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Schwarzschild-like solutions in Finsler–Randers gravity

A. Triantafyllopoulos[1,a] , S. Basilakos[2,3,b], E. Kapsabelis[1,c], P. C. Stavrinos[4,d]

[1] Section of Astrophysics, Astronomy and Mechanics, Department of Physics, National and Kapodistrian University of Athens, Panepistimiopolis, 15784 Athens, Greece
[2] Academy of Athens, Research Center for Astronomy and Applied Mathematics, Soranou Efessiou 4, 11527 Athens, Greece
[3] National Observatory of Athens, Lofos Nymfon, 11852 Athens, Greece
[4] Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis, 15784 Athens, Greece

Abstract In this work, we extend for the first time the spherically symmetric Schwarzschild and Schwarzschild–De Sitter solutions with a Finsler–Randers-type perturbation which is generated by a covector A_γ. This gives a locally anisotropic character to the metric and induces a deviation from the Riemannian models of gravity. A natural framework for this study is the Lorentz tangent bundle of a spacetime manifold. We apply the generalized field equations to the perturbed metric and derive the dynamics for the covector A_γ. Finally, we find the timelike, spacelike and null paths on the Schwarzschild–Randers spacetime, we solve the timelike ones numerically and we compare them with the classic geodesics of general relativity. The obtained solutions are new and they enrich the corresponding literature.

1 Introduction

In the context of research on generalized metric spaces, Finsler, Lagrange and Finsler-like geometries have played an important role in modified theories of gravity, general relativity and cosmology. In the last two decades, a development of these promising topics of research has extended the limits of general relativity and cosmology by including locally-anisotropic approaches. Over the last two decades field equations have been thoroughly studied in the context of Finsler, Lagrange, generalized Finsler and Finsler-like geometries as well as in scalar–tensor theories. The basic feature of these theories is the presence of extra terms in the equations of motion due to the intrinsic geometrical spacetime anisotropy. In this framework the term "spacetime anisotropy" is related to the Lorentz violation feature of the geometry. This approach may provide the necessary platform in understanding one of the most crucial problems in cosmology which is related with the underlying mechanism of cosmic acceleration and thus of dark energy. There is a lot of work in the literature on "spacetime anisotropic" geometries and below we briefly present some relevant works.

Generalized Einstein field equations have been studied in the Finsler, Lagrange, generalized Finsler and Finsler-like spaces, for an osculating gravitational approach in which the second variable $y(x)$ is a tangent/vector field [1–3] and in Finsler cosmology [4–8]. Different sets of generalized Einstein field equations were derived for the aforementioned spaces in the framework of a tangent bundle [9–17] and for the momentum space on the cotangent bundle [18–23]. Additionally, Lorentz invariance violation in Finsler/Finsler-like spacetime and in Finsler cosmology in very special relativity has also been studied in a large series of papers [5,24–27]. Investigations on generalized scalar–tensor theories with Finsler-like structure modeled on a vector bundle with two-internal fibers have also been done [28–30]. Also, the causality problem and light cones for different types of Finsler spaces have been investigated in [31–34]. In this context the Raychaudhuri equations with locally anisotropic internal variables which are providing extra effective terms have been derived in [16,18,35–37]. Investigations of the extended Friedmann equations in Finsler spaces with extra internal degrees of freedom [4] and dynamical analysis (critical points) [38,39] provide a better understanding of the dynamical properties of the Finsler–Randers cosmological models. To this end, articles in the framework of the weak field and pp-waves in Finsler spacetime can be found in [16,35,40] and potentially they can be used in order to test the performance of the Finslerian gravitational theory against current observations of gravitational waves.

[a] e-mail: alktrian@phys.uoa.gr (corresponding author)

[b] e-mail: svasil@academyofathens.gr

[c] e-mail: manoliskapsabelis@yahoo.gr

[d] e-mail: pstavrin@math.uoa.gr

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It is well known in gravitation and cosmology that the Schwarzschild metric constitutes a fundamental ingredient of general relativity. This metric describes the most general spherical symmetric solution of the Einstein field equations in a region of spacetime where the energy–momentum tensor vanishes.

In the context of a Finsler/generalized Finsler space, an extension of a locally anisotropic perturbation of a Finsler type Schwarzschild metric has been studied by different authors. In some of these works, possible observational predictions are given based on the direction-dependent structure of spacetime. We would like to point out that our study is realized in a Schwarzschild–Randers model which is different from the works of other researchers on the Finslerian extensions of a classic Schwarzschild metric [41–51]. In our approach, we use sufficiently generalized Einstein field equations on a Lorentz tangent bundle of a spacetime manifold.

In Sect. 2 we introduce the basic framework and geometric structures of our model.

In Sect. 3 we study a Schwarzschild metric in a special Finsler-like spacetime of Randers type. This study provides a locally anisotropic perturbation of the classical Schwarzschild metric of the Riemannian structure in a natural way, induced by a covector field of the base manifold. In addition, the geometrical setting that we use, namely the framework of a Lorentz tangent bundle of a Riemannian spacetime, contains additional degrees of freedom compared to classic gravity. The generalized field equations which have been derived in [17] are applied on the perturbed metric of our Schwarzschild–Randers spacetime and are solved for the covector field.

In Sect. 4, we study particle paths for our generalized spacetime. We follow the approach in [17] which takes into account the effect of internal degrees of freedom on the point particle dynamics. We apply the solution of the covector derived in the previous sections and we obtain an explicit form for the path equations which is an extension of classical geodesics of general relativity. Finally, as an application, we solve the timelike paths numerically and compare them with the geodesics of general relativity.

2 Preliminaries

The natural background space for a locally anisotropic gravity is the tangent bundle of a differentiable Lorentzian spacetime manifold called a Lorentz Tangent Bundle (we will refer to it as TM hereafter) [15,22]. TM is itself an 8-dimensional differentiable manifold, so we can define coordinate charts and tensors on it in the usual way. We briefly present the basics for this structure, for more details see Sect. 2.

The Lorentz tangent bundle TM is locally covered by a coordinate map $\{x^\mu, y^\alpha\}$ where the range of values for

the indices of the x variables is $\kappa, \lambda, \mu, \nu, \ldots = 0, \ldots, 3$ and the range of values for the indices of the y variables is $\alpha, \beta, \ldots, \theta = 4, \ldots, 7$. An adapted basis on TM is $\left\{ \delta_\mu = \frac{\partial}{\partial x^\mu} - N_\mu^\beta \frac{\partial}{\partial y^\beta}, \dot{\partial}_\alpha = \frac{\partial}{\partial y^\alpha} \right\}$ and its dual basis is $\left\{ dx^\mu, \delta y^\alpha = dy^\alpha + N_\nu^\alpha dx^\nu \right\}$.

The bundle TM is equipped with a Sasaki-type metric $\mathcal{G}$:

$$\mathcal{G} = g_{\mu\nu}(x, y) \, dx^\mu \otimes dx^\nu + v_{\alpha\beta}(x, y) \, \delta y^\alpha \otimes \delta y^\beta \tag{1}$$

where the metric of the horizontal space (h-space) $g_{\mu\nu}$ and the metric of the vertical space (v-space) $v_{\alpha\beta}$ are defined to be of Lorentzian signature $(-, +, +, +)$. In the rest of this work, the following homogeneity conditions will be assumed: $g_{\mu\nu}(x, ky) = g_{\mu\nu}(x, y)$, $v_{\alpha\beta}(x, ky) = v_{\alpha\beta}(x, y)$, $k > 0$. These conditions are met when the following relations hold:

$$g_{\alpha\beta} = \pm \frac{1}{2} \frac{\partial^2 F_g^2}{\partial y^\alpha \partial y^\beta} \tag{2}$$

$$v_{\alpha\beta} = \pm \frac{1}{2} \frac{\partial^2 F_v^2}{\partial y^\alpha \partial y^\beta} \tag{3}$$

where the functions F_g, F_v satisfy the following conditions:

1. F_m, $m = g, v$, is continuous on TM and smooth on $\widetilde{TM} \equiv TM \setminus \{0\}$ i.e. the tangent bundle minus the null set $\{(x, y) \in TM \mid F_m(x, y) = 0\}$
2. F_m is positively homogeneous of first degree on its second argument:

$$F_m(x^\mu, ky^\alpha) = k F_m(x^\mu, y^\alpha), \quad k > 0 \tag{4}$$

3. The form

$$f_{\alpha\beta}(x, y) = \frac{1}{2} \frac{\partial^2 F_m^2}{\partial y^\alpha \partial y^\beta} \tag{5}$$

defines a non-degenerate matrix:

$$\det \left[f_{\alpha\beta} \right] \neq 0 \tag{6}$$

with $g_{\alpha\beta} = \delta_\alpha^{\tilde\mu} \delta_\beta^{\tilde\nu} g_{\mu\nu}$ and the sign in the rhs of (2), (3) is chosen so that the resulting metric has the corrrect signature.

When the above conditions are met, our metric is called a (pseudo-)Finsler metric. Details about the connection, curvature and torsion structures can be found in Sect. 2.

3 Field equations

In this chapter we consider a general metric (1) on the tangent bundle of a spacetime manifold constituted from a horizontal and a vertical part. In this consideration, we set the

horizontal part to be the classical Schwarzschild metric and the vertical part to be a Randers-type perturbation of the Schwarzschild metric, so the form of our metric will be a Schwarzschild–Randers metric. We also consider a similar case for a Schwarzschild–De Sitter–Randers metric. This is a different type of metric than the Finsler–Randers one which has been considered in the framework of cosmological study by different authors [4,8,37,39,52–56]. In these cases, the horizontal part is a Friedmann–Robertson–Walker model. In both models the velocity and the vertical part play the role of intrinsic anisotropic perturbation of the traditional metrics. We already know the form of the Schwarzschild metric, so we need to find an explicit form for the Randers-type perturbation which contains the additional information of local anisotropy.

3.1 Field equations on the Lorentz tangent bundle

In this paragraph, we present a set of field equations for the dynamic variables of our generalized framework. These equations are:

$$\overline{R}_{\mu\nu} - \frac{1}{2}(\overline{R} + S) g_{\mu\nu}$$
$$+ (\delta_\nu^{(\lambda} \delta_\mu^{\kappa)} - g^{\kappa\lambda} g_{\mu\nu})(\mathcal{D}_\kappa \mathcal{T}_{\lambda\beta}^\beta - \mathcal{T}_{\kappa\gamma}^\gamma \mathcal{T}_{\lambda\beta}^\beta) = \kappa T_{\mu\nu} \quad (7)$$

$$S_{\alpha\beta} - \frac{1}{2}(\overline{R} + S) v_{\alpha\beta}$$
$$+ (v^{\gamma\delta} v_{\alpha\beta} - \delta_\alpha^{(\gamma} \delta_\beta^{\delta)})(\mathcal{D}_\gamma C_{\mu\delta}^\mu - C_{\nu\gamma}^\nu C_{\mu\delta}^\mu) = \kappa Y_{\alpha\beta} \quad (8)$$

$$g^{\mu[\kappa} \dot{\partial}_\alpha L_{\mu\nu}^{\nu]} + 2\mathcal{T}_{\mu\beta}^\beta g^{\mu[\kappa} C_{\lambda\alpha}^{\lambda]} = \frac{\kappa}{2} \mathcal{Z}_\alpha^\kappa \quad (9)$$

with

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{|\mathcal{G}|}} \frac{\Delta(\sqrt{|\mathcal{G}|} \mathcal{L}_M)}{\Delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}} \frac{\Delta(\sqrt{-g} \mathcal{L}_M)}{\Delta g^{\mu\nu}}$$
$$(10)$$

$$Y_{\alpha\beta} \equiv -\frac{2}{\sqrt{|\mathcal{G}|}} \frac{\Delta(\sqrt{|\mathcal{G}|} \mathcal{L}_M)}{\Delta v^{\alpha\beta}} = -\frac{2}{\sqrt{-v}} \frac{\Delta(\sqrt{-v} \mathcal{L}_M)}{\Delta v^{\alpha\beta}}$$
$$(11)$$

$$\mathcal{Z}_\alpha^\kappa \equiv -\frac{2}{\sqrt{|\mathcal{G}|}} \frac{\Delta(\sqrt{|\mathcal{G}|} \mathcal{L}_M)}{\Delta N_\kappa^\alpha} = -2\frac{\Delta \mathcal{L}_M}{\Delta N_\kappa^\alpha} \quad (12)$$

where $\mathcal{L}_M$ is the Lagrangian of the matter fields, δ_ν^μ and δ_β^α are the Kronecker symbols, $|\mathcal{G}|$ is the absolute value of the determinant of the total metric (1), and

$$\mathcal{T}_{\nu\beta}^\alpha = \dot{\partial}_\beta N_\nu^\alpha - L_{\beta\nu}^\alpha \quad (13)$$

are torsion components, where $L_{\beta\nu}^\alpha$ is defined in (A.14). From the form of (1) it follows that $\sqrt{|\mathcal{G}|} = \sqrt{-g}\sqrt{-v}$, with g, v the determinants of the metrics $g_{\mu\nu}$, $v_{\alpha\beta}$ respectively. This relation was used in (10)–(12).

Equations (7)–(9) are derived from an extension of the Hilbert-Einstein action on the eight-dimensional Lorentz tangent bundle and constitute a generalization of the Einstein field equations of general relativity. They are appropriate for the study of locally anisotropic models of gravity with internal degrees of freedom. For details on their derivation from a Hilbert-like action on the Lorentz tangent bundle see [17].

We will make some comments in order to give a physical interpretation in relation to the Eqs. (10)–(12). Lorentz violations produce anisotropies in the space and the matter sector. These act as a source of local anisotropy and can contribute to the torsion, connection, curvature components and to the energy–momentum tensors of the horizontal and vertical space $T_{\mu\nu}$ and $Y_{\alpha\beta}$. As a result, the curvatures $\overline{R}_{\mu\nu}$ and R as well as the energy–momentum tensor $T_{\mu\nu}$ contain the additional information of local anisotropy of the metric and the matter fields. $S_{\alpha\beta}$, S and $Y_{\alpha\beta}$, on the other hand, are objects with no equivalent in Riemannian gravity. They contain more information about local anisotropy which is produced from the metric $v_{\alpha\beta}$ which includes additional internal structure of spacetime. Finally, the nonlinear connection N_μ^α, a structure which induces an interaction between internal and external spaces [29], can also contribute to all the above-mentioned objects, while the energy–momentum tensor $\mathcal{Z}_\alpha^\kappa$ shows the variation of $\mathcal{L}_M$ with respect to N_μ^α and it reflects the dependence of matter fields on the nonlinear connection [17]. This is different from $T_{\mu\nu}$ and $Y_{\alpha\beta}$ which depend on just the external or internal structure respectively.

Notice that the field equations (7)–(9) reduce to the usual Einstein field equations of general relativity (GR) in the limit:

$$g_{\mu\nu}(x, y) \rightarrow g_{\mu\nu}(x) \quad (14)$$

$$v_{\alpha\beta}(x, y) \rightarrow g_{\alpha\beta}(x) = \tilde{\delta}_\alpha^\mu \tilde{\delta}_\beta^\nu g_{\mu\nu}(x) \quad (15)$$

$$N_\nu^\alpha(x, y) \rightarrow \frac{1}{2} y^\gamma g^{\alpha\beta} \partial_\nu g_{\beta\gamma}. \quad (16)$$

We observe that when the metric $g_{\mu\nu}(x, y)$ is reduced to a Riemannian one according to (14) then the connection coefficients $L_{\kappa\lambda}^\mu(x, y)$ (eq. (A.13)) reduce to the Christoffel symbols:

$$\gamma_{\kappa\lambda}^\mu(x) = \frac{1}{2} g^{\mu\nu}(\partial_\kappa g_{\nu\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\nu g_{\kappa\lambda}) \quad (17)$$

and the Cartan torsion tensor (A.15) vanishes:

$$C_{\nu\alpha}^\mu = 0. \quad (18)$$

From (15) it becomes clear that the internal structure (y variable) of spacetime doesn't provide additional information about the geometry, since the metric $v_{\alpha\beta}$ of the internal structure reduces to the metric $g_{\mu\nu}(x)$ of base spacetime (x variable).

Moreover, we see that the Cartan-type [22] nonlinear connection (16) ensures that $\mathcal{T}_{\nu\alpha}^\alpha = 0$. Indeed, from Eq. (13) and

from the conditions (14–16) we get

$$
\begin{aligned}
\mathcal{T}_{\nu\alpha}^{\alpha} &= \dot{\partial}_{\alpha} N_{\nu}^{\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_{\nu} g_{\alpha\beta} \\
&= \frac{1}{2} g^{\alpha\beta} \partial_{\nu} g_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \partial_{\nu} g_{\alpha\beta} = 0.
\end{aligned}
\tag{19}
$$

Now, as long as conditions (14) and (15) are satisfied, the metric tensor (1) takes the Sasaki form [57]

$$
\mathcal{G} = g_{\mu\nu}(x) \, dx^{\mu} \otimes dx^{\nu} + g_{\alpha\beta}(x) \, \delta y^{\alpha} \otimes \delta y^{\beta}
\tag{20}
$$

Based on (20), it is straightforward to calculate the curvatures $\overline{R}_{\mu\nu}$ and $\overline{R}$ from (A.29) and (A.30) respectively in the GR limit of the Lorentz tangent bundle gravity and we find that they reduce to the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R of general relativity for the metric $g_{\mu\nu}$. As expected, the curvatures $S_{\alpha\beta}$ and S (Eqs. (A.26) and (A.28)) both vanish in this limit.

From the above relations (14–20) it follows that, in the GR limit, the corresponding field equations (7–9) boil down to

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}
\tag{21}
$$

$$
-\frac{1}{2} R g_{\alpha\beta} = \kappa Y_{\alpha\beta}
\tag{22}
$$

$$
Z_{\alpha}^{\kappa} = 0
\tag{23}
$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar of general relativity for the metric $g_{\mu\nu}$, as we mentioned above. Of course in Eq. (21) G denotes the Newton's constant and $\kappa = \frac{8\pi G}{c^4}$, while the energy momentum tensor $T_{\mu\nu}$ is given by (10). From the the trace of Eq. (21) we obtain $R = -\kappa T$, hence Eq. (22) gives

$$
Y_{\alpha\beta} = \frac{1}{2} T g_{\alpha\beta}.
\tag{24}
$$

This is the GR limit for the energy–momentum tensor $Y_{\alpha\beta}$.

Finally, from (12) and (23) we conclude that in the GR limit the matter fields have no direct dependence on the nonlinear connection.

3.2 Schwarzschild–Randers spacetime

As we mentioned above, the horizontal part $g_{\mu\nu}$ of the metric (1) will be taken to be the Schwarzschild metric so that

$$
\begin{aligned}
g_{\mu\nu} dx^{\mu} dx^{\nu} = &-\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} \\
&+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2
\end{aligned}
\tag{25}
$$

where $R_s = 2GM$ is the Schwarzschild radius (we have set the speed of light constant $c = 1$).

In the following, we assume a function F_v of Randers type from which we will derive $v_{\alpha\beta}$ by using (3):

$$
F_v = \sqrt{-g_{\alpha\beta}(x) y^{\alpha} y^{\beta}} + A_{\gamma}(x) y^{\gamma}
\tag{26}
$$

where $g_{\alpha\beta} = g_{\mu\nu} \tilde{\delta}_{\alpha}^{\mu} \tilde{\delta}_{\beta}^{\nu}$ is the Schwarzschild metric and $A_{\gamma}(x)$ is a covector which will be determined by the equations. We focus on the timelike subspace of the internal y-space with respect to the Schwarzschild metric ($g_{\alpha\beta}(x) y^{\alpha} y^{\beta} < 0$) hence the minus sign under the square root. We take $A_{\gamma}(x)$ to be a weak term ($|A_{\gamma}(x)| \ll 1$), hence we neglect high order terms from the calculations. In addition, we consider as the appropriate form for the nonlinear connection the one given in (16) i.e. its GR limit value on the tangent bundle:

$$
N_{\mu}^{\alpha} = \frac{1}{2} y^{\beta} g^{\alpha\gamma} \partial_{\mu} g_{\beta\gamma}
\tag{27}
$$

This choice will give us a locally anisotropic gravitational model which deviates minimally from general relativity due to the extra Randers term $A_{\gamma} y^{\gamma}$ in (26).

We calculate the metric tensor $v_{\alpha\beta}$ of (26) from (3):

$$
v_{\alpha\beta} = -\frac{1}{2} \frac{\partial^2 F_v^2}{\partial y^{\alpha} \partial y^{\beta}}
\tag{28}
$$

and we find

$$
\begin{aligned}
v_{\alpha\beta} = &\, g_{\alpha\beta}(x) + \frac{1}{a} (A_{\beta} g_{\alpha\gamma} y^{\gamma} + A_{\gamma} g_{\alpha\beta} y^{\gamma} + A_{\alpha} g_{\beta\gamma} y^{\gamma}) \\
&+ \frac{1}{a^3} A_{\gamma} g_{\alpha\epsilon} g_{\beta\delta} y^{\gamma} y^{\delta} y^{\epsilon},
\end{aligned}
\tag{29}
$$

where we have set $a = \sqrt{-g_{\alpha\beta} y^{\alpha} y^{\beta}}$. From (29) we see that the metric $v_{\alpha\beta}$ takes the form

$$
v_{\alpha\beta}(x, y) = g_{\alpha\beta}(x) + w_{\alpha\beta}(x, y)
\tag{30}
$$

where we have set

$$
\begin{aligned}
w_{\alpha\beta} = &\, \frac{1}{a} (A_{\beta} g_{\alpha\gamma} y^{\gamma} + A_{\gamma} g_{\alpha\beta} y^{\gamma} + A_{\alpha} g_{\beta\gamma} y^{\gamma}) \\
&+ \frac{1}{a^3} A_{\gamma} g_{\alpha\epsilon} g_{\beta\delta} y^{\gamma} y^{\delta} y^{\epsilon}
\end{aligned}
\tag{31}
$$

Its inverse is $v^{\beta\gamma} = g^{\beta\gamma} - w^{\beta\gamma}$ so that $v_{\alpha\beta} v^{\beta\gamma} = g_{\alpha\beta} g^{\beta\gamma} = \delta_{\alpha}^{\gamma}$ to first order in $w_{\alpha\beta}$. The total metric over the tangent bundle is then written as

$$
\mathcal{G} = g_{\mu\nu}(x) \, dx^{\mu} \otimes dx^{\nu} + [g_{\alpha\beta}(x) + w_{\alpha\beta}(x, y)] \, \delta y^{\alpha} \otimes \delta y^{\beta}
\tag{32}
$$

We remark that, as we can see from (30), the metric $v_{\alpha\beta}(x, y)$ is a Finslerian perturbation of the Riemannian metric $g_{\alpha\beta}(x)$. We observe that if we let $w_{\alpha\beta} \to 0$ then the field equations (7)–(9) reduce to the Einstein field equations of general relativity.

Next we will calculate the terms for (7) and (8). From the definitions (A.29) and (A.30) we see that when $g_{\mu\nu}$ has no explicit dependence on y then $\overline{R}_{\mu\nu}$ and $\overline{R}$ reduce to the classical Ricci tensor and scalar of general relativity. Additionally, since $g_{\mu\nu}(x)$ is the Schwarzschild metric, both $\overline{R}_{\mu\nu}$ and $\overline{R}$ are zero. Here we have assumed vacuum solutions, so the energy momentum tensors are zero and the Eqs. (7) and (8) become

$$-\frac{1}{2}Sg_{\mu\nu} + (\delta_\nu^{(\lambda}\delta_\mu^{\kappa)} - g^{\kappa\lambda}g_{\mu\nu})(\mathcal{D}_\kappa \mathcal{T}_{\lambda\beta}^\beta - \mathcal{T}_{\kappa\gamma}^\gamma \mathcal{T}_{\lambda\beta}^\beta) = 0 \tag{33}$$

$$S_{\alpha\beta} - \frac{1}{2}S\,v_{\alpha\beta}$$
$$+ (v^{\gamma\delta}v_{\alpha\beta} - \delta_\alpha^{(\gamma}\delta_\beta^{\delta)})(\mathcal{D}_\gamma C_{\mu\delta}^\mu - C_{\nu\gamma}^\nu C_{\mu\delta}^\mu) = 0 \tag{34}$$

Field equation (9) gives us no additional information since all three terms vanish identically in our case. We can simplify Eq. (34) by calculating $C_{\mu\delta}^\mu$ from (A.15) and we find that it is zero since the metric $g_{\mu\nu}$ depends only on x. Then by taking the trace of the remaining terms in (34) we can show that $S_{\alpha\beta}$ and S are also zero, so the field equation (33) becomes

$$\left(\delta_\nu^{(\lambda}\delta_\mu^{\kappa)} - g^{\kappa\lambda}g_{\mu\nu}\right)\left(\mathcal{D}_\kappa \mathcal{T}_{\lambda\beta}^\beta - \mathcal{T}_{\kappa\gamma}^\gamma \mathcal{T}_{\lambda\beta}^\beta\right) = 0. \tag{35}$$

We substitute (A.14) and (27) in (13) and after some calculations we get

$$\mathcal{T}_{\nu\alpha}^\alpha = -\frac{1}{2}\delta_\nu w \tag{36}$$

with $w = g_{\alpha\beta}w^{\alpha\beta}$. The above relation (36) shows us that the torsion is of first order on $w_{\alpha\beta}$ so the terms $\mathcal{T}_{\kappa\gamma}^\gamma \mathcal{T}_{\lambda\beta}^\beta$ from (35) are omitted. Then by taking the trace of the remaining terms in (35) we have the equation that follows:

$$g^{\mu\nu}\mathcal{D}_\mu \mathcal{T}_{\nu\alpha}^\alpha = 0 \tag{37}$$

Substituting the latter equation to (35) we find

$$\mathcal{D}_{(\mu}\mathcal{T}_{\nu)\alpha}^\alpha = 0 \tag{38}$$

By the definition of the covariant derivative in (A.10), Eq. (38) becomes

$$\delta_{(\mu}\mathcal{T}_{\nu)\alpha}^\alpha - L_{\mu\nu}^\kappa \mathcal{T}_{\kappa\alpha}^\alpha = 0 \tag{39}$$

Using Eqs. (31) and (36) we get $\mathcal{T}_{\nu\alpha}^\alpha$ in terms of A_β:

$$\mathcal{T}_{\nu\alpha}^\alpha = -\frac{5}{2}\delta_\nu \left(A_\beta \frac{y^\beta}{a}\right). \tag{40}$$

It is straightforward to show that

$$\delta_\mu \left(\frac{y^\alpha}{a}\right) = -E_{\beta\mu}^\gamma \frac{y^\beta}{a} \tag{41}$$

with

$$E_{\beta\mu}^\gamma(x) \equiv \frac{1}{2}g^{\alpha\gamma}\partial_\mu g_{\alpha\beta} \tag{42}$$

Using (40) and (41) we get

$$\mathcal{T}_{\nu\alpha}^\alpha = -\frac{5}{2}K_{\nu\gamma}\frac{y^\gamma}{a} \tag{43}$$

with

$$K_{\nu\gamma}(x) \equiv \partial_\nu A_\gamma - A_\beta E_{\gamma\nu}^\beta. \tag{44}$$

from relation (43) we can calculate (39):

$$\left(\partial_{(\mu}K_{\nu)\gamma} - E_{\gamma(\mu}^\beta K_{\nu)\beta} - L_{\mu\nu}^\lambda K_{\lambda\gamma}\right)\frac{y^\gamma}{a} = 0. \tag{45}$$

Relation (45) must hold for every y. Since the expression in parentheses does not depend on y, we conclude that it must identically vanish:

$$\partial_{(\mu}K_{\nu)\gamma} - E_{\gamma(\mu}^\beta K_{\nu)\beta} - L_{\mu\nu}^\lambda K_{\lambda\gamma} = 0 \tag{46}$$

Remark. By comparing (A.14) and (42) we get a relation of the form $L_{\beta\mu}^\alpha = E_{\beta\mu}^\alpha + \mathcal{O}(A)$. If we use this in (44) and (46) we get the equation $\mathcal{D}_{(\mu}\mathcal{D}_{\nu)}A_\gamma = 0$, which is equivalent to the system of Eqs. (44) and (46). Contracting this with $g^{\mu\nu}$ gives

$$\Box A_\gamma = 0 \tag{47}$$

with $\Box \equiv g^{\mu\nu}\mathcal{D}_\mu \mathcal{D}_\nu$.

In order to fully determine the metric (32) for the respective subspace of the internal space, we need $g_{\mu\nu}(x)$ and $A_\gamma(x)$. The first is already defined in (25), so we need to solve (46) for $A_\gamma(x)$ to get a full expression for the metric in our space. If we use the definitions (42) and (44) on relation (46), we get the equation that we need to solve for $A(x)$:

$$\partial_\mu \partial_\nu A_\gamma - \frac{1}{2}g^{\beta\delta}\partial_\nu g_{\delta\gamma}\partial_\mu A_\beta - \frac{1}{2}g^{\beta\delta}\partial_\mu g_{\delta\gamma}\partial_\nu A_\beta$$
$$+ \frac{1}{4}A_\beta \left(\frac{1}{2}g^{\beta\epsilon}g^{\delta\zeta}\partial_\nu g_{\epsilon\delta}\partial_\mu g_{\gamma\zeta} + \frac{1}{2}g^{\beta\epsilon}g^{\delta\zeta}\partial_\mu g_{\epsilon\delta}\partial_\nu g_{\gamma\zeta}\right.$$
$$- \partial_\mu g^{\beta\delta}\partial_\nu g_{\gamma\delta} - \partial_\nu g^{\beta\delta}\partial_\mu g_{\gamma\delta} - 2g^{\beta\delta}\partial_\mu \partial_\nu g_{\gamma\delta}\Big)$$
$$- \frac{1}{2}g^{\kappa\lambda}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu})$$
$$\times \left(\partial_\lambda A_\gamma - \frac{1}{2}A_\beta g^{\beta\delta}\partial_\lambda g_{\gamma\delta}\right) = 0, \tag{48}$$

where $g_{\mu\nu}(x)$ is the Schwarzschild metric (25). Once we get $A(x)$ from (48), we can calculate $w_{\alpha\beta}(x, y)$ from (31) and then use the result to calculate the full metric (32).

A similar analysis holds for the spatial subspace of the internal space. In that case, instead of (26) and (28) we have

$$F_\nu = \sqrt{g_{\alpha\beta}y^\alpha y^\beta} + A_\gamma y^\gamma \tag{49}$$

and

$$v_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 F_\nu^2}{\partial y^\alpha \partial y^\beta}$$

$$= g_{\alpha\beta}(x) + \frac{1}{a}(A_\beta g_{\alpha\gamma} y^\gamma + A_\gamma g_{\alpha\beta} y^\gamma + A_\alpha g_{\beta\gamma} y^\gamma)$$
$$- \frac{1}{a^3} A_\gamma g_{\alpha\epsilon} g_{\beta\delta} y^\gamma y^\delta y^\epsilon, \tag{50}$$

where $a = \sqrt{g_{\mu\nu} y^\mu y^\nu}$ for the spacelike sector of $g_{\mu\nu}$ taking into consideration the signature of the metric. Following the same steps as above, we reach the same equation for A_γ, i.e. Eq. (48). Therefore, solving this equation will give us the metric for both the timelike and spacelike (with respect to $g_{\alpha\beta}$) sub-spaces of the internal space.

We will solve (48) analytically with separation of variables, see Appendix B for more details. After calculations we find the solution

$$A_\gamma(x) = \left[\tilde{A}_4 \left| 1 - \frac{R_S}{r} \right|^{1/2}, 0, 0, 0 \right] \tag{51}$$

with $\tilde{A}_4$ a constant. This is a timelike covector since $g^{\alpha\beta} A_\alpha A_\beta = -(\tilde{A}_4)^2 < 0$. It is interesting to mention that the horizon of the Schwarzschild–Randers metric is correlated with that of Schwarzschild. Practically, the quantity A_γ can be seen as a distortion factor which quantifies the deviation from the pure Schwarzschild solution. Obviously, the solution (51) on small spherical scales ($r \sim R_S$) tends to zero. On the other hand, for $r \gg R_s$ the Schwarzschild - Randers metric tends asymptotically to Minkowski. Finally, we see that this metric has a singularity in $r = 0$ similarly with the classic Schwarzschild one. The Schwarzschild–Randers model will be further studied for intrinsic singularities and horizons in a future research.

3.3 Schwarzschild–De Sitter–Randers spacetime

We will follow the same procedure as in the previous paragraph but for a Schwarzschild–Randers spacetime with a cosmological horizon, namely a Schwarzschild–De Sitter–Randers spacetime. In this scenario, we take the horizontal part of the metric (1) to be:

$$g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{R_s}{r} - \frac{\Lambda}{3} r^2\right) dt^2$$
$$+ \frac{dr^2}{1 - \frac{R_s}{r} - \frac{\Lambda}{3} r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{52}$$

while, as before, the metric tensor $v_{\alpha\beta}$ will be derived from the Lagrangian (26) and relation (28), where $g_{\alpha\beta} = \tilde{\delta}_\alpha^\mu \tilde{\delta}_\beta^\nu$ is now given by (52). The latter is a static spherically symmetric vacuum solution for the classical Einstein field equations with a cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0 \tag{53}$$

with $R_{\mu\nu}$ and R the Ricci tensor and scalar of general relativity. In accordance, we introduce a cosmological constant term to the field equations (7) in vacuum:

$$\overline{R}_{\mu\nu} - \frac{1}{2}(\overline{R} + S) g_{\mu\nu}$$
$$+ (\delta_\nu^{(\lambda} \delta_\mu^{\kappa)} - g^{\kappa\lambda} g_{\mu\nu})(\mathcal{D}_\kappa \mathcal{T}_{\lambda\beta}^\beta - \mathcal{T}_{\kappa\gamma}^\gamma \mathcal{T}_{\lambda\beta}^\beta) + g_{\mu\nu}\Lambda = 0 \tag{54}$$

The tensors $\overline{R}_{\mu\nu}$ and $\overline{R}$ reduce to the standard Ricci tensor and scalar of general relativity for the metric (52) since the latter has no direct dependence on y. Additionally, from (8) in vacuum we get $S = 0$ the same way as in the previous paragraph. Therefore, using (53) in Eq. (54) we get relation (35) again. It is obvious that the procedure is the same as before and only the explicit form of $g_{\mu\nu}(x)$ changes. As such, we reach the same equation for A_γ, namely relation (48) with $g_{\mu\nu}$ given by (52). Again, by separation of variables, one finds (see Appendix B):

$$A_\gamma(x) = \left[\tilde{A}_4 \left| 1 - \frac{R_S}{r} - \frac{\Lambda}{3} r^2 \right|^{1/2}, 0, 0, 0 \right] \tag{55}$$

4 Paths in the Schwarzschild–Randers spacetime

Now that we have A_γ and hence the full metric, we can study particle trajectories in TM. A Lagrangian for point particles in the total space of the Lorentz tangent bundle has been proposed in [17]:

$$L(x, \dot{x}, y) = (a g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + b \tilde{\delta}_\mu^\alpha v_{\alpha\beta} \dot{x}^\mu y^\beta + c v_{\alpha\beta} y^\alpha y^\beta)^{1/2} \tag{56}$$

with a, b, c constants. The associated equations of motion are

$$(g_{\kappa\nu} + z \tilde{\delta}_\kappa^\alpha \tilde{\delta}_\nu^\beta v_{\alpha\beta}) \ddot{x}^\kappa + (\gamma_{\nu\kappa\lambda} + z \sigma_{\nu\kappa\lambda}) \dot{x}^\kappa \dot{x}^\lambda = 0 \tag{57}$$

and

$$y^\alpha = \tilde{\delta}_\mu^\alpha \dot{x}^\mu \tag{58}$$

with

$$\sigma_{\nu\kappa\lambda} = \frac{1}{2}(\tilde{\delta}_\nu^\alpha \tilde{\delta}_\lambda^\beta \partial_\kappa v_{\alpha\beta} + \tilde{\delta}_\nu^\alpha \tilde{\delta}_\kappa^\beta \partial_\lambda v_{\alpha\beta} - \tilde{\delta}_\kappa^\alpha \tilde{\delta}_\lambda^\beta \partial_\nu v_{\alpha\beta}) \tag{59}$$

and $z = -b^2/4ac$ is a constant. The Christoffel symbols of the first kind for the metric $g_{\kappa\nu}(x)$ are

$$\gamma_{\nu\kappa\lambda} = \frac{1}{2}(\partial_\kappa g_{\nu\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\nu g_{\kappa\lambda}) \tag{60}$$

The term $\tilde{\delta}_\kappa^\alpha \tilde{\delta}_\nu^\beta v_{\alpha\beta}$ is the metric of the v-space lowered down to the h-space via the generalized Kronecker symbols which are defined as $\delta_\alpha^{\tilde{\mu}} = \tilde{\delta}_\mu^\alpha = 1$ for $a = \mu + 4$ and equal to

zero otherwise.[1] We will write for convenience $\tilde{\delta}^\alpha_\kappa \tilde{\delta}^\beta_\nu v_{\alpha\beta} = v_{\kappa\nu}$ and similarly $\tilde{\delta}^\alpha_\kappa \tilde{\delta}^\beta_\nu w_{\alpha\beta} = w_{\kappa\nu}$.

We define $\overline{g}_{\kappa\nu} = g_{\kappa\nu} + z v_{\kappa\nu}$ and we observe that its inverse is $\overline{g}^{\mu\nu} = (1+z)^{-2}(g^{\mu\nu} + z v^{\mu\nu})$ in the sense that $\overline{g}_{\kappa\nu}\overline{g}^{\mu\nu} = \delta^\mu_\kappa$ to first order in $w_{\mu\nu}$. Contracting (57) with $\overline{g}^{\mu\nu}$ gives

$$
\overline{g}^{\mu\nu}\overline{g}_{\kappa\nu}\ddot{x}^\kappa + (\overline{g}^{\mu\nu}\gamma_{\nu\kappa\lambda} + z\overline{g}^{\mu\nu}\sigma_{\nu\kappa\lambda})\dot{x}^\kappa\dot{x}^\lambda = 0
$$
$$
\Leftrightarrow \ddot{x}^\mu + (1+z)^{-2}[\gamma^\mu_{\kappa\lambda} + z\sigma^\mu_{\kappa\lambda}
$$
$$
+ zv^{\mu\nu}(\gamma_{\nu\kappa\lambda} + z\sigma_{\nu\kappa\lambda})]\dot{x}^\kappa\dot{x}^\lambda = 0 \tag{61}
$$

where $\gamma^\mu_{\kappa\lambda} = g^{\mu\nu}\gamma_{\nu\kappa\lambda}$ and $\sigma^\mu_{\kappa\lambda} = g^{\mu\nu}\sigma_{\nu\kappa\lambda}$. After some straightforward calculations, eq. (61) gives

$$
\ddot{x}^\mu + \gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = -\frac{z}{1+z}(\tilde{\sigma}^\mu_{\kappa\lambda} - w^{\mu\nu}\gamma_{\nu\kappa\lambda})\dot{x}^\kappa\dot{x}^\lambda, \tag{62}
$$

with

$$
\tilde{\sigma}^\mu_{\kappa\lambda} \equiv \frac{1}{2}g^{\mu\nu}\left(\partial_\kappa w_{\nu\lambda} + \partial_\lambda w_{\nu\kappa} - \partial_\nu w_{\kappa\lambda}\right) \tag{63}
$$

The horizontal part of the tangent vector on the paths is $\dot{x}^\mu = dx^\mu/ds$ with s an affine parameter along the path defined as [17]:

$$
s = s_0 + \int_{\lambda_0}^{\lambda_1} \sqrt{\pm\overline{g}_{\mu\nu}(x, y)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda \tag{64}
$$

with s_0, λ_0 and λ_1 constants and λ is an arbitrary parameter of the path. The sign of $\overline{g}_{\mu\nu}(x, y)$ is determined by the tangent vector of the path, specifically if $dx^\nu/d\lambda$ is timelike with respect to $\overline{g}_{\mu\nu}(x, y)$ ($\overline{g}_{\mu\nu}(x, y)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0$) then we get "$-$", likewise for a spacelike tangent vector with respect to $\overline{g}_{\mu\nu}(x, y)$ ($\overline{g}_{\mu\nu}(x, y)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} > 0$) we get "$+$".

The paths (62) will play the role for our model that the geodesics play for general relativity. As is the case for the latter, we need a classification of path segments with respect to their character i.e. timelike, null and spacelike. We define:

– Timelike segment: $g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu < 0$ at every point
– Null segment: $g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu = 0$ at every point
– Spacelike segment: $g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu > 0$ at every point

Therefore, the character of the path is determined by the metric tensor $g_{\mu\nu}(x)$ of the horizontal subspace. We define the proper time τ as

$$
\tau = \tau_0 + \int_{\lambda_0}^{\lambda_1} \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda, \tag{65}
$$

---

[1] In general, $\tilde{\delta}^\alpha_\mu$ and $\tilde{\delta}^\nu_\beta$ can be used to lift an object from the horizontal to the vertical subspace of TTM or lower down one from the vertical to the horizontal subspace. This allows us to perform algebraic operations between components of tensors belonging to different sub-spaces of TTM.

where τ_0 is constant. By comparing relations (64) and (65) we see that the parameter s on the paths (62) cannot be written as an affine transformation of the proper time in general.

We remark that Eq. (62) reduces to the classic geodesics equation of general relativity when the perturbation $w_{\alpha\beta}$ goes to zero, as it should.

4.1 Timelike paths

To begin, we rewrite the perturbation (31) as

$$
w_{\nu\lambda} = g_{\lambda\rho}A_\nu u^\rho + g_{\nu\rho}A_\lambda u^\rho + (g_{\nu\lambda} + g_{\nu\sigma}g_{\lambda\tau}u^\sigma u^\tau)A_\rho u^\rho \tag{66}
$$

with

$$
u^\nu \equiv \frac{y^\nu}{a} \tag{67}
$$

where $a = \sqrt{-g_{\mu\nu}y^\mu y^\nu}$ and we have lowered down A_γ and y^γ using the generalized Kronecker deltas. It is straightforward to show

$$
\partial_\mu u^\nu = -\frac{1}{2}\partial_\mu g_{\kappa\lambda}u^\kappa u^\lambda u^\nu \tag{68}
$$

Using (63), (66) and (68) we calculate

$$
\tilde{\sigma}^\mu_{\kappa\lambda}y^\kappa y^\lambda = a^2 g^{\mu\nu}\left\{\left(g_{\lambda\sigma}\partial_\kappa A_\nu + 2g_{\nu\sigma}\partial_\kappa A_\lambda - \frac{3}{2}g_{\lambda\sigma}\partial_\nu A_\kappa\right.\right.
$$
$$
\left.+ 2A_\sigma\partial_\kappa g_{\nu\lambda} + A_\nu\partial_\kappa g_{\lambda\sigma} - \frac{3}{2}A_\sigma\partial_\nu g_{\kappa\lambda}\right)u^\sigma u^\kappa u^\lambda
$$
$$
+ \frac{1}{2}(-A_\nu g_{\lambda\rho}\partial_\kappa g_{\sigma\tau} + A_\lambda g_{\nu\rho}\partial_\kappa g_{\sigma\tau}
$$
$$
+ 2A_\rho g_{\lambda\tau}\partial_\kappa g_{\nu\sigma} + 2g_{\nu\sigma}g_{\lambda\tau}\partial_\kappa A_\rho - A_\kappa g_{\lambda\rho}\partial_\nu g_{\sigma\tau}
$$
$$
- g_{\kappa\sigma}g_{\lambda\tau}\partial_\nu A_\rho)u^\sigma u^\tau u^\rho u^\kappa u^\lambda
$$
$$
- \frac{3}{4}A_\rho(2g_{\nu\sigma}g_{\kappa\tau}\partial_\lambda g_{\xi\pi}
$$
$$
\left.- g_{\kappa\sigma}g_{\lambda\tau}\partial_\nu g_{\xi\pi})u^\sigma u^\tau u^\rho u^\kappa u^\lambda u^\xi u^\pi\right\} \tag{69}
$$

Now, if we take into account that $g_{\mu\nu}u^\mu u^\nu = -1$, the above relation gives

$$
\tilde{\sigma}^\mu_{\kappa\lambda}y^\kappa y^\lambda = a^2\left\{g^{\mu\nu}\left(\partial_\nu A_\kappa - \partial_\kappa A_\nu\right)u^\kappa\right.
$$
$$
+ g^{\mu\nu}\left(g_{\nu\sigma}\partial_\kappa A_\lambda + A_\sigma\partial_\kappa g_{\nu\lambda} + \frac{3}{2}A_\nu\partial_\kappa g_{\lambda\sigma}\right.
$$
$$
\left.- \frac{1}{4}A_\kappa\partial_\nu g_{\sigma\lambda}\right)u^\sigma u^\kappa u^\lambda
$$
$$
\left.+ A_\lambda\partial_\kappa g_{\sigma\tau}u^\sigma u^\tau u^\mu u^\kappa u^\lambda\right\}. \tag{70}
$$

Substituting the relations (58), (60), (66) and (70) into (62) we get

$$\ddot{x}^\mu + \gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda$$
$$= -\frac{z}{1+z}\left\{ ag^{\mu\nu}(\partial_\nu A_\kappa - \partial_\kappa A_\nu)\dot{x}^\kappa \right.$$
$$+ \frac{1}{a}\left[A^\nu\left(\partial_\kappa g_{\nu\lambda} - \frac{1}{2}\partial_\nu g_{\kappa\lambda}\right) + \partial_\kappa A_\lambda \right]\dot{x}^\mu\dot{x}^\kappa\dot{x}^\lambda$$
$$+ \frac{1}{a}\left(\frac{1}{4}g^{\mu\nu}A_\kappa\partial_\nu g_{\sigma\lambda} + g^{\mu\nu}A_\kappa\partial_\lambda g_{\sigma\nu} + A^\mu\partial_\kappa g_{\lambda\sigma}\right)\dot{x}^\sigma\dot{x}^\kappa\dot{x}^\lambda$$
$$\left. + \frac{1}{2a^3}A_\lambda\partial_\kappa g_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau\dot{x}^\mu\dot{x}^\kappa\dot{x}^\lambda \right\} \tag{71}$$

This is the generalized path equation for the timelike sector of the metric $g_{\mu\nu}(x)$.

Remark. If we set $a = 1$ at some fixed point then (71) can be written as

$$\ddot{x}^\mu + \gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda - \frac{e}{m}F^\mu_\kappa\dot{x}^\kappa$$
$$= \frac{e}{m}\left\{ \left[A^\nu\left(\partial_\kappa g_{\nu\lambda} - \frac{1}{2}\partial_\nu g_{\kappa\lambda}\right) + \partial_\kappa A_\lambda \right]\dot{x}^\mu\dot{x}^\kappa\dot{x}^\lambda \right.$$
$$+ \left(\frac{1}{4}g^{\mu\nu}A_\kappa\partial_\nu g_{\sigma\lambda} + g^{\mu\nu}A_\kappa\partial_\lambda g_{\sigma\nu} + A^\mu\partial_\kappa g_{\lambda\sigma}\right)\dot{x}^\sigma\dot{x}^\kappa\dot{x}^\lambda$$
$$\left. + A_\lambda\partial_\kappa g_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau\dot{x}^\mu\dot{x}^\kappa\dot{x}^\lambda \right\} \tag{72}$$

with $F_{\kappa\nu} = \partial_\nu A_\kappa - \partial_\kappa A_\nu$ the field strength tensor of A_ν and we have set $\frac{z}{1+z} := -\frac{e}{m}$ where e the electric charge and m the mass of the particle. If we ignore the r.h.s of the above equation then (72) will have the same form as the equation of a charged particle subject to the Lorentz force with an electromagnetic vector potential A_γ in the Riemannian setting. A similar equation which is derived from a Finsler–Randers Lagrangian and contains a Lorentz force term has been studied in [37]. However, in our more generalized setting we also get the r.h.s. perturbation term which depends on A_ν and its first derivatives. Therefore, a possible relation between our Schwarzschild–Randers metric and the Lorentz force requires further investigation and goes beyond the scope of this work.

Now, it is known that we can always approach a time-like path (geodesic) with a proper time parameter broken null path with the same endpoints [58]. In this approximation it is considered that the number of null path segments with infinitesimal distance between two neighboring points increases following the timelike path. Therefore, the final null path of zero length (with respect to $g_{\mu\nu}$) approaches the timelike path (71), however the parameter along them is replaced by an appropriate affine one.

Substituting to (71) the solution (51) we get the explicit form of the timelike paths components for $r > R_s$:

$$\ddot{t} + \frac{1-f}{rf}\dot{r}\dot{t}$$
$$= -\frac{z}{1+z}\tilde{A}_4\left\{ \left(\frac{1}{2}af^{-3/2}\dot{r} + \frac{2}{a}f^{-1/2}\dot{r}\dot{t}^2\right.\right.$$
$$\left. + \frac{1}{a}f^{-5/2}\dot{r}^3\right)\frac{1-f}{r}$$
$$- \frac{2r}{a}f^{-1/2}\left(\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{\theta}\dot{\phi}^2\right)$$
$$+ \dot{t}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.$$
$$+ \frac{1}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.$$
$$\left.\left.\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right] \right\} \tag{73}$$

$$\ddot{r} + \frac{f(1-f)}{2r}\dot{t}^2 - \frac{1-f}{2rf}\dot{r}^2 - rf\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)$$
$$= -\frac{z}{1+z}\tilde{A}_4\left\{ \left(\frac{1}{2}af^{1/2}\dot{t} - \frac{1}{4a}f^{3/2}\dot{t}^3\right.\right.$$
$$\left. - \frac{5}{4a}f^{-1/2}\dot{r}^2\dot{t}\right)\frac{1-f}{r}$$
$$+ \frac{1}{2a}f^{3/2}r\left(\dot{t}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{\phi}^2\right)$$
$$+ \dot{r}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.$$
$$+ \frac{1}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.$$
$$\left.\left.\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right] \right\} \tag{74}$$

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \frac{1}{2}\sin 2\theta\,\dot{\phi}^2$$
$$= -\frac{z}{1+z}\tilde{A}_4\left\{ \frac{1}{a}\frac{1}{r}\left(\frac{1}{4}f^{1/2}\sin 2\theta\,\dot{t}\dot{\phi}^2 + 2\dot{t}\dot{r}\dot{\theta}\right)\right.$$
$$+ \dot{\theta}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.$$
$$+ \frac{1}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.$$
$$\left.\left.\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right] \right\} \tag{75}$$

$$\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\cot\theta\,\dot{\theta}\dot{\phi}$$

$$
= -\frac{z}{1+z}\tilde{A}_4\Bigg\{\frac{1}{a}\left(\frac{1}{r}\dot{t}\dot{r}\dot{\phi}+\cot\theta\,\dot{t}\dot{\theta}\dot{\phi}\right)
$$
$$
+\dot{\phi}\left[\frac{1}{a}\left(-f^{-3/2}+\frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\,\frac{1-f}{r}\right.
$$
$$
+\frac{1}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r}+f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.
$$
$$
\left.\left.+2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2+\sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2+\frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right]\Bigg\}
$$
$$
\tag{76}
$$

with $f=1-\frac{R_S}{r}$, R_S the Schwarzschild radius.

4.1.1 Application

As an application for our model we present a numerical solution (using the differential equation solver of Mathematica) of the timelike path equations (73–76) for an appropriate choice of parameters and initial values. For this application we consider that $\theta=\frac{\pi}{2}$ while t, r and ϕ are the variables. Below we present two figures which clarify the difference between Schwarzschild–Finsler–Randers (S–F–R) paths (red line) and general relativity (GR) geodesics (blue line). Notice, that the full analysis of timelike paths as well as the applications to Astrophysics will be studied in a forthcoming paper.

From the figures we observe that in the case of S–F–R model the timelike path reaches a higher maximum distance which is somewhat larger than the path provided by General Relativity. We notice that the time it takes for the S–F–R path to reach the Schwarzschild radius is more than in GR (Fig. 1). From these observations we see that in our model
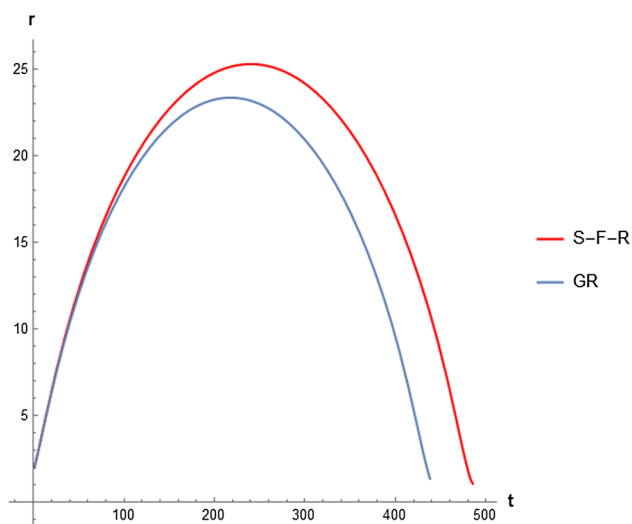
the maximum radial distance of the orbit is greater than that of GR and also the rate at which the particle falls is slower (Fig. 2). As expected, this deviation from the GR geodesic is produced from the right hand side of (73–76), where the extra terms in the S–F–R model act as a force that opposes gravity. Since the extra terms of the rhs of (73–76) are taken to be small the corresponding deviation from the GR geodesic is relatively small.

4.2 Spacelike paths

For completeness, we will find the spacelike paths from (62) following the same procedure as above. From (50) we get

$$
w_{\nu\lambda}=g_{\lambda\rho}A_\nu u^\rho+g_{\nu\rho}A_\lambda u^\rho+(g_{\nu\lambda}-g_{\nu\sigma}g_{\lambda\tau}u^\sigma u^\tau)A_\rho u^\rho
\tag{77}
$$

where, as before, we have lowered down $w_{\alpha\beta}$ to the horizontal space using the generalized Kronecker symbols and we have set $u^\nu=y^\nu/a$ where $a=\sqrt{g_{\mu\nu}y^\mu y^\nu}$. Following the same steps as for the timelike section of $g_{\mu\nu}$ and taking into account that $g_{\mu\nu}u^\mu u^\nu=1$, (62) gives

$$
\ddot{x}^\mu+\gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda
$$
$$
=-\frac{z}{1+z}\Bigg\{-ag^{\mu\nu}\left(\partial_\nu A_\kappa-\partial_\kappa A_\nu\right)\dot{x}^\kappa
$$
$$
+\frac{1}{a}\left[A^\nu\left(\partial_\kappa g_{\nu\lambda}-\frac{1}{2}\partial_\nu g_{\kappa\lambda}\right)+\partial_\kappa A_\lambda\right]\dot{x}^\mu\dot{x}^\kappa\dot{x}^\lambda
$$
$$
+\frac{1}{a}\left(-\frac{1}{4}g^{\mu\nu}A_\kappa\partial_\nu g_{\sigma\lambda}+g^{\mu\nu}A_\kappa\partial_\lambda g_{\sigma\nu}\right)\dot{x}^\sigma\dot{x}^\kappa\dot{x}^\lambda
$$
$$
-\frac{5}{2a^3}A_\lambda\partial_\kappa g_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau\dot{x}^\mu\dot{x}^\kappa\dot{x}^\lambda\Bigg\}
\tag{78}
$$

Fig. 1 This is an r, t graph of the timelike paths that we find using our theoretical model Schwarzschild–Finsler–Randers (S–F–R) in comparison to the geodesics of general relativity (GR)
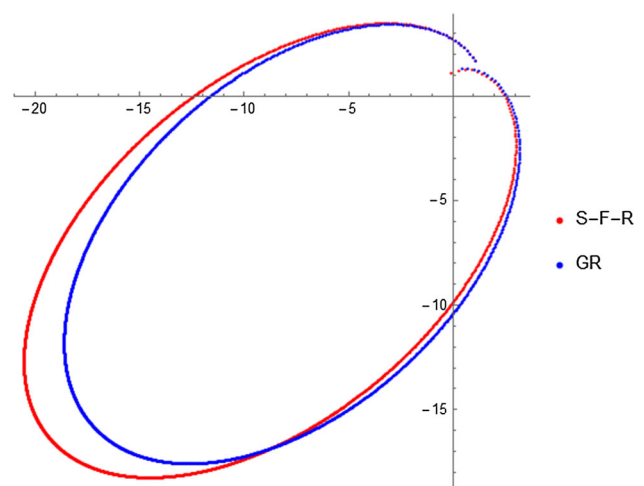

Fig. 2 This is an r, ϕ polar graph of the timelike paths that we find using our theoretical Schwarzschild–Finsler–Randers (S–F–R; red curve) model in comparison to the geodesics of general relativity (GR; blue line)

Substituting to (78) the solution (51) we get the explicit form of the spacelike paths components for $r > R_S$:

$$
\ddot{t} + \frac{1-f}{rf}\dot{r}\dot{t}
$$

$$
= -\frac{z}{1+z}\tilde{A}_4\left\{\left(-\frac{1}{2}af^{-3/2}\dot{r} + \frac{1}{a}f^{-1/2}\dot{r}\dot{t}^2\right)\frac{1-f}{r}\right.
$$

$$
+ \dot{t}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.
$$

$$
- \frac{5}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.
$$

$$
\left.\left.\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right]\right\}
$$
(79)

$$
\ddot{r} + \frac{f(1-f)}{2r}\dot{t}^2 - \frac{1-f}{2rf}\dot{r}^2 - rf\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)
$$

$$
= -\frac{z}{1+z}\tilde{A}_4\left\{\left(-\frac{1}{2}af^{1/2}\dot{t} + \frac{1}{4a}f^{3/2}\dot{t}^3\right.\right.
$$

$$
\left.-\frac{3}{4a}f^{-1/2}\dot{r}^2\dot{t}\right)\frac{1-f}{r}
$$

$$
-\frac{1}{2a}f^{3/2}r\left(\dot{t}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{\phi}^2\right)
$$

$$
+ \dot{r}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.
$$

$$
-\frac{5}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.
$$

$$
\left.\left.\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right]\right\}
$$
(80)

$$
\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \frac{1}{2}\sin 2\theta\,\dot{\phi}^2
$$

$$
= -\frac{z}{1+z}\tilde{A}_4\left\{\frac{1}{a}\frac{1}{r}\left(-\frac{1}{4}f^{1/2}\sin 2\theta\,\dot{t}\dot{\phi}^2 + 2\dot{t}\dot{r}\dot{\theta}\right)\right.
$$

$$
+ \dot{\theta}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.
$$

$$
-\frac{5}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.
$$

$$
\left.\left.\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right]\right\}
$$
(81)

$$
\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\cot\theta\,\dot{\theta}\dot{\phi}
$$

$$
= -\frac{z}{1+z}\tilde{A}_4\left\{\frac{1}{a}\left(\frac{1}{r}\dot{t}\dot{r}\dot{\phi} + \cot\theta\,\dot{t}\dot{\theta}\dot{\phi}\right)\right.
$$

$$
+ \dot{\phi}\left[\frac{1}{a}\left(-f^{-3/2} + \frac{1}{2}f^{-1/2}\right)\dot{t}\dot{r}\frac{1-f}{r}\right.
$$

$$
-\frac{5}{2a^3}\left(-\{f^{1/2}\dot{t}^3\dot{r} + f^{-3/2}\dot{t}\dot{r}^3\}\frac{1-f}{r}\right.
$$

$$
\left. + 2f^{1/2}r\{\dot{t}\dot{r}\dot{\theta}^2 + \sin^2\theta\,\dot{t}\dot{r}\dot{\phi}^2 + \frac{r}{2}\sin 2\theta\,\dot{t}\dot{\theta}\dot{\phi}^2\}\right)\right]\right\}
$$
(82)

5 Conclusion

In this paper we derived for the first time the gravitational field as a solution of the spherically symmetric Schwarzschild–Randers and Schwarzschild–De Sitter–Randers metric. In this framework we used generalized Einstein field equations on the tangent bundle of a spacetime with zero horizontal energy–momentum tensor in which we get more degrees of freedom. In addition, we specified an appropriate timelike covector which plays a significant role in this theory differentiating our model from the traditional Schwarzschild one, giving an intrinsic anisotropic character to the ordinary Schwarzschild metric as well as for the particle paths. Moreover, we studied the correlation of the Schwarzschild–De Sitter model with the Randers one as becomes apparent from (55). We also studied the forms of paths in our spacetime and we obtained more generalized forms than the ordinary geodesic paths of the classical Schwarzschild spacetime. In this context, we provided some numerical solutions of the timelike paths of our model and we found small but not negligible deviations from general relativity. This difference can been seen as a result of the local anisotropy which creates an effective force that affects the corresponding geodesics.

It is obvious that when the covector A_γ of our theory vanishes then we recover the ordinary form of a Schwarzschild metric and Schwarzschild–De Sitter metric respectively and their derived geodesics.

Such an approximation can be considered compatible with some current observational data and parameters with anisotropic character in cosmological models of Schwarzschild–Randers and Schwarzschild–De Sitter–Randers spacetimes. These features mean that Finsler–Randers gravity can be interesting at the astrophysical level. This study will be the goal of our next work.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Based on the numerical data that we have received regarding our application, we displayed the paths equations for the S-F-R model and compared them graphically with the classic GR geodesics. The results that we presented

in this work constitute a first qualitative astrophysical application of our model. We plan to make a more detailed study on the S-F-R model's predictions in a future work in which the full data will be provided.]

Appendix A: Basic structures on the Lorentz tangent bundle

The eight-dimensional Lorentz tangent bundle TM is equipped with local coordinates $\{\mathcal{U}^A\} = \{x^\mu, y^\alpha\}$ where x^μ are the local coordinates on the base manifold M around $\pi(\sigma)$, $\sigma \in TM$, and y^α are the coordinates on the fiber. The range of values for the indices is $\kappa, \lambda, \mu, \nu, \ldots = 0, \ldots, 3$ and $\alpha, \beta, \ldots, \theta = 4, \ldots, 7$.

The adapted basis on the total space TTM is defined as $\{E_A\} = \{\delta_\mu, \dot{\partial}_\alpha\}$ where

$$\delta_\mu = \frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N_\mu^\alpha(x, y)\frac{\partial}{\partial y^\alpha} \tag{A.1}$$

and

$$\dot{\partial}_\alpha = \frac{\partial}{\partial y^\alpha} \tag{A.2}$$

where N_μ^α are the components of a nonlinear connection. The curvature of the nonlinear connection is defined as

$$\Omega_{\nu\kappa}^\alpha = \frac{\delta N_\nu^\alpha}{\delta x^\kappa} - \frac{\delta N_\kappa^\alpha}{\delta x^\nu} \tag{A.3}$$

The nonlinear connection induces a split of the total space TTM into a horizontal distribution $T_H TM$ and a vertical distribution $T_V TM$. The above-mentioned split is expressed with the Whitney sum:

$$TTM = T_H TM \oplus T_V TM. \tag{A.4}$$

The horizontal distribution or h-space is spanned by δ_μ, while the vertical distribution or v-space is spanned by $\dot{\partial}_\alpha$. Under a local coordinate transformation on the base manifold, the adapted basis vectors transform as:

$$\delta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\delta_\mu, \quad \dot{\partial}_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\dot{\partial}_\alpha. \tag{A.5}$$

with $x^\alpha = \tilde{\delta}_\mu^\alpha x^\mu$.[2] The adapted dual basis of the adjoint total space T^*TM is $\{E^A\} = \{dx^\mu, \delta y^\alpha\}$ with the definition

$$\delta y^\alpha = dy^\alpha + N_\nu^\alpha dx^\nu. \tag{A.6}$$

The transformation rule for $\{dx^\mu, \delta y^\alpha\}$ is:

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}dx^\mu, \quad \delta y^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\delta y^\alpha \tag{A.7}$$

The bundle TM

In this work, we consider a distinguished connection (d-connection) D on TM. This is a linear connection with coefficients $\{\Gamma_{BC}^A\} = \{L_{\nu\kappa}^\mu, L_{\beta\kappa}^\alpha, C_{\nu\gamma}^\mu, C_{\beta\gamma}^\alpha\}$ which preserves by parallelism the horizontal and vertical distributions:

$$D_{\delta_\kappa}\delta_\nu = L_{\nu\kappa}^\mu(x, y)\delta_\mu, \quad D_{\dot{\partial}_\gamma}\delta_\nu = C_{\nu\gamma}^\mu(x, y)\delta_\mu \tag{A.8}$$

$$D_{\delta_\kappa}\dot{\partial}_\beta = L_{\beta\kappa}^\alpha(x, y)\dot{\partial}_\alpha, \quad D_{\dot{\partial}_\gamma}\dot{\partial}_\beta = C_{\beta\gamma}^\alpha(x, y)\dot{\partial}_\alpha \tag{A.9}$$

From these, the definitions for partial covariant differentiation follow as usual, e.g. for $X \in TTM$ we have the definitions for covariant h-derivative

$$X_{|\nu}^A \equiv D_\nu X^A \equiv \delta_\nu X^A + L_{B\nu}^A X^B \tag{A.10}$$

and covariant v-derivative

$$X^A|_\beta \equiv D_\beta X^A \equiv \dot{\partial}_\beta X^A + C_{B\beta}^A X^B \tag{A.11}$$

A d-connection can be uniquely defined given that the following conditions are satisfied:

- The d-connection is metric compatible
- Coefficients $L_{\nu\kappa}^\mu, L_{\beta\kappa}^\alpha, C_{\nu\gamma}^\mu, C_{\beta\gamma}^\alpha$ depend solely on the quantities $g_{\mu\nu}$, $v_{\alpha\beta}$ and N_μ^α
- Coefficients $L_{\kappa\nu}^\mu$ and $C_{\beta\gamma}^\alpha$ are symmetric on the lower indices, i.e. $L_{[\kappa\nu]}^\mu = C_{[\beta\gamma]}^\alpha = 0$

We use the symbol $\mathcal{D}$ instead of D for a connection satisfying the above conditions, and call it a canonical and distinguished d-connection. Metric compatibility translates into the conditions:

$$\mathcal{D}_\kappa g_{\mu\nu} = 0, \quad \mathcal{D}_\kappa v_{\alpha\beta} = 0, \quad \mathcal{D}_\gamma g_{\mu\nu} = 0, \quad \mathcal{D}_\gamma v_{\alpha\beta} = 0. \tag{A.12}$$

The coefficients of canonical and distinguished d-connection are

$$L_{\nu\kappa}^\mu = \frac{1}{2}g^{\mu\rho}(\delta_k g_{\rho\nu} + \delta_\nu g_{\rho\kappa} - \delta_\rho g_{\nu\kappa}) \tag{A.13}$$

$$L_{\beta\kappa}^\alpha = \dot{\partial}_\beta N_\kappa^\alpha + \frac{1}{2}v^{\alpha\gamma}(\delta_\kappa v_{\beta\gamma} - v_{\delta\gamma}\dot{\partial}_\beta N_\kappa^\delta - v_{\beta\delta}\dot{\partial}_\gamma N_\kappa^\delta) \tag{A.14}$$

---

[2] The generalized Kronecker symbols are defined as: $\tilde{\delta}_\alpha^\mu = \tilde{\delta}_\mu^\alpha = 1$ for $a = \mu + 4$ and equal to zero otherwise.

$$C^{\mu}_{\nu\gamma} = \frac{1}{2} g^{\mu\rho} \dot{\partial}_{\gamma} g_{\rho\nu} \tag{A.15}$$

$$C^{\alpha}_{\beta\gamma} = \frac{1}{2} v^{\alpha\delta} (\dot{\partial}_{\gamma} v_{\delta\beta} + \dot{\partial}_{\beta} v_{\delta\gamma} - \dot{\partial}_{\delta} v_{\beta\gamma}). \tag{A.16}$$

Curvature and torsion in TM can be defined as multi-linear maps:

$$\mathcal{R}(X, Y)Z = [\mathcal{D}_X, \mathcal{D}_Y]Z - \mathcal{D}_{[X,Y]}Z \tag{A.17}$$

and

$$\mathcal{T}(X, Y) = \mathcal{D}_X Y - \mathcal{D}_Y X - [X, Y], \tag{A.18}$$

where $X, Y, Z \in TTM$. We use the definitions

$$\mathcal{R}(\delta_{\lambda}, \delta_{\kappa})\delta_{\nu} = R^{\mu}_{\nu\kappa\lambda}\delta_{\mu} \tag{A.19}$$

$$\mathcal{R}(\dot{\partial}_{\delta}, \dot{\partial}_{\gamma})\dot{\partial}_{\beta} = S^{\alpha}_{\beta\gamma\delta}\dot{\partial}_{\alpha} \tag{A.20}$$

$$\mathcal{T}(\delta_{\kappa}, \delta_{\nu}) = T^{\mu}_{\nu\kappa}\delta_{\mu} + T^{\alpha}_{\nu\kappa}\dot{\partial}_{\alpha} \tag{A.21}$$

$$\mathcal{T}(\dot{\partial}_{\gamma}, \dot{\partial}_{\beta}) = T^{\mu}_{\beta\gamma}\delta_{\mu} + T^{\alpha}_{\beta\gamma}\dot{\partial}_{\alpha} \tag{A.22}$$

The h-curvature tensor of the d-connection in the adapted basis and the corresponding h-Ricci tensor have, respectively, the components

$$R^{\mu}_{\nu\kappa\lambda} = \delta_{\lambda}L^{\mu}_{\nu\kappa} - \delta_{\kappa}L^{\mu}_{\nu\lambda} + L^{\rho}_{\nu\kappa}L^{\mu}_{\rho\lambda} - L^{\rho}_{\nu\lambda}L^{\mu}_{\rho\kappa} + C^{\mu}_{\nu\alpha}\Omega^{\alpha}_{\kappa\lambda} \tag{A.23}$$

$$R_{\mu\nu} = R^{\kappa}_{\mu\nu\kappa} = \delta_{\kappa}L^{\kappa}_{\mu\nu} - \delta_{\nu}L^{\kappa}_{\mu\kappa} + L^{\rho}_{\mu\nu}L^{\kappa}_{\rho\kappa} - L^{\rho}_{\mu\kappa}L^{\kappa}_{\rho\nu} + C^{\kappa}_{\mu\alpha}\Omega^{\alpha}_{\nu\kappa} \tag{A.24}$$

The v-curvature tensor of the d-connection in the adapted basis and the corresponding v-Ricci tensor have, respectively, the components

$$S^{\alpha}_{\beta\gamma\delta} = \dot{\partial}_{\delta}C^{\alpha}_{\beta\gamma} - \dot{\partial}_{\gamma}C^{\alpha}_{\beta\delta} + C^{\epsilon}_{\beta\gamma}C^{\alpha}_{\epsilon\delta} - C^{\epsilon}_{\beta\delta}C^{\alpha}_{\epsilon\gamma} \tag{A.25}$$

$$S_{\alpha\beta} = S^{\gamma}_{\alpha\beta\gamma} = \dot{\partial}_{\gamma}C^{\gamma}_{\alpha\beta} - \dot{\partial}_{\beta}C^{\gamma}_{\alpha\gamma} + C^{\epsilon}_{\alpha\beta}C^{\gamma}_{\epsilon\gamma} - C^{\epsilon}_{\alpha\gamma}C^{\gamma}_{\epsilon\beta}. \tag{A.26}$$

The generalized Ricci scalar curvature in the adapted basis is defined as

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} + v^{\alpha\beta}S_{\alpha\beta} = R + S. \tag{A.27}$$

where

$$R = g^{\mu\nu}R_{\mu\nu}, \quad S = v^{\alpha\beta}S_{\alpha\beta} \tag{A.28}$$

In the main text, we use the more convenient definitions

$$\overline{R}_{\mu\nu} = R_{\mu\nu} - C^{\kappa}_{\mu\alpha}\Omega^{\alpha}_{\nu\kappa} = \delta_{\kappa}L^{\kappa}_{\mu\nu} - \delta_{\nu}L^{\kappa}_{\mu\kappa} + L^{\rho}_{\mu\nu}L^{\kappa}_{\rho\kappa} - L^{\rho}_{\mu\kappa}L^{\kappa}_{\rho\nu} \tag{A.29}$$

$$\overline{R} = g^{\mu\nu}\overline{R}_{\mu\nu} \tag{A.30}$$

Appendix B: Calculation of A_{γ}

Appendix B.1: Solution for the Schwarzschild–Randers spacetime

In order to calculate A_{γ} we will give values to μ, ν, γ of (48) and solve the resulting equations. For $\mu = 0, \nu = 0$ and $\gamma = 4$ we get:

$$\partial_0^2 A_4 + \frac{1}{2} f \partial_1(-f) \left(\partial_1 A_4 + \frac{1}{2f} A_4 \partial_1(-f) \right) = 0, \tag{B.31}$$

where we have set $f = 1 - \frac{R_s}{r}$. After some calculations and by separation of variables $A_4 = R_4(r)T_4(t)$ we get two equations:

$$\partial_0 T_4(t) = -c^2_{(4)} T_4(t) \tag{B.32}$$

$$\partial_1 R_4(r) = \frac{1-f}{2rf} R_4(r) - c^2_{(4)} \frac{2r}{f(1-f)} \tag{B.33}$$

where $c^2_{(4)}$ is the separation constant. For $\mu = 0, \nu = 1$ and $\gamma = 4$ we get

$$\partial_0 \partial_1 A_4 + \frac{1}{f} \partial_1(-f) \partial_0 A_4 = 0. \tag{B.34}$$

After rearranging the terms and again separating variables, for $\partial_0 T_4 \neq 0$ we have

$$\partial_1 R_4(r) = \frac{\partial_1 f}{f} R_4(r) \tag{B.35}$$

So we get $R_4(r) = k_4 f(r)$ with k_4 being a constant resulting from the integration. By substituting this to (B.33) we find that the separation constant $c^2_{(4)}$ must be zero. That means that in order to satisfy (B.32) and (B.33), $T_4(t)$ must be constant and $R_4(r) = \tilde{R}_4 f^{1/2}(r)$ with $\tilde{R}_4$ a constant. By calculating the remaining equations for $\mu = 2, \nu = 2, \gamma = 4$ and $\mu = 3, \nu = 3, \gamma = 4$ we find that A_4 has no dependence on θ or ϕ. Therefore, we end up with

$$A_4 = \tilde{A}_4 |f|^{1/2}(r) \tag{B.36}$$

with $\tilde{A}_4$ being a constant. For $\mu = 0, \nu = 0, \gamma = 5$ we get

$$\partial_0^2 A_5 - \frac{f(1-f)}{2r} \left(\partial_1 A_5 - \frac{1}{2} A_5 f \partial_1(f^{-1}) \right) = 0. \tag{B.37}$$

After calculations and by separating variables like before we end up with equations

$$\partial_0^2 T_5(t) = -c^2_{(5)} T_5(t) \tag{B.38}$$

$$\partial_1 R_5(r) = -\frac{1-f}{2rf} R_5(r) - c^2_{(5)} \frac{2r}{f(1-f)} \tag{B.39}$$

with $c^2_{(5)}$ being the separation constant. For $\mu = 0, \nu = 1, \gamma = 5$ we get

$$\partial_0\partial_1 A_5 - \frac{1}{2}f\partial(f^{-1})\partial_0 A_5 - \frac{1}{2}(-f^{-1})\partial_1(-f)\partial_0 A_5 = 0$$
(B.40)

After calculations we end up with $c_{(5)} = 0$ and by substitution to (B.38) and (B.39) we find

$$R_5(r) = k_5|f|^{-1/2}$$
(B.41)

with k_5 being a constant. Also, like before, if we calculate the $\mu = 2, \nu = 2, \gamma = 5$ and $\mu = 3, \nu = 3, \gamma = 5$ we get no dependence on θ and ϕ. Therefore, we find

$$A_5 = \tilde{A}_5|f|^{-1/2}(r)$$
(B.42)

with $\tilde{A}_5$ a constant of integration. If we put this solution in the $\mu = 1, \nu = 1$ equation, we get $\tilde{A}_5 = 0$. For $\mu = 0, \nu = 0, \gamma = 6$ we separate variables like before $A_6 = R_6(r)T_6(t)$ and we get two equations:

$$\partial_1 R_6(r) = \frac{R_6(r)}{r} - c_{(6)}^2\frac{2r}{f(1-f)}$$
(B.43)

$$\partial_0^2 T_6(t) = -c_{(6)}^2 T_6(t)$$
(B.44)

with $c_{(6)}^2$ the separation constant. For $\mu = 0, \nu = 1, \gamma = 6$ like before we find that $c_{(6)} = 0$ and by substitution to (B.43) we find that $R_6(r) = \tilde{R}_6 r$ with $\tilde{R}_6$ a constant of integration. We set $\tilde{R}_6$ to zero to keep our solution finite at infinity, so we end up with $A_6 = 0$. For A_7 we set $\mu = 0, \nu = 0, \gamma = 7$ and we find the same equations as for A_6. That leads to $A_7 = 0$ as well.

To sum up, we have found the following solution for A_γ from Eq. (48):

$$A_\gamma(x) = \left[\tilde{A}_4\left|1 - \frac{R_S}{r}\right|^{1/2}, 0, 0, 0\right]$$
(B.45)

with $\tilde{A}_4$ a constant.

Solution for the Schwarzschild–De Sitter–Randers spacetime

We will modify the solution (51) and see if it satisfies (48) for the metric (52). An obvious ansatz is to replace the term $1 - \frac{R_S}{r}$ in (51) with $1 - \frac{R_S}{r} - \frac{\Lambda}{3}r^2$. Doing this we get

$$A_\gamma(x) = \left[\tilde{A}_4\left|1 - \frac{R_S}{r} - \frac{\Lambda}{3}r^2\right|^{1/2}, 0, 0, 0\right]$$
(B.46)

which is verified to be a solution of (48) for the metric (52).

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