



The Hamiltonian dynamics of Hořava gravity

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Abstract We consider the Hamiltonian formulation of Hořava gravity in *arbitrary* dimensions, which has been proposed as a renormalizable gravity model for quantum gravity without the ghost problem. We study the *full* constraint analysis of the *non-projectable* Hořava gravity whose potential, $\mathcal{V}(R)$, is an arbitrary function of the (intrinsic) Ricci scalar R but without the extension terms which depend on the proper acceleration a_i . We find that there exist generally three distinct cases of this theory, **A**, **B**, and **C**, depending on (i) whether the Hamiltonian constraint generates new (second-class) constraints or just fixes the associated Lagrange multipliers, or (ii) whether the IR Lorentz-deformation parameter λ is at the conformal point or not. It is found that, for Cases **A** and **C**, the dynamical degrees of freedom are the same as in general relativity, while, for Case **B**, there is *one additional phase-space degree of freedom*, representing an extra (odd) scalar graviton mode. This would achieve the dynamical consistency of a restricted model at the *fully non-linear* level and be a positive result in resolving the long-standing debates about the extra graviton modes of the Hořava gravity. Several exact solutions are also studied as some explicit examples of the new constraints. The structure of the newly obtained, “extended” constraint algebra seems to be generic to Hořava gravity and its general proof would be a challenging problem. Some other challenging problems, which include the path integral quantization and the Dirac bracket quantization are discussed also.

1 Introduction

11 years ago, Hořava proposed a renormalizable, higher-derivative gravity theory, without the ghost problem in the usual *tensor* graviton modes, which reduces to Einstein gravity in low energy (IR) at the full action level but with improved

high-energy (UV) behaviors, by abandoning the Lorentz symmetry from non-equal-footing treatment of space and time [1, 2]. However, due to absence of the full diffeomorphisms (*Diff*), extra graviton modes can be expected generally and there have been questions about the recovery of general relativity (GR) in IR, and more generally, the consistency of Hořava gravity [3–16].

In this paper, we reconsider the Hamiltonian formulation of the *non-projectable* Hořava gravity whose potential, $\mathcal{V}(R)$, is an arbitrary function of the (intrinsic) Ricci scalar R in arbitrary dimensions but without the extension terms which depend on the proper acceleration $a_i = \partial_i N / N$, for simplicity. We study the *full* constraint analysis and find that there exist generally three distinct cases of this theory, **A**, **B**, and **C**, depending on (i) whether the Hamiltonian constraint generates new (second-class) constraints, or just fixes the associated Lagrange multipliers, according to Dirac’s method [17], or (ii) whether the IR Lorentz-deformation parameter λ is at the conformal point, $\lambda = 1/D$, or not. It is found that, for Cases **A** and **C**, the dynamical degrees of freedom are the same as in general relativity, while, for Case **B**, there is *one additional phase-space degree of freedom*, representing an extra (odd) scalar graviton mode. This would achieve the dynamical consistency of a restricted model at the *fully non-linear* level and be a positive result in resolving the long-standing debates about the extra graviton modes of the Hořava gravity.

The organization of the paper is as follows. In Sect. 2, we consider the set-up for the Hamiltonian formulation of the non-projectable Hořava gravity in arbitrary dimensions. In Sect. 3, we consider Dirac’s constraint analysis when $\lambda \neq 1/D$ for the IR Lorentz-deformation parameter λ and study two cases, **A** and **B**, depending on whether the Hamiltonian constraint generates new (second-class) constraints or just fixes the associated Lagrange multipliers. In Sect. 4, we consider the third case, **C**, when $\lambda = 1/D$, which is at a conformal point and generates new (second-class) constraints similar to Case **A**. We show that the dynamical degrees of

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freedom are the same as in GR for Cases **A** and **C**, while there is one additional phase-space degree of freedom for Case **B** which representing an extra (odd) scalar graviton mode. In Sect. 5, we study several exact solutions as some explicit examples of the new constraints. In Sect. 6, we conclude with remarks on several challenging theoretical problems, which include the path integral quantization and the Dirac bracket quantization.

2 The Hamiltonian formulation in arbitrary dimensions: set-up

In this section, we consider the Hamiltonian formulation of the non-projectable Hořava gravity in arbitrary dimensions. To this ends, we start by considering the ADM decomposition of the metric [18]

$$ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt) \quad (1)$$

with the arbitrary space-time dependent lapse and shift functions¹, N , N^i , and induced metric g_{ij} ($i, j = 1, 2, \dots, D$) on a time-slicing hypersurface Σ_t . Then, the action on a $(D+1)$ -dimensional manifold \mathcal{M} with the boundary $\partial\mathcal{M}$ is given by

$$S = \int_{\mathcal{M}} dt d^D x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \mathcal{V}[g_{ij}] \right\} + S_{\partial\mathcal{M}} \quad (2)$$

with an appropriate boundary action² $S_{\partial\mathcal{M}}$ and the potential $\mathcal{V}[g_{ij}]$, which depends only on the metric g_{ij} and its spatial derivatives [1, 2]³. Here,

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \quad (3)$$

is the extrinsic curvature (the overdot $\dot{}$ denotes the time derivative) of the hypersurface Σ_t and $K \equiv g_{ij} K^{ij}$ denotes its trace.

¹ For the projectable case, where the lapse function depends only on time coordinate, i.e., $N(t)$, there is no “local” Hamiltonian constraint so that there is no smooth way to recover the usual constraint algebra in GR [1, 6]. Later, we will comment about other alternative formulation which can provide a smooth limit to the non-projectable case.

² The explicit form of boundary terms is not essential in this paper and will not be considered in detail. But, due to the similarity to GR or its Lorentz invariant higher-curvature gravity theories, the required boundary terms are also quite close [19], whose IR limit agrees with those of [20, 21], for example.

³ The UV Lorentz violation due to higher-spatial-derivative terms in the potential part is originated from the works of Lifshitz [22] in the study of, so called, “Lifshitz field theories”. On the other hand, the IR Lorentz violation due to the deformation parameter $\lambda \neq 1$ in the kinetic part was first studied by DeWitt in the study of “canonical quantum gravity” [23]. So, it would be desirable to call the bulk part of action (2) as *DeWitt-Hořava-Lifshitz (DHL)*’s action, more precisely.

It is well known that GR is not renormalizable in the conventional way. Its Lorentz invariant higher-curvature modifications have some improvements of UV behaviors but there are more dynamical degrees of freedom than those of GR generally, and the existence of *negative* kinetic energy modes in the additional degrees of freedom, called *ghost* modes, is unavoidable due to higher-time derivatives [24–26]. In order to avoid the *possible* problems of ghost degrees of freedom, we do not simply consider the higher-time-derivative terms, like K^4 , $(K_{ij} K^{ij})^2$, etc., but only consider the second-order time-derivative terms, like $K_{ij} K^{ij}$ and K^2 , in the kinetic part of the action (2). Whereas, in order to achieve the renormalizable theory with the improved UV behaviors, we consider the higher-spatial-derivative terms, like R^2 , $R_{ij} R^{ij}$, etc., in the potential part $\mathcal{V}[g_{ij}]$ with the intrinsic Ricci curvature R_{ij} and its trace $R \equiv g_{ij} R^{ij}$. In order that the theory be *power-counting* renormalizable, the potential part needs to contain “ $2 \times D$ ” (spatial) derivatives at least, which is sometimes represented by the dynamical critical exponent, $z = D$ [2]. In order that this construction of a renormalizable action is *not* spoiled by the mixing of space and time (derivatives) in the general coordinate transformations, we need to further constrain the allowed coordinate transformations into the foliation-preserving diffeomorphisms (*Diff_F*),

$$\begin{aligned} \delta x^i &= -\zeta^i(t, \mathbf{x}), \quad \delta t = -f(t), \\ \delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f \dot{g}_{ij}, \\ \delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + f \dot{N}_i + \dot{f} N_i, \\ \delta N &= \zeta^j \partial_j N + f \dot{N} + \dot{f} N. \end{aligned} \quad (4)$$

In *Diff_F*, each term in the kinetic part is invariant separately and λ can be arbitrary⁴ [23]. If we consider $\lambda = 1$ and $\mathcal{V}[g_{ij}] = -(2/\kappa^2)R - \Lambda$ as in GR, then there is an “*accidental*” symmetry enhancement which mixes each term in the action so that the full *Diff* is recovered [8]. So, there are two sources of the Lorentz violations, either from the higher-spatial-derivative (UV) terms in the potential part or from the deformation of kinetic part with an arbitrary λ in IR, generally.

For the potential part, one may consider any function having $2D$ spatial derivatives for the power-counting renormalizability, but in this paper we consider only the function of Ricci curvature scalar, R , i.e., $\mathcal{V}[g_{ij}] \equiv \mathcal{V}(R)$, for simplicity⁵. Then, the first-order formulation of the action (2) is given by

⁴ For the case $\lambda = 1/D$, where the theory becomes singular, a separate consideration is needed [10, 21]. We will consider this case later, in Sect. 4.

⁵ The terms of $\nabla^2 R$, $R_{ij} R^{ij}$, etc. could also produce some other peculiar UV behaviors due to ingenious combinations of terms depending on space-time dimensions, but we will not consider this possibility in this paper.

$$S = \int_{\mathcal{M}} dt d^D x \left\{ \pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N_i \mathcal{H}^i - \partial_i \gamma^i \right\} \quad (5)$$

with appropriate boundary terms, $\partial_i \gamma^i$, the conjugate momenta,

$$\pi^{ij} \equiv \frac{\delta S}{\delta \dot{g}_{ij}} = \frac{2\sqrt{g}}{\kappa^2} \left(K^{ij} - \lambda K g^{ij} \right), \quad (6)$$

and

$$\mathcal{H} \equiv \frac{\kappa^2}{2\sqrt{g}} \left[\pi^{ij} \pi_{ij} - \left(\frac{\lambda}{D\lambda - 1} \right) \pi^2 \right] + \sqrt{g} \mathcal{V}, \quad (7)$$

$$\mathcal{H}^i \equiv -2\nabla_j \pi^{ij}, \quad (8)$$

where $\pi \equiv g_{ij} \pi^{ij}$. Here, we first consider the case $\lambda \neq 1/D$ so that \mathcal{H} in (7) and the first-order action (5) are not singular.

The Poisson brackets for the canonical variables are given by

$$\{g_{ij}(x), \pi^{kl}(y)\} = \delta_{ij}^{kl} \delta^D(x - y) \quad (9)$$

with $\delta_{ij}^{kl} \equiv (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k)/2$.

3 Constraint analysis à la Dirac ($\lambda \neq 1/D$)

The primary constraints of the action (2) are given by

$$\Phi^t \equiv \pi_N \approx 0, \quad \Phi^i \equiv \pi^i \approx 0, \quad (10)$$

from the definition of conjugate momenta, $\pi_N \equiv \delta S / \delta \dot{N}$ and $\pi^i \equiv \delta S / \delta \dot{N}^i$. Here, the weak equality ‘ \approx ’ means that the constraint equations are used only after calculating the Poisson brackets.

The preservation of the primary constraints, $\Phi^\mu \equiv (\Phi^t, \Phi^i)$, i.e., $\dot{\Phi}^\mu = \{\Phi^\mu, H_C\} \approx 0$, as being required by the consistency of the constraints, with the canonical Hamiltonian,

$$H_C = \int_{\Sigma_t} d^D x \left\{ N\mathcal{H} + N_i \mathcal{H}^i \right\} + H_B \quad (11)$$

produces the secondary constraints,

$$\mathcal{H} \approx 0, \quad \mathcal{H}^i \approx 0. \quad (12)$$

Here, H_B is an appropriate boundary Hamiltonian, $H_B = \oint_{\mathcal{B}} d^D x \hat{n}_i \gamma^i$, for the intersection \mathcal{B} of an arbitrary time-like boundary (\hat{n}_i is its unit normal) with a time slice Σ_t so that the total boundaries are $\partial \mathcal{M} = \Sigma_{t_f} \cup \Sigma_{t_i} \cup \mathcal{B}$. On the other hand, the dynamical equations for g_{ij} and π^{kl} are obtained as follows, neglecting boundary contributions (see Appendix A for the details),

$$\begin{aligned} \dot{g}_{ij} &= \{g_{ij}, H_C\} = \frac{\delta H_C}{\delta \pi^{ij}} \\ &= \left(\frac{\kappa^2}{2} \right) \frac{2N}{\sqrt{g}} (\pi_{ij} - \tilde{\lambda} g_{ij} \pi) + \nabla_i N_j + \nabla_j N_i, \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{\pi}^{ij} &= \{\pi^{ij}, H_C\} = -\frac{\delta H_C}{\delta g_{ij}} \\ &= \left(\frac{\kappa^2}{2} \right) \frac{N}{\sqrt{g}} \left[\frac{1}{2} g^{ij} (\pi_{mn} \pi^{mn} - \tilde{\lambda} \pi^2) \right. \\ &\quad \left. - 2 (\pi^{im} \pi_m^j - \tilde{\lambda} \pi \pi^{ij}) \right] \\ &\quad - N \sqrt{g} \left[\frac{1}{2} g^{ij} \mathcal{V}(R) - R^{ij} \mathcal{V}'(R) \right] \\ &\quad - \sqrt{g} \left[\nabla^i \nabla^j (N \mathcal{V}'(R)) - g^{ij} \nabla_m \nabla^m (N \mathcal{V}'(R)) \right] \\ &\quad + \nabla_m (N^m \pi^{ij}) - (\nabla_m N^i) \pi^{jm} - (\nabla_m N^j) \pi^{im}, \end{aligned} \quad (14)$$

where $\tilde{\lambda} \equiv \lambda / (D\lambda - 1)$ and $(\cdot)' \equiv d(\cdot) / dR$.

With the primary constraints in (10), one can consider the extended Hamiltonian with the Lagrange multipliers u_μ ,

$$H_E = H_C + \int_{\Sigma_t} d^D x (u_\mu \Phi^\mu), \quad (15)$$

from the arbitrariness in the equations of motion, due to the primary constraints.

Then, after tedious computations, we obtain the following constraint algebra (see Appendix A for the details),

$$\begin{aligned} \{\mathcal{H}(x), \mathcal{H}(y)\} &= C^i(x) \nabla_i^x \delta^D(x - y) \\ &\quad - C^i(y) \nabla_i^y \delta^D(x - y), \end{aligned} \quad (16)$$

$$\{\mathcal{H}(x), \mathcal{H}_i(y)\} = -\mathcal{H}(y) \nabla_i^y \delta^D(x - y), \quad (17)$$

$$\begin{aligned} \{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_i(y) \nabla_j^x \delta^D(x - y) \\ &\quad + \mathcal{H}_j(x) \nabla_i^x \delta^D(x - y), \end{aligned} \quad (18)$$

or, for the smeared constraints, $\langle \eta \mathcal{H} \rangle \equiv \int d^D x \eta \mathcal{H}$, etc., with the smearing functions η and η_i ,

$$\{\langle \eta \mathcal{H} \rangle, \langle \zeta \mathcal{H} \rangle\} = \langle (\eta \nabla_i \zeta - \zeta \nabla_i \eta) C^i \rangle, \quad (19)$$

$$\{\langle \eta \mathcal{H} \rangle, \langle \zeta^i \mathcal{H}_i \rangle\} = -\langle \zeta^i \nabla_i \eta \rangle, \quad (20)$$

$$\{\langle \eta^i \mathcal{H}_i \rangle, \langle \zeta^j \mathcal{H}_j \rangle\} = \left\langle \left(\eta^i \nabla_i \zeta^j - \zeta^i \nabla_i \eta^j \right) \mathcal{H}_j \right\rangle, \quad (21)$$

where⁶

$$\begin{aligned} C^i &\equiv \sqrt{g} \mathcal{V}'^2(R) \nabla_j \left(\frac{K^{ij}}{\mathcal{V}'(R)} \right) \\ &= \left(\frac{\kappa^2}{2} \right) 2\mathcal{V}'^2(R) \nabla_j \left(\frac{\pi^{ij} - \tilde{\lambda} g^{ij} \pi}{\mathcal{V}'(R)} \right) \end{aligned}$$

⁶ From the Jacobi's identity, one may obtain some non-trivial relations about C^i . For example, from $\{\langle \eta \mathcal{H} \rangle, \langle \zeta \mathcal{H} \rangle\}, \langle \rho \mathcal{H} \rangle + \{\langle \zeta \mathcal{H} \rangle, \langle \rho \mathcal{H} \rangle\}, \langle \eta \mathcal{H} \rangle + \{\langle \rho \mathcal{H} \rangle, \langle \eta \mathcal{H} \rangle\}, \langle \zeta \mathcal{H} \rangle = 0$, one obtains “ $\langle (\eta \nabla_i \zeta - \zeta \nabla_i \eta) \Theta^i |_{N \rightarrow \rho} \rangle + (\text{cyclic permutations about } \eta, \zeta, \rho) = 0$ ”, where Θ^i is defined by $\Theta^i \equiv \{C^i, \langle N \mathcal{H} \rangle\}$ and its explicit form is given by (34), wherein the momentum constraint, $\mathcal{H}_i \approx 0$, is imposed.

$$= \left(-\frac{\kappa^2}{2}\right) \left[\left(\mathcal{H}^i + 2\hat{\lambda} \nabla^i \pi \right) \mathcal{V}'(R) + 2 \left(\pi^{ij} - \hat{\lambda} g^{ij} \pi \right) \nabla_j \mathcal{V}'(R) \right] \quad (22)$$

with $\hat{\lambda} \equiv (\lambda - 1)/(D\lambda - 1)$. If we consider $-\mathcal{V}(R) \equiv \Lambda + \xi R + \alpha R^n$ as a typical example with an arbitrary power n , we can obtain

$$C^i = \left(\frac{\kappa^2}{2}\right) \left[\left(\mathcal{H}^i + 2\hat{\lambda} \nabla^i \pi \right) \left(\xi + \alpha n R^{n-1} \right) + 2 \left(\pi^{ij} - \hat{\lambda} g^{ij} \pi \right) \alpha n \nabla_j R^{n-1} \right]. \quad (23)$$

Note that, when the higher-derivative contributions are absent, i.e., $\alpha = 0$, (23) reduces to $C^i = (\kappa^2/2)\xi \mathcal{H}^i \approx 0$ and the Hamiltonian constraint, $\mathcal{H} \approx 0$, becomes the first-class constraint as in GR, for either (i) $\lambda = 1$, i.e., $\hat{\lambda} = 0$, or (ii) $\pi \approx 0$, representing the *maximal* slicing, for an arbitrary λ [18]. However, when the higher-derivative terms are present, C^i does not vanish generally, so that “the Hamiltonian constraint, $\mathcal{H} \approx 0$, becomes the second-class constraint” even for the maximal slicing, $\pi \approx 0$, due to the genuine dynamical degrees of freedom in π^{ij} , other than the trace part, π .

On the other hand, we note that the momentum constraint, $\mathcal{H}_i \approx 0$, in (12) has the same *canonical* form as in GR with no higher-derivative corrections⁷ so that we also have the same constraint algebra for \mathcal{H}_i , (18) or (21), which representing the spatial *Diff* generator,

$$\delta_\zeta g_{ij} = \{g_{ij}, \langle \zeta^k \mathcal{H}_k \rangle\} = \frac{\delta \langle \zeta^k \mathcal{H}_k \rangle}{\delta \pi^{ij}} = \nabla_i \zeta_j + \nabla_j \zeta_i, \quad (24)$$

$$\delta_\zeta \pi^{ij} = \{\pi^{ij}, \langle \zeta^k \mathcal{H}_k \rangle\} = -\frac{\delta \langle \zeta^k \mathcal{H}_k \rangle}{\delta g_{ij}} = \nabla_m \left(\zeta^m \pi^{ij} \right) - \left(\nabla_m \zeta^i \right) \pi^{jm} - \left(\nabla_m \zeta^j \right) \pi^{im}, \quad (25)$$

as in GR. Moreover, the algebras (17) and (18) show that

$$\delta_\zeta \mathcal{H} = \{\mathcal{H}, \langle \zeta^k \mathcal{H}_k \rangle\} = \nabla_k \left(\zeta^k \mathcal{H} \right), \quad (26)$$

$$\delta_\zeta \mathcal{H}_i = \{\mathcal{H}_i, \langle \zeta^k \mathcal{H}_k \rangle\} = \nabla_k \left(\zeta^k \mathcal{H}_i \right) + \left(\nabla_i \zeta^k \right) \mathcal{H}_k, \quad (27)$$

which tells that \mathcal{H} and \mathcal{H}_i behave as, under the spatial *Diff*, the scalar and vector densities, respectively, as in GR.

Using the above constraint algebra, one can easily find that the preservation of the secondary constraints give

$$\begin{aligned} \dot{\mathcal{H}} &= \{\mathcal{H}, H_E\} \\ &= \frac{1}{N} \nabla_i (N^2 C^i) + \nabla_i (N^i \mathcal{H}) \approx \frac{1}{N} \nabla_i (N^2 \tilde{C}^i), \end{aligned} \quad (28)$$

⁷ This is a key observation for the proof of Birkhoff's theorem in Hořava gravity [27], in contrast to other general higher-curvature gravities [25, 26]. But the IR Lorentz-deformation parameter λ enters still in the momentum constraint through the relation (6).

$$\begin{aligned} \dot{\mathcal{H}}_i &= \{\mathcal{H}_i, H_E\} \\ &= \mathcal{H} \nabla_i N + \nabla_j (N^j \mathcal{H}_i) + \mathcal{H}_j \nabla_i N^j \approx 0, \end{aligned} \quad (29)$$

which produces the *tertiary* constraint,

$$\tilde{\Omega} \equiv \nabla_i (N^2 \tilde{C}^i) \approx 0, \quad (30)$$

where

$$\begin{aligned} \tilde{C}^i &\equiv C^i|_{\mathcal{H}_i \approx 0} = \left(\frac{\kappa^2}{2}\right) \left[2\hat{\lambda} \nabla^i \pi \left(\xi + \alpha n R^{n-1} \right) + 2 \left(\pi^{ij} - \hat{\lambda} g^{ij} \pi \right) \alpha n \nabla_j R^{n-1} \right] \end{aligned} \quad (31)$$

from the preservation of $\mathcal{H} \approx 0$ in (28), excluding the trivial case of $N = 0$ for all space-time. Here we note that, in the above computations, there are no contributions from the multiplier terms in H_E so that we need to consider further steps of preserving the constraints until the multipliers are determined.

Now, one more step of preserving the new tertiary constraint, $\tilde{\Omega} \approx 0$, gives

$$\begin{aligned} \dot{\tilde{\Omega}} &= \{\tilde{\Omega}, H_E\} \\ &= \{\tilde{\Omega}, H_C\} + 2\tilde{\Omega} \left(\frac{u_t}{N} \right) + 2\tilde{C}^i N^2 \nabla_i \left(\frac{u_t}{N} \right) \\ &\approx \{\tilde{\Omega}, H_C\} + 2\tilde{C}^i N^2 \nabla_i \left(\frac{u_t}{N} \right) \approx 0. \end{aligned} \quad (32)$$

Then, there are two different constraint systems, with different subsequent procedures, depending on whether $\tilde{C}^i = 0$ or $\tilde{C}^i \neq 0$.

A. Case $\tilde{C}^i = 0$: In this case⁸, (32) does not determine the multiplier u_t but reduces to

$$\begin{aligned} \dot{\tilde{\Omega}} &\approx \{\tilde{\Omega}, H_C\} \\ &\approx \nabla_i \left(N^2 \tilde{\Theta}^i \right) \equiv \tilde{\Xi} \approx 0 \end{aligned} \quad (33)$$

for preserving the tertiary constraint $\tilde{\Omega}$ with

$$\begin{aligned} \tilde{\Theta}^i &\equiv \{\tilde{C}^i, \langle N \mathcal{H} \rangle\} \\ &= -\left(\frac{\kappa^2}{2}\right)^2 \frac{2\hat{\lambda}}{(\lambda D - 1)\sqrt{g}} \mathcal{V}'(R) \left[\left[(2\lambda + 1) g^{ij} \pi - 2(\lambda D - 1) \pi^{ij} \right] N \nabla_j \pi + \pi^2 \nabla^i N \right] \\ &\quad - \left(\frac{\kappa^2}{2}\right) 2\hat{\lambda} (D - 1) \sqrt{g} \mathcal{V}'(R) \nabla^i \left[\left(\xi R + \frac{\Lambda D}{D - 1} + \frac{D - n}{D - 1} \alpha R^n \right) N + \nabla^2 (N \mathcal{V}') \right] \end{aligned}$$

⁸ Here, the condition $\tilde{C}^i = 0$ does not necessarily mean the constraint equation, which is stronger than the original constraint (30). Actually, the constraint (30) implies that all the components of \tilde{C}^i are not independent so that the condition $\tilde{C}^i = 0$ may be subject to more fundamental conditions or constraints.

$$- \left(\frac{\kappa^2}{2} \right) \left[2\hat{\lambda} \nabla_i \pi \{ \mathcal{V}'(R), \langle N \mathcal{H} \rangle \} \right. \\ \left. + \{ 2(\pi^{ij} - \hat{\lambda} g^{ij} \pi) \nabla_j \mathcal{V}'(R), \langle N \mathcal{H} \rangle \} \right], \quad (34)$$

where $-\mathcal{V}'(R) \equiv (\xi + \alpha n R^{n-1})$. Here, the higher-derivative contributions come, either from the constraint (30), $\tilde{\Omega}$, i.e., \tilde{C}^i in (31), or from the Hamiltonian constraint \mathcal{H} in (7) and (11). One can compute the explicit forms of the higher-derivative contributions with the help of (13) and (14) (see Appendix B for some more details; see also [15] for the case of $n = 2$) but, due to its messy expression, we will not consider the explicit forms in the analysis below unless it is crucial.

Then, one more time-evolution of the new constraint, $\tilde{\Xi} \approx 0$, will read,

$$\tilde{\Xi} = \{ \tilde{\Xi}, H_E \} \\ = \{ \tilde{\Xi}, H_C \} + \nabla_i \left(N^2 \tilde{\Theta}^i|_{N \rightarrow u_t} + 2N u_t \tilde{\Theta}^i \right) \approx 0. \quad (35)$$

After a long computation, we obtain (see also Appendix B for the details)⁹

$$\nabla_i \left[N^2 \tilde{\Theta}^i|_{N \rightarrow (u_t - N_i \nabla^i N)} + 2(u_t - N_i \nabla^i N) N \tilde{\Theta}^i \right] \\ \approx -\{ \tilde{\Xi}, \bar{H}_C \} = \left(\frac{\kappa^2}{2} \right)^2 4\xi \hat{\lambda} (D-1) \nabla_i \\ \times \left\{ N^2 \left[2N \pi^{ij} \nabla_j \left[\left(\xi R + \frac{\Lambda D}{D-1} \right) N - \xi \nabla^2 N \right] \right. \right. \\ \left. \left. + \xi \nabla^i \left[\pi^{jk} \left(R_{jk} N^2 - 2N \nabla_j \nabla_k N - \nabla_j N \nabla_k N \right) \right] \right] \right\} \\ - \nabla_j \left(N^j \tilde{\Xi} \right) \\ + (\pi, \nabla_i \pi - \text{dependent terms}) \\ + (\text{higher-derivative contributions}). \quad (36)$$

Here, it is important to note that the multipliers, u_t , $\nabla^i u_t$, etc., have generally non-vanishing coefficients in the left-hand side so that (36) may provide the equation for determining the multiplier u_t . However, the similar equation for u_i does not exist and u_i is still undetermined but this is just a reflection of the first-class nature of the constraint, $\pi^i \approx 0$, in (10). This would now complete the Dirac's procedure for finding the complete set of constraints, though we would not try to solve for the explicit solution of the multiplier u_t . Then, the full set of constraints are given by $\chi_A \equiv (\pi_N, \mathcal{H}, \tilde{\Omega}, \tilde{\Xi}) \approx 0$, $\Gamma_B \equiv (\pi^i, \mathcal{H}_i) \approx 0$. Here, the constraints $\chi_A \approx 0$ are the

⁹ For convenience, we consider the modified momentum constraint [28], $\bar{\mathcal{H}}^i \equiv \mathcal{H}^i + \pi_N \nabla^i N$, by redefining the multiplier, $u_t \rightarrow u_t - N_i \nabla^i N$ in (15) so that $H_E = \bar{H}_C + \langle (u_t - N_i \nabla^i N) \pi_N + u_i \pi^i \rangle$, $\bar{H}_C \equiv \langle N \mathcal{H} + N_i \bar{\mathcal{H}}^i \rangle$ and $\{ \bar{\Xi}, \langle N_j \mathcal{H}^j \rangle \} = \nabla_j (N^j \tilde{\Xi})$. In this way, one can compactly collect all N_i -dependent terms in the left hand side, up to the weakly vanishing term, $\nabla_j (N^j \tilde{\Xi}) \approx 0$. Interestingly, this modified constraint $\bar{\mathcal{H}}^i$ satisfies the same constraint algebra (16)–(18) or (19)–(21).

second-class constraints with the constraint algebra,

$$\{ \pi_N(x), \mathcal{H}(y) \} = 0, \\ \{ \pi_N(x), \tilde{\Omega}(y) \} = -2 \nabla_i^y \left(N \tilde{C}^i(y) \delta^D(x-y) \right) \approx 0, \\ \{ \pi_N(x), \tilde{\Xi}(y) \} = \Delta(x-y), \\ \{ \mathcal{H}(x), \mathcal{H}(y) \} = C^i(x) \nabla_i^x \delta^D(x-y) \\ - C^i(y) \nabla_i^y \delta^D(x-y) \approx 0, \\ \{ \mathcal{H}(x), \tilde{\Omega}(y) \} \approx \{ \pi_N(x), \tilde{\Xi}(y) \}, \text{ etc.}, \quad (37)$$

whose determinant, $\det\{\chi_A, \chi_B\}$, is generally non-vanishing,

$$\det\{\chi_A(x), \chi_B(y)\} \\ \approx (\det\{\mathcal{H}(x), \tilde{\Omega}(y)\})^2 (\det\{\tilde{\Omega}(x), \mathcal{H}(y)\})^2 \\ \approx (\det[\Delta(x-y) \Delta(y-x)])^2, \quad (38)$$

where

$$\Delta(x-y) \equiv -\nabla_i^y \left[2N \tilde{\Theta}^i(y) \delta^D(x-y) \right. \\ \left. + N^2(y) \left(\frac{\delta \tilde{\Theta}^i(y)}{\delta N(x)} \right) \right] \\ = -2 \tilde{\Xi} \delta^D(x-y) - 2N^2 \tilde{\Theta}^i(x) \nabla_i^x \left(\frac{\delta^D(x-y)}{N(x)} \right) \\ - \nabla_i^y \left[N^2(y) \left(\frac{\delta \tilde{\Theta}^i(y)}{\delta N(x)} \right) \right] \quad (39)$$

with

$$\left(\frac{\delta \tilde{\Theta}^i(y)}{\delta N(x)} \right) = - \left(\frac{\kappa^2}{2} \right)^2 \frac{2\hat{\lambda} \mathcal{V}'}{(\lambda D - 1) \sqrt{g}} \left\{ [(2\lambda + 1) g^{ij} \pi \right. \\ \left. - 2(\lambda D - 1) \pi^{ij}] \left(\nabla_j \pi(y) + \pi^2 \nabla_j^y \right) \delta^D(x-y) \right\} \\ - \left(\frac{\kappa^2}{2} \right) 2\hat{\lambda} (D-1) \sqrt{g} \mathcal{V}' \nabla_j^y \left[\left(\xi R + \frac{\Lambda D}{D-1} \right) \right. \\ \left. + \frac{D-n}{D-1} \alpha R^n \right] \delta^D(x-y) + \nabla^2 \left(\mathcal{V}' \delta^D(x-y) \right) \\ - \left(\frac{\kappa^2}{2} \right) \left[2\hat{\lambda} \nabla_i \pi \{ \mathcal{V}'(R)(y), \mathcal{H}(x) \} \right. \\ \left. + \{ 2(\pi^{ij} - \hat{\lambda} g^{ij} \pi) \nabla_j \mathcal{V}'(R)(y), \mathcal{H}(x) \} \right]. \quad (40)$$

On the other hand, the constraints, $\Gamma_A \equiv (\pi^i, \mathcal{H}_i) \approx 0$, are the first-class constraints with the vanishing determinant, $\det(\{\Gamma_A, \Gamma_B\}) = 0$. Then, the resulting number of dynamical degrees of freedom in the “configuration” space is given by

$$s = \frac{1}{2} (P - 2N_1 - N_2) \\ = \frac{1}{2} [(D+1)(D+2) - 2 \times 2D - “4”] \\ = \frac{1}{2} (D+1)(D-2), \quad (41)$$

where $P = (D + 1)(D + 2)$ is the number of canonical variables in the “phase” space $(N, \pi_N, N_i, \pi^i, g_{ij}, \pi_{ij})$, $N_1 = 2D$ is the number of the first-class constraints $(\pi^i, \mathcal{H}_i) \approx 0$, and $N_2 = “4”$ is the number of the second-class constraints, $(\pi_N, \mathcal{H}, \tilde{\Omega}, \tilde{\Xi}) \approx 0$. Note that, for Case **A**, the dynamical degrees of freedom are the same as that of GR (in arbitrary dimensions) though the constraint structure is different:¹⁰ Actually, in GR, i.e., $\hat{\lambda} = 0$, $\alpha = 0$ or the λ -deformed GR (λR model) with the condition, $\pi = 0$, we have $N_1 = 2(D + 1)$, $N_2 = 0$ so that the 2 *first-class* constraints, $(\pi_N, \mathcal{H}) \approx 0$, in GR or λ -deformed GR, transform into the 4 *second-class* constraints, $(\pi_N, \mathcal{H}, \tilde{\Omega}, \tilde{\Xi}) \approx 0$, in the Case **A** of full Hořava gravity, with maintaining the same dynamical degrees of freedom s . This completes the previous linear analysis in [10,33], but now at the “fully non-linear” level. (cf. [14,27]).

B. Case $\tilde{C}^i \neq 0$: This is the more generic case where the conjugate momenta π^{ij} and the (scalar) curvature R are arbitrary, with the generic higher-derivative potential, $\mathcal{V}(R)$. In this case, (32) does not yield new constraints but determines the multiplier u^i generally¹¹ so that the Dirac’s procedure may be completed, without further iterations. Then, in contrast to Case **A**, there are the second-class constraints, $\tilde{\chi}_A \equiv (\pi_N, \mathcal{H}, \Omega) \approx 0$, whose determinant $\det(\{\tilde{\chi}_A, \tilde{\chi}_B\})$ is non-vanishing, generally,

$$\begin{aligned} \det\{\tilde{\chi}_A(x), \tilde{\chi}_B(y)\} &= -\det\{\pi_N(x), \tilde{\Omega}(y)\} \\ &\quad \times \det\{\mathcal{H}(x), \mathcal{H}(y)\} \det\{\tilde{\Omega}(x), \pi_N(y)\} \\ &\approx 4 \det(N\tilde{C}^j(y)\nabla_j^y \delta^D(x-y)) \\ &\quad \times \det(N\tilde{C}^k(x)\nabla_k^x \delta^D(x-y)) \det(C^i(x)\nabla_i^x \delta^D(x-y)) - (x \leftrightarrow y), \end{aligned} \quad (42)$$

whereas the first-class constraints, $\Gamma_A \equiv (\pi^i, \mathcal{H}_i)$, are the same as in Case **A**. Hence, the resulting number of dynamical degrees of freedom is

$$\begin{aligned} s &= \frac{1}{2} [(D + 1)(D + 2) - 2 \times 2D - “3”] \\ &= \frac{1}{2} (D + 1)(D - 2) + \frac{1}{2}, \end{aligned} \quad (43)$$

with $N_1 = 2D$ and $N_2 = “3”$, which shows *one extra degree of freedom in phase space*, in addition to the usual $(D + 1)(D - 2)$ graviton (transverse traceless) modes in arbitrary $(D + 1)$ -dimensions.¹² In particular, in $(2 + 1)$ -dimensions,

¹⁰ Recent constructions of, so called, “minimally-modified” gravity theories [29–32] may correspond to this case also.

¹¹ For some detailed discussion about the determination of the multiplier u^i , see [15] (see also [13] for an earlier discussion).

¹² Here, we do not consider the extension terms which depend on the proper acceleration, $a_i = \partial_i N/N$, for simplicity. If we include these terms, in addition to the standard action (2) [1,2], the extra modes have “two” phase space degrees of freedom, like the ordinary scalar fields

the extra mode is the only dynamical degree of freedom. This result supports the previous case-by-case results [15,27] but in a more generic set-up with arbitrary dimensions and cosmological constant.

The usual increase of dynamical degrees of freedom with higher-time derivatives is the result of the competition between the increased canonical momenta for the higher-time derivative fields and their increased, associated new constraints [25,26]. However, for Hořava gravity, there are no increased canonical momenta but exist only the increased *second-class* constraints: For Case **A**, the increased second-class constraints are enough to preserve the dynamical degrees of freedom of GR, whereas for Case **B**, they are not enough and one extra degree of freedom persists in the phase-space.

4 Constraint analysis at the conformal point, $\lambda = 1/D$: Case C

The kinetic part of Hořava action (2) can be written formally as,

$$S_K \equiv \left(\frac{2}{\kappa^2}\right) \int dt d^D x \sqrt{g} N (K_{ij} G^{ijkl} K_{kl}), \quad (44)$$

where $G^{ijkl} \equiv \delta^{ijkl} - \lambda g^{ij} g^{kl}$ is the (generalized) DeWitt metric [1,2,23]¹³. In the previous sections, we have considered the case, $\lambda \neq 1/D$, so that the DeWitt metric is not degenerated. Then the first-order action (5) can be obtained by considering the Legendre transformation of S_K ,

$$S_K = \left(\frac{\kappa^2}{2}\right) \int dt d^D x \frac{N}{2\sqrt{g}} (\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl}), \quad (45)$$

with the conjugate momenta, $\pi^{ij} = (2\sqrt{g}/\kappa^2) G^{ijkl} K_{kl}$, and the *inverse* DeWitt metric, $\mathcal{G}_{ijkl} = \delta_{ijkl} - \lambda g_{ij} g_{kl}/(D\lambda - 1)$, satisfying $G^{ijmn} \mathcal{G}_{mnkl} = \mathcal{G}_{klmn} G^{mni} = \delta_{kl}^{ij}$ [1,2].

On the other hand, for $\lambda = 1/D$, the DeWitt metric is degenerated and we need to project out the non-degenerate parts only when considering the appropriate inverse of the DeWitt metric. Actually, using the fact that $\hat{G}^{ijkl} \equiv G^{ijkl}|_{\lambda=1/D}$ has a null eigenvector g_{ij} ,

$$\hat{G}^{ijkl} g_{ij} = 0, \quad (46)$$

it is easy to see that its inverse, $\hat{\mathcal{G}}_{ijkl}$, is given by

$$\hat{\mathcal{G}}_{ijkl} = \delta_{ijkl} - \frac{1}{D} g_{ij} g_{kl},$$

[7,20,21,34], and this may become another different case, say, Case **D**. Actually, this corresponds to an alternative formulation of the projectable case but now a smooth limit to the non-projectable case exists [7].

¹³ $\lambda_{\text{Hořava}} = -\lambda_{\text{DeWitt}}/2$.

$$\widehat{G}_{ijkl}g^{ij} = 0, \quad \widehat{G}_{ijmn}\widehat{G}^{mnkl} = \widetilde{\delta}_{ij}^{kl} \quad (47)$$

with the (projected) Kronecker-delta, $\widehat{\delta}_{ij}^{kl} = \delta_{ij}^{kl} - g_{ij}g^{kl}/D$, satisfying $\widehat{\delta}_{ij}^{kl}g^{ij} = \widehat{\delta}_{ij}^{kl}g_{kl} = 0$ [10]. (See also [25, 26] for the corresponding analyses in *Lorentz invariant* higher-curvature gravities.)

In the Hamiltonian formulation, the existence of a null eigenvector in g_{ij} is reflected in the primary constraint,

$$\chi \equiv \widehat{\pi}_i^i \equiv g_{ij}\widehat{\pi}^{ij} \approx 0 \quad (48)$$

for the momenta,

$$\widehat{\pi}^{ij} \equiv \frac{\delta S}{\delta \dot{g}_{ij}} = \frac{2\sqrt{g}}{\kappa^2}\widehat{G}^{ijkl}K_{kl}, \quad (49)$$

in addition to the usual primary constraints, Φ_μ , in (10).

Then, one can find the canonical Hamiltonian, up to boundary terms,

$$\widehat{H}_C = \int_{\Sigma_t} d^Dx \left\{ N\widehat{\mathcal{H}} + N_i\widehat{\mathcal{H}}^i \right\} \quad (50)$$

with

$$\widehat{\mathcal{H}} \equiv \frac{\kappa^2}{2\sqrt{g}}\widehat{\pi}^{ij}\widehat{\pi}_{ij} + \sqrt{g}\mathcal{V}, \quad (51)$$

$$\widehat{\mathcal{H}}^i \equiv -2\nabla_j\widehat{\pi}^{ij}. \quad (52)$$

Now, the preservation of the additional primary constraint (48),

$$\begin{aligned} \dot{\chi} &= \{\chi, \widehat{H}_C\} \\ &\equiv \Psi \approx 0 \end{aligned} \quad (53)$$

produces a new secondary constraint,

$$\begin{aligned} \Psi &\equiv \left(\frac{\kappa^2}{2}\right)\left(\frac{D}{2}\right)\frac{N}{\sqrt{g}}\widehat{\pi}_{mn}\widehat{\pi}^{mn} \\ &\quad - N\sqrt{g}\left[\left(\frac{D}{2}\right)\mathcal{V}(R) - R\mathcal{V}'(R)\right] \\ &\quad + \sqrt{g}(D-1)\nabla^2(N\mathcal{V}'(R)) \\ &= \frac{D}{2}N\widehat{\mathcal{H}} - N\sqrt{g}[D\mathcal{V}(R) - R\mathcal{V}'(R)] \\ &\quad + \sqrt{g}(D-1)\nabla^2(N\mathcal{V}'(R)) \approx 0 \end{aligned} \quad (54)$$

from (A18), in addition to the usual (reduced) Hamiltonian and momentum constraints in (12),

$$\widehat{\mathcal{H}} \approx 0, \quad \widehat{\mathcal{H}}^i \approx 0. \quad (55)$$

The extended Hamiltonian is then given by

$$\widehat{H}_E = \widehat{H}_C + \int_{\Sigma_t} d^Dx (u_\mu \Phi^\mu + v\chi) \quad (56)$$

with a new Lagrange multiplier v . The constraint algebra, (16)–(18), are reduced to

$$\begin{aligned} \{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\} &= \widehat{C}^i(x)\nabla_i^x\delta^D(x-y) \\ &\quad - \widehat{C}^i(y)\nabla_i^y\delta^D(x-y), \end{aligned} \quad (57)$$

$$\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}_i(y)\} = -\widehat{\mathcal{H}}(y)\nabla_i^y\delta^D(x-y), \quad (58)$$

$$\begin{aligned} \{\widehat{\mathcal{H}}_i(x), \widehat{\mathcal{H}}_j(y)\} &= \widehat{\mathcal{H}}_i(y)\nabla_j^y\delta^D(x-y) \\ &\quad + \widehat{\mathcal{H}}_j(x)\nabla_i^x\delta^D(x-y), \end{aligned} \quad (59)$$

where

$$\begin{aligned} \widehat{C}^i &= \left(-\frac{\kappa^2}{2}\right)\left[\left(\widehat{\mathcal{H}}^i + 2\nabla^i\widehat{\pi}\right)\mathcal{V}'(R) \right. \\ &\quad \left. + 2\left(\widehat{\pi}^{ij} - g^{ij}\widehat{\pi}\right)\nabla_j\mathcal{V}'(R)\right]. \end{aligned} \quad (60)$$

Using the above reduced constraint algebra, one can find that

$$\begin{aligned} \widehat{\mathcal{H}} &= \{\widehat{\mathcal{H}}, \widehat{H}_E\} \\ &= \frac{1}{N}\nabla_i(N^2\widehat{C}^i) + \nabla_i(N^i\widehat{\mathcal{H}}) + \{\widehat{\mathcal{H}}, \langle v\pi \rangle\}, \end{aligned} \quad (61)$$

$$\begin{aligned} \widehat{\mathcal{H}}_i &= \{\widehat{\mathcal{H}}_i, \widehat{H}_E\} \\ &= \widehat{\mathcal{H}}\nabla_i N + \nabla_j(N^j\widehat{\mathcal{H}}_i) + \widehat{\mathcal{H}}_j\nabla_i N^j + \widehat{\pi}\nabla_i v \approx 0, \end{aligned} \quad (62)$$

where

$$\begin{aligned} \{\widehat{\mathcal{H}}, \langle v\pi \rangle\} &= \left[-\frac{v}{N}\Psi + \sqrt{g}(D-1)\left(\frac{v}{N}\nabla^2(N\mathcal{V}'(R)) \right. \right. \\ &\quad \left. \left. - \mathcal{V}'(R)\nabla^2 v\right)\right]. \end{aligned} \quad (63)$$

For the potential, $-\mathcal{V}(R) \equiv \Lambda + \xi R + \alpha R^n$, (63) becomes

$$\begin{aligned} \{\widehat{\mathcal{H}}, \langle v\pi \rangle\} &= -\left[\frac{v}{N}\Psi + \sqrt{g}(D-1)\left(-\frac{v}{N}\nabla^2\left(N\left(\xi \right. \right. \right. \right. \\ &\quad \left. \left. \left. + \alpha n R^{n-1}\right)\right) + \left(\xi + \alpha n R^{n-1}\right)\nabla^2 v\right)\right]. \end{aligned} \quad (64)$$

Since we are considering the non-trivial case of $-\mathcal{V}'(R) \equiv \xi + \alpha n R^{n-1} \neq 0$, preserving the Hamiltonian constraint, $\widehat{\mathcal{H}} \approx 0$, i.e., $\widehat{\mathcal{H}} \approx 0$ in (61), does not produce new constraints but determines the Lagrange multiplier v : For the λ -deformed GR ($\alpha = 0$), where $\widehat{C}^i = 0$, (61) and (64) determine $v = N$.

On the other hand, for the preservation of the secondary constraint, $\widetilde{\Psi} \equiv \Psi_{\widehat{\mathcal{H}} \approx 0}$,

$$\begin{aligned} \widetilde{\Psi} &\equiv N\sqrt{g}[D\Lambda + (D-1)\xi R + (D-n)\alpha R^n] \\ &\quad + \sqrt{g}(D-1)\nabla^2(N\mathcal{V}'), \end{aligned} \quad (65)$$

one can find that

$$\begin{aligned} \widetilde{\Psi} &= \{\widetilde{\Psi}, \widehat{H}_E\} \\ &= \{\widetilde{\Psi}, \widehat{H}_C\} + \{\widetilde{\Psi}, \langle u_t \pi_N \rangle\} + \{\widetilde{\Psi}, \langle v\pi \rangle\}, \end{aligned} \quad (66)$$

where¹⁴

$$\{\widetilde{\Psi}, \langle u_t \pi_N \rangle\} = \frac{u_t}{N}\widetilde{\Psi} + \sqrt{g}(D-1)\left(\nabla^2(u_t\mathcal{V}')\right)$$

¹⁴ Useful relations for these computations are $\{\sqrt{g}(x), \pi(y)\} = (D/2)\sqrt{g}(x)\delta^D(x-y)$ and $\{R(x), \pi(y)\} = -R(x)\delta^D(x-y) - (D-1)\nabla_x^2\delta^D(x-y)$ from (A7).

$$\begin{aligned}
 & -\frac{u_t}{N}\nabla^2(N\mathcal{V}')) , \\
 \{\tilde{\Psi}, \langle v\pi \rangle\} = & vN\sqrt{g}\left[\frac{D^2}{2}\Lambda + \frac{(D-1)(D-2)}{2}\xi R \right. \\
 & \left. + \frac{(D-2)(D-2n)}{2}\alpha R^n\right] \\
 & + \sqrt{g}(D-1)\left[\frac{(D-2)}{2}\nabla_i(v\nabla^i(N\mathcal{V}')) \right. \\
 & \left. - N\nabla^2 v((D-1)\xi \right. \\
 & \left. + (D-n)\alpha n R^{n-1})\right] \\
 & + \alpha n(n-1)(D-1)\sqrt{g}\nabla^2\left[R^{n-2}N(Rv \right. \\
 & \left. + (D-1)\nabla^2 v)\right]. \quad (68)
 \end{aligned}$$

Now, (61) and (66) determine the multipliers v and u_t , respectively, so that one can finish the Dirac's procedure without generating further constraints. Then, the complete set of constraints are $\widehat{\chi}_A \equiv (\pi_N, \widehat{\mathcal{H}}, \pi, \tilde{\Psi}) \approx 0$ and $\Gamma_A \equiv (\pi^i, \mathcal{H}_i) \approx 0$. Here, the constraints, $\widehat{\chi}_A \approx 0$, are the *second-class* constraints with the algebra,

$$\begin{aligned}
 \{\pi_N(x), \widehat{\mathcal{H}}(y)\} &= 0, \quad \{\pi_N(x), \chi(y)\} = 0, \\
 \{\pi_N(x), \tilde{\Psi}(y)\} &= \widehat{\Delta}(x-y), \\
 \{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\} &= \widehat{C}^i(x)\nabla_i^x\delta^D(x-y) \\
 & \quad - \widehat{C}^i(y)\nabla_i^y\delta^D(x-y), \\
 \{\widehat{\mathcal{H}}(x), \pi(y)\} &\approx \{\pi_N(x), \tilde{\Psi}(y)\}, \text{ etc.}, \quad (69)
 \end{aligned}$$

whose determinant, $\det(\{\widehat{\chi}_A, \widehat{\chi}_B\})$, is generally non-vanishing,

$$\begin{aligned}
 \det\{\widehat{\chi}_A, \widehat{\chi}_B\} &\approx (\det\{\widehat{\mathcal{H}}(x), \pi(y)\})^2 (\det\{\pi(x), \widehat{\mathcal{H}}(y)\})^2 \\
 &\approx (\det[\widehat{\Delta}(x-y)\widehat{\Delta}(y-x)])^2, \quad (70)
 \end{aligned}$$

where

$$\begin{aligned}
 \widehat{\Delta}(x-y) &\equiv -\sqrt{g}\left[D\Lambda + (D-1)\xi R + (D-n)\alpha R^n\right]\delta^D(x-y) \\
 & \quad + \sqrt{g}(D-1)\nabla_y^2\left[\delta^D(x-y)\left(\xi + \alpha n R^{n-1}\right)(y)\right]. \quad (71)
 \end{aligned}$$

On the other hand, the constraints, $\Gamma_A \equiv (\pi^i, \mathcal{H}_i) \approx 0$, are the first-class constraints, as in Cases **A** and **B**. So, the resulting number of dynamical degrees of freedom is the same as in Case **A**,

$$\begin{aligned}
 s &= \frac{1}{2}(P - 2N_1 - N_2) \\
 &= \frac{1}{2}(D+1)(D-2), \quad (72)
 \end{aligned}$$

which is the same as in GR. This provides the fully *non-perturbative* proof of the previous perturbative analysis (see also [35] for an earlier work), which does not show the extra degrees of freedom at the *linear* level [10]. There may exist some similarities with Case **A** due to the same physical degrees of freedom. Actually, one can consider the maximal slicing condition, i.e., $\pi = 0$, for λ -deformed GR (with an arbitrary λ), as an example satisfying the condition, $\tilde{C}^i = 0$ for Case **A**. But, an importance difference is that Case **C** does not depend whether $\widehat{C}^i = 0$, i.e., *commuting* Hamiltonian constraint $\widehat{\mathcal{H}}$, or $\widehat{C}^i \neq 0$, i.e., *non-commuting* Hamiltonian constraint $\widehat{\mathcal{H}}$: If we consider the maximal slicing, $\pi = 0$, for Case **B** with an arbitrary $\lambda \neq 1/D$, we have basically the same results as Case **C**! This implies that the constraint structure of Case **C** and so its number of dynamical degrees of freedom do not depend on spatially-higher-derivative terms in the potential, which are important for distinguishing Cases **A** and **B**. In other words, Case **C** does not depend on the UV conformal symmetry for the Cotton square term, $C^{ij}C_{ij}$, in the Hořava's original potential [1,2] and this is essentially due to the kinematic origin of the constraint, $\chi = \widehat{\pi} \approx 0$, in (48).¹⁵

5 Examples

In this section, we consider several exact solutions for our typical potential, $-\mathcal{V}(R) \equiv \xi R + \alpha R^n + \Lambda$, as some explicit examples of the constraint analysis.

5.1 Non-rotating black holes in arbitrary dimensions

For non-rotating, spherically symmetric black holes with $N_i \equiv 0$, the extrinsic curvature, K_{ij} , and the conjugate momentum, π_{ij} , vanish so that the Hamiltonian constraint becomes simply $\mathcal{H} = \sqrt{g}\mathcal{V}(R) \approx 0$. The general solution is ' $R = \text{constant}$ ', whose constant value depends on the theory parameters, ξ , α , Λ , and n . If we consider, as an explicit example, $D = 3$, $n = 2$, i.e., $z = 2$ case in $3+1$ dimensions [36],¹⁶ then the solution is given by

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (73)$$

¹⁵ This case corresponds to the $\dot{\beta} = 0$ case in the Birkhoff's analysis of spherically symmetric system since $\pi \sim \dot{\beta}$ and the results are in a good agreement [27].

¹⁶ This can be also the solution for $z = 3$ case with the Cotton square term, $C_{ij}C^{ij}$, since the Cotton tensor vanishes, $C^{ij} \equiv \epsilon^{ikl}\nabla_k(R^j{}_l - \delta^j{}_l R/4) = 0$, for the spherically symmetric cases.

with

$$N^2 = f = 1 - \frac{2M}{r} + \left(\frac{\xi - \sqrt{\xi^2 - 4\alpha\Lambda}}{12\alpha} \right) r^2, \quad (74)$$

$$R = \frac{-\xi + \sqrt{\xi^2 - 4\alpha\Lambda}}{2\alpha}. \quad (75)$$

In this case, even though there are higher-derivative contributions, we have $\tilde{C}^i = 0$, trivially, due to $\pi_{ij} = 0$ in (31). This would be a trivial example though its result is generally valid for arbitrary power n and dimension D . If we consider the more general, spherically symmetric solutions with $N_r \neq 0$ (cf. [37]), it would be a more non-trivial example with the extrinsic curvature and conjugate momenta.

5.2 Black string solutions in 3 + 1 dimensions

For stationary black strings in 3 + 1 dimensions, the general ansatz is given by

$$ds^2 = (-N^2 + N_r N^r + N_\phi N^\phi) dt^2 + 2(N_r dr + N_\phi d\phi) dt + \frac{dr^2}{f} + r^2 d\phi^2 + g dz^2, \quad (76)$$

where all the metric functions, N , N_i , f , and g depend on the radial coordinate r . For $z = 2$ black string solution which satisfies the vanishing Cotton tensor, $C_{ij} = 0$, for simplicity, a simple solution with $g = \text{constant}$, $N_\phi = 0$, and $\lambda = 1$, which is called BTZ-type black string, is given by (with $g \equiv 1$)

$$N^2 = f = \eta r^2 - m, \quad (77)$$

$$N_r^2 = f^{-1} \left[\delta + \frac{\kappa^4 \mu^2}{64} \left(\eta^2 - 2(\Lambda_W + \omega) - 3\Lambda_W^2 \right) \right], \quad (78)$$

where η , m , δ are integration constants [38] and the other parameters κ , μ , Λ_W , ω are introduced by the usual parametrization,

$$\begin{aligned} \xi &\equiv \frac{\kappa^4 \mu^2 (\Lambda_W + \omega)}{8(1 - 3\lambda)}, \\ \alpha &\equiv \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)}, \\ \Lambda &\equiv -\frac{2\kappa^2 \mu^2 \Lambda_W^2}{8(1 - 3\lambda)}. \end{aligned} \quad (79)$$

In this case, even though the extrinsic curvatures and conjugate momenta are non-vanishing, it has a constant curvature, $R \sim f'/r = 2\eta = \text{constant}$. Since this is the solution for $\lambda = 1$, one can easily find that $\tilde{C}^i = 0$ in (31) is trivially satisfied¹⁷. The first term in (31) vanishes due to $\lambda = 1$ and the second term vanishes due to $R = \text{constant}$. If we consider the more general solutions with $\lambda \neq 1$ or the $\lambda = 1$

solution without the condition, $C_{ij} = 0$, it would be more non-trivial examples with the *non-constant* extrinsic curvatures and conjugate momenta.

5.3 Rotating black holes in 3 + 1 dimensions

The exact solutions for rotating black holes in 3 + 1 dimensions has not been found yet. However, for slowly rotating black holes, one can consider the ansatz [39,40],

$$ds^2 = -N^2(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + 2ag(r) \sin^2 \theta dt d\phi. \quad (80)$$

At the linear order in the rotation parameter a , the solution of $g(r)$ for $n = 2$ is given by

$$g(r) = \sigma r^2 + \frac{\gamma}{r} \quad (81)$$

with the integration parameters, σ , γ , and the same solution of $N^2 = f(r)$ as in (74). In this case, the extrinsic curvature has a non-vanishing component, $K_{r\phi} = \mathcal{O}(a)$ but $K = 0$, $\pi = 0$, at the linear order in ' a '. Since the curvature scalar is constant as in (75), we will have the case $\tilde{C}^i = 0$ again and this result is valid for arbitrary power n and dimensions D .¹⁸ If we consider higher orders in the rotation parameter a so that K or π is non-vanishing or R is non-constant, it would be a more non-trivial example.

5.4 Rotating black holes in 2 + 1 dimensions

For stationary black holes in 2 + 1 dimensions, the most general ansatz is given by

$$ds^2 = (-N^2 + N_r N^r + N_\phi N^\phi) dt^2 + 2(N_r dr + N_\phi d\phi) dt + \frac{dr^2}{f} + r^2 d\phi^2, \quad (82)$$

where all the metric functions, N , N_i , f , depend on the radial coordinate r , as in (76), due to the symmetry property in 2 + 1 dimensions. The general solution for the metric (82) is not known yet. For the simple case, however, of $N_r = 0$, one can find that there is non-vanishing conjugate momentum, $\pi^{\phi r} = (f/2N)(N^\phi)'$, whose trace is vanishing trivially, $\pi = 0$, with the solution (for the $n = 2$ case) [41,42]

¹⁷ This result is still valid with the angular shift vector, N_ϕ .

¹⁸ In [39,40], $\lambda = 1$, $\Lambda_W = 0$ case for the IR-modified Hořava gravity was considered. But our result is generally valid for arbitrary λ and Λ_W .

$$f = -\mathcal{M} + \frac{br^2}{2} \left[1 - \sqrt{a + \frac{c}{r^4}} + \sqrt{\frac{c}{r^4}} \ln \left(\sqrt{\frac{c}{ar^4}} + \sqrt{1 + \frac{c}{ar^4}} \right) \right],$$

$$N^2/f = 1 / \left(1 + \frac{c}{ar^4} \right),$$

$$N^\phi = -\frac{\mathcal{J}}{2} \sqrt{\frac{a}{c}} \ln \left[\sqrt{\frac{c}{ar^4}} + \sqrt{1 + \frac{c}{ar^4}} \right], \quad (83)$$

where

$$a = 1 - \frac{4\alpha\Lambda}{\xi^2}, \quad b = \frac{\xi}{2\alpha}, \quad c = \frac{2\alpha\mathcal{J}^2}{\xi^2}. \quad (84)$$

In this case, even though $K = 0$, $\pi = 0$ trivially, the curvature scalar is non-constant,

$$R = -\frac{f'}{r} = -b \left(1 - \sqrt{a + \frac{c}{r^4}} \right), \quad (85)$$

so that we have a non-vanishing component of \tilde{C}^i ,

$$\tilde{C}^\phi \sim \alpha \pi^{\phi r} \partial_r R \neq 0. \quad (86)$$

But, it easy to find that the constraint $\tilde{\Omega} \approx 0$ is satisfied again

$$\tilde{\Omega} \sim \partial_\phi \left(N^2 \tilde{C}^\phi \right) = 0, \quad (87)$$

due to the spherical symmetry, i.e., no ϕ -dependence, in the solution. It is important to note that the non-vanishing \tilde{C}^i is the genuine result of higher-derivative terms ($\alpha \neq 0$) with a rotation ($\pi^{\phi r} \neq 0$). In other words, if we turn off the higher-derivative term, i.e., $\alpha = 0$, similarly to BTZ black hole in GR, the system is reduced to the case with the vanishing $\tilde{C}^i \sim \xi \hat{\lambda} \nabla^i \pi = 0$. Of course, this does not correspond to Case **B** but Case **C**, due to the fact of $\pi = 0$ for the solution (83), as can be seen by checking the constraint, $\tilde{\Psi} \approx 0$, in (65). However, if we consider the $N_r \neq 0$ case, one obtains $\pi \neq 0$ generally so that it may correspond to a “genuine” case of Case **B**. This example would show the importance of higher-derivative terms for Case **B**, where the extra scalar graviton mode is involved.

6 Concluding remarks

We study the *full* constraint analysis of the *non-projectable* Hořava gravity whose potential, $\mathcal{V}(R)$, is an arbitrary function of the (intrinsic) Ricci scalar R in arbitrary dimensions but without the extension terms which depend on the proper acceleration a_i , for simplicity. We find that there are generally three distinct cases in this theory, depending on

- (i) whether the Hamiltonian constraint, $\mathcal{H} \approx 0$, generates new (second-class) constraints (Cases **A**, **C**) or just fixes the associated Lagrange multipliers (Case **B**), or

- (ii) whether the IR Lorentz-deformation parameter λ is at the conformal point, $\lambda = 1/D$ (Case **C**), or not (Cases **A**, **B**).

We find that, for Cases **A** and **C**, the dynamical degrees of freedom of Hořava gravity are the same as in GR, while for Case **B**, there is *one extra phase-space degree of freedom*, representing an extra (odd) scalar graviton mode. This would achieve the dynamical consistency of a restricted model at the *fully non-linear* level and be positive in resolving the long-standing debates about the extra graviton modes of the Hořava gravity. Several further remarks about other challenging problems are in order.

1. We have obtained the new “extended” constraint algebra for the Hamiltonian and momentum constraints, (16)–(18) ((57)–(59) for Case **C**) or (19)–(21), for the *non-projectable* Hořava gravity whose potential is an arbitrary function of the Ricci scalar R . The structure of the newly obtained “extended” constraints algebra seems to be generic to Hořava gravity itself, analogous to that of general higher-curvature gravities [25,26], where C^i becomes the momentum constraint \mathcal{H}^i with higher-curvature corrections. An important difference is that the momentum constraint \mathcal{H}^i in Hořava is the same form as in GR with no higher-derivative corrections and satisfies the same algebra as in GR [23] but the full algebra with the Hamiltonian constraint, \mathcal{H} , is *not closed*, whereas the momentum constraints, $\mathcal{H}^i \equiv C^i$, in generic higher-curvature gravities satisfy exactly the same *closed* algebra as in GR, “as has been argued generically in [43]”, even with the generic higher-curvature terms which include Riemann tensors also [25,26,44]. We suspect that a similar general argument in Hořava gravity exists also so that the extended constraint algebra reflects the generic space-time structure of our Lorentz violating gravities with Ricci and Riemann tensors, R_{ij} , R_{ijkl} , etc., i.e., with the potential, $\mathcal{V}(R, R_{ij}, R_{ijkl}, \dots)$,¹⁹ and its general proof would be a challenging problem.

2. Our constraint analysis shows the dynamical degrees of freedom in a restricted model of the Hořava gravity at the *fully non-linear* level. Comparison to the previous linear perturbation analyses [8,10,33], which do *not* show the extra degrees of freedom, implies that the extra modes would be the *genuine consequence of non-linear effect (around the homogeneous background)* with Lorentz-violating higher-derivative terms [27]: For a direct proof in the $n = 2$ constraint algebra, see [15]. On the other hand, it has been also argued that the extra mode may appear when considering (even linear) perturbations around the spatially-*inhomogeneous* and time-dependent background [5], even with the IR Lorentz-violating terms only, as anticipated from the usual connection

¹⁹ This implies the *generic* absence of the third and second-derivative terms in (2.24) of [4], which is the case of $\lambda = 1$ and $\mathcal{V} \sim C_{ij}C^{ij}$.

between non-linear perturbations for a *homogenous* background and linear perturbations for a corresponding *inhomogeneous* background.²⁰ This may be in contrast to the Birkhoff's theorem analysis in *spherically symmetric, vacuum* configuration which does *not* show time-dependent solutions, representing the “extra gravitational” modes [27]. However, this might be due to its high symmetry, i.e., lower inhomogeneity, and there might exist still some possibilities for higher inhomogeneities: This might be consistent with a related analysis in [15] which does not show the extra mode only up to the first order of inhomogeneity, L^{-1} . So, a direct proof of the argument of [5] in the generic constraint algebra would be an interesting open problem. The role of non-linear, UV scalar graviton in Big Bang cosmology and gravitational wave physics for compact objects with strong gravities, like black holes and neutron stars, would be also a challenging problem.

3. In the literature, there have been claims of inconsistencies of the Hořava gravity, in its original form. It seems that some originate from the limited analysis, like linear approximations [7], or incomplete constraint analysis [4, 11]. Here, we consider the second case,²¹ especially about the claim in [11], which seems to be the most rigorous criticism against the (non-projectable) Hořava gravity. The basic claim in [11] is that “ $N = 0$ is the only possible solution” for the constraint (4.1), $\nabla_i(N^2\nabla^i\pi) \approx 0$, which corresponds to our constraint (30), $\tilde{\Omega} \equiv \nabla_i(N^2\tilde{C}^i) \approx 0$, for the asymptotically flat ($\Lambda = 0$) and λ -deformed GR, otherwise N blows up at infinity, generically.²² But, the elaborate analysis in [11] is just another proof of an adequacy of the condition, $\pi = 0$, in that example, which is the only remaining solution for the constraint (4.1) in [11] or (30) in this paper, as has been argued also in [14, 15]. Actually, contrary to the argument in [11], the perturbations from $\pi = 0$ are not arbitrary but restricted by another constraint (33), $\tilde{\Xi} \approx 0$ in this paper (or (20) in [14]), when $\lambda \neq 1$. Moreover, when the metric is not asymptotically flat ($\Lambda \neq 0$), the argument of [14] (and possibly of [11] or [5] also) would not be valid generally, as can be seen in the general, $(2+1)$ -dimensional solution, (82) with $N_r \neq 0$, which shows the non-vanishing π and N .

4. With the full set of constraints, we can now consider Feynman's path-integral for the S-matrix elements [45, 46], whose Hamiltonian expression for Case A, for example, is given by

$$Z = \int \mathcal{D}g_{ij} \mathcal{D}\pi^{ij} \mathcal{D}N \mathcal{D}N_i \delta(\mathcal{H}_i) \delta(\gamma_j) \delta(\mathcal{H}) \delta(\tilde{\Omega}) \\ \times \int \mathcal{D}c_j \mathcal{D}\bar{c}_i \exp \left\{ i \int dt d^D x \bar{c}_i [\{\mathcal{H}_i, \gamma_j\}|_{\pi^i \approx 0}] c_j \right\} \\ \times \int \mathcal{D}b \mathcal{D}\bar{b} \exp \left\{ i \int dt d^D x \bar{b} [\{\mathcal{H}, \tilde{\Omega}\} \{\tilde{\Omega}, \mathcal{H}\}|_{\pi_N \approx 0}] b \right. \\ \left. + iS/\hbar \right\}, \quad (88)$$

where we have integrated out for the primary constraints $\pi_N, \pi^i \approx 0$ with the Faddeev–Popov's anti-commuting fields, $c_j, \bar{c}_i, b, \bar{b}$, the gauge-fixing conditions, $\gamma_j = 0$, for the first-class constraint, $\mathcal{H}_i \approx 0$, and the determinant factor for the second-class constraints, $(\det\{\chi_A, \chi_B\})^{1/2} \approx \det(\{\mathcal{H}, \tilde{\Omega}\} \{\tilde{\Omega}, \mathcal{H}\})$. One can obtain similarly the path integral for Cases B and C also. It would be desirable to study the renormalizability for the non-projectable Hořava gravity, based on the above S-matrix elements, beyond the recent proof of (perturbative) renormalizability for the projectable cases [47–49].²³

5. In the canonical quantization with the second-class constraints, we need to compute the Dirac brackets [17],²⁴ whose definition for Case A, for example, is given by, for any variable P, Q ,

$$\{P, Q\}^* \equiv \{P, Q\} \\ - \int d^D z \int d^D w \{P, \chi_A(z)\} C_{AB}^{-1}(z, w) \{\chi_B(w), Q\}, \quad (89)$$

where C_{AB}^{-1} is defined as $\int d^D z C_{AB}^{-1}(x, z) C_{BC}(z, y) = \int d^D z C_{CB}(y, z) C_{BA}^{-1}(z, x) = \delta_{AC} \delta^D(x - y)$ for the Poisson brackets of the second-class constraints $C_{AB} \equiv \{\chi_A, \chi_B\}$, and given by

$$C_{AB}^{-1}(x, z) = \begin{pmatrix} 0 & -\Delta^{-1}(z - x) & 0 & 0 \\ \Delta^{-1}(x - z) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Delta^{-1}(z - x) \\ 0 & 0 & \Delta^{-1}(x - z) & 0 \end{pmatrix}$$

Δ^{-1} , defined by $\int d^D z \Delta^{-1}(x - z) \Delta(z - y) = \int d^D z \Delta(y - z) \Delta^{-1}(z - x) = \delta^D(x - y)$. The bracket satisfies $\{\chi_A, Q\}^* = 0$ for any variable Q so that the second-class constraints, $\chi_A \approx 0$, can be imposed consistently in the Hamiltonian dynamics, i.e., $\{\mathcal{H}, \mathcal{H}\}^* = 0, \{\mathcal{H}, \mathcal{H}_i\}^* = 0$, etc.,²⁵ which corresponds to the delta-function insertions for the constraints, $\chi_A \approx 0$, in the path integral, (88). If we consider the gauge-fixing conditions as in the path integral formalism, we

²⁰ MIP thank Koyama and Gumrukcuoglu for discussion about this matter.

²¹ The first case has been discussed in several places, like [15] (footnote 3) and [4] (Note added).

²² This result corresponds to exactly what has been argued in [5], which shows “instabilities for perturbations around a non-vanishing K background”, though given at the linear level.

²³ For an earlier work on non-projectable case in $(2+1)$ -dimensions, see also [50] (Case C).

²⁴ After this paper appeared in the archive, we became aware of [52] in which the Dirac brackets were computed for the linearized Hořava gravity. We thank Ghosh for informing us about his work.

²⁵ This algebra looks like that of the “ultra-local” theory of gravity [1, 2, 51].

can obtain further corrections to the Dirac brackets. Then the Dirac's quantization rule is given by $[\hat{P}, \hat{Q}] \equiv (i/\hbar)\{P, Q\}^*$ for the quantum operators, \hat{P} and \hat{Q} , corresponding to classical variables P and Q , respectively, with the "appropriate" operator orderings. One can consider also Cases **B** and **C** similarly, but it would be more involved for the former case.

Note added: After finishing this paper, a related paper [53] appeared which is overlapping with ours for $D = 2$ case. But due to the (full) a_i extensions in [53], it shows a different constraint structure, as noted in our footnote No. 9.

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Appendix A: Computing $\{\mathcal{H}, \mathcal{H}\}$, $\{\mathcal{H}, \mathcal{H}_i\}$, and $\{\mathcal{H}_i, \mathcal{H}_j\}$

In this Appendix, we compute the constraint algebra, $\{\mathcal{H}, \mathcal{H}\}$, $\{\mathcal{H}, \mathcal{H}_i\}$, and $\{\mathcal{H}_i, \mathcal{H}_j\}$ in (16)–(18). To this ends, it is useful to consider the variations of the smeared constraints, $\langle \eta \mathcal{H} \rangle \equiv \int d^D x \eta \mathcal{H} \approx 0$, etc., with the smearing functions, η, η_i (neglecting the boundary terms),

$$\delta \langle \eta \mathcal{H} \rangle = \langle A^{kl} \delta g_{kl} + B_{kl} \delta \pi^{kl} \rangle, \quad (\text{A1})$$

$$\delta \langle \eta^i \mathcal{H}_i \rangle = \langle C^{kl} \delta g_{kl} + D_{kl} \delta \pi^{kl} \rangle. \quad (\text{A2})$$

After long computations, one can obtain the coefficients, A, B, C, D as follows:

$$\begin{aligned} A^{kl} &\equiv \frac{\delta \langle \eta \mathcal{H} \rangle}{\delta g_{kl}} = \left(\frac{\kappa^2}{2} \right) \frac{\eta}{\sqrt{g}} \left[-\frac{1}{2} g^{kl} (\pi_{mn} \pi^{mn} - \tilde{\lambda} \pi^2) \right. \\ &\quad \left. + 2 (\pi^{km} \pi_m^l - \tilde{\lambda} \pi \pi^{kl}) \right] \\ &\quad + \eta \sqrt{g} \left[\frac{1}{2} g^{kl} \mathcal{V}(R) - R^{kl} \mathcal{V}'(R) \right] \end{aligned}$$

$$+ \sqrt{g} \left[\nabla^k \nabla^l (\eta \mathcal{V}'(R)) - g^{kl} \nabla_m \nabla^m (\eta \mathcal{V}'(R)) \right], \quad (\text{A3})$$

$$B_{kl} \equiv \frac{\delta \langle \eta \mathcal{H} \rangle}{\delta \pi^{kl}} = \left(\frac{\kappa^2}{2} \right) \frac{2\eta}{\sqrt{g}} (\pi_{kl} - \tilde{\lambda} g_{kl} \pi), \quad (\text{A4})$$

$$\begin{aligned} C^{kl} &\equiv \frac{\delta \langle \eta^i \mathcal{H}_i \rangle}{\delta g_{kl}} = -\nabla_m (\eta^m \pi^{kl}) \\ &\quad + (\nabla_m \eta^k) \pi^{lm} + (\nabla_m \eta^l) \pi^{km}, \end{aligned} \quad (\text{A5})$$

$$D_{kl} \equiv \frac{\delta \langle \eta^i \mathcal{H}_i \rangle}{\delta \pi^{kl}} = \nabla_k \eta_l + \nabla_l \eta_k. \quad (\text{A6})$$

As a byproduct, one can also obtain a useful formula,

$$\begin{aligned} \{R(x), \pi^{kl}(y)\} &= \frac{\delta R(x)}{\delta g_{kl}(y)} = -R^{kl}(x) \delta^D(x-y) \\ &\quad + \nabla_x^k \nabla_x^l \delta^D(x-y) - g^{kl} \nabla_x^2 \delta^D(x-y). \end{aligned} \quad (\text{A7})$$

Then, after some manipulations, one can find the Poisson bracket algebras for the smeared constraints as follows:

$$\begin{aligned} \{\langle \eta \mathcal{H} \rangle, \langle \zeta \mathcal{H} \rangle\} &= \int d^D z \left[\frac{\delta \langle \eta \mathcal{H} \rangle}{\delta g_{kl}(z)} \frac{\delta \langle \zeta \mathcal{H} \rangle}{\delta \pi^{kl}(z)} - \frac{\delta \langle \eta \mathcal{H} \rangle}{\delta \pi^{kl}(z)} \frac{\delta \langle \zeta \mathcal{H} \rangle}{\delta g_{kl}(z)} \right] \\ &= \langle (\eta \nabla_i \zeta - \zeta \nabla_i \eta) C^i \rangle, \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} \{\langle \eta \mathcal{H} \rangle, \langle \zeta^i \mathcal{H}_i \rangle\} &= \int d^D z \left[\frac{\delta \langle \eta \mathcal{H} \rangle}{\delta g_{kl}(z)} \frac{\delta \langle \zeta^i \mathcal{H}_i \rangle}{\delta \pi^{kl}(z)} - \frac{\delta \langle \eta \mathcal{H} \rangle}{\delta \pi^{kl}(z)} \frac{\delta \langle \zeta^i \mathcal{H}_i \rangle}{\delta g_{kl}(z)} \right] \\ &= -\langle \zeta^i \nabla_i \eta \mathcal{H} \rangle, \end{aligned} \quad (\text{A9})$$

$$\begin{aligned} \{\langle \zeta^i \mathcal{H}_i \rangle, \langle \zeta^j \mathcal{H}_j \rangle\} &= \int d^D z \left[\frac{\delta \langle \zeta^i \mathcal{H}_i \rangle}{\delta g_{kl}(z)} \frac{\delta \langle \zeta^j \mathcal{H}_j \rangle}{\delta \pi^{kl}(z)} - \frac{\delta \langle \zeta^i \mathcal{H}_i \rangle}{\delta \pi^{kl}(z)} \frac{\delta \langle \zeta^j \mathcal{H}_j \rangle}{\delta g_{kl}(z)} \right] \\ &= \langle (\zeta^i \nabla_i \zeta^j - \zeta^i \nabla_j \zeta^j) \mathcal{H}_j \rangle, \end{aligned} \quad (\text{A10})$$

where

$$\begin{aligned} C^i &= \left(-\frac{\kappa^2}{2} \right) \left[(\mathcal{H}^i + 2\tilde{\lambda} \nabla^i \pi) \mathcal{V}'(R) \right. \\ &\quad \left. + 2 (\pi^{ij} - \tilde{\lambda} g^{ij} \pi) \nabla_j \mathcal{V}'(R) \right]. \end{aligned} \quad (\text{A11})$$

Now, one can easily check that the Poisson algebra for the local constraints are given by (16)–(18):

$$\begin{aligned} \{\mathcal{H}(x), \mathcal{H}(y)\} &= C^i(x) \nabla_i^x \delta^D(x-y) \\ &\quad - C^i(y) \nabla_i^y \delta^D(x-y), \end{aligned} \quad (\text{A12})$$

$$\{\mathcal{H}(x), \mathcal{H}_i(y)\} = -\mathcal{H}(y) \nabla_i^y \delta^D(x-y), \quad (\text{A13})$$

$$\begin{aligned} \{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_i(y) \nabla_j^x \delta^D(x-y) \\ &\quad + \mathcal{H}_j(x) \nabla_i^x \delta^D(x-y). \end{aligned} \quad (\text{A14})$$

Moreover, from (A1)–(A6), one can easily obtain the dynamical equations of motion (13)–(14) as follows,

$$\begin{aligned}
\dot{g}_{ij} &= \{g_{ij}, \langle N\mathcal{H} + N_i\mathcal{H}^i \rangle\} = \frac{\delta\langle N\mathcal{H} \rangle}{\delta\pi^{ij}} + \frac{\delta\langle N_i\mathcal{H}^i \rangle}{\delta\pi^{ij}} \\
&= \left(\frac{\kappa^2}{2}\right) \frac{2N}{\sqrt{g}} (\pi_{ij} - \tilde{\lambda}g_{ij}\pi) + \nabla_i N_j + \nabla_j N_i, \quad (\text{A15}) \\
\dot{\pi}^{ij} &= \{\pi^{ij}, \langle N\mathcal{H} + N_i\mathcal{H}^i \rangle\} = -\frac{\delta\langle N\mathcal{H} \rangle}{\delta g_{ij}} - \frac{\delta\langle N_i\mathcal{H}^i \rangle}{\delta g_{ij}} \\
&= \left(\frac{\kappa^2}{2}\right) \frac{N}{\sqrt{g}} \left[\frac{1}{2} g^{ij} (\pi_{mn}\pi^{mn} - \tilde{\lambda}\pi^2) \right. \\
&\quad \left. - 2(\pi^{im}\pi_m^j - \tilde{\lambda}\pi\pi^{ij}) \right] \\
&\quad - N\sqrt{g} \left[\frac{1}{2} g^{ij} \mathcal{V}(R) - R^{ij}\mathcal{V}'(R) \right] \\
&\quad - \sqrt{g} \left[\nabla^i \nabla^j (N\mathcal{V}'(R)) - g^{ij} \nabla_m \nabla^m (N\mathcal{V}'(R)) \right] \\
&\quad + \nabla_m (N^m \pi^{ij}) - (\nabla_m N^i) \pi^{jm} - (\nabla_m N^j) \pi^{im}. \quad (\text{A16})
\end{aligned}$$

Combining (A15) and (A16), one can also obtain the dynamical equation for \sqrt{g} and the momentum trace π as follows,

$$\begin{aligned}
\dot{\sqrt{g}} &= \left(\frac{\kappa^2}{2}\right) \frac{N\pi}{1-D\lambda} + 2\nabla_i N^i, \quad (\text{A17}) \\
\dot{\pi} &= \left(\frac{\kappa^2}{2}\right) \left(\frac{D}{2}\right) \frac{N}{\sqrt{g}} (\pi_{mn}\pi^{mn} - \tilde{\lambda}\pi^2) + \nabla_m (N^m \pi) \\
&\quad - N\sqrt{g} \left[\left(\frac{D}{2}\right) \mathcal{V}(R) - R\mathcal{V}'(R) \right] \\
&\quad + \sqrt{g}(D-1)\nabla^2 (N\mathcal{V}'(R)) \\
&= \frac{D}{2} N\mathcal{H} - N\sqrt{g} [D\mathcal{V}(R) - R\mathcal{V}'(R)] \\
&\quad + \sqrt{g}(D-1)\nabla^2 (N\mathcal{V}'(R)) + \nabla_m (N^m \pi). \quad (\text{A18})
\end{aligned}$$

Appendix B: More details of computations in Sect. 3 A

In this Appendix, we present some more details of the computations in Sect. 3 A.

First, we consider the variations of the smeared constraints, $\langle \eta \tilde{\Omega} \rangle \equiv \int d^D x \eta \tilde{\Omega}$ and $\langle \zeta \tilde{\Xi} \rangle \equiv \int d^D x \zeta \tilde{\Xi}$ with the smearing functions η and ζ ,

$$\delta\langle \eta \tilde{\Omega} \rangle = \langle E^{mn} \delta g_{mn} + F_{mn} \delta \pi^{mn} \rangle, \quad (\text{B1})$$

$$\delta\langle \zeta \tilde{\Xi} \rangle = \langle G^{mn} \delta g_{mn} + H_{mn} \delta \pi^{mn} \rangle. \quad (\text{B2})$$

After quite tedious computations, we obtain the coefficients, E, F, G, H as follows²⁶:

$$E^{mn} \equiv \frac{\delta\langle \eta \tilde{\Omega} \rangle}{\delta g_{mn}} = -\left(\frac{\kappa^2}{2}\right) 2\hat{\lambda} \mathcal{V}'(R) \left[\pi^{mn} \nabla_i (N^2 \nabla^i \eta) \right.$$

$$\begin{aligned}
&\quad \left. - \frac{1}{2} g^{mn} \nabla_i (N^2 \pi \nabla^i \eta) + N^2 \nabla^{(m} \pi \nabla^{n)} \eta \right] \\
&\quad - \left(\frac{\kappa^2}{2}\right) 2\hat{\lambda} \left[\eta \nabla_i \left[N^2 \nabla^i \pi \frac{\delta \mathcal{V}'(R)}{\delta g_{mn}} \right. \right. \\
&\quad \left. \left. + N^2 (\pi^{ij} - \hat{\lambda} g^{ij} \pi) \frac{\delta \nabla_j \mathcal{V}'(R)}{\delta g_{mn}} \right. \right. \\
&\quad \left. \left. - N^2 (\delta^{ij}_{mn} \pi + g^{ij} \pi^{mn}) \nabla_j \mathcal{V}'(R) \right] \right], \quad (\text{B3})
\end{aligned}$$

$$\begin{aligned}
F_{mn} &\equiv \frac{\delta\langle \eta \tilde{\Omega} \rangle}{\delta \pi^{mn}} = -\left(\frac{\kappa^2}{2}\right) 2\hat{\lambda} \mathcal{V}'(R) \nabla_i \left[g_{mn} N^2 \nabla^i \eta \right] \\
&\quad - \left(\frac{\kappa^2}{2}\right) 2 \left[\eta \nabla_i \left[N^2 (\delta^{ij}_{mn} - \hat{\lambda} g^{ij} g_{mn}) \nabla_j \mathcal{V}'(R) \right] \right], \quad (\text{B4})
\end{aligned}$$

$$\begin{aligned}
G^{mn} &\equiv \frac{\delta\langle \zeta \tilde{\Xi} \rangle}{\delta g_{mn}} = \left(\frac{\kappa^2}{2}\right) 2(D-1) \xi^2 \hat{\lambda} \sqrt{g} \left\{ \frac{1}{2} g^{mn} N^2 \right. \\
&\quad \times \nabla_i \zeta (\nabla^i \nabla^j \nabla_j N - N \nabla^i R - R \nabla^i N) \\
&\quad + N^2 \left[\nabla_i (N R^{mn}) \nabla^i \zeta - \nabla^{(m} \nabla_i \nabla^j N \nabla^{n)} \zeta \right. \\
&\quad \left. + \nabla^{(m} (N R) \nabla^{n)} \zeta - \nabla_i \nabla^m \nabla^n N \nabla^i \zeta \right] \\
&\quad + \frac{\Lambda D}{2\xi(D-1)} \left[2N^2 \nabla^{(m} N \nabla^{n)} \zeta - g^{mn} N^2 \nabla^i N \nabla_i \zeta \right] \\
&\quad + \nabla_i \left[N^2 (X^{imnk} l_j \nabla_j \nabla_k N \nabla^j \zeta \right. \\
&\quad \left. + 2X^{imnkjl} \nabla_l \nabla_k N \nabla_j \zeta - N R^{mn} \nabla^i \zeta) \right] \\
&\quad + \nabla_k \nabla_i \left[N^2 (X^{kmnl} l^j - X^{kmnl} l_j) \nabla_j N \nabla^i \zeta \right. \\
&\quad \left. - N^2 X^{kmnjl} l_j \nabla_j N \nabla^i \zeta \right] \\
&\quad + \nabla_k \nabla_j \nabla_i \left[N^3 \nabla^i \zeta (X^{kmnjl} l_j - X^{kmnl} l^j) \right] \Big\} \\
&\quad + \left(\frac{\kappa^2}{2}\right)^2 \frac{\xi \hat{\lambda}}{(\lambda D - 1) \sqrt{g}} \left\{ (2\lambda + 1) [2\pi^{mn} \pi \right. \\
&\quad \times \nabla_i (N^3 \nabla^i \zeta) - g^{mn} \nabla_i (N^3 \pi^2 \nabla^i \zeta) \\
&\quad + g^{mn} N^3 \pi \nabla^i \pi \nabla_i \zeta + 2N^3 \pi \nabla^{(m} \pi \nabla^{n)} \zeta \Big] \\
&\quad + N^2 \left[g^{mn} \pi^2 \nabla_i N \nabla^i \zeta + 2\pi^2 \nabla^{(m} N \nabla^{n)} \zeta \right. \\
&\quad \left. - 4\pi^{mn} \pi \nabla_i N \nabla^i \zeta \right] + 4(\lambda D - 1) \\
&\quad \times \left[\left(\frac{1}{2} g^{mn} \pi^{ij} \pi - \pi^{mn} \pi^{ij} \right) \nabla_j (N^3 \nabla_i \zeta) \right] \Big\} \\
&\quad + (\text{higher-derivative contributions}), \quad (\text{B5})
\end{aligned}$$

$$\begin{aligned}
H_{mn} &\equiv \frac{\delta\langle \zeta \tilde{\Xi} \rangle}{\delta \pi^{mn}} = \left(\frac{\kappa^2}{2}\right)^2 \frac{4\xi \hat{\lambda}}{\sqrt{g}} \left\{ \right. \\
&\quad \left. N^3 \nabla^{(m} \pi \nabla^{n)} \zeta - g_{mn} \nabla_i (N^3 \pi^{ij} \nabla_j \zeta) \right. \\
&\quad \left. + \frac{N^2 g_{mn}}{2(\lambda D - 1)} \left[(2\lambda + 1) N \pi \nabla_i \nabla^i \zeta \right. \right. \\
&\quad \left. \left. + (6\lambda + 1) \pi \nabla_i N \nabla^i \zeta \right] \right\}
\end{aligned}$$

²⁶ Due to the messy expressions, we have used “xAct” for cross-checking our computations.

$$+(\text{higher-derivative contributions}), \quad (\text{B6})$$

where $f_{(ij)} \equiv (f_{ij} + f_{ji})/2$, $-\mathcal{V}'(R) \equiv (\xi + \alpha n R^{n-1})$, and

$$\begin{aligned} -\frac{\delta \mathcal{V}'(R)(y)}{\delta g_{mn}(x)} &= \alpha n(n-1) R^{n-2} \frac{\delta R(y)}{\delta g_{mn}(x)} \\ &= -\alpha n(n-1) R^{n-2} \left(R^{mn} \delta^D(x-y) \right. \\ &\quad \left. + A^{mnpq}{}_{kl} g^{kl} \nabla_p \nabla_q \delta^D(x-y) \right) \\ &= -\alpha n(n-1) R^{n-2} \left[R^{mn} \delta^D(x-y) \right. \\ &\quad \left. - (\nabla^n \nabla^m - g^{mn} \nabla^2) \delta^D(x-y) \right], \quad (\text{B7}) \end{aligned}$$

$$\begin{aligned} A^{mnpq}{}_{kl} &\equiv g^{pi} X^{qmn}{}_{ikl} - g^{ij} X^{qmn}{}_{ij(k} \delta^{p)}{}_l, \\ X^{mij}{}_{nkl} &\equiv \frac{1}{2} \left[g^m{}_k g^{(i} l g^{j)}{}_n + g^m{}_l g^{(i} n g^{j)}{}_k \right. \\ &\quad \left. - g^m{}_n g^{(i} k g^{j)}{}_l \right]. \quad (\text{B8}) \end{aligned}$$

Then, after some manipulations one can find the Poisson bracket algebras as follows:

$$\begin{aligned} \{\langle \eta \tilde{\Omega} \rangle, \langle \zeta \tilde{\Omega} \rangle\} &= \left(\frac{\kappa^2}{2} \right) \hat{\lambda} (D-2) \int d^D z \left[\nabla_i \eta \nabla_j (N^2 \nabla^j \zeta) \right. \\ &\quad \left. - \nabla_i \zeta \nabla_j (N^2 \nabla^j \eta) \right] \mathcal{V}'(R) N^2 \tilde{C}^i \\ &\quad + (\text{higher-derivative contributions}), \\ \{\langle \eta \tilde{\Omega} \rangle, \langle \zeta \mathcal{H} \rangle\} &\approx - \int d^D z (\nabla_i \eta) \zeta N \tilde{\Theta}^i, \\ \{\langle \eta \tilde{\Omega} \rangle, \langle \zeta^k \mathcal{H}_k \rangle\} &= - \int d^D z \zeta^k \left[(\nabla_k \eta) \tilde{\Omega} - (\nabla_k N^2) \tilde{C}^i \nabla_i \eta \right], \\ \{\langle \eta \tilde{\Xi} \rangle, \langle \zeta \mathcal{H} \rangle\} &\approx - \int d^D z (\nabla_i \eta) \zeta N \tilde{\Sigma}^i \\ &\quad + (\text{higher-derivative contributions}), \\ \{\langle \eta \tilde{\Xi} \rangle, \langle \zeta^k \mathcal{H}_k \rangle\} &= - \int d^D z \zeta^k \left[(\nabla_k \eta) \tilde{\Xi} - (\nabla_k N^2) \tilde{\Theta}^i \nabla_i \eta \right. \\ &\quad \left. - \nabla_k N \left\langle N^2 \left(\frac{\delta \tilde{\Theta}^i}{\delta N(z)} \right) \nabla_i \eta \right\rangle \right], \\ &\quad + (\text{higher-derivative contributions}) \quad (\text{B9}) \end{aligned}$$

where,

$$\begin{aligned} \tilde{\Sigma}^i &= \left(\frac{\kappa^2}{2} \right) 2(D-1) N \left[\frac{(2\lambda+1)}{(\lambda D-1)} \pi \tilde{\Theta}^i - 2\pi^i{}_k \tilde{\Theta}^k \right] \\ &\quad + \left(\frac{\kappa^2}{2} \right)^2 2\xi^2 (D-1) \hat{\lambda} \left[\nabla^i \mathcal{U} + \frac{2\mathcal{V}^i}{(\lambda D-1)} \right] \\ &\quad + \left(\frac{\kappa^2}{2} \right)^3 \frac{8\xi \hat{\lambda}}{\sqrt{g}} \mathcal{W}^i, \quad (\text{B10}) \end{aligned}$$

$$\begin{aligned} \mathcal{U} &\equiv (2N \nabla_j \nabla_k N + \nabla_k N \nabla_j N) \left[2\pi^{jk} - \frac{(2\lambda-1)}{(\lambda D-1)} \pi g^{jk} \right] \\ &\quad + 2N^2 \left[\frac{\lambda}{(\lambda D-1)} \pi R - \pi^{jk} R_{jk} \right], \end{aligned}$$

$$\mathcal{V}^i \equiv \pi \left(-N^2 \nabla^i R - \nabla^i N \nabla_j \nabla^j N + N \nabla^i \nabla^j \nabla_j N \right)$$

$$\begin{aligned} &+ \nabla^i \pi \left[\frac{\Lambda(D+2)}{2\xi(D-1)} N^2 \right. \\ &\quad \left. - \frac{(2\lambda-D-1)}{2(D-1)} \left(N^2 R - N \nabla_j \nabla^j N \right) \right] \\ &\quad + \frac{(\lambda D-1)}{(D-1)} \nabla_j \pi \left(N^2 R^{ij} - N \nabla^j \nabla^i N \right) \\ &\quad - (\lambda-1) \nabla^i (N^2 \nabla_j \nabla^j \pi) - \frac{(6\lambda-5)}{2} \nabla^i (\nabla^j \pi N \nabla_j N), \\ \mathcal{W}^i &\equiv - \frac{(2\lambda+1)(D-2)}{2(\lambda D-1)^2} \pi^3 N \nabla^i N \\ &\quad - \frac{(2\lambda+1)(D(2\lambda+1)-3\lambda-2)}{2(\lambda D-1)^2} N^2 \pi^2 \nabla^i \pi \\ &\quad + \frac{(D-2)}{(\lambda D-1)} \pi^{ij} \pi^2 N \nabla_j N - (2D-3) N^2 \pi^{ik} \pi_{jk} \nabla^j \pi \\ &\quad + \frac{(4D(2\lambda+1)-12\lambda-7)}{2(\lambda D-1)} N^2 \pi^{ij} \pi \nabla_j \pi. \quad (\text{B11}) \end{aligned}$$

The Poisson algebra for the local constraints are given by

$$\begin{aligned} \{\tilde{\Omega}(x), \tilde{\Omega}(y)\} &= \left(\frac{\kappa^2}{2} \right) \hat{\lambda} (D-2) \nabla_i^x \left[\mathcal{V}'(R) N^2 \tilde{C}^i(x) \nabla_j^y \right. \\ &\quad \left. \times \left(N^2 \nabla_y^j \delta^D(x-y) \right) \right] \\ &\quad + (\text{higher-derivative contributions}), \\ \{\tilde{\Omega}(x), \mathcal{H}(y)\} &\approx N \tilde{\Theta}^i(y) \nabla_i^x \delta^D(x-y), \\ \{\tilde{\Omega}(x), \mathcal{H}_k(y)\} &= -\tilde{\Omega}(y) \nabla_k^y \delta^D(x-y) \\ &\quad - \tilde{C}^i \nabla_i^x \delta^D(x-y) \nabla_k N^2(y), \\ \{\tilde{\Xi}(x), \mathcal{H}(y)\} &\approx N \tilde{\Sigma}^i(y) \nabla_i^x \delta^D(x-y) \\ &\quad + (\text{higher-derivative contributions}), \\ \{\tilde{\Xi}(x), \mathcal{H}_k(y)\} &= -\tilde{\Xi}(y) \nabla_k^y \delta^D(x-y) - \tilde{\Theta}^i(y) \nabla_i^x \delta^D(x-y) \\ &\quad \nabla_k N^2(y) \\ &\quad - \nabla_i^x \left[N^2(x) \left(\frac{\delta \tilde{\Theta}^i}{\delta N(y)} \right) \right] \nabla_k N(y) \\ &\quad + (\text{higher-derivative contributions}). \quad (\text{B12}) \end{aligned}$$

Here, we have used

$$\{\tilde{\Omega}(x), \pi_N(y)\} = \nabla_i^x \left[2N \tilde{C}^i(x) \delta^D(x-y) \right], \quad (\text{B13})$$

$$\begin{aligned} \{\tilde{\Xi}(x), \pi_N(y)\} &= \nabla_i^x \left[2N \tilde{\Theta}^i(x) \delta^D(x-y) \right. \\ &\quad \left. + N^2(x) \left(\frac{\delta \tilde{\Theta}^i(x)}{\delta N(y)} \right) \right], \quad (\text{B14}) \end{aligned}$$

where

$$\begin{aligned} &\left(\frac{\delta \tilde{\Theta}^i(x)}{\delta N(y)} \right) \\ &= - \left(\frac{\kappa^2}{2} \right)^2 \frac{2\hat{\lambda}}{(\lambda D-1)\sqrt{g}} \mathcal{V}' \left\{ \left[(2\lambda+1) g^{ij} \pi \right. \right. \\ &\quad \left. \left. - 2(\lambda D-1) \pi^{ij} \right] \nabla_j \pi(x) \delta^D(x-y) \right\} \end{aligned}$$

$$\begin{aligned}
& + \pi^2(x) \nabla_x^i \delta^D(x-y) \Big\} - \left(\frac{\kappa^2}{2} \right) 2\hat{\lambda}(D-1) \\
& \times \sqrt{g} \mathcal{V}' \nabla_x^i \left[\left(\xi R + \frac{\Lambda D}{D-1} + \frac{D-n}{D-1} \alpha R^n \right) \delta^D(x-y) \right. \\
& \left. + \nabla^2 \left(\mathcal{V}' \delta^D(x-y) \right) \right] \\
& - \left(\frac{\kappa^2}{2} \right) [2\hat{\lambda} \nabla_i \pi \{ \mathcal{V}'(R)(x), \mathcal{H}(y) \} \\
& + \{ 2(\pi^{ij} - \hat{\lambda} g^{ij} \pi) \nabla_j \mathcal{V}'(R)(x), \mathcal{H}(y) \}]. \quad (\text{B15})
\end{aligned}$$

From (B9) or (B12), one can now compute

$$\begin{aligned}
\{ \tilde{\Omega}, H_C \} &= \{ \tilde{\Omega}, \langle N\mathcal{H} + N_i \mathcal{H}^i \rangle \} \\
&\approx \nabla_i \left[N^2 \tilde{\Theta}^i - N_k (\nabla_k N^2) \tilde{C}^i \right] + \nabla_k (N^k \tilde{\Omega}), \quad (\text{B16})
\end{aligned}$$

$$\begin{aligned}
\{ \tilde{\Xi}(x), H_C \} &= \{ \tilde{\Xi}(x), \langle N\mathcal{H} + N_i \mathcal{H}^i \rangle \} \\
&\approx \nabla_i \left[N^2 \tilde{\Sigma}^i - N_k (\nabla_k N^2) \tilde{\Theta}^i \right] + \nabla_k (N^k \tilde{\Xi}) \\
&\quad - \nabla_i^x \left[N^2(x) \left(\left(\frac{\delta \tilde{\Theta}^i(x)}{\delta N} \right) N^k \nabla_k N \right) \right] \\
&\quad + (\text{higher-derivative contributions}), \quad (\text{B17})
\end{aligned}$$

which reduce to (33) and (36), respectively, for the case $\tilde{C}^i = 0$.

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