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State-dependent graviton noise in the equation of geodesic deviation

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Abstract We consider an equation of the geodesic deviation appearing in the problem of gravitational wave detection in an environment of gravitons. We investigate a state-dependent graviton noise (as discussed in a recent paper by Parikh,Wilczek and Zahariade) from the point of view of the Feynman integral and stochastic differential equations. The evolution of the density matrix and the transition probability in an environment of gravitons is obtained. We express the time evolution by a solution of a stochastic geodesic deviation equation with a noise dependent on the quantum state of the gravitational field.

1 Introduction

The study of the effect of an environment, which is not directly observable, on the motion of macroscopic bodies, began with the phenomenon of Brownian motion. The mathematical theory of Brownian motion allowed to confirm the presence of invisible molecules (see [1]). In a recent paper Parikh, Wilczek and Zahariade (PWZ) [2] suggest that in a similar way we can detect the environment of gravitons in spite of the negative conclusions of Dyson [3]. According to PWZ the noise resulting from the ground state or coherent state of gravitons is weak and practically undetectable. However, the high temperature states and squeezed states of the graviton can lead to a substantial increase of noise which can be detected in the gravitational wave detection experiments. The authors [2] derive their results by an investigation of the geodesic deviation equation in the environment of the quantized gravitational field. They apply the influence functional method [4] in order to transform the evolution of the transition probability of macroscopic bodies into an expectation value with respect to the noise disturbing the geodesic deviation equation. Studies on the effect of gravitons on the motion of other particles appeared earlier [5–10], but these papers did not tackle directly the geodesic deviation equation. The quantum noise from the environment has been investigated in other fields of physics [11–13]. It can be detected in quantum optics (Ref. [14], sec. 14) and solid state physics.

The evolution of the density matrix in an environment of unobservable particles is usually treated by means of the influence functional [4] which starts from the Feynman integral. As a result one can express the evolution of the density matrix as a solution of the master equation. One can also express the density matrix by the Wigner function and derive a stochastic equation for the evolution of the Wigner function. We have derived such equations for the thermal graviton environment by means of the Feynman integral in [7]. In this paper we apply the method to the quantum evolution of a particle following the geodesic deviation equation when the quantum gravitational field is in an arbitrary Gaussian state. We apply a method of transforming the Feynman integral into an expectation value with respect to the Brownian motion developed in [15,16]. In such a case the dependence of the evolution of the density matrix on the state of the gravitational field is exhibited explicitly.

The plan of the paper is the following. In Sect. 2we derive in a novel way the transformation of the Feynman integral in quantum mechanics into an expectation value over a state-dependent noise. In Sect. 3 we show how this transformation applies to quantum field theory. In Sect. 4 we discuss in detail perturbations by noise resulting from the time-dependent Gaussian states. We explain the calculations of the density matrix and the transition probability in a linear coupling to the environment in Sect. 5. We apply the method to the PWZ model in Sect. 6 (as introduced in [2]) in the one mode approximation. In Sect. 7 the influence of infinite number of modes of the thermal gravitational field upon the geodesic deviation equation is treated by the method of the forward-backward Feynman integral as previously applied in [7] to a geodesic motion. In Sect. 8 we apply our method of the state-dependent noise to derive a stochastic deviation equa-

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tion which governs the evolution of the density matrix and the transition probability. Section 9 contains a summary of the results. In the Appendix we discuss the stochastic deviation equation which results from the assumption that the initial values of the classical gravitational field have the thermal Gibbs distribution.

2 State-dependent transformation of the Feynman integral

Let us consider first a simple model of the Schrödinger equation of quantum mechanics in one dimension with the time-dependent potential U_t

$$i\hbar\partial_t\psi_t=\left(-\frac{\hbar^2}{2m}\nabla_x^2+U_t\right)\psi_t. \tag{1}$$

Assume that we have a solution ψ_t^g of another Schrödinger equation (with a potential $\tilde{U}_t$)

$$i\hbar\partial_t\psi_t^g=\left(-\frac{\hbar^2}{2m}\nabla_x^2+\tilde{U}_t\right)\psi_t^g. \tag{2}$$

Let us write the solution of Eq. (1) in the form

$$\psi_t=\psi_t^g\chi_t. \tag{3}$$

Inserting χ_t from Eq. (3) into Eqs. (1)–(2) we find that χ_t satisfies the equation

$$\partial_t\chi_t=\frac{i\hbar}{2m}\nabla_x^2\chi_t+\frac{i\hbar}{m}(\nabla_x\ln\psi_t^g)\nabla_x\chi_t-\frac{i}{\hbar}(U_t-\tilde{U}_t)\chi_t \tag{4}$$

with the initial condition

$$\chi_0=\psi_0(\psi_0^g)^{-1} \tag{5}$$

expressed by the initial conditions for ψ_t and ψ_t^g.

We apply Eq. (4) either with $\tilde{U}_t=U_t$ or with $U_t=V_t+\frac{m\omega^2x^2}{2}$ and $\tilde{U}_t=\frac{m\omega^2x^2}{2}$. In the first case Eq. (4) reads

$$\partial_t\chi_t=\frac{i\hbar}{2m}\nabla_x^2\chi_t+\frac{i\hbar}{m}(\nabla_x\ln\psi_t^g)\nabla_x\chi_t. \tag{6}$$

In the second case

$$\partial_t\chi_t=\frac{i\hbar}{2m}\nabla_x^2\chi_t+\frac{i\hbar}{m}(\nabla_x\ln\psi_t^g)\nabla_x\chi_t-\frac{i}{\hbar}V_t\chi_t \tag{7}$$

Equation (6) can be considered as the diffusion equation with the imaginary diffusion constant $\frac{i\hbar}{m}$ and a time-dependent drift $\frac{i\hbar}{m}\nabla_x\ln\psi_t^g$. If we extend the diffusion theory [1,17] to a complex domain then we can conclude that the solution of Eq. (6) is determined by the solution of the Langevin equation

$$dq_s=\frac{i\hbar}{m}\nabla\ln\psi_{t-s}^g(q_s)ds+\sqrt{\frac{i\hbar}{m}}db_s. \tag{8}$$

Here, the Brownian motion b_s is defined as the Gaussian process with the covariance

$$E[b_tb_s]=min(t,s). \tag{9}$$

The solution of Eq. (6) is

$$\chi_t(x)=E[\chi_0(q_t(x))], \tag{10}$$

where $q_t(x)$ is the solution of Eq. (8) with the initial condition $q_0(x)=x$ and the expectation value is over the paths of the Brownian motion.

The solution of Eq. (7) is expressed [17] by the Feynman–Kac formula (where the Feynman paths are replaced by the paths of the diffusion process $q_s(x)$ of Eq. (8))

$$\chi_t(x)=E\Big[\exp\Big(-\frac{i}{\hbar}\int_0^t dsV_{t-s}(q_s(x))\Big)\chi_0(q_t(x))\Big] \tag{11}$$

We could derive Eq. (10) from the Feynman integral. The solution ψ_t of Eq. (1) with the initial condition ψ_0 at $t=0$ can be expressed by the Feynman integral

$$\psi_t(x)=\int\mathcal{D}q\\ \times\exp\left(\frac{i}{\hbar}\int_0^t\left(\frac{1}{2}m\left(\frac{dq_s}{ds}\right)^2-U_{t-s}(q_s)\right)ds\right)\psi_0(q_t(x)), \tag{12}$$

where the integral is over paths $q_s(x)$ starting from x (i.e.$q_0(x)=x$; we shall also denote $q_s(x)$ by q_s when there is no danger of confusion).

We show (first for the formula (10)) that if in the Feynman formula (12) we take as Feynman paths the paths of the diffusion process (8) then we obtain Eq. (10). The proof is based on the representation of $\psi_0^g(q_t)=\exp\ln\left(\psi_0^g(q_t)\right)$ as an integral

$$\ln\psi_0^g(q_t(x))=\ln\psi_t^g(x)+\int_0^t d\ln\psi_{t-s}^g(q_s(x)) \tag{13}$$

and the identity (for the process (8))

$$\frac{i}{\hbar}\int_0^t\left(\frac{1}{2}m(\frac{dq_s}{ds})^2-U_{t-s}(q_s)\right)+\ln(\psi_0^g(q_t(x))\\ =-\frac{1}{2}\int_0^t(\frac{db}{ds})^2+\ln\psi_t^g(x). \tag{14}$$

Equation (14) shows that the integration over q in the Feynman formula (12) can be replaced by an average over Brownian paths b_t because we arrive at Eq. (10) (which still will be proved in another way by means of the stochastic calculus).

It is crucial for the derivation that functionals of the Brownian motion satisfy a modified differential formula (Ito formula [18], for an elementary version see [19]) following from the non-differentiability of the Brownian paths b_t

$$\begin{aligned} df_s(b) &= \nabla f_s \circ db + \partial_s f_s ds \\ &= \nabla f_s db + \frac{1}{2}\nabla^2 f_s \langle (db)^2\rangle + \partial_s f_s ds \\ &= \nabla f_s db + \frac{1}{2}\nabla^2 f_s ds + \partial_s f_s ds, \end{aligned} \tag{15}$$

where $\circ$ denotes the Stratonovitch differential (the differential db without circle is called Ito differential [18]) and on the rhs of Eq. (15) we insert $\langle (db)^2\rangle = ds$ (we denote $E[..]$ by $\langle ..\rangle$ as a shorthand). If the Langevin equation (8) is satisfied and if f_s is a function of q then

$$\begin{aligned} df_s &= \nabla f_s \circ dq + \partial_s f_s ds \\ &= \nabla f_s dq + \frac{i\hbar}{2m}\nabla^2 f_s ds + \partial_s f_s ds. \end{aligned}$$

We insert $f_s = \ln \psi^g_{t-s}$. Then

$$\begin{aligned} d\ln \psi^g_{t-s}(q_s) &= \nabla \ln \psi^g_{t-s} dq_s + \frac{i\hbar}{2m}\nabla^2 \ln \psi^g_{t-s} ds \\ &\quad + \partial_s \ln \psi^g_{t-s} ds. \end{aligned} \tag{16}$$

We have

$$\nabla^2 \ln \psi^g_{t-s} = (\nabla^2 \psi^g_{t-s})(\psi^g_{t-s})^{-1} - (\psi^g_{t-s})^{-2}(\nabla \psi^g_{t-s})^2.$$

We use the fact that ψ^g_{t-s} satisfies the Schrödinger equation (2) (with $\tilde{U} = U$) . Then, in equation (16) we can express $\nabla^2\psi_{t-s}$ by U_{t-s} and $\partial_s \psi^g_{t-s}$. We can check after an insertion of Eqs. (13) and (16) in Eq. (14) that in the exponential of the Feynman integral there remains solely $-\frac{1}{2}\int ds(\frac{db}{ds})^2$. This means that the functional integral in the Feynman formula (12) is reduced to an average over the Brownian motion as in Eq. (10).

The formula (10) can be proved directly by differentiation (using the Ito formula). As q_s (8) is a Markov process Eq. (10) defines a semigroup. Then, in order to show that the formula (10) solves Eq. (6) it is sufficient to calculate the generator of the semigroup at $t = 0$. We have from Eq. (10)

$$d\chi_t = E[\nabla \chi_0 dq_s + \tfrac{1}{2}\nabla^2 \chi_0 dq dq] \tag{17}$$

In this formula we insert dq_t from Eq. (8). Note that $E[F db_t] = 0$(if F depends on time less or equal to t) and $dqdq = \frac{i\hbar}{m}dt$. After the calculation of the differentials we may let $t \to 0$ to convince ourselves that the rhs of $d\chi$ is $\frac{i\hbar}{2m}\nabla^2\chi + \frac{i\hbar}{m}\nabla \ln \psi^g_t \nabla\chi$.

If in Eq. (2) we used $\tilde{U} = \frac{m\omega^2 x^2}{2}$ in Eq. (16) with $U = V + \frac{m\omega^2 x^2}{2}$ then after an insertion of Eq. (16) in Eq. (12) the potential $U - \tilde{U} = V$ would remain in the Feynman integral. Hence, we would obtain the result (11). This equation can be proved directly using the stochastic calculus (Eq. (11) defines a semigroup, hence it is sufficient to calculate its generator at $t = 0$)

$$d\chi_t = E\left[-\tfrac{i}{\hbar}V_t(q_t)\chi_0(q_t)dt + \nabla\chi_0 dq + \tfrac{1}{2}\nabla^2\chi_0 dq dq\right]. \tag{18}$$

When we insert dq_t from Eq. (8) then we obtain the rhs of Eq. (7).

The imaginary time version of our formulas belongs to the standard theory of stochastic differential equations [17]. The derivation of the Feynman integral has been discussed earlier in a different form in [15,16]. The stochastic formulas for the Feynman integral are also discussed in [20–23]. There are some mathematical subtleties concerning analytic properties of functions of the stochastic processes as our formalism is an analytic continuation of the one well known for the imaginary time. For a large class of analytic potentials the replacement of the real paths of Feynman by complex diffusion processes driven by Brownian paths is completely equivalent (this is a mathematical theory of the Feynman integral). As follows from the derivation of the Schrödinger equation by means of the stochastic calculus the formula (11) holds true for arbitrary particular solution ψ^t_g and arbitrary initial condition $\psi^g_0\chi_0$ if these functions have a holomorphic extension from the real line.

If we take ψ^g_t in the WKB form $\exp(\frac{i}{\hbar}S_t)$ then in the limit $\hbar \to 0$ of the Schrödinger equation the function S satifies the Hamilton–Jacobi equation. The stochastic equation (8)

$$dq_s = -\frac{1}{m}\nabla S_{t-s}(q_s)ds + \sqrt{\frac{i\hbar}{m}}db_s$$

together with Eq. (10) can be considered as a Hamilton-Jacobi version of a sum over trajectories.

The formalism can be extended to arbitrary number of dimensions and to Lagrangians of the form

$$\mathcal{L} = \frac{1}{2}g_{lk}\frac{dy^l}{ds}\frac{dy^k}{ds} - U. \tag{19}$$

Then the Schrödinger equation (1) holds with the Hamiltonian

$$H = \frac{1}{2}g^{-\frac{1}{2}}p_j g^{\frac{1}{2}}g^{jk}p_k + U, \tag{20}$$

where $g = \det[g_{jk}]$, g^{jk} are the matrix elements of the inverse matrix and

$$p_j = g_{jk}\frac{dy^k}{ds} \tag{21}$$

is the momentum in classical mechanics . In quantum mechanics

$$p_j = -i\hbar\frac{\partial}{\partial y^j}. \tag{22}$$

The corresponding stochastic equation (an analog of Eq. (8)) reads

$$dq^j = \frac{i\hbar}{m}g^{jk}\partial_k \ln(\psi^g_{t-s})ds + \sqrt{\frac{i\hbar}{m}}e^j_k \circ db^k_s, \tag{23}$$

where $e^j_l e^k_l = g^{jk}$.

We consider as a simple example $\tilde{U} = \frac{m\omega^2 x^2}{2}$. Then the ground state solution of Eq. (2) is

$$\psi_g(x) = \left(\frac{\pi\hbar}{m\omega}\right)^{-\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{24}$$

The stochastic equation (8) reads

$$dq = -i\omega q dt + \sqrt{\frac{i\hbar}{m}}db. \tag{25}$$

A simple calculation gives

$$\int dx|\psi^g_s(x)|^2 E[q_s(x)q_{s'}(x)] = \frac{\hbar}{2m\omega}\exp(-i\omega|t-t'|). \tag{26}$$

The rhs of Eq. (26) is the expectation value in the ground state of the time-ordered products of Heisenberg picture position operators of the harmonic oscillator. In this sense the stochastic process q_t has a physical meaning as its correlation functions coincide with quantum expectation values.

3 The relativistic quantum field theory

In the canonical field theory with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int d\mathbf{x}(\Pi^2 + (\omega\Phi)^2) + \int d\mathbf{x}V(\Phi), \tag{27}$$

where $\Pi(\mathbf{x})$ is the canonical momentum

$$\omega = \sqrt{-\triangle + m^2}$$

and

$$[\Phi(\mathbf{x}), \Pi(\mathbf{y})] = i\hbar\delta(\mathbf{x}-\mathbf{y})$$

consider the Schrödinger equation

$$i\hbar\partial_t\psi = \mathcal{H}\psi \tag{28}$$

and its imaginary time version

$$-\hbar\partial_t\psi = \mathcal{H}\psi. \tag{29}$$

Let

$$\psi_0 = \psi_g\chi_0, \tag{30}$$

where ψ^g is the ground state. Then, χ satisfies the equation (an infinite dimensional version of Eq. (6))

$$\hbar\partial_t\chi = \frac{1}{2}\int d\mathbf{x}\Big(-\Pi^2 - 2\big(\Pi\ln\psi_g\big)\Pi\Big)\chi, \tag{31}$$

where

$$\Pi(\mathbf{x}) = -i\hbar\frac{\delta}{\delta\Phi(\mathbf{x})}. \tag{32}$$

Equation (31) is a diffusion equation in infinite dimensional spaces. An approach to Euclidean field theory based on this equation has been developed in [24]. It follows that the solution of Eq. (31) can be expressed as

$$\chi_t(\Phi) = E[\chi_0(\Phi_t(\Phi))], \tag{33}$$

where $\Phi_t(\Phi)$ is the solution of the stochastic equation

$$d\Phi_t(\mathbf{x}) = \hbar\frac{\delta}{\delta\Phi(\mathbf{x})}\ln\psi_g dt + \sqrt{\hbar}dW_t(\mathbf{x}) \tag{34}$$

with the initial condition Φ. $E[...]$ denotes an expectation value with respect to the Wiener process (Brownian motion) defined by the covariance

$$E[W_t(\mathbf{x})W_s(\mathbf{y})] = min(t,s)\delta(\mathbf{x}-\mathbf{y}). \tag{35}$$

The correlation functions of the Euclidean field in the ground state ψ_g can be expressed by the correlation functions of the stochastic process Φ_t.

Let us consider the simplest example:the free field. Then, the ground state is

$$\psi_g = \left(\det\big(\frac{\pi\hbar}{\omega}\big)\right)^{-\frac{1}{4}} \exp\big(-\frac{1}{2\hbar}\Phi\omega\Phi\big). \tag{36}$$

Equation (34) reads

$$d\Phi = -\omega\Phi dt + \sqrt{\hbar}dW \tag{37}$$

with the solution (with the initial condition Φ at t_0)

$$\Phi_t = \exp(-\omega(t-t_0))\Phi + \sqrt{\hbar}\int_{t_0}^t \exp(-\omega(t-s))dW_s. \tag{38}$$

We can calculate

$$\int \mathcal{D}\Phi|\psi_g(\Phi)|^2 E[\Phi_t(\mathbf{x})\Phi_{t'}(\mathbf{x}')] = \hbar\left(\frac{1}{2\omega}\exp(-|t-t'|\omega)\right)(\mathbf{x},\mathbf{x}'). \tag{39}$$

In Eq. (39) the rhs denotes the kernel of the operator.

If the ground state has some analycity property then we can extend the time in Eqs. (28)–(38) $t \to it$ from imaginary time to the real time [15,16]. An analytic continuation to real time of Eq. (37) reads

$$d\Phi_t = -i\omega\Phi_t dt + \sqrt{i\hbar}dW. \tag{40}$$

The solution is

$$\Phi_t(\Phi) = \exp(-i\omega(t-t_0))\Phi + \sqrt{i\hbar}\int_{t_0}^t \exp(-i\omega(t-s))dW_s. \tag{41}$$

The solution of the Schrödinger equation (28) has the same form (33). We can see that

$$\left(\psi_g, T\left(\exp\left(\int dt\Phi_t^q(f_t)\right)\psi_g\right)\right) = \int d\Phi\exp\left(-\frac{1}{\hbar}\Phi\omega\Phi\right)E\left[\exp\left(\int dtd\mathbf{x}f_t(\mathbf{x})\Phi_t(\mathbf{x})\right)\right] = \exp\left(\frac{\hbar}{2}\int dtdt'\left(f_t,(2\omega)^{-1}\exp(-i\omega|t-t'|)f_{t'}\right)\right), \tag{42}$$

where $\Phi_t^q(f) = \int d\mathbf{x}\Phi_t^q(\mathbf{x})f(\mathbf{x})$ is the quantum field and T denotes the time-ordered product.

We can transform the complex equation (40) into two real equations defining

$$\Phi = \phi_1 + i\phi_2 \tag{43}$$

and

$$\phi_+ = \phi_1 + \phi_2 \tag{44}$$

$$\phi_- = \phi_1 - \phi_2. \tag{45}$$

Then it follows from Eq. (40) that

$$\phi_+ = \omega^{-1}\phi_- \tag{46}$$

and ϕ_- satisfies the random wave equation

$$\partial_t^2\phi_- = -\omega^2\phi_- + \sqrt{2\hbar}\omega\partial_t W, \tag{47}$$

which has the solution

$$\phi_-(t) = \cos(\omega t)\phi_-(0) + \omega^{-1}\sin(\omega t)\partial_t\phi_-(0) + \sqrt{2\hbar}\int_0^t \sin(\omega(t-s))dW_s, \tag{48}$$

where $\partial_t\phi_-(0) = \omega\phi_+(0)$.

We further develop the theory of stochastic wave equations for the quantum theory of fields in expanding universes in [26] as an extension of Starobinsky stochastic inflation [27].

4 Time-dependent reference state

There are time-dependent solutions in the Gaussian form of the Schrödinger equation for the free field theory

$$\psi_t^g(\Phi) = A(t)\exp\left(\frac{i}{2\hbar}(\Phi\Gamma(t)\Phi + 2J_t\Phi)\right). \tag{49}$$

In the time-dependent case Eq. (8) in quantum field theory takes the form [17]

$$d\Phi_s = -\Gamma(t-s)\Phi_s ds - J_{t-s}ds + \sqrt{i\hbar}dW_s. \tag{50}$$

Let $\Phi_s(\Phi)$ be the solution of Eq. (50) with the initial condition Φ then the solution of the Schrödinger equation (28) with the initial condition $\psi_0^g\chi$ is

$$\psi_t = \psi_t^g E[\chi(\Phi_t(\Phi))]. \tag{51}$$

In the model (27) with a time-dependent interaction V_t it would be difficult to find any explicit solution ψ_t^g. However, we can use the solution ψ_t^g of the free field theory and take interaction into account by means of the Feynman–Kac formula. According to Eq. (11) the solution of the Schrödinger equation with the interaction V_t reads

$$\psi_t(\Phi) = \psi_t^g E\left[\exp\left(-\frac{i}{\hbar}\int_0^t V_{t-s}(\Phi_s)ds\right)\chi\left(\Phi_t(\Phi)\right)\right]. \tag{52}$$

We shall apply Eq. (52) for V which is linear in Φ (Φ could be the gravitational potential in de Donder gauge as in [25]; it will have many components when describing gravitons in the transverse-traceless gauge in Sects. 7, 8).

As an application to the particle motion in a (quantized squeezed) gravitational wave let us consider the Hamiltonian of a harmonic oscillator($\tilde{U}=\frac{\omega^2x^2}{2}$ in Eq. (2) corresponding to a single mode of the gravitational field)

$$H=-\frac{\hbar^2}{2}\nabla^2+\frac{1}{2}\omega^2x^2. \tag{53}$$

Let us look for a time-dependent solution of the Schrödinger equation (2) in the form

$$\psi_t^g=A(t)\exp\left(\frac{i}{2\hbar}(x\Gamma(t)x+2J_tx)\right). \tag{54}$$

Then, ψ^g is a solution of the Schrödinger equation (2) if

$$i\hbar\partial_t\ln A=-\frac{i\hbar}{2}\Gamma+\frac{1}{2}J^2, \tag{55}$$

$$-\partial_t J=\Gamma J \tag{56}$$

and

$$\partial_t\Gamma+\Gamma^2+\omega^2=0. \tag{57}$$

It can be seen that the Riccati equation (57) is equivalent to

$$\frac{d^2u}{dt^2}+\omega^2u=0, \tag{58}$$

where

$$u=\exp\left(\int^t\Gamma\right). \tag{59}$$

i.e.$\Gamma=u^{-1}\partial_tu$. Then, $J_t=J_0u_0u_t^{-1}$. The general solution of Eq. (58) is

$$u=\sigma\cos(\omega t)+\delta\sin(\omega t), \tag{60}$$

where σ,δ are complex numbers.

Equation (8) reads

$$dq_s=-\partial_t\ln u_{t-s}q_sds-J_{t-s}ds+\sqrt{i\hbar}db_s \tag{61}$$

with the solution

$$q_s(q)=\frac{u_{t-s}}{u_t}q+\sqrt{i\hbar}u_{t-s}\int_0^s u_{t-\tau}^{-1}db_\tau \\ -u_0u_{t-s}J_0\int_0^s u_{t-\tau}^{-2}d\tau. \tag{62}$$

We have (as $E[(\int f_sdb_s)^2]=\int f_s^2ds$)

$$E[(q_s-\langle q_s\rangle)(q_{s'}-\langle q_{s'}\rangle)] \\ =i\hbar u_{t-s}u_{t-s'}\int_0^{min(s,s')}u_{t-\tau}^{-2}d\tau \\ =(-i\hbar)\omega^{-2}u_{t-s}u_{t-s'} \\ \times(\sigma^2+\delta^2)^{-1}\Big(\Gamma(t)-\Gamma(t-min(s,s'))\Big). \tag{63}$$

If $\delta a=i\sigma$ then $\Gamma(0)=ia^{-1}\omega$ and the solution ψ_t starts from a real $i\Gamma$. Choosing $J_0=a^{-1}\omega x_0+ip$ we obtain the squeezed state

$$\psi_0^g=\exp\left(-\frac{\omega(x-x_0)^2}{2a\hbar}+\frac{i}{\hbar}px\right)$$

with the squeezing a of the coordinate (for a time evolution of squeezed states see [28]). The case $\delta=i\sigma$ corresponds to the ground state (24) ($m=1$). It can be shown that the formula (63) is continuous with respect to the limit $\delta\to i\sigma$.

5 Linear coupling to an oscillator environment

Let us consider a model of a system with a Lagrangian $\mathcal{L}_\xi$ described by a coordinate ξ linearly interacting with an oscillator. We have the Lagrangian

$$\mathcal{L}=\mathcal{L}_\xi+\frac{1}{2}\left(\left(\frac{dq}{ds}\right)^2-\omega^2q^2\right)+qf_s(\xi). \tag{64}$$

In the model with a linear coupling and the quadratic Lagrangian for the q variable (64) the functional integral in Eq. (11) can be reduced to the Gaussian integral

$$\chi_t(x)=\int dk\chi_0(k)E\Big[\exp\Big(\tfrac{i}{\hbar}\int_0^t dsq_s(x)f_{t-s}(\xi_s)+ikq_t(x)\Big)\Big] \tag{65}$$

where we used the Fourier representation of $\chi_0(x)$ (we use the same notation for a function and its Fourier transform)

$$\chi_0(x)=\int dk\chi_0(k)\exp(ikx)$$

For a Gaussian variable q_s we have (for any number α_s)

$$E[\exp(\alpha_sq_s)] \\ =\exp\left(\alpha_s\langle q_s\rangle+\tfrac{1}{2}\langle(\alpha_sq_s-\alpha_s\langle q_s\rangle)^2\rangle\right) \tag{66}$$

The formula (66) can easily be generalized to $\int ds\alpha_sq_s$ with q_s of Eq. (62)) and to an infinite number of modes q (as will be done in Sects. 7, 8).

We are interested in the scattering amplitude from an initial state $\psi_0^g(x)\chi_i(x)\phi_i(\xi)$ to the final state $\psi_0^g(x)\chi_f(x)\phi_f(\xi)$. We apply the transformation of Sect. 2 only to the oscillator path integral. Then, according to Eq. (11) the amplitude a_{fi} is (where $(\psi_0^g\chi_i(x)\phi_i)_t$ means the unitary evolution of the wave function)

$$a_{fi}=(\psi_0^g\chi_f\phi_f,(\psi_0^g\chi_i(x)\phi_i)_t)$$
$$=\int dxd\xi\int\mathcal{D}\xi\exp(\frac{i}{\hbar}\int ds\mathcal{L}_\xi)\overline{\psi_0^g(x)}\overline{\chi_f(x)\phi_f(\xi)}$$
$$\times\psi_t^g\phi_i(\xi_t(\xi))E\Big[\exp(\frac{i}{\hbar}\int_0^t q_sf_{t-s}(\xi_s))\chi_i(q_t(x))\Big], \tag{67}$$

where the initial state of the oscillator is $\psi_0^g\chi_i$.

If we do not observe the final states of the oscillator and average the probability $P(i,f)$ of the transition $\phi_i\to\phi_f$ over these states using the completeness relation

$$\sum_f\overline{\psi_t^g(x)\chi_f(x)}\psi_t^g(x')\chi_f(x')=\delta(x-x') \tag{68}$$

then we obtain

$$P(i,f)=\sum_f|a_{fi}|^2=\int dxd\xi d\xi'\mathcal{D}\xi\mathcal{D}\xi'|\psi_t^g(x)|^2$$
$$\times\phi_f(\xi)\overline{\phi_f}(\xi')\overline{\phi_i}(\xi_t(\xi))\phi_i(\xi_t(\xi'))$$
$$\times\exp\Big(-\frac{i}{\hbar}\int ds\mathcal{L}_\xi+\frac{i}{\hbar}\int ds\mathcal{L}_{\xi'}\Big)$$
$$\times E\Big[\exp\Big(\frac{i}{\hbar}\int_0^t q_sf_{t-s}(\xi'_s)\Big)\chi_i(q_t(x))\Big)$$
$$\times\exp\Big(-\frac{i}{\hbar}\int_0^t q_s^*f_{t-s}(\xi_s)\Big)\overline{\chi_i}(q_t^*(x))\Big]. \tag{69}$$

where q^* denotes a complex conjugation of an independent version of the process q_t and $\xi'_s(\xi')$ is another realization of the path ξ_s.

If we define the density matrix ρ as an average over the environment of the oscillator then the density matrix of the ξ system is

$$\rho_t(\xi,\xi')$$
$$=\int dx\mathcal{D}\xi\mathcal{D}\xi'|\psi_t^g(x)|^2\exp\Big(-\frac{i}{\hbar}\int ds\mathcal{L}_\xi+\frac{i}{\hbar}\int ds\mathcal{L}_{\xi'}\Big)$$
$$\times\overline{\phi_i}(\xi_t(\xi))\phi_i(\xi_t(\xi'))E\Big[\exp\Big(\frac{i}{\hbar}\int_0^t q_sf_{t-s}(\xi'_s)\Big)\chi_i(q_t(x))$$
$$\times\exp\Big(-\frac{i}{\hbar}\int_0^t q_s^*f_{t-s}(\xi_s)\Big)\overline{\chi_i}(q_t^*(x))\Big], \tag{70}$$

It can be seen from Eqs. (69), (70) that the calculations of the density matrix and the transition probability are closely related. When calculating the transition probability we need to perform an extra (ξ,ξ') integral over the final states $\overline{\phi_f(\xi)}\phi_f(\xi')$ of the ξ system in comparison to the calculation of the density matrix in Eq. (70).

If $\chi_i=1$ then according to Eq. (66) the expectation value (67) is

$$E\Big[\exp\Big(\frac{i}{\hbar}\int_0^t q_sf_{t-s}(\xi_s)\Big)\Big]$$
$$=\exp\Big(\frac{i}{\hbar}\int_0^t\langle q_s\rangle f_{t-s}(\xi_s)\Big)$$
$$\times\exp\Big(-\frac{1}{2\hbar^2}\int_0^t dsds'$$
$$\times E[(q_s-\langle q_s\rangle)(q_{s'}-\langle q_{s'}\rangle)]f_{t-s}(\xi_s)f_{t-s'}(\xi_{s'})\Big). \tag{71}$$

6 Particle interacting with gravitons: PWZ model in one mode approximation

Parikh,Wilczek and Zahariade [2] consider two masses M and m_0 interacting with gravitational field q_ω. In a one mode approximation the Lagrangian describing the geodesic deviation of the m_0 mass (in the free falling frame and $M>>m_0$) is [2,29]

$$\mathcal{L}=\frac{1}{2}((\frac{dq_\omega}{ds})^2-\omega^2q_\omega^2)+\frac{1}{2}m_0(\frac{d\xi}{ds})^2$$
$$-m_0\lambda\frac{dq_\omega}{ds}\frac{d\xi}{ds}\xi-V(\xi)$$
$$=\frac{1}{2}g_{jk}\frac{dy^j}{ds}\frac{dy^k}{ds}-\frac{1}{2}\omega^2q_\omega^2-V(\xi), \tag{72}$$

where $\lambda=\sqrt{8\pi G}$, $y=(q_\omega,\xi)$, ξ is the geodesic deviation (the potential $V(\xi)$ is absent in [2] but we add it here for further applications in the next sections).

$$(g_{jk})=\begin{bmatrix}1 & -m_0\lambda\xi\\ -m_0\lambda\xi & m_0\end{bmatrix}. \tag{73}$$

The Lagrange equation is

$$\frac{d^2\xi}{ds^2}=m_0\lambda\frac{d^2q_\omega}{ds^2}\xi.$$

The Hamiltonian is determined by Eq. (20) with

$$[g^{jk}]=(m_0-\lambda^2\xi^2)^{-1}\begin{bmatrix}m_0 & m_0\lambda\xi\\ m_0\lambda\xi & 1\end{bmatrix}. \tag{74}$$

We could quantize the model (72) with the explicit quantum version (20) of the Hamiltonian. However, the averaging over gravitons is simple only in the influence functional approach.

In the PWZ model (72), where the gravitational field is described by one mode q_ω, when we apply the transformation

(25) to q_s with the ground state (24) ($m=1$) then from Eq. (70) we obtain

$$\rho_t(\xi,\xi')=\int dx\exp\Big(-\frac{\omega}{\hbar}x^2\Big)\times E\Big[\exp\Big(\frac{i}{\hbar}\int_0^t\Big(\frac{m_0}{2}\frac{d\xi}{ds}\frac{d\xi}{ds}-\frac{m_0}{2}\frac{d\xi'}{ds}\frac{d\xi'}{ds}-V(\xi)+V(\xi')\Big)\times\exp\Big(-\frac{i\lambda m_0}{2\hbar}\int_0^t\Big(q_{t-s}(x)\frac{d^2\xi^2}{ds^2}-q^*_{t-s}(x)\frac{d^2\xi'^2}{ds^2}\Big)\Big)\times\chi_i(q_t(x))\overline{\chi_i}(q_t^*(x))\overline{\phi_i}(\xi_t(\xi))\phi_i(\xi_t(\xi'))\Big]\equiv K_t\rho_0, \tag{75}$$

where q^* is another realization of the process q (and the complex conjugation of this realization). We have changed $s\to t-s$ in the integral in the exponential and inserted $f_s=\frac{m_0}{2}\frac{d^2\xi^2}{ds^2}$. We have denoted the evolution kernel of the density matrix by K_t .

The calculation of the expectation value (75) according to Eq. (71) gives (we use the solution (41), where $\Phi\to q$, of Eq. (25) and assume that the initial condition $\chi_i=1$)

$$\rho_t\simeq\int dx\mathcal{D}\xi\mathcal{D}\xi'\exp(-\frac{\omega x^2}{\hbar})\times E\Big[\exp\Big(\frac{i}{\hbar}\int_0^t\Big(\frac{m_0}{2}\frac{d\xi}{ds}\frac{d\xi}{ds}-\frac{m_0}{2}\frac{d\xi'}{ds}\frac{d\xi'}{ds}-V(\xi)+V(\xi')-\frac{i\lambda}{\hbar}\int_0^t(q_{t-s}f_s-q^*_{t-s}f'_s)ds\Big)\Big]$$
$$\simeq\int dx\mathcal{D}\xi\mathcal{D}\xi'\exp(-\tfrac{\omega x^2}{\hbar})\times\exp\Big(-\frac{i\lambda}{\hbar}\int_0^t(q\exp(-i\omega(t-s))f_s-q\exp(i\omega(t-s))f'_s)ds\Big)\times\exp\Big(-\frac{\lambda^2}{2\hbar^2}\int_0^t dsds'\times\Big(E[(q_{t-s}-\langle q_{t-s}\rangle)(q_{t-s'}-\langle q_{t-s'}\rangle)]f_sf_{s'}+E[(q^*_{t-s}-\langle q^*_{t-s}\rangle)(q^*_{t-s'}-\langle q^*_{t-s'}\rangle)]f'_sf'_{s'}\Big)\Big), \tag{76}$$

where $f=\frac{m_0}{2}\frac{d^2\xi^2}{ds^2}$, $f'=\frac{m_0}{2}\frac{d^2\xi'^2}{ds^2}$. In Eq. (76) we have (this is the special case of Eq. (63) with $u_s=\exp(i\omega s)$)

$$E[(q_{t-s}-\langle q_{t-s}\rangle)(q_{t-s'}-\langle q_{t-s'}\rangle)]=\frac{\hbar}{2\omega}(\exp(-i\omega|s-s'|)-\exp(-i\omega(2t-s-s'))). \tag{77}$$

If the oscillator is in a time-dependent state then we should insert the solution (62) in the Feynman formula (75). Hence, instead of Eq. (76) we have (for typographical reasons from now on we identify $\overline{\Gamma}=\Gamma^*$, when acting on q_t the star $*$ has an extra meaning:it means complex conjugation and an independent realization of the process q_t)

$$\int dx|\exp(i\tfrac{\Gamma(t)x^2}{2\hbar})|^2E\Big[\exp\Big(\tfrac{i\lambda}{\hbar}\int_0^t(q_sf_{t-s}-q^*_sf'_{t-s})ds\Big)\Big]=\int dx\exp(i\frac{\Gamma(t)x^2}{2\hbar})\exp(-i\frac{\Gamma^*(t)x^2}{2\hbar})\times\exp\Big(\frac{-i\lambda}{\hbar}\int_0^t(\langle q_{t-s}\rangle f_s-\langle q^*_{t-s}\rangle f'_s)ds\Big)\times\exp\Big(-\frac{\lambda^2}{2\hbar^2}\int_0^t dsds'\times\Big(E[(q_{t-s}-\langle q_{t-s}\rangle)(q_{t-s'}-\langle q_{t-s'}\rangle)]f_sf_{s'}+E[(q^*_{t-s}-\langle q^*_{t-s}\rangle)(q^*_{t-s'}-\langle q^*_{t-s'}\rangle)]f'_sf'_{s'}\Big)\Big), \tag{78}$$

where

$$E[(q_{t-s}-\langle q_{t-s}\rangle)(q_{t-s'}-\langle q_{t-s'}\rangle)]=i\hbar u_su_{s'}\int_0^{min(t-s,t-s')}d\tau u(t-\tau)^{-2}=-i\hbar\omega^{-2}u_su_{s'}(\sigma^2+\delta^2)^{-1}\times(\Gamma(t)-\Gamma(max(s,s'))). \tag{79}$$

After calculation of the x integral in Eqs. (76) and (78) we obtain a quadratic functional of f_s and $f'_{s'}$ in the exponential. In the simplest case (76) of the ground state of the oscillator we obtain

$$\rho_t\simeq\int\mathcal{D}\xi\mathcal{D}\xi'\times\exp\Big(\frac{i}{\hbar}\int_0^t(\frac{m_0}{2}\frac{d\xi}{ds}\frac{d\xi}{ds}-\frac{m_0}{2}\frac{d\xi'}{ds}\frac{d\xi'}{ds}-V(\xi)+V(\xi'))\Big)\times\exp\Big(-\frac{\lambda^2}{4\hbar\omega}\int_0^t dsds'\Big(f_sf_{s'}\exp(-i\omega|s-s'|)+f'_sf'_{s'}\exp(i\omega|s-s'|)-2f_{s'}f'_s\exp(i\omega(s-s')\Big)\Big). \tag{80}$$

We write

$$X=\frac{1}{2}(\xi+\xi')$$
$$y=\xi-\xi'. \tag{81}$$

We expand the exponential in Eq. (80) in y. Then, the terms independent of y cancel and there remains (till the terms quadratic in y)

$$\rho_t\simeq\int\mathcal{D}X\mathcal{D}y\exp\Big(\frac{i}{\hbar}\int_0^t y(m_0\frac{d^2X}{ds^2}-V'(X))\Big)\times\exp\Big(-\frac{\lambda^2m_0^2}{8\hbar\omega}\int_0^t dsds'\Big(-i\sin(\omega(s-s'))(\frac{d^2Xy}{ds'^2}\frac{d^2X^2}{ds^2}-\frac{d^2Xy}{ds^2}\frac{d^2X^2}{ds'^2})+\frac{d^2Xy}{ds^2}\frac{d^2Xy}{ds'^2}\cos(\omega(s-s'))\Big)\Big)\rho_0(X_t,y_t). \tag{82}$$

The term linear in y gives a modification of the equation of motion of the ξ coordinate whereas the term quadratic in y is a noise acting upon the particle [7].

In the expression (78) of the time-dependent reference state we obtain

$$\rho_t \simeq \exp\Big(-\frac{i}{2\hbar}(\Gamma(t)-\Gamma^*(t))^{-1} \times\Big(\lambda\int_0^t (u_t^{-1}u_s f_s - u_t^{*-1}u_s^* f_s')ds\Big)^2 -\frac{i\lambda^2}{2\hbar\omega^2}\int_0^t\Big(u_s u_{s'}(\sigma^2+\delta^2)^{-1}(\Gamma(t) -\Gamma(max(s,s')))f_s f_{s'} - u_s^* u_{s'}^*(\sigma^{*2} +\delta^{*2})^{-1}(\Gamma^*(t)-\Gamma^*(max(s,s')))f_s' f_{s'}'\Big)dsds'\Big). \tag{83}$$

We expand (83) in y again. Let us consider the general expression appearing after an expansion in y till the second order terms in Eqs. (82), (83)

$$\int \mathcal{D}X\mathcal{D}y \times\exp\Big(\frac{i}{\hbar}\int_0^t y(m_0\frac{d^2X}{ds^2}+V'(X)+L(X)+\frac{i}{2\hbar}My)\Big)\rho_0(X,y) \equiv \int \mathcal{D}X\mathcal{D}y\exp\Big(\frac{i}{\hbar}\int_0^t(y\tilde{L}+\frac{i}{2\hbar}yMy)\Big)\rho_0(X,y) = \int \mathcal{D}X\mathcal{D}y\exp\Big(-\frac{1}{2\hbar^2}(y-i\hbar M^{-1}\tilde{L})M(y-i\hbar M^{-1}\tilde{L}) -\frac{1}{2}\tilde{L}M^{-1}\tilde{L}\Big)\rho_0(X,y),$$

where by L we denote a functional of X, M is an operator and by $\tilde{L}$ we denote the term proportional to y. If we introduce $\tilde{X}=M^{-\frac{1}{2}}\tilde{L}$ then $\tilde{X}$ becomes a Gaussian variable with the white noise distribution which can be represented as $\partial_s b_s$. If ρ_0 depends only on X then the factor depending on y is integrated out contributing just a constant. The calculation of ρ_t is reduced to an average over solutions of the stochastic equation

$$m_0\frac{d^2X}{ds^2}+V'(X)+L(X)=M^{\frac{1}{2}}\partial_s b_s. \tag{84}$$

In general, there still will be the Gaussian integral over y so that the expression for the density matrix can be obtained in the form of an expectation value over the solutions of the stochastic equation (84) and the y terms resulting from an expansion in y of $\rho_0(X,y)$ (this is an expansion in $\hbar$).

In the next section dealing with a thermal state of the gravitational field we approximate L and M by local functions of X.

7 Infinite number of modes:thermal state of gravitons

We are to generalize the results of Sect. 6 to an infinite number of modes of the gravitational field. We could do it in the formulation of Sect. 6 by means of a representation of the thermal state as a sum over eigenstates with a proper weight factor. We have studied earlier [7] an analogous model of a particle geodesic motion in an environment of quantum gravitational waves in a thermal state. There are minor changes from the setting of Parikh, Wilczek and Zahariade [2]where the effect of gravitons on geodesic deviation is considered. The Lagrangian (72) with an infinite number of modes in the coordinate space is [29]

$$\mathcal{L}=\frac{1}{4}\int d\mathbf{x}h_\alpha(s,\mathbf{x})(-\partial_s^2+\triangle)h_\alpha(s,\mathbf{x})+\frac{1}{2}m_0\frac{d\xi_r}{ds}\frac{d\xi_r}{ds} -\frac{1}{4}m_0\lambda\partial_s^2 h_{rl}^w(s,\xi_s)\xi_s^r\xi_s^l-\frac{1}{4}m_0\lambda\partial_s^2 h_{rl}^q(s,\xi_s)\xi_s^r\xi_s^l, \tag{85}$$

where the metric perturbation h_{rl} of the Minkowski metric $\eta_{\mu\nu}$ ($\eta_{\mu\nu}\to\eta_{\mu\nu}+h_{\mu\nu}$) is in the transverse-traceless gauge, the geodesic deviation ξ in a gravitational transverse traceless gauge has only the spatial components ξ_r. We apply the decomposition of $h_{rl}=h_{rl}^w+h_{rl}^q$ into the classical wave solution h_{rl}^w and the quantized (graviton) part h_{rl}^q. h_{rl}^q is decomposed in the amplitudes h_α (where $\alpha=+,\times$,in the linear polarization) by means of the polarization tensors e_{rl}^α (as $h_{rl}=e_{rl}^\alpha h_\alpha$) [30,31,37]

$$h_{rl}(t,\mathbf{x})=h_{rl}^w(t,\mathbf{x}) +(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}(h_\alpha e_{rl}^\alpha\exp(-i\mathbf{kx})+h_\alpha^* e_{rl}^\alpha\exp(i\mathbf{kx})).$$

With the infinite number of gravitational wave modes in the model (72) of Sect. 6 we shall have $\omega=|\mathbf{k}|$ (we set the velocity of light $c=1$).$H=H_+ + H_\times$ is the sum of the independent Hamiltonians (27) with $V=m=0$.

The h_{rl}^q are quantized at finite temperature T with the Gibbs distribution $\exp(-\beta H)$, $\beta^{-1}=k_BT$ where k_B is the Boltzman constant. The classical part of the particle-wave interaction can be considered as a time-dependent external potential $V(\xi)$ with

$$V=\frac{m_0}{4}\partial_s^2 h_{kn}^w(s,\xi_s)\xi_s^k\xi_s^n. \tag{86}$$

Denote

$$f^{rl}=\frac{m_0}{2}\frac{d^2}{ds^2}\xi^r\xi^l \tag{87}$$

$$f'^{rl}=\frac{m_0}{2}\frac{d^2}{ds^2}\xi'^l\xi'^r. \tag{88}$$

Then, as in [7] (there are some misprints of signs in [7]; for the derivation of the thermal formula see [32], sec. 18, see also [5,8,10]; the gravitational case is analogous to the electromagnetic one treated in [33]) we obtain for the density matrix evolution kernel

$$K_t(\xi,\xi')=\int D\xi D\xi' \exp\left(\frac{im_0}{2\hbar}\int_0^t ds\left(\frac{d\xi_r}{ds}\frac{d\xi_r}{ds}-\frac{d\xi'_r}{ds}\frac{d\xi'_r}{ds}\right)\right.$$
$$\times\exp\left(\frac{i}{\hbar}\int_0^t (V_{t-s}(\xi_s)-V_{t-s}(\xi'_s)\right)$$
$$\times\exp\left(\frac{\lambda^2}{\hbar^2}\int_0^t ds\int_0^s ds'\Big((f^{rl}-f'^{rl})C_{rl:mn}(f^{mn}+f'^{mn})\right.$$
$$\left.-(f^{rl}-f'^{rl})A_{rl:mn}(f^{mn}-f'^{mn})\Big)\right). \quad (89)$$

In (89) we have a decomposition of the finite temperature transverse-traceless graviton propagator D into the real and imaginary parts $D=A+iC$

$$A_{rl;mn}(\mathbf{x}-\mathbf{x}',s-s')=2\beta\hbar(2\pi)^{-3}\int\frac{\mathbf{dk}}{2k}\Lambda_{rl;mn}$$
$$\cos\left(\mathbf{k}(\mathbf{x}-\mathbf{x}')\right)\cos(k(s-s'))\coth\left(\frac{\hbar\beta k}{2}\right), \quad (90)$$
$$C_{rl;mn}(\mathbf{x}-\mathbf{x}',s-s')=2\hbar(2\pi)^{-3}\int\frac{\mathbf{dk}}{2k}\Lambda_{rl;mn}$$
$$\cos(\mathbf{k}(\mathbf{x}-\mathbf{x}'))\sin(k(s-s')), \quad (91)$$

where

$$2\Lambda_{ij;mn}=2e^\alpha_{ij}e^\alpha_{mn}=(\delta_{im}-k^{-2}k_ik_m)(\delta_{jn}-k^{-2}k_jk_n)$$
$$+(\delta_{in}-k^{-2}k_ik_n)(\delta_{jm}-k^{-2}k_jk_m)$$
$$-\frac{2}{3}(\delta_{ij}-k^{-2}k_ik_j)(\delta_{nm}-k^{-2}k_nk_m).$$

We neglect the dependence on $\mathbf{x}$ in Eqs. (90)–(91). Then the angular average over $\mathbf{k}k^{-1}$ gives

$$\frac{1}{4\pi}\langle\Lambda_{ij;mn}\rangle=\frac{1}{5}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})-\frac{2}{15}\delta_{ij}\delta_{nm}. \quad (92)$$

When we neglect the $\mathbf{x}$ dependence of the propagators ($\xi^r-\xi'^r$ in Eq. (89) should be inserted as $x^r-x'^r$ and we assume that ξ is small in comparison with the gravitational wave length, hence it can be set to zero) and average over the angles then the k-integral $d\mathbf{k}\simeq 4\pi dkk^2$ in the high temperature limit $\beta\hbar\to 0$ of $A_{rl;mn}$ in Eq. (90) gives $\delta(s-s')$. In $C_{rl;mn}$ (91) we write (as in [7]) $\sin(k(s-s'))=-k^{-1}\partial_s\cos(k(s-s'))$. Then, integrating over k we obtain $\partial_s\delta(s-s')$. In such a case the evolution kernel of the density matrix reads

$$K_t(\xi;\xi')=\int D\xi D\xi'\exp\left(\frac{im_0}{2\hbar}\int_0^t ds\left(\frac{d\xi_r}{ds}\frac{d\xi_r}{ds}\right.\right.$$
$$\left.\left.-\frac{d\xi'_r}{ds}\frac{d\xi'_r}{ds}\right)+\frac{i}{\hbar}\int_0^t(V_{t-s}(\xi_s)-V_{t-s}(\xi'_s))\right)$$
$$\times\exp\left(-i\frac{\gamma}{2\hbar}\int_0^t ds\Big((q^{rl}-q'^{rl})\partial_s(q^{rl}+q'^{rl})\right.$$
$$\left.-\frac{w}{2\hbar^2}(q^{rl}-q'^{rl})(q^{rl}-q'^{rl})\Big)\right), \quad (93)$$

where

$$q^{rl}=\frac{1}{2}\frac{d^2}{ds^2}\left(\xi^r\xi^l-\frac{1}{3}\delta^{rl}\xi_j\xi_j\right) \quad (94)$$

results from a resummation of f^{rl} with $\langle\Lambda_{ij;rl}\rangle$

$$\gamma=\frac{8\pi Gm_0^2}{10\pi} \quad (95)$$
$$w=\frac{2\gamma}{\beta} \quad (96)$$

We expand Eq. (93) around X (81) (now with three spatial indices)

$$q^{rl}-q'^{rl}=\frac{d^2}{ds^2}\left(X^ry^l+X^ly^r-\frac{2}{3}\delta^{rl}X^jy^j\right)$$

In the exponential (93) the term linear in y becomes

$$y_n\left(-\frac{d^2X_n}{ds^2}+\frac{\lambda}{2}\frac{d^2h^w_{nr}}{ds^2}X_r\right.$$
$$\left.+\frac{8\pi Gm_0}{10\pi}X_l\frac{d^5}{ds^5}\left(\frac{1}{3}X_rX_r\delta_{nl}-X_nX_l\right)\right) \quad (97)$$

The term quadratic in y is the noise term. For low temperature we obtain in general the non-local and non-Markovian stochastic equation (84). In the high temperature limit $\beta\hbar\to 0$ the calculation of the evolution kernel is reduced to an expectation value over the solutions of the stochastic equation

$$-\frac{d^2X_n}{ds^2}+\frac{\lambda}{2}h^w_{nr}X_r+\frac{8\pi Gm_0}{10\pi}X^l\frac{d^5}{ds^5}\left(\frac{1}{3}X_rX_r\delta_{nl}-X_nX_l\right)$$
$$=m_0^{-1}\sqrt{w}(M^{\frac{1}{2}})_{nr}\partial_sb^r_s. \quad (98)$$

As explained in the derivation of Eq. (84) the term quadratic in y defines the operator M. From Eq. (93) we obtain that M is an operator defined by the bilinear form (on the rhs of Eq. (98) we have the square root of the matrix M)

$$wy^rM^{rl}y^l=2w\int ds\left(\frac{d^2}{ds^2}(X^jy^l)\frac{d^2}{ds^2}(X^jy^l)\right.$$
$$\left.+\frac{d^2}{ds^2}(X^jy^l)\frac{d^2}{ds^2}(X^ly^j)-\frac{2}{3}\frac{d^2}{ds^2}(X^jy^j)\frac{d^2}{ds^2}(X^ly^l)\right)$$
$$=\frac{\lambda^2}{4\pi}\beta^{-1}y^rX^k\mathcal{M}_{rk;ln}y^lX^n$$

where

$$\mathcal{M}_{rk;ln}(s,s')=\langle\Lambda_{rk;ln}\rangle\partial_s^2\partial_{s'}^2\delta(s-s') \quad (99)$$

We derive Eq. (98) with the noise (99) in the Appendix working directly with the classical model of thermal gravitational waves. The modification of the deviation equation coincides with the one of Ref. [29] (up to an infinite renormalization term). It is discussed already in [37] (sec. 36.8) and in more detail in [38] (see in particular the last section of this paper). The spectrum of the noise (of the quadrupole q^{rl}) as seen from Eqs. (89), (90) is $8\pi Gk\hbar\coth(\frac{\hbar\beta k}{2})$ which at low temperature is $8\pi Gk$ and at high temperature $8\pi G\beta^{-1}$ (the spectrum of the coordinates ξ, as considered by [29] in Eqs. (3.13)–(3.14), is multiplied by k^4 as a result of the fourth order derivative in Eq. (99)).

8 General Gaussian state of the graviton

We generalize the results of Sect. 6 on averaging over the oscillator modes in a squeezed state to infinite number of modes of the gravitational field. The gravitational perturbation in the transverse-traceless gauge is decomposed into the amplitudes h^α by means of the polarization tensors e^α_{rl},$\alpha = +, \times$, $h_{rl} = e^\alpha_{rl}h_\alpha$. The gravitational Hamiltonian $H = H_+ + H_\times$ has the same form as for two independent scalar fields $\Phi \to h_\alpha$ (Eq. (27) with $m = U = 0$). In such a case we have two independent equations (57) for Γ_α and two independent u_α. The Lagrangian (72) in multimode case takes the form of a sum over modes (we omit the classical part h^w and neglect the dependence of $h^q(s,\mathbf{x})$ on spatial coordinates as in Eq. (93) of the previous section))

$$\mathcal{L} = \frac{1}{4}\int d\mathbf{k}h^*_\alpha(s,\mathbf{k})(-\partial_s^2 - k^2)h_\alpha(s,\mathbf{k}) + \frac{1}{2}m_0\frac{d\xi_r}{ds}\frac{d\xi_r}{ds}$$
$$-\frac{1}{4}m_0\lambda(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\partial_s^2 h^q_{rl}(s,\mathbf{k})\xi_r\xi_l,$$

The last term will be expressed as

$$\frac{\lambda m_0}{2}(2\pi)^{-\frac{3}{2}}\int dsd\mathbf{k}\partial_s^2 h^q_{kl}\xi^k\xi^l = \lambda(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}dsh^q_\alpha f^\alpha,$$

where $f^\alpha = e^\alpha_{rl}f^{rl}$.

We consider a solution of the Schrödinger equation (28) ($V = 0$) in the Gaussian form (if we did not split the gravitational field into h^w and h^q then the classical part would be obtained from the term Jh in Eq. (54))

$$\psi^g_t(h) = \exp\left(\frac{i}{2\hbar}\int d\mathbf{k}h^q_\alpha\Gamma(t)^\alpha h^q_\alpha\right). \tag{100}$$

In Eq. (100) $(\Gamma^\alpha - \Gamma^{*\alpha})^{-1}$ has a physical meaning as the squeezing of the amplitude h^α in the uncertainty relations (squeezed states are produced during inflation [35,36]). In general, Γ^+ and $\Gamma^\times$ are independent solutions of Eq. (57). We restrict our discussion to the case $\Gamma^+ = \Gamma^\times \equiv \Gamma$ (and $u_+ = u_\times$). Then, the gravitational field resulting from the state (100) is covariant under rotations. In the expectation values (80) and (83) we shall have sums of the form

$$f^\alpha_s f^\alpha_{s'} = \Lambda_{mn;rl}f^{mn}_s f^{rl}_{s'} \tag{101}$$

which after averaging over $\mathbf{k}k^{-1}$ will be expressed by q^{rl} (94) in the way analogous to the thermal case of Sect. 7 ($\Lambda_{mn;rl}f^{mn}_s f^{rl}_{s'} \to \langle\Lambda_{mn;rl}\rangle f^{mn}_s f^{rl}_{s'}$). In detail

$$\rho_t \simeq \exp\Bigg(-\frac{i\lambda^2 m_0^2}{2\hbar}\int d\mathbf{k}$$
$$\times(\Gamma(t)-\Gamma(t)^*)^{-1}\left(\int_0^t (u_t^{-1}u_s f^\alpha_s - u_t^{*-1}u^*_s f'^\alpha_s)ds\right)^2$$
$$-\frac{i\lambda^2 m_0^2}{2\hbar}\frac{4\pi}{5}\int dk\int_0^t\Big(u_s u_{s'}(\sigma^2+\delta^2)^{-1}$$
$$\times(\Gamma(t)-\Gamma(max(s,s')))q^{rl}_s q^{rl}_{s'}$$
$$-u^*_s u^*_{s'}(\sigma^{*2}+\delta^{*2})^{-1}(\Gamma^*(t)$$
$$-\Gamma^*(max(s,s')))q'^{rl}_s q'^{rl}_{s'}\Big)dsds'\Bigg). \tag{102}$$

In Eq. (102) the $f_\alpha f_\alpha$ term of Eq. (83) has been expressed by q^{rl} and $u(k)$ is defined in Eq. (60) with $\omega = k$ (the coefficients δ and σ may depend on k).

For the final result the integral over k is crucial. Its exact value depends on the complex functions $\sigma(k)$ and $\delta(k)$ in Eq. (60). In the thermal case the effective action at high temperature in the exponential (93)was local in time (for a small β) owing to the k^{-1} factor coming from $\coth(\frac{1}{2}\hbar\beta k)$. We do not have such a factor here. Nevertheless, we can see that the non-local final noise can be large owing to the squeezing factor $(\Gamma - \Gamma^*)^{-1}$ in Eq. (102).Explicitly, this term is

$$\exp\Bigg(-\frac{i\lambda^2 m_0^2}{2\hbar}\frac{8\pi}{5}\int dkk^2$$
$$\times(\Gamma(t)-\Gamma(t)^*)^{-1}\int_0^t dsds'\Big(u_t^{-2}u_s u_{s'}q^{rm}_s q^{rm}_{s'}$$
$$+u_t^{*-2}u^*_s u^*_{s'}q'^{rm}_s q'^{rm}_{s'} - u_t^{-1}u_t^{*-1}u^*_s u_{s'}q'^{rm}_s q^{rm}_{s'}$$
$$-u_t^{-1}u_t^{*-1}u_s u^*_{s'}q^{rm}_s q'^{rm}_{s'}\Big)\Bigg). \tag{103}$$

The exponential in Eq. (102) is of the form

$$\exp(i\alpha_1 qq + i\alpha_2 q'q' + i\alpha_3 qq'). \tag{104}$$

When $\frac{\delta}{\sigma} = i$, $\Gamma = ik$, $u_s = \exp(iks)$ then in the integral (102) we obtain (this is an infinite mode version of Eq. (80))

$$\rho_t \simeq \exp\Bigg(-\frac{\lambda^2 m_0^2}{2\hbar}\frac{4\pi}{5}\int dkk\int_0^t dsds'$$
$$\times\Big(\exp(-ik|s-s'|)q_{rl}(s)q_{rl}(s')$$
$$+\exp(ik|s-s'|)q'_{rl}(s)q'_{rl}(s')+$$
$$-\exp(-ik(s+s'))(q_{rl}(s))q'_{rl}(s') + q'_{rl}(s)q_{rl}(s')\Big)\Bigg). \tag{105}$$

Representing $\sin(kt)$ as $-k^{-1}\partial_t\cos(kt)$ we integrate over k obtaining $\partial_s\delta(s-s')$ in a similar way as we did it in the thermal case arriving at the phase factor

$$\exp\left(-i\frac{\gamma}{2\hbar}\int_0^t(q_{rl}-q'_{rl})\partial_s(q_{rl}(s)+q'_{rl}(s)\right). \tag{106}$$

We obtain the same lhs of the stochastic equation (98) as in the thermal state , but the noise resulting from Eq. (105)is different than the one of Eq. (98) (non-local in time and non-Markovian).

In Eq. (102) the exponential of the density matrix can be written in the form

$$\rho_t\simeq\int D\xi D\xi'\exp\Big(\frac{i}{2\hbar}\int_0^t ds(m_0\frac{d\xi_r}{ds}\frac{d\xi_r}{ds}-m_0\frac{d\xi'_r}{ds}\frac{d\xi'_r}{ds}$$
$$+2V(\xi)-2V(\xi'))+i\int dsds'\Big((q^{rl}-q'^{rl})C(q^{rl}+q'^{rl})$$
$$+(q^{rl}+q'^{rl})A(q^{rl}+q'^{rl})+(q^{rl}-q'^{rl})B(q^{rl}-q'^{rl})\Big). \tag{107}$$

If the exponential (102), (103) is written in the form (104) then $\alpha_1=A+B+C, \alpha_2=A+B-C, \alpha_3=2(A-B-C)$. The functions A, B, C can be read from Eqs. (102), (103). Expanding in y we obtain the non-Markovian stochastic equation as derived in Eq. (84). In Eq. (107) the C-term is proportional to y whereas the A and B terms are quadratic in y. So the C term gives the modification of the equation for the geodesic deviation whereas the A, B terms contribute to the noise.

Equation (102) simplifies if $\Gamma(t)\simeq const$. Let us set in Eq. (60) $a\delta=i\sigma$ (a may depend on k). Then $\Gamma(0)=ika^{-1}$. We have a real Gaussian function in Eq. (100) as an initial state. This squeezed state can still be approximated by an initial real function if a is large and $(kt)^{-1}>>a>>kt$ with $kt<<1$. In such a case to Eq. (102) only the term (103) contributes, where $u_s\simeq\sigma\cos(ks)$. Hence

$$\rho_t\simeq$$
$$\exp\left(-\frac{\lambda^2m_0^2}{4\hbar}\int d\mathbf{k}\frac{a}{k}\left(\int_0^t(\cos(kt))^{-1}\cos(ks)(f_s^\alpha-f_s'^\alpha)ds\right)^2\right). \tag{108}$$

The integration over the angles $k^{-1}\mathbf{k}$ of $\epsilon^\alpha_{rl}\epsilon^\alpha_{mn}$ is expressed by $\langle\Lambda_{rl;mn}\rangle$ (92). Hence, finally

$$\rho_t\simeq\exp\Big(-\frac{\lambda^2m_0^2}{16\hbar}\int dkka\Big(\int_0^t dsds'\langle\Lambda_{rl;mn}\rangle$$
$$\times(\cos(kt))^{-2}\cos(ks)\cos(ks')$$
$$\times\frac{d^2}{ds^2}(X^ry^l+X^ly^r)\frac{d^2}{ds'^2}(X^my^n+X^ny^m)\Big). \tag{109}$$

Equation (109) gives the spectrum of the noise of the quadrupole q^{rl} as $8\pi Gak$ (defined in [39]) or the spectrum of the noise of the measured coordinate ξ (if we perform the differentiation by parts over s contained in q^{rl}) as $8\pi Gak^5$ in agreement with [29] (Eqs. (3.13)–(3.14)). The kinematic form of the noise (109) is the same as the one for the thermal noise discusssed at the end of Sect. 7 and in the Appendix. However, because of a different spectrum of the noise (109) it is not of the local form (99).

9 Summary

We have calculated the density matrix resulting from an average over the gravitational field in a thermal state and in a general Gaussian state. The result shows that for gravitons in high temperature or in a highly squeezed state the modification of the geodesic deviation equation applied in a gravitational wave detection can have large noise amplitude. In general, the perturbation of the geodesic deviation equation is non-local and non-Markovian. Our results on the stochastic geodesic deviation equation show some differences in comparison to PWZ [2] which may come from approximations used by those authors. The formula for the noise (although quite complicated) can be useful in order to distinguish the contribution of the graviton noise from other sources of noise in the gravitational wave detection. Using Eq. (69) we can calculate the transition probability between the initial and final states of the detector. In the expansion in $\hbar$ the calculation is reduced to expectation values of the noise. The noise could be detectable if gravitational waves come from an inflationary stage of universe evolution (squeezing) or from a merge of hot neutron stars.

Remark After a submission to arXiv of the first version of this paper there appeared extended versions [39,40] of PWZ paper [2]. Their modification of the deviation equation (Eq.(125) of Ref. [40]) is different from (98). The difference comes from the approximations discussed in [40] at the end of Sect. 2 where it is assumed that only one quantum mode of ξ^r is excited.

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Appendix: High temperature (classical) limit

We can treat the system in an environment of thermal gravitons by means of the same procedure as applied in [34]. The equation for the gravitational field resulting from the Lagrangian (85) is

$$\frac{d^2h^{rl}(\mathbf{k})}{dt^2}+k^2h^{rl}(\mathbf{k})=(2\pi)^{-\frac{3}{2}}\lambda f^{rl}. \tag{110}$$

We write the solution of Eq. (110) (assuming that when $t\leq t_0$, where t_0 is the initial time, h_{rl} behaves as free wave) in the form (in the transverse-traceless gauge)

$$\begin{aligned}h_{rl}(\mathbf{k})&=h^w_{rl}(\mathbf{k})+h^{th}_{rl}(\mathbf{k})+h^I_{rl}(\mathbf{k})\\&=h^w_{rl}+e^\alpha_{rl}(h^\alpha_0\cos(kt)+k^{-1}\Pi^\alpha_0\sin(kt))\\&\quad+\lambda(2\pi)^{-\frac{3}{2}}\Lambda_{rl;mn}\int_{t_0}^t k^{-1}\sin(k(t-t'))f_{mn}(t')dt',\end{aligned} \tag{111}$$

here

$$h^w_{rl}(t,\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\exp(i\mathbf{kx})h^w_{rl}(\mathbf{k},t)$$

is the gravitational wave, h^{th} describes the thermal modes distributed with the Gibbs equilibrium measure, h^α_0 and Π^α_0 are random initial conditions. h^I_{rl} is the gravitational field created by the motion of m_0. h^w_{rl} and h^{th}_{rl} satisfy the homogeneous equation

$$\frac{d^2h_{rl}(\mathbf{k})}{dt^2}+k^2h_{rl}(\mathbf{k})=0.$$

The equation of motion for the coordinate ξ is

$$\begin{aligned}\frac{d^2\xi_r}{dt^2}&=\frac{\lambda}{2}\frac{d^2h^w_{rl}(\mathbf{k})}{dt^2}\xi^l+\frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\frac{d^2h^{th}_{rl}(\mathbf{k})}{dt^2}\xi^l\\&\quad+\frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\frac{d^2h^I_{rl}(\mathbf{k})}{dt^2}\xi^l.\end{aligned} \tag{112}$$

We insert the gravitational field from Eq. (111) into Eq. (112). We obtain

$$\frac{d^2\xi_r}{dt^2}=\frac{\lambda}{2}\frac{d^2h^w_{rl}(\mathbf{k})}{dt^2}\xi^l+N_{rl}(t)\xi^l+F_r(\xi,t), \tag{113}$$

where F is a non-linear interaction resulting from the interaction with the environment and

$$N^{rl}(t)=\int d\mathbf{k}N^{rl}(k,t)=-\frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}k^2e^\alpha_{rl}\times(h^\alpha_0\cos(kt)+k^{-1}\Pi^\alpha_0\sin(kt)). \tag{114}$$

The classical Gibbs distribution (a classical limit of the quantum Gibbs distribution of Sect. 7) is (Π_α has the meaning of the initial canonical momentum)

$$d\Pi^\alpha_0dh^\alpha_0\exp\left(-\frac{\beta}{2}\int d\mathbf{k}(|\Pi_\alpha|^2+k^2|h_\alpha|^2)\right). \tag{115}$$

Calculating the correlation functions of the noise (assuming the Gibbs distribution of the initial values) we obtain

$$\langle N^{rl}(\mathbf{k},t)N^{mn}(\mathbf{k}',t')\rangle=\frac{\lambda^2}{4}\beta^{-1}(2\pi)^{-3}\delta(\mathbf{k}+\mathbf{k}')k^2\Lambda_{rl:mn}\times\cos(k(t-t')). \tag{116}$$

So that

$$\langle N^{rl}(t)N^{mn}(t')\rangle=\langle\Lambda_{rl:mn}\rangle\frac{\lambda^2}{4\pi}\beta^{-1}\partial_t^2\partial_{t'}^2\delta(t-t'). \tag{117}$$

The non-linear force resulting from the graviton environment is

$$\begin{aligned}F^r&=\frac{1}{2}\lambda^2(2\pi)^{-3}\partial_t^2\Big(\int d\mathbf{k}\\&\quad\times\Lambda_{rl;mn}\int_{t_0}^t k^{-1}\sin(k(t-t'))f_{mn}(t')dt'\Big)\xi^l\\&=\frac{1}{2}\lambda^2m_0(2\pi)^{-3}\int dkk^2\\&\quad\times\langle\Lambda_{rl;mn}\rangle(\int_{t_0}^t\partial_t\cos(k(t-t'))f_{mn}(t')dt'\xi^l\\&\quad+f_{mn}(t)\xi^l).\end{aligned} \tag{118}$$

We write the last factor as

$$\begin{aligned}&\left(\int_{t_0}^t\partial_t\cos(k(t-t'))f_{mn}(t')dt'\xi^l(t)+f_{mn}(t)\xi^l(t)\right)\\&=(-\int_{t_0}^t\partial_{t'}\cos(k(t-t'))f_{mn}(t')dt'\xi^l(t)+f^{mn}(t)\xi^l(t))\\&=\int_{t_0}^t\cos(k(t-t'))\partial_{t'}f_{mn}(t')dt'\xi^l(t)\\&\quad+\cos(k(t-t_0))f_{mn}(t_0)\xi^l(t).\end{aligned} \tag{119}$$

Then

$$\begin{aligned}&\int_{t_0}^t\int dkk^2\cos(k(t-t'))\partial_{t'}f_{mn}(t')dt'\\&=-2\pi\int_{t_0}^t\partial_{t'}^2\delta(t-t')\partial_{t'}f_{mn}(t')dt'=2\pi\partial_t^3f_{mn}(t),\end{aligned} \tag{120}$$

where we assumed that at t_0 the first and the second derivatives of f_{mn} are zero. If we assume that in (119) $f_{mn}(t_0)=0$

then we can write Eq. (113) in the form

$$\frac{d^2\xi^r}{dt^2} = \frac{\lambda}{2}\frac{d^2h_w^{rl}}{dt^2}\xi_l - \frac{\lambda^2}{5\pi}(\delta_{rm}\delta_{ln} - \frac{1}{3}\delta_{rl}\delta_{mn})\xi^l\partial_t^3 f^{mn} + N^{rl}(t)\xi_l. \quad (121)$$

Equation (121) coincides with Eq. (98) as the noise (117) is the same as the one defined by Eqs. (98), (99).

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