow \mathbb{C} \\ z \longmapsto -\big(\log(\sqrt{1-z} + i\sqrt{z})\big)^2 \end{cases} \tag{B.1}$$

is analytic, in view of the two cuts on the real axis. The distribution $f(\rho)$ is defined by

$$f : [0, \infty) \longrightarrow \mathbb{C} : \rho \longmapsto f(\rho) := \tilde f(\rho + i0). \tag{B.2}$$

We claim that

$$f(\rho) = (\arcsin\sqrt{\rho})^2 = \left[\arctan\frac{\rho}{\sqrt{1-\rho^2}}\right]^2 \quad \text{for} \quad 0 \le \rho \le 1, \tag{B.3}$$

$$f(\rho) = -\frac{1}{4}\left[\log\frac{1+\sqrt{1-\rho^{-1}}}{1-\sqrt{1-\rho^{-1}}} - i\pi\right]^2 \quad \text{for} \quad \rho \ge 1, \tag{B.4}$$

from which one easily obtains the following formula for the imaginary part:

$$\begin{aligned} \Im f(\rho) &= \theta(\rho-1)\,\frac{\pi}{2}\log\frac{\sqrt{\rho}+\sqrt{\rho-1}}{\sqrt{\rho}-\sqrt{\rho-1}} \\ &= -\theta(\rho-1)\,\frac{\pi}{2}\log\big(2\rho - 2\sqrt{\rho(\rho-1)} - 1\big). \end{aligned} \tag{B.5}$$

The first claim (B.3) follows immediately from the identity

$$\arcsin\sqrt{\rho} = -i\log\big(\sqrt{1-\rho} + i\sqrt{\rho}\big) \quad \text{for} \quad \rho \in [0,1],$$

which is obvious from $\exp(i\arcsin x) = \sqrt{1-x^2} + ix, \quad |x| \le 1$.

To prove the second claim (B.4), first note that one has $\sqrt{1-(\rho+i0)} = -i\sqrt{\rho-1}$ for $\rho \ge 1$. Hence, there holds:

$$\begin{aligned}
&\log\big(\sqrt{1-(\rho+i0)} + i\sqrt{\rho}\big) = \log\big(\sqrt{\rho} - \sqrt{\rho-1}\big) + i\pi/2 \\
&= \frac{1}{2}\big(\log((\sqrt{\rho}-\sqrt{\rho-1})^2) + i\pi\big) \\
&= \frac{1}{2}\left(\log\frac{\sqrt{\rho}-\sqrt{\rho-1}}{\sqrt{\rho}+\sqrt{\rho-1}} + i\pi\right) - \frac{1}{2}\left(\log\frac{1+\sqrt{1-\rho^{-1}}}{1-\sqrt{1-\rho^{-1}}} - i\pi\right),
\end{aligned}$$

from which assertion (B.4) follows.

We point out that, for $\rho \in [0,1]$, in the distribution $F_0(\rho) = \rho^{-1}\big(1-\rho^{-1}f(\rho)\big)$ in Eq. (1.3) the terms $\sim \rho^{-1}$ cancel. We bring in the power series expansion

$$\begin{aligned}
&\arcsin x = x + \frac{x^3}{2\cdot 3} + \frac{3\,x^5}{2\cdot 4\cdot 5} \\
&\qquad + \frac{3\cdot 5\,x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots \quad \text{for} \quad |x| \le 1, \quad \text{yielding} \\
&f(\rho) = (\arcsin\sqrt{\rho})^2 = \rho + \frac{\rho^2}{3} \\
&\qquad + \frac{8\rho^3}{45} + \cdots \quad \text{so that} \quad F_0(\rho) = -\frac{1}{3} - \frac{8}{45}\rho + \cdots.
\end{aligned} \tag{B.6}$$

Appendix C: Bogoliubov–Epstein–Glaser normalization

Epstein and Glaser [21,54] started from Bogoliubov's functional $\mathbb{S}[g]$-matrix [55, Sect. 21], based on [56] and on previous work by Stückelberg and Rivier [57]. That is an expansion of operator-valued distributions (OVD) on configuration space, of the form

$$\begin{aligned}
&\mathbb{S}[g] = 1 \\
&\quad + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n\, T_n(x_1,\dots,x_n)\, g(x_1)\cdots g(x_n), \\
&\quad g \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}).
\end{aligned} \tag{C.1}$$

We have taken $\hbar = 1$. The g's are multiplets of coupling functions which work as adiabatic cutoffs. The T_n, *symmetric* in their arguments, are identified with chronological or time-ordered n-products. This is Bogoliubov's version of the summands in the formal Dyson expansion for the scattering matrix in the interaction picture. One tries to recursively build the T_n from natural postulates: the ultraviolet problem is solved in that construction. In the "adiabatic limit" $g \uparrow 1$ the functional scattering matrix (C.1) is expected to converge to the physical $\mathbb{S}$ in suitable senses [58].

C.1: The Epstein–Glaser postulates

Beginning of induction: The procedure is perturbative, the basic building blocks being finite sets of quantum free fields on their corresponding Fock spaces. Precisely, $T_1(x)$ is a Wick polynomial in those and their derivatives – a well-defined OVD.[16] The coupling constants of the model are included in the T_n, the expansion being a power series on them. The other postulates shall enable us to construct the T_n from T_1 by induction on n.

Causality: This is the key requirement, for which the Epstein–Glaser manufacturing of TOPs is also called "causal perturbation theory". Let $V_\pm$ and $\overline{V}_\pm$ respectively denote the open forward and backward lightcones and their closures. If g_1, g_2 are such that

$$\begin{aligned}
&\operatorname{supp} g_2 \cap \big(\operatorname{supp} g_1 + \overline{V}_-\big) = \emptyset, \quad \text{then} \\
&\quad \mathbb{S}[g_1+g_2] = \mathbb{S}[g_2]\,\mathbb{S}[g_1]; \\
&\quad \text{equivalently,} \quad T_n(x_1,\dots,x_n) \\
&\quad = T_r(x_1,\dots,x_r)\,T_{n-r}(x_{r+1},\dots,x_n) \quad \text{whenever} \\
&\quad \{x_1,\dots,x_r\} \cap \big(\{x_{r+1},\dots,x_n\} + \overline{V}_-\big) = \emptyset, \\
&\quad \text{for all } r \text{ and } n \text{ with} 1 \le r \le n-1.
\end{aligned}$$

This is a powerful postulate, called *causal factorization*. It means that on large open sets of the n-point Minkowski space $(\mathbb{M}_4)^{\times n} \equiv \mathbb{M}^n$ the TOP T_n can be built up from its lower-order counterparts. In the inductive step of the Epstein–Glaser method, this requirement uniquely determines T_n on the set of Schwartz functions $\mathcal{S}(\mathbb{M}^n \backslash \Delta_n)$, in terms of the given T_k at lower orders $k \le n-1$, where Δ_n is the "thin" diagonal $\Delta_n := \{\,(x_1,\dots,x_n) : x_1 = x_2 = \cdots = x_n\,\}$. Perturbative normalization is the *extension* of the operator-valued distribution T_n from $\mathcal{S}'(\mathbb{M}^n \setminus \Delta_n)$ to $\mathcal{S}'(\mathbb{M}^n)$. The gist of BEG normalization is that in local quantum field theory this problem finds a solution, the induction process going through. So there is no need to deal with infinities. The solution of the extension problem is non-unique: in principle one may add any OVD which is supported on Δ_n. All further postulates of Epstein–Glaser have the purpose of giving guidance for this problem; hence they may be called "normalization conditions".

Causal Wick expansion: The TOPs are required to satisfy the Wick expansion formula. We display the latter in terms of the interaction $T_1(x) = \varphi^k(x)$, for φ a real

[16] One can think of T_1 as an "interaction Lagrangian". However, the Lagrangian mindset is inessential here.

scalar field:

$$T_n\big(\varphi^k(x_1), \dots, \varphi^k(x_n)\big) = \sum_{l_1,\dots,l_n=0}^{k} \binom{k}{l_1} \cdots \binom{k}{l_n} \langle\!\langle T_n(\varphi^{k-l_1}(x_1), \dots, \times \varphi^{k-l_n}(x_n)) \rangle\!\rangle \, \varphi^{l_1}(x_1) \cdots \varphi^{l_n}(x_n)$$

with $\langle\!\langle \cdots \rangle\!\rangle$ denoting vacuum expectation value. This postulate reduces the extension problem for the OVD $T_n(\cdots)$ to one of numerical distributions – a simpler task.

Poincaré Covariance: Let there be given the standard lifting $U(a, \Lambda)$ to Fock space of the Poincaré unitary irreducible representations (unirreps) on 1-particle subspaces. Then

$$U(a, \Lambda)\, \mathbb{S}[g]\, U^\dagger(a, \Lambda) = \mathbb{S}\big[(a, \Lambda) \cdot g\big],$$

where $((a, \Lambda) \cdot g)(x) = g(\Lambda^{-1}(x - a))$. In particular, translation invariance implies that the coefficients in the causal Wick expansion depend only on the relative coordinates. Therefore, the extension problem for the numerical distributions is step by step simplified to an extension to one point, namely from $\mathcal{S}'(\mathbb{R}^{4(n-1)} \setminus \{0\})$ to $\mathcal{S}'(\mathbb{R}^{4(n-1)})$.

Unitarity (conservation of probability):

$$\mathbb{S}[g]\, \mathbb{S}^\dagger[g] = \mathbb{S}^\dagger[g]\, \mathbb{S}[g] = 1; \quad \text{here we denote:}$$

$$\mathbb{S}^{-1}[g] =: 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n\, \overline{T}_n(x_1, \dots, x_n) \times g(x_1) \cdots g(x_n).$$

Divergence degree: Heuristically, this is the requirement that normalization does not make the T-product "more singular" (in the UV-region). This is expressed in terms of the **scaling degree** of the coefficients (i.e., the numerical distributions) in the causal Wick expansion of the T-product: that degree may *not* be increased by the extension. The standard definitions of the scaling degree $\mathrm{sd}(t)$ and the singular order $\omega(t)$ of a distribution $t \in \mathcal{S}'(\mathbb{R}^k)$ or $t \in \mathcal{S}'(\mathbb{R}^k \setminus \{0\})$ – see, e.g., [23, Sect. 3.2.2] – are as follows:

$$\mathrm{sd}(t) := \inf\{\, r \in \mathbb{R} : \lim_{\lambda \downarrow 0} \lambda^r\, t(\lambda x) = 0 \,\}, \quad \omega(t) := \mathrm{sd}(t) - k, \tag{C.2}$$

where $\inf \emptyset := \infty$ and $\inf \mathbb{R} := -\infty$. For instance, for a translation-invariant distribution $d(x_1 - x_3, x_2 - x_3) \in \mathcal{S}'(\mathbb{R}^8)$ fulfilling $\mathrm{sd}(d) = 8$, equivalently $\omega(d) = 0$, we say that the amplitude superficially is "logarithmically divergent".

Other invariance rules and physical requirements: Discrete symmetries can be accomodated in the Epstein–Glaser construction [59]. A Ward identity playing a paramount role in this paper corresponds to EGI – see Sects. 3.1 and 4.2 for this. For different types of requirements, consult Sects. 4.4 and 4.5.

C.2: Iterative building of the time-ordered products

To assemble the T_n outside of the thin diagonal Δ_n from the inductively known $(T_k)_{1\le k\le n-1}$ directly by causal factorization, one would need a partition of unity subordinate to an open cover of $\mathbb{M}^n \setminus \Delta_n$ – see [60] and [23, Sect. 3.3]. This is problematic for practical computations. For this reason the original Epstein–Glaser construction [21,39] is less direct: it introduces an intermediate D_n-distribution having causal support; and the crucial step is the *splitting* of D_n into its advanced and retarded parts. This splitting corresponds precisely to the above-mentioned extension problem, that is, to perturbative normalization. A decisive advantage of the method is that the problem is solved in momentum space by a dispersion integral.

To explain the construction, we first express the antichronological product $\overline{T}_n$ in terms of the TOPs $(T_k)_{1\le k\le n}$. Let $N = \{x_1, \dots, x_n\}$ and $I \subseteq N$ with $|I| \ne 0$ elements. Define $T_{|I|}(I) = T_{|I|}(x_i : x_i \in I)$. By the standard inversion of a formal power series with noncommuting terms in terms of set compositions, we obtain

$$\overline{T}_{|N|}(N) = \sum_{k=1}^{n} (-)^{n+k} \sum_{I_1 \uplus \cdots \uplus I_k = N} T_{|I_1|}(I_1) \cdots T_{|I_k|}(I_k), \tag{C.3}$$

where the disjoint union is over nonempty *blocks* I_j. The terminology of antichronological products is appropriate, since if $I \cap (J + \overline{V}_-) = \emptyset$, then $\overline{T}(I \cup J) = \overline{T}(J)\, \overline{T}(I)$.

Retarded and advanced products, denoted by R_n and A_n respectively, are the coefficients in the perturbative expansion of the respective retarded and advanced interacting fields. For them we follow the convention in the book [23], identical to that of [21] except that R_n and A_n have an extra factor i^{n-1}. In general, Bogoliubov's definitions read:

$$R_{n+1}(x_1, \dots, x_{n+1}) := i^n \sum_{I \subset \{1,\dots,n\}} (-1)^{|I|}\, \overline{T}_{|I|}(I)\, T_{|I^c|+1}(I^c, x_{n+1}), \tag{C.4}$$

$$A_{n+1}(x_1, \dots, x_{n+1}) := i^n \sum_{I \subset \{1,\dots,n\}} (-1)^{|I|}\, T_{|I^c|+1}(I^c, x_{n+1})\, \overline{T}_{|I|}(I), \tag{C.5}$$

where $I^c := \{1, \dots, n\}\backslash I$. Epstein and Glaser [21] prove that $A_{n+1},\ R_{n+1}$ have advanced or retarded support, respectively:

$$\operatorname{supp} A_{n+1} \subseteq \{\, x \in \mathbb{M}^{n+1} : x_j - x_{n+1} \in \overline{V}_+ \ \forall j\,\};$$
$$\operatorname{supp} R_{n+1} \subseteq \{\, x \in \mathbb{M}^{n+1} : x_j - x_{n+1} \in \overline{V}_- \ \forall j\,\}.$$

In the induction step $n \to n+1$ neither the T_{n+1} nor the R_{n+1} nor the A_{n+1} are known. But by the induction hypothesis the *difference* D_{n+1}, defined by $D_{n+1} := A_{n+1} - R_{n+1}$, only *depends on known quantities*. For instance, in D_3 the unknown T_3 has dropped out – and $\overline{T}_1, \overline{T}_2$ are uniquely given in terms of T_1 and T_2. It follows that D_{n+1} has causal support:

$$\operatorname{supp} D_{n+1} \subseteq \{\, x \in \mathbb{M}^{n+1} : x_j - x_{n+1} \in \overline{V}_+ \ \forall j\,\}$$
$$\cup \{\, x \in \mathbb{M}^{n+1} : x_j - x_{n+1} \in \overline{V}_- \ \forall j\,\}.$$

If one finds a way to extract the advanced part A_{n+1} of D_{n+1}, that is, to *split* the OVD D_{n+1} into A_{n+1} and $-R_{n+1}$ in such a way that the latter two satisfy the just given support properties, then one can construct a candidate for T_{n+1}.[17]

For the sake of normalization conditions, at this stage we may add to T_{n+1} any OVD supported on Δ_{n+1} which is symmetric in $x_1, \dots, x_{n+1}$. The D_{n+1} fulfils all the normalization conditions, in particular the 'Causal Wick expansion' and 'Translation invariance', because of the validity of those for the inductively given $(T_k)_{1\le k\le n}$. Therefore, the splitting problem for D_{n+1} translates into a consonant problem for the coefficients $d(x_1 - x_{n+1}, \dots, x_n - x_{n+1}) \in \mathcal{S}'(\mathbb{R}^{4n}, \mathbb{C})$ in the Wick expansion of D_{n+1}, yielding $a, r(x_1 - x_{n+1}, \dots, x_n - x_{n+1}) \in \mathcal{S}'(\mathbb{R}^{4n}, \mathbb{C})$, which are the coefficients in the Wick expansion of A_{n+1} and R_{n+1}, respectively.

In fine, by the induction process, one specifies the ambiguity in the vacuum expectation value of each T_{n+1} by adding to it a *contact term*, that is,

$$t(x_1 - x_{n+1}, \dots, x_n - x_{n+1}) + \sum_{|a|\le\omega} c_a\, \partial^a \delta(x_1 - x_{n+1}, \dots, x_n - x_{n+1}), \tag{C.6}$$

where ω is the singular order of the pertinent $d(x_1 - x_{n+1}, \dots)$ and the coefficients $c_a \in \mathbb{C}$ depending on the multi-index a **are arbitrary**, up to restrictions coming from the 'Poincaré covariance' and 'Other invariance rules' requirements.

C.3: Dispersion integrals from splitting in BEG normalization: the central solution

For simplicity, here we restrict ourselves to the case of two four-variables, relevant for this paper. For the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^8)$ we employ the following convention:

$$f(y_1, y_2) = (2\pi)^{-4} \int dk_1\, dk_2\, e^{-i(k_1 y_1 + k_2 y_2)}\, \hat{f}(k_1, k_2). \tag{C.7}$$

Let $\Gamma_\pm := \overline{V}_\pm \times \overline{V}_\pm$ henceforth. Given a "causal distribution", that is, $d \in \mathcal{S}'(\mathbb{R}^8)$ with

$$\operatorname{supp} d \subseteq \Gamma_+ \cup \Gamma_- \quad \text{and} \quad \operatorname{sd}(d) < \infty, \tag{C.8}$$

by a *splitting solution* of d we mean a distribution $a \in \mathcal{S}'(\mathbb{R}^8)$ with

$$(a-d)\big|_{\mathcal{S}(\mathbb{R}^8\setminus\Gamma_-)} = 0, \quad \operatorname{supp} a \subseteq \Gamma_+ \quad \text{and} \quad \operatorname{sd}(a) \le \operatorname{sd}(d). \tag{C.9}$$

In what follows we assume that the Fourier transform $\hat{d}$ of the causal d-distribution we wish to split vanishes in an open ball $\mathcal{R} \subset \mathbb{R}^8$ centered at $k = 0$. This holds if all propagators contributing to d are massive, as it is the case in this paper – see [21, Sect. 5.2]. Also in [21] it is shown for any splitting solution a that $\hat{d}|_{\mathcal{R}} = 0$ entails analyticity of $\hat{a}(k)$ on $\mathcal{R}$. In this case there exists a distinguished splitting solution, the so-called *central solution* a^c, characterized by the conditions

$$\partial^a \hat{a}^c(0) = 0, \quad \text{for all} \quad |a| \le \omega(d). \tag{C.10}$$

As indicated in Eq. (C.6), for $\operatorname{sd}(d) \ge 8$ – i.e., for $\omega(d) \ge 0$, as defined in Eq. (C.2) – the splitting solution of d is not uniquely determined. Any two solutions a_1 and a_2 differ by

$$a_1(y) - a_2(y) = \sum_{|a|=0}^{\omega(d)} C_a\, \partial^a \delta(y) \quad \text{or equivalently,}$$

$$\hat{a}_1(k) - \hat{a}_2(k) = \frac{1}{(2\pi)^4} \sum_{|a|=0}^{\omega(d)} C_a\, (-ik)^a,$$

with arbitrary constants $C_a \in \mathbb{C}$.

Essential for dealing with our situation is that the central solution of the splitting problem in momentum space *can be computed by a dispersion integral*. Now we sketch the derivation of a few versions of this distinguished splitting integral. [18] The naive way to extract the advanced part a of d is to multiply the latter by a θ-function:

$$a_{\text{naive}}(y_1, y_2) := d(y_1, y_2)\, \chi(y_1, y_2) \quad \text{with}$$
$$\chi(y_1, y_2) := \theta\big((y_1 v_1) + (y_2 v_2)\big),$$

where $v := (v_1, v_2) \in V_+ \times V_+$ is arbitrary. But for $\operatorname{sd}(d) \ge 8$ the pointwise product $d\chi$ exists only as an element of $\mathcal{S}'(\mathbb{R}^8 \setminus \{0\})$. Therefore, the splitting problem *is an*

[17] That sometimes needs to be symmetrized, by adding a suitable OVD supported on Δ_{n+1}.

[18] For further detail we refer to [39, Sect. 3.2], which relies on [21, Sect. 6.5].

extension problem: we have to extend $d\chi \in \mathcal{S}'(\mathbb{R}^8 \setminus \{0\})$ to an $a \in \mathcal{S}'(\mathbb{R}^8)$ such that $\mathrm{sd}(a) = \mathrm{sd}(d\chi) = \mathrm{sd}(d)$.[19]

The problem is studied in its particulars in [23, Sect. 3.2.2]. Given a with singular order ω, there exists an obvious extension a^ω, belonging in the dual space $\mathcal{S}'_\omega(\mathbb{R}^8)$ of

$$\mathcal{S}_\omega(\mathbb{R}^8) := \{\, f \in \mathcal{S}(\mathbb{R}^8) : \partial^b f(0) = 0 \ \text{ for all } \ |b| \leq \omega \,\},$$

uniquely determined by the requirement that $\mathrm{sd}(a^\omega) = \mathrm{sd}(d)$. Next, a projection is introduced:

$$W_\omega : \mathcal{S}(\mathbb{R}^8) \longrightarrow \mathcal{S}_\omega(\mathbb{R}^8);$$
$$W_\omega f(y) := f(y) - w(y) \sum_{|b|=0}^{\omega} \frac{y^b}{b!}\, \partial^b f(0), \qquad \text{(C.11)}$$

where the suitably decaying function w must fulfil $w(0) = 1$ and $\partial^b w(0) = 0$ for $1 \leq |b| \leq \omega$. One verifies that a solution a_w (depending on the choice of the function w) of the splitting problem (C.9) is obtained by setting

$$\langle a_w \mid f \rangle := \langle a^\omega \mid W_\omega f \rangle. \qquad \text{(C.12)}$$

The a^ω involved here is $d\chi$ with enlarged domain. If furthermore assumption $d|_{\mathcal{R}} = 0$ is satisfied, the infrared behaviour of $d(y)$ is harmless. Hence, one may simply choose $w(y) = 1$ for all $y \in \mathbb{R}^4$. Then the correspondent splitting solution $a_{w=1}$ is actually the central solution a^c (C.10). Substituting $d\chi$ for a^ω and further using the convolution formula

$$\hat{f} \star \hat{\chi}(k) = (2\pi)^4 \frac{i}{2\pi} \int_{\mathbb{R}} \frac{dt}{t + i0}\, \hat{f}(k - tv)$$

and $\widehat{fg} = (2\pi)^{-4} \hat{f} \star \hat{g}$, we see that

$$\hat{a}^c(k) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{dt}{t + i0} \left[\hat{d}(k - tv) - \sum_{|b|=0}^{\omega} \frac{k^b}{b!}\, \partial^b \hat{d}(-tv) \right]. \qquad \text{(C.13)}$$

This splitting integral does not depend on the choice of $v \in V_+ \times V_+$. Moreover, for $k \in V_\eta \times V_\eta$, where $\eta \in \{+, -\}$, we may choose $v := \eta k$ – that v vary with k is admissible. With some extra work [39, Prop. 3.4], this formula is then simplified into a *convergent* dispersion integral:

$$\hat{a}^c(k) = \frac{i\eta}{2\pi} \int_{\mathbb{R}} dt\, \frac{\hat{d}(tk)}{(t - \eta i0)^{\max\{\omega+1,\,0\}}(1 - t + i\eta 0)} \quad \text{for} \quad k \in V_\eta \times V_\eta\,. \qquad \text{(C.14)}$$

In the applications treated in this paper, $d(tk)$ is of the form

$$\hat{d}(tk) = \eta\ \mathrm{sgn}(t)\, \theta(t^2 - t^2_{\min})\, f\big(t^2 k_1^2, t^2 k_2^2, t^2 (k_1 + k_2)^2\big) \quad \text{for} \quad k \in V_\eta \times V_\eta, \qquad \text{(C.15)}$$

for some $f \in \mathcal{S}'(\mathbb{R}^3)$, where $t_{\min} > 0$ depends on the squares of the momenta. So finally, introducing the new integration variable $u := t^2$, the integral (C.14) goes over into

$$\hat{a}^c(k) = \frac{i}{2\pi} \int_{t^2_{\min}}^{\infty} du\, \frac{f(uk_1^2, uk_2^2, u(k_1 + k_2)^2)}{u^{\max\{\lfloor \omega/2 \rfloor + 1,\,0\}}(1 - u + i\eta 0)} \quad \text{for} \quad k \in V_\eta \times V_\eta, \qquad \text{(C.16)}$$

where $\lfloor \cdot \rfloor$ denotes the integer part.

[19] A priori, it might happen that $\mathrm{sd}(d\chi) < \mathrm{sd}(d)$; but in the applications to Epstein–Glaser normalization known to us one always finds $\mathrm{sd}(d\chi) = \mathrm{sd}(d)$. Hence we assume the latter relation to hold true.

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