



Fix the dual geometries of $T\bar{T}$ deformed CFT_2 and highly excited states of CFT_2

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Abstract In previous works, we have developed an approach to fix the leading behaviors of the pure AdS_3 and BTZ black hole from the entanglement entropies of the free CFT_2 and finite temperature CFT_2 , respectively. We exclusively use holographic principle only and make no restriction about the bulk geometry, not only the kinematics but also the dynamics. In order to verify the universality and correctness of our method, in this paper, we apply it to the $T\bar{T}$ deformed CFT_2 , which breaks the conformal symmetry. In terms of the physical arguments of the $T\bar{T}$ deformed CFT_2 , the derived metric is a deformed BTZ black hole. The requirement that the CFT_2 lives on a conformally flat boundary leads to $r_c^2 = 6R_{\text{AdS}}^4/(\pi c\mu)$ naturally, in perfect agreement with previous conjectures in literature. The energy spectrum and propagation speed calculated with this deformed BTZ metric are the same as these derived from $T\bar{T}$ deformed CFT_2 . We furthermore fix the dual geometry of highly excited states with our approach. The result contains the descriptions for the conical defect and BTZ black hole.

1 Introduction and summary

The $T\bar{T}$ deformed CFT_2 , discovered by the remarkable works [1–3], attracts increasing attention recently. It is a solvable example of quantum field theory. A conformal field theory (CFT), which is conformally invariant at different fixed points along the RG flow, can be deformed by relevant, irrelevant and marginal deformations. The irrelevant deformation only affects the physics in the UV region but not the IR region. $T\bar{T}$ deformation supports a class of solvable irrelevant deformations of CFT_2 by turning on the $T\bar{T}$ term. The integrability implies the theory contains an infinite element of

conserved charges. The class of $T\bar{T}$ deformed CFT_2 is characterized by a length squared parameter $\mu \geq 0$ as $dS^\mu/d\mu = \int d^2x (T\bar{T})_\mu$, where $(T\bar{T})_\mu = \frac{1}{8} (T^{\alpha\beta} T_{\alpha\beta} - (T^\alpha_\alpha)^2)_\mu$ is defined in terms of the stress tensor of the deformed theory. As $\mu \rightarrow 0$, the simplest example is

$$S^\mu = S_{\text{CFT}} + \mu \int d^2x T\bar{T}, \quad (1)$$

It is clear that $T\bar{T}$ deformation is Lorentz invariant and breaks the conformal symmetry. McGough, Mezei, and Verlinde [4] proposed that the $T\bar{T}$ deformed CFT_2 is no longer located on the asymptotic boundary of AdS_3 but lives at the finite radial position r_c ¹,

$$r_c^2 = \frac{6R_{\text{AdS}}^4}{\pi c\mu}, \quad (2)$$

where R_{AdS} is the radius of AdS_3 and $c = \frac{3R_{\text{AdS}}}{2G}$ is the central charge of the CFT_2 . Based on this picture, a new way to extend and study the AdS/CFT correspondence emerges. In a recent work [5], Chen and collaborators calculated the entanglement entropy of the $T\bar{T}$ deformed CFT_2 by the replica trick. They found that for a finite size system, there is no leading correction. But for a finite temperature system, they obtained

$$S_{EE}^\mu = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \left(\frac{\pi \Delta x}{\beta} \right) \right) - \mu \frac{\pi^4 c^2 \Delta x}{9\beta^3} \coth \left(\frac{\pi \Delta x}{\beta} \right). \quad (3)$$

It is curious why there is no correction to the finite size system. Without a correction, how do we tell the CFT lives at a finite radial distance for a finite size system by the entanglement entropy? We speculate the reason is that the entanglement entropy is defined for spacelike intervals only, which is

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¹ Note that to compare with [4], our μ should be replaced by $\mu/(4\pi^2)$ for notation difference.

Table 1 The symbol \checkmark marks a correction to the entanglement entropy caused by the $T\bar{T}$ deformation

	Space-like	Time-like
Finite size		\checkmark
Finite temperature	\checkmark	

not consistent with the Lorentz covariance. Since the finite size and finite temperature systems have the same cylindrical topology under exchanging $t \leftrightarrow x$, we are inspired to speculate that there exists a temporal entanglement entropy for a temporal interval (indicating entangling between the big bang and big crunch?), and the finite size system receives a correction in this temporal entanglement entropy. Unfortunately, up to now, there is no well-defined temporal entanglement entropy. We are thus led to believe, once a temporal entanglement entropy is correctly introduced, one would find the results as shown in Table (1):

In the same paper, Chen etc. verified the holographic method (RT formula [6]) by calculating the geodesic length in the BTZ background. To match the entanglement entropy of the $T\bar{T}$ deformed CFT_2 , they found two endpoints of the geodesics must be fixed on the finite cut-off r_c . The relevant paper [7] calculated entanglement entropy of the $T\bar{T}$ deformed CFT_2 between two antipodal points on a sphere. More recent works on the holographic studies of the $T\bar{T}$ deformed CFT_2 can be found in the Refs. [8–19].

In previous works, we have developed a method to fix the leading behaviors of the dual geometries from the entanglement entropies of the CFT_2 without using AdS/CFT or imposing any constraints on the bulk geometry. It must be understood that neither kinematics nor dynamics (Einstein equation) can be used in the derivation, especially the latter one. Out of the general geometries, gravity is those respecting the Einstein equation. Therefore, using the Einstein equation in any place makes the derivations circular. On the other hand, it is easy to understand that the symmetry argument is a necessary but not a sufficient condition to determine the dual geometry of a CFT. Furthermore, obviously, the $SO(2, 2)$ symmetry argument is no help for the $T\bar{T}$ deformed CFT_2 considered in this paper, since this particular CFT breaks the $SO(2, 2)$ symmetry. Under these considerations, we successfully derived the asymptotic AdS_3 from the infinitely long system of CFT_2 [20], and 3D BTZ black hole from the finite temperature system of CFT_2 [21], both with time direction included as it should be.

One of the distinct ingredients of our approach is to use Sygne's world function [22] to extract the spacetime metric from given geodesic lengths:

$$g_{ij} = - \lim_{x \rightarrow x'} \partial_{x^i} \partial_{x'^j} \left[\frac{1}{2} L^2(x, x') \right]. \quad (4)$$

where $x^i = (x, y)$, and $L(x, x')$ is the length of the geodesic connecting two points x^i and x'^i in the unknown bulk geometry. It turns out that, in order to determine the function form of the bulk geodesic length $L(x, x')$, we need to use the entanglement entropies in UV and IR (mass gap) regions. These entanglement entropies can fix the behaviors along the boundary directions x, t where the CFT lives, and the holographic direction y which is created by the energy cut-off of the CFT. In order to verify the universality and correctness of our method, we apply it to the $T\bar{T}$ deformed CFT_2 at finite temperature to fix the leading behaviors of its dual geometry in this paper. The fixed metric turns out to be a deformed BTZ metric in the planar coordinate, where “deformed” refers to a prefactor of dx^2 in contrast to the standard BTZ metric:

$$ds^2 = \frac{R_{AdS}^2}{y^2} \left(- \left(1 - \frac{y^2}{\beta_H^2} \right) dt^2 + \left(1 - \frac{\pi \mu c}{12 \beta_H^2} + \dots \right)^2 dx^2 + \left(1 - \frac{y^2}{\beta_H^2} \right)^{-1} dy^2 \right), \quad (5)$$

$$= \frac{R_{AdS}^2}{y^2} \left(- \left(1 - \frac{y^2}{\beta_H^2} \right) dt^2 + d\tilde{x}^2 + \left(1 - \frac{y^2}{\beta_H^2} \right)^{-1} dy^2 \right), \quad (6)$$

where $\beta_H \equiv T^{-1} = \frac{\beta}{2\pi}$. It should be emphasized that x and t in Eq. (5) are the physical (proper) arguments of the $T\bar{T}$ deformed CFT_2 . Though after a redefinition $\tilde{x} = (1 - \frac{\pi \mu c}{12 \beta_H^2} + \dots)x$, the metric (6) is the standard BTZ, \tilde{x} is the physical argument of the original undeformed CFT. The advantage of the metric (5) is that, once requiring the CFT lives on the flat boundary of the dual geometry, we immediately deduce that the geometry covers $y_c \leq y \leq \beta_H$, with $y_c^2 = \frac{\pi \mu c}{6}$. After transforming y_c to the global coordinate, we get $r_c^2 = \frac{6 R_{AdS}^4}{\pi c \mu}$, which is in perfect agreement with Eq. (2). In [4], in order to check the duality, quantities such as energy spectrum, thermodynamic properties and propagation speeds were calculated based on the standard BTZ metric (6), and they were compared with the results from the $T\bar{T}$ deformed CFT_2 to determine the value of r_c . Nevertheless, using the deformed metric (5), in addition to determining $r_c^2 = \frac{6 R_{AdS}^4}{\pi c \mu}$ immediately, we straightforwardly calculate the energy spectrum and propagation speeds, which are the same as those from the $T\bar{T}$ deformed CFT_2 . It is important to note that all these conclusions cannot be obtained without the metric component g_{tt} .

As a further check, we next apply our approach to the highly excited states of CFT_2 with large c and sparse spec-

trum of low dimensional operators. Using the entanglement entropies of such states of a finite size system, given in [23–25], we fix the dual geometry

$$ds^2 = \frac{R_{AdS}^2}{y^2} \left(- \left(1 - \frac{y^2}{\beta_\Psi^2} \right) dt^2 + \left(1 - \frac{y^2}{\beta_\Psi^2} \right)^{-1} \left(1 + \mathcal{O} \left(\frac{y}{\beta_\Psi} \right) \right) dy^2 + dx^2 \right), \quad (7)$$

where $\beta_\Psi = (\frac{12\Delta_h}{c} - 1)^{-1/2}$. When $\Delta_h < \frac{c}{12}$, the metric represents a conical defect in the center of the global AdS_3 . On the other hand, when $\Delta_h > \frac{c}{12}$, the inserted operator is heavy enough to produce a BTZ black hole at temperature $T = \beta_\Psi^{-1}$.

In literature, there is an alternative method to extract the bulk geometries from geodesics, namely, *kinematic space*. It shows that the geodesics of AdS_3 can be viewed as points in the kinematic space by calculating the Crofton form from the entanglement entropy. The Crofton form is defined as the second derivatives with respect to two different points of the entanglement entropy (not taking the limits afterwards!). Then the points-grouped one-dimensional object, say point curve (codimension-1 locus), in the kinematic space corresponds to a set of geodesics which intersect with each other at a single point of AdS_3 . Therefore, geodesic length between any two points in the bulk of AdS_3 equals the integral of Crofton form over the area between two corresponding point curves in the kinematic space [26]. The advantage of kinematic space is that the correspondence between Crofton's formula and field theory is clear. Then, it is possible to obtain the OPE blocks from the local operators in the AdS by Radon transformation [27,28]. However, it is not easy and straightforward to fix the leading behaviors of the spacetime metric of previously unknown bulk geometry. One of the main differences and advantages of our approach is to impose the limit $x \rightarrow x'$ after taking the derivatives, which enables us to get the metric immediately from the geodesics, as shown in Eq. (4).

The reminder of this paper is outlined as follows. In Sect. 2, we show how to fix the leading behaviors of the deformed BTZ spacetime from the entanglement entropies of the $T\bar{T}$ deformed CFT_2 , and calculate the energy spectrum and propagation speeds with the deformed metric. In Sect. 3, we address the highly excited states of CFT_2 and fix the leading behaviors of its dual geometry from the entanglement entropy. Section 4 is for conclusion.

2 The dual geometry of the $T\bar{T}$ deformed CFT_2

To the leading order correction, the UV entanglement entropy of $T\bar{T}$ deformed CFT_2 at finite temperature was calculated in [5],

$$S_{EE}^\mu = \frac{c}{3} \log \left(\frac{2\beta_H}{a} \sinh \left(\frac{\Delta x}{2\beta_H} \right) \right) - \pi \mu \frac{c^2 \Delta x}{72\beta_H^3} \coth \left(\frac{\Delta x}{2\beta_H} \right), \quad (8)$$

where $\beta_H \equiv T^{-1} = \frac{\beta}{2\pi}$ and a is a UV cut-off. As $\mu/\beta_H^2 \rightarrow 0$, it reduces to the entanglement entropy of the undeformed CFT_2 at finite temperature:

$$S_{EE} = \frac{c}{3} \log \left(\frac{2\beta_H}{a} \sinh \left(\frac{\Delta x}{2\beta_H} \right) \right). \quad (9)$$

Since the correction is a function of $\Delta x/\beta_H$ and μ/β_H^2 , it is tempting to check if it can be captured by an effective correction to Δx . This step is consistent with the fact that irrelevant deformations only affect the physics in the UV region. Therefore, we set

$$S_{EE}^\mu = \frac{c}{3} \log \left(\frac{2\beta_H}{a} \sinh \left(\frac{\Delta x}{2\beta_H} (1 + F(\mu, \Delta x)) \right) \right), \quad (10)$$

where $F(\mu, \Delta x)$ is a correction to Δx . It is remarkable to notice for small $F(\mu, \Delta x)$, the Taylor expansion is

$$S_{EE}^\mu = \frac{c}{3} \log \left(\frac{2\beta_H}{a} \sinh \left(\frac{\Delta x}{2\beta_H} \right) + F(\mu, \Delta x) \frac{c \Delta x}{6\beta_H} \coth \left(\frac{\Delta x}{2\beta_H} \right) + \dots \right). \quad (11)$$

Therefore, comparing with Eq. (8), we can get

$$F(\mu, \Delta x) = -\frac{\pi \mu c}{12\beta_H^2} + \mathcal{O}(\mu^2) \quad (12)$$

So, the UV entanglement entropy of $T\bar{T}$ deformed CFT_2 can be written as:

$$S_{EE}^\mu = \frac{c}{3} \log \left(\frac{2\beta_H}{a} \sinh \left(\frac{\Delta x}{2\beta_H} \left(1 - \frac{\pi \mu c}{12\beta_H^2} + \dots \right) \right) \right). \quad (13)$$

The time-dependent entanglement entropy of the undeformed CFT_2 at finite temperature is:

$$S_{EE}(t) = \frac{c}{3} \log \left(\frac{\beta_H}{a} \left[\sqrt{2 \cosh \left(\frac{\Delta x}{\beta_H} \right) - 2 \cosh \left(\frac{\Delta t}{\beta_H} \right)} \right] \right). \quad (14)$$

As $\Delta t = 0$, it reduces to Eq. (9). For the $T\bar{T}$ deformation, we only need to replace Δx by $\Delta x \left(1 - \frac{\pi \mu c}{12 \beta_H^2} + \dots \right)$ to get the $T\bar{T}$ deformed one

$$S_{EE}^\mu(t) = \frac{c}{3} \log \left(\frac{\beta_H}{a} \left[\sqrt{2 \cosh \left(\frac{\Delta x}{\beta_H} \left(1 - \frac{\pi \mu c}{12 \beta_H^2} + \dots \right) \right) - 2 \cosh \left(\frac{\Delta t}{\beta_H} \right)} \right] \right). \quad (15)$$

Setting $R \equiv \frac{2G_N^{(3)}c}{3}$ and identifying this entanglement entropy with the length of the geodesic anchored on the boundary in the dual geometry, one has

$$\frac{L_{\text{boundary}}}{R} = \log \left(\frac{2\beta_H^2}{a^2} \left[\cosh \left(\frac{\Delta x}{\beta_H} \left(1 - \frac{\pi \mu c}{12 \beta_H^2} + \dots \right) \right) - \cosh \left(\frac{\Delta t}{\beta_H} \right) \right] \right). \quad (16)$$

From the holographic principle, the energy cut-off a generates a holographic dimension y , say $a^2 \rightarrow yy' (1 + \dots)$. In order to apply Eq. (4) to get the metric, this ending on boundary geodesic length Eq. (16) is obviously not sufficient and we need to push the endpoints into the bulk, with dependence on the holographic dimension y , to describe bulk geodesic connecting arbitrary endpoints (t, x, y) and (t', x', y') . On the other hand, as shown by Eq. (8), when $\beta_H \rightarrow \infty$, the entanglement entropy becomes the free CFT₂ one, whose dual geometry is the pure AdS₃ in Poincare patch, as we derived in [20] by our approach. The geodesic length of pure AdS₃ in Poincare patch connecting (t, x, y) and (t', x', y') is

$$\cosh \left(\frac{L_{\text{bulk}}}{R} \right) = \frac{(\Delta x)^2 - (\Delta t)^2 + y^2 + y'^2}{2yy'}. \quad (17)$$

To meet these two requirements (16) and (17) under different limits, we only have one possibility to generalize the boundary anchored expression (16) to the general bulk geodesic length:

$$\cosh \left(\frac{L_{\text{bulk}}}{R} \right) = \frac{\beta_H^2}{yy'} \left[f \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right) \times \cosh \left(\frac{\Delta x}{\beta_H} \left(1 - \frac{\mu c}{12 \beta_H^2} + \dots \right) \right) \right]$$

$$- g \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right) \times \cosh \left(\frac{\Delta t}{\beta_H} \right)]. \quad (18)$$

where $f \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right)$ and $g \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right)$ are arbitrary regular functions to keep the generalization

universal. The next steps are determining the behaviors of $f \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right)$ and $g \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right)$, and then we can apply Eq. (4) to get the metric.

Step 1: When $\beta_H \gg y = y' = a$ (geodesics anchored on the boundary), L_{bulk} in Eq. (18) must reduce to L_{boundary} , given by equation (16),

$$L_{\text{bulk}} = R \log \left(\frac{2\beta_H^2}{yy'} \left[f \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right) \times \cosh \left(\frac{\Delta x}{\beta_H} \left(1 - \frac{\mu c}{12 \beta_H^2} + \dots \right) \right) - g \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right) \cosh \left(\frac{\Delta t}{\beta_H} \right) \right] \right) \quad (19)$$

$$\rightarrow R \log \left(\frac{2\beta_H^2}{a^2} \left[\cosh \left(\frac{\Delta x}{\beta_H} \right) - \cosh \left(\frac{\Delta t}{\beta_H} \right) \right] \right). \quad (20)$$

So as $\beta_H \gg y$ and y' , we have

$$\begin{aligned} & f \left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t' \right) \\ &= 1 + \varrho_1 \left(x, x', \frac{\mu}{\beta_H^2}; t, t' \right) \left(\frac{y}{\beta_H} + \frac{y'}{\beta_H} \right) \\ &+ \varrho_2 \left(x, x', \frac{\mu}{\beta_H^2}; t, t' \right) \left(\frac{y}{\beta_H} + \frac{y'}{\beta_H} \right)^2 + \dots \\ &+ \rho_1 \left(x, x', \frac{\mu}{\beta_H^2}; t, t' \right) \left(\frac{yy'}{\beta_H^2} \right) \\ &+ \rho_2 \left(x, x', \frac{\mu}{\beta_H^2}; t, t' \right) \left(\frac{yy'}{\beta_H^2} \right)^2 + \dots, \end{aligned}$$

$$\begin{aligned}
& g\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) \\
&= 1 + \bar{q}_1\left(x, x', \frac{\mu}{\beta_H^2}; t, t'\right)\left(\frac{y}{\beta_H} + \frac{y'}{\beta_H}\right) \\
&+ \bar{q}_2\left(x, x', \frac{\mu}{\beta_H^2}; t, t'\right)\left(\frac{y}{\beta_H} + \frac{y'}{\beta_H}\right)^2 + \dots \\
&+ \bar{\rho}_1\left(x, x', \frac{\mu}{\beta_H^2}; t, t'\right)\left(\frac{yy'}{\beta_H^2}\right) \\
&+ \bar{\rho}_2\left(x, x', \frac{\mu}{\beta_H^2}; t, t'\right)\left(\frac{yy'}{\beta_H^2}\right)^2 + \dots, \quad (21)
\end{aligned}$$

where q_i , ρ_i , \bar{q}_i and $\bar{\rho}_i$ are regular and bounded functions.

Step 2: As $\beta_H \rightarrow \infty$, or $\mu/\beta_H^2 \ll 1$ and $\beta_H \gg \Delta x, \Delta t$, y and y' , the general expression (18) must match the pure AdS₃ background (17). From step 1, we know the leading term of f and g is the unit. So, we have

$$\begin{aligned}
& \cosh\left(\frac{L_{\text{bulk}}}{R}\right) \\
& \simeq \frac{\beta_H^2}{yy'} \left[f\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) \left(1 + \frac{(\Delta x)^2}{2\beta_H^2} - \frac{\pi\mu c\Delta x}{12\beta_H^3} + \dots\right) \right. \\
& \quad \left. - g\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) \left(1 + \frac{(\Delta t)^2}{2\beta_H^2} + \dots\right) \right] \\
&= \frac{1}{2yy'} \left[f\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) (\Delta x)^2 \right. \\
& \quad \left. - g\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) (\Delta t)^2 \right. \\
& \quad \left. + 2\beta_H^2 \left(f\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) \right. \right. \\
& \quad \left. \left. - g\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right) \right) + \dots \right], \quad (22) \\
& \rightarrow \frac{(\Delta x)^2 - (\Delta t)^2 + y^2 + y'^2}{2yy'}, \quad (23)
\end{aligned}$$

where we only keep the leading order. When keeping the higher orders, we suppose to get the metric of asymptotic AdS by using equation (4). Or equivalently speaking, since $f\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right)$ and $g\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right)$ lead to the coefficients of dt^2 and dx^2 , to be an asymptotic AdS, the regular functions $f\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right)$ and $g\left(x, x', \frac{\mu}{\beta_H^2}; y, y'; t, t'\right)$ can only depend on y and y' . Thus the general expression of the bulk geodesic length takes the form

$$\cosh\left(\frac{L_{\text{bulk}}}{R}\right) = \frac{\beta_H^2}{yy'} \left[f\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right) \cosh\left(\frac{\Delta x}{\beta_H} \left(1 - \frac{\pi\mu c}{12\beta_H^2} + \dots\right)\right) \right.$$

$$\left. - g\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right) \cosh\left(\frac{\Delta t}{\beta_H}\right) \right], \quad (24)$$

where

$$\begin{aligned}
f\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right) &= 1 + q_1\left(\frac{y}{\beta_H} + \frac{y'}{\beta_H}\right) + q_2\left(\frac{y}{\beta_H} + \frac{y'}{\beta_H}\right)^2 + \dots \\
&+ \rho_1\left(\frac{yy'}{\beta_H^2}\right) + \rho_2\left(\frac{yy'}{\beta_H^2}\right)^2 + \dots, \\
g\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right) &= 1 + \bar{q}_1\left(\frac{y}{\beta_H} + \frac{y'}{\beta_H}\right) + \bar{q}_2\left(\frac{y}{\beta_H} + \frac{y'}{\beta_H}\right)^2 + \dots \\
&+ \bar{\rho}_1\left(\frac{yy'}{\beta_H^2}\right) + \bar{\rho}_2\left(\frac{yy'}{\beta_H^2}\right)^2 + \dots \quad (25)
\end{aligned}$$

and

$$f\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right) - g\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right) = \frac{1}{2\beta_H^2} (y^2 + y'^2) + \mathcal{O}\left(\frac{1}{\beta_H^4}\right), \quad (26)$$

given by the last two terms of (22).

Since the correction $\frac{\Delta x}{\beta_H} \left(1 - \frac{\pi\mu c}{12\beta_H^2} + \dots\right)$ appears in the cosh-function only, it does not affect determining the behaviors of functions $f\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right)$ and $g\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right)$, we are free to set $\mu = 0$ in the following steps and the derivations are same as what were done in our previous work [21] for the case of finite temperature CFT. After fixing $f\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right)$ and $g\left(\frac{y}{\beta_H}, \frac{y'}{\beta_H}\right)$, we restore $\mu \neq 0$.

Step 3: When two endpoints of a geodesic coincide, the geodesic length vanishes exactly. Plugging $x = x'$, $y = y'$ and $t = t'$ into Eq. (24), we get

$$f\left(\frac{y}{\beta_H}, \frac{y}{\beta_H}\right) - g\left(\frac{y}{\beta_H}, \frac{y}{\beta_H}\right) = \frac{y^2}{\beta_H^2}. \quad (27)$$

Step 4: In Ref. [29], T. Takayanagi proposed a new holographic dual of the boundary conformal field theory (BCFT). It presents that the phase transitions of entanglement entropy relate to the topological change of the RT surface in the bulk. Based on this realization, the boundary of BCFT will extend into the bulk and play a role in the end-of-the-world (ETW) brane [30]. The brane's tension corresponds to the boundary entropy of BCFT, and the RT surface which is anchored between the ETW in the bulk and the BCFT on the boundary relates to the entanglement entropy of BCFT. The region enclosed by ETW brane in the bulk and BCFT on the boundary is asymptotically AdS. Therefore, the dual RT surfaces of BCFT can be simply calculated between one point in the bulk and another point on the boundary in the AdS background without placing any new configuration, such as virtual branes

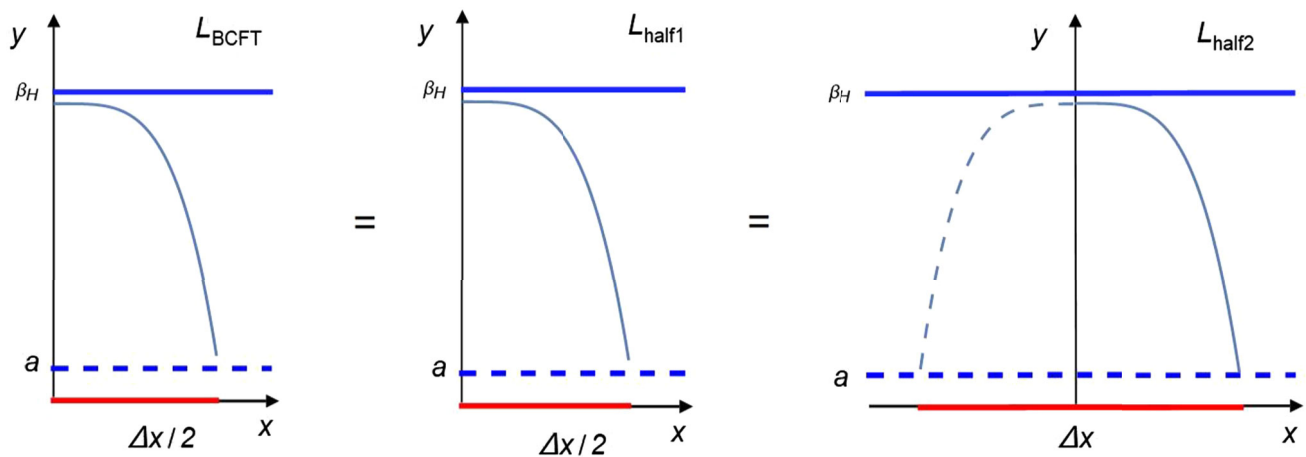


Fig. 1 The left picture is given by BCFT with $\Delta x \rightarrow \infty$. The middle picture is obtained from L_{bulk} by setting $y = a$, $y' = \beta_H$ and $\Delta x/2 \rightarrow \infty$. The right picture is also given by L_{bulk} from a different point of view, by setting $y = y' = a$ and $\Delta x \rightarrow \infty$. The solid lines in all three pictures describe the same object

which modify the bulk geometry [29]. We are going to use this conclusion in this step, since our method only depends on the RT surfaces and aims to fix the leading behaviors of bulk geometry.

We now consider the length of the segment in Fig. 1 as $\Delta x \rightarrow \infty$. There are three ways to calculate it. The left picture is given in [31,32] from boundary conformal field theory (BCFT),

$$L_{\text{BCFT}} = R \log \left[\frac{\beta_H}{a} \exp \left(\frac{\Delta x}{2\beta_H} \right) \right]. \quad (28)$$

On the other hand, by using Eq. (24), we have two other ways to calculate it. The first way is to straightforwardly substitute $y = a$, $y' = \beta_H$, $\Delta x/2 \rightarrow \infty$ into (24) to get

$$L_{\text{bulk}} \rightarrow L_{\text{half1}} = R \log \left(\frac{\beta_H}{a} f \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) \exp \left(\frac{\Delta x}{2\beta_H} \right) \right). \quad (29)$$

It is easy to understand that this length is one half of that connecting $y = y' = a$, $\Delta x \rightarrow \infty$. So the second way is

$$L_{\text{bulk}} \rightarrow L_{\text{half2}} = \frac{1}{2} R \log \left(\frac{\beta_H^2}{a^2} f \left(\frac{a}{\beta_H}, \frac{a}{\beta_H} \right) \exp \left(\frac{\Delta x}{\beta_H} \right) \right). \quad (30)$$

These three lengths (28), (29) and (30) ought to be identical. We thus obtain $f \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right)^2 = f \left(\frac{a}{\beta_H}, \frac{a}{\beta_H} \right) = 1$. The derivation of this constraint does not require $\beta_H \gg a$. As long as β_H is the upper bound of y , the derivation is justified. Since a is a varying cut-off not beyond β_H , satisfying $0 < a/\beta_H \leq 1$, we can safely replace $\frac{a}{\beta_H}$ by $\frac{y}{\beta_H}$ to get:

$$f \left(\frac{y}{\beta_H}, \frac{\beta_H}{\beta_H} \right)^2 = f \left(\frac{y}{\beta_H}, \frac{y}{\beta_H} \right) = 1. \quad (31)$$

Step 5: An important lesson we learned from the free CFT₂ case in [20] is that, in order to completely determine the dual geometry, we need to know the geodesic length between a and β_H with $\Delta x = 0$, i.e. the vertical geodesic. To be consistent, this particular geodesic length must be provided by the CFT₂ entanglement entropy. In the free CFT₂, the IR entanglement entropy precisely fits the requirement.

Remarkably, we know that the finite temperature CFT₂ and the finite size CFT₂ have the same geometry $\mathbb{R} \times \mathbb{S}^1$. We can either interpret it as a CFT on a compact spatial interval of size L_S , or view it as a thermal CFT on the real line with the Euclidean time along the circle with the period $\beta = L_S$, as explained in [31,33] in detail. So these two CFTs are basically the same scenario and have the same bulk dual. We are allowed to use the results from both CFTs to construct the dual bulk geometry with the identification

$$\beta_H = \frac{\beta}{2\pi} \Leftrightarrow \frac{L_S}{2\pi}. \quad (32)$$

Therefore, the geodesic length between a and $\beta_H = \frac{\beta}{2\pi}$ in the finite temperature system can be obtained from the geodesic length connects a and $\frac{L_S}{2\pi}$ in the finite size system,

$$\begin{array}{cc} \text{finite temperature} & \text{finite size} \\ L_{\text{geodesic}} \left(a, \frac{\beta}{2\pi} \right) & \xrightarrow{\beta \leftrightarrow L_S} L_{\text{geodesic}} \left(a, \frac{L_S}{2\pi} \right). \end{array} \quad (33)$$

Noting that $\frac{L_S}{2\pi}$ is the radius of the finite size system with the circumference L_S , therefore, this geodesic goes from boundary to the center of the circle, as illustrated in Fig. 2.

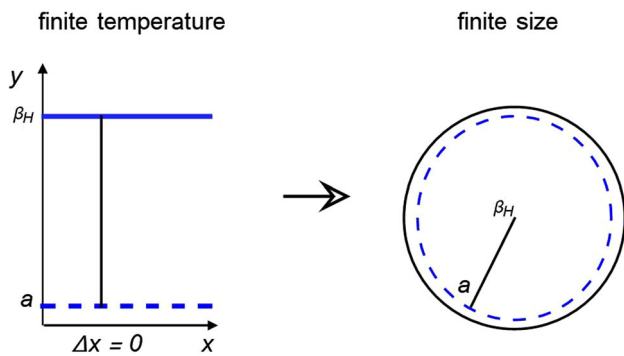


Fig. 2 The geodesic between $y = a$ and $y' = \beta_H$ at a finite temperature system is mapped to a finite size system, corresponding to the radius of a circle from $y = a$ to $y = \frac{L_S}{2\pi}$

We know that the entanglement entropy of a finite size system is²

$$S_{EE} = \frac{c}{3} \log \left(\frac{L_S}{\pi a} \sin \left(\frac{\pi \Delta x}{L_S} \right) \right), \quad (34)$$

The maximal entanglement entropy is achieved by splitting the circle into two equal regions, $\Delta x = L_S/2$. The corresponding geodesic is nothing but a diameter

$$S_{EE} = \frac{c}{3} \log \left(\frac{L_S}{\pi a} \right), \quad L_{\text{boundary}} = 2R \log \left(\frac{L_S}{\pi a} \right). \quad (35)$$

It is then easy to get what we need

$$L_{\text{radius}} = \frac{1}{2} L_{\text{boundary}} = R \log \left(\frac{L_S}{\pi a} \right). \quad (36)$$

We now map $L_S \rightarrow \beta$ to get the geodesic length between a and $\beta_H = \frac{\beta}{2\pi}$ in the finite temperature system:

$$L = R \log \left(\frac{\beta}{\pi a} \right) = R \log \left(\frac{2\beta_H}{a} \right), \quad (37)$$

which agrees with that obtained through holographic duality in [34]. Therefore, from the general expression (24), as $x = x'$, $t = t'$, $y = a$ and $y' = \beta_H$, we have

$$\begin{aligned} L_{\text{boundary}} &= R \log \left(\frac{2\beta_H}{a} \left[f \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) - g \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) \right] \right) \\ &\rightarrow R \log \frac{2\beta_H}{a}. \end{aligned} \quad (38)$$

We thus obtain

$$f \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) - g \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) = 1. \quad (39)$$

² According to [5], the entanglement entropy of a finite size system receives no correction.

For convenience, we summarize all the constraints we have obtained for the general expression (24) of bulk geodesic length:

$$\begin{aligned} f \left(\frac{y}{\beta_H}, \frac{y'}{\beta_H} \right) - g \left(\frac{y}{\beta_H}, \frac{y'}{\beta_H} \right) \\ = \frac{1}{2\beta_H^2} (y^2 + y'^2) + \mathcal{O} \left(\frac{y^4}{\beta_H^4} \right), \quad \beta_H \gg y, y', \end{aligned} \quad (40)$$

$$f \left(\frac{y}{\beta_H}, \frac{\beta_H}{\beta_H} \right)^2 = f \left(\frac{y}{\beta_H}, \frac{y}{\beta_H} \right) = 1, \quad (41)$$

$$f \left(\frac{y}{\beta_H}, \frac{y}{\beta_H} \right) - g \left(\frac{y}{\beta_H}, \frac{y}{\beta_H} \right) = \frac{y^2}{\beta_H^2}, \quad (42)$$

$$f \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) - g \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) = 1. \quad (43)$$

From Eqs. (41) and (43), we get

$$g \left(\frac{a}{\beta_H}, \frac{\beta_H}{\beta_H} \right) = 0. \quad (44)$$

Since a is a varying quantity, y or $y' = \beta_H$ must be a zero of $g(y/\beta_H, y'/\beta_H)$. Moreover, g must be symmetric for y and y' . So, the function form must be

$$g \left(\frac{y}{\beta_H}, \frac{y'}{\beta_H} \right) \propto \left(1 - \frac{y^n}{\beta_H^n} \right)^\kappa \left(1 - \frac{y'^n}{\beta_H^n} \right)^\kappa (\dots) \quad (45)$$

On the other hand, from Eqs. (41) and (42), one gets

$$g \left(\frac{y}{\beta_H}, \frac{y}{\beta_H} \right) = 1 - \frac{y^2}{\beta_H^2} \quad (46)$$

It then easy to fix $n = 2$ and $\kappa = 1/2$ and

$$\begin{aligned} g \left(\frac{y}{\beta_H}, \frac{y'}{\beta_H} \right) &= \sqrt{\left(1 - \left(\frac{y}{\beta_H} \right)^2 \right) \left(1 - \left(\frac{y'}{\beta_H} \right)^2 \right)} \\ &\times \left[1 + \left(\frac{\Delta y}{\beta_H} \right)^2 \left(\sigma_1 + \mathcal{O} \left(\frac{y}{\beta} \right) \right) \right]. \end{aligned} \quad (47)$$

Similarly, from Eq. (41), we get

$$\begin{aligned} f \left(\frac{y}{\beta_H}, \frac{y'}{\beta_H} \right) &= 1 + \left(\frac{\Delta y}{\beta_H} \right)^2 \left(1 - \frac{y^m}{\beta_H^m} \right)^\delta \\ &\times \left(1 - \frac{y'^m}{\beta_H^m} \right)^\delta \left[\theta_1 + \mathcal{O} \left(\frac{y}{\beta} \right) + \dots \right], \end{aligned} \quad (48)$$

where $m, \delta > 0$ are some numbers. The story is not over yet. In order to match Eq. (40), there must be $\sigma_1 = \theta_1$. When applying Eq. (4) to calculate the metric, noting a limit $\Delta x, \Delta y, \Delta t \rightarrow 0$ is going to be imposed after making the

derivatives, it is easy to see that terms proportional to $\left(\frac{\Delta y}{\beta_H}\right)^2$ only contribute to g_{yy} , but not to g_{xx} and g_{tt} . So, looking at Eqs. (24), (47) and (48), without altering the fixed metric, equivalently, we are free to pack all the corrections into $g(y, y')$ and simply set $f(y, y') = 1$. Finally, we substitute $f(y, y')$ and $g(y, y')$ into Eq. (24) to get the bulk geodesic length,

$$\cosh\left(\frac{L_{\text{bulk}}}{R}\right) = \frac{\beta_H^2}{yy'} \left[\cosh\left(\frac{\Delta x}{\beta_H} \left(1 - \frac{\pi\mu c}{12\beta_H^2} + \dots\right)\right) - \sqrt{\left(1 - \left(\frac{y}{\beta_H}\right)^2\right) \left(1 - \left(\frac{y'}{\beta_H}\right)^2\right)} \times \left(1 + \left(\frac{\Delta y}{\beta_H}\right)^2 \cdot \mathcal{O}\left(\frac{y}{\beta}\right)\right) \cosh\left(\frac{\Delta t}{\beta_H}\right) \right]. \quad (49)$$

Using (4), we obtain the deformed BTZ metric,

$$ds^2 = \frac{R_{AdS}^2}{y^2} \left(- \left(1 - \frac{y^2}{\beta_H^2}\right) dt^2 + \left(1 - \frac{\pi\mu c}{12\beta_H^2} + \dots\right) dx^2 + \left(1 - \frac{y^2}{\beta_H^2}\right)^{-1} \left(1 + \mathcal{O}\left(\frac{y}{\beta_H}\right)\right) dy^2 \right). \quad (50)$$

Note the coordinate x and t here are the physical arguments of the $T\bar{T}$ deformed CFT₂. In contrast, in the Refs. [4, 5], the authors started with the standard BTZ metric, and then the boundary coordinates in their works are the physical arguments of the undeformed CFT. To have the deformed CFT living on a conformally flat boundary, from this deformed BTZ metric (50), there ought to be

$$\left(1 - \frac{y_c^2}{\beta_H^2}\right) = \left(1 - \frac{\pi\mu c}{12\beta_H^2} + \dots\right)^2. \quad (51)$$

So the boundary of the dual geometry is located at $y_c^2 = \frac{\pi c \mu}{6} + \dots$, which is finite. Transforming the planar coordinate to the global one by $y = R_{AdS}/r$, we get

$$y_c^2 = \frac{\pi c \mu}{6} + \dots, \quad \text{or} \quad r_c^2 = \frac{6R_{AdS}^4}{\pi c \mu} + \dots, \quad (52)$$

which completely agrees with (2) as conjectured in Ref. [4]. One of the advantages of the deformed metric (50) is that it naturally indicates that the dual geometry is a finite region $y_c \leq y \leq \beta_H$. For simplicity, we rewrite the metric by using Eq. (52) as

$$ds^2 = \frac{R_{AdS}^2}{y^2} \left(- \left(1 - \frac{y^2}{\beta_H^2}\right) dt^2 + \left(1 - \frac{y_c^2}{\beta_H^2}\right) dx^2 + \left(1 - \frac{y^2}{\beta_H^2}\right)^{-1} \times \left(1 + \mathcal{O}\left(\frac{y}{\beta_H}\right)\right) dy^2 \right). \quad (53)$$

It is worth noting that, in a recent development of $T\bar{T}$ deformed CFT [35], the authors obtained a remarkable all-orders result for the entanglement entropy in the Poincare disk at finite μ . That nonperturbative result might be of help to fix the identification in (51) nonperturbatively, and eliminate the higher order corrections in (y/β) of the metric component g_{yy} in Eq. (53). The tricky part is that that nonperturbative result obtained in [35] from the field theory side is $dS_{EE}^\mu/d(\Delta x)$, but not the entanglement entropy itself. When we integrate it to obtain $S_{EE}^\mu(\Delta x)$, the integration constant contains information about the holographic direction y and we must be very careful to explore its role. We wish to study this question in future work.

To study the propagation speed, let us consider the induced metric at some point $y = y_*$:

$$ds^2|_{y=y_*} = \frac{R_{AdS}^2}{y_*^2} \left(- \left(1 - \frac{y_*^2}{\beta_H^2}\right) dt^2 + \left(1 - \frac{y_c^2}{\beta_H^2}\right) dx^2 \right) = \frac{R_{AdS}^2}{y_*^2} (-dt_*^2 + dx_*^2), \quad (54)$$

where we define

$$dt_* = \left(1 - \frac{y_*^2}{\beta_H^2}\right)^{\frac{1}{2}} dt, \quad dx_* = \left(1 - \frac{y_c^2}{\beta_H^2}\right)^{\frac{1}{2}} dx. \quad (55)$$

The right and left propagation speeds of the wave are given by

$$v_{\pm} = \pm \frac{dx_*}{dt_*} = \frac{\left(1 - \frac{y_c^2}{\beta_H^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{y_*^2}{\beta_H^2}\right)^{\frac{1}{2}}}, \quad \mu \geq 0. \quad (56)$$

Considering an observer sitting at $y_c^2 = \frac{\mu\pi c}{6}$ (or $r_c^2 = \frac{6R_{AdS}^4}{\pi c \mu}$), when $0 < y_*^2 < y_c^2 = \frac{\mu\pi c}{6} + \dots$, the speed is less than the speed of light (subluminal). When $y_*^2 > y_c^2 = \frac{\mu\pi c}{6} + \dots$, the speed is superluminal, which again agrees with the results of $T\bar{T}$ deformed CFT [4]. More discussions can be found in [36, 37].

As a further check, we compute the proper energy by using the deformed BTZ metric Eq. (50). Transforming the planar coordinate to the global one, setting $R_{AdS} = 1$ for simplicity, we then have

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)^2} + r^2 d\phi^2, \quad (57)$$

with

$$f(r)^2 = r^2 - 8GM, \quad r_H^2 = 8GM, \quad \phi \sim \phi + 2\pi. \quad (58)$$

Since we already determined the boundary of the geometry at r_c , when calculating the quasi-local gravitational energy, we should integrate along $r = r_c$. Using $\frac{3}{2Gc} = R_{AdS} = 1$, the proper energy [38–40] is

$$\begin{aligned} \mathcal{E} = EL &= \frac{L}{2\pi} \int_0^{2\pi} d\phi \sqrt{g_{\phi\phi}} \mu^i \mu^j T_{ij} \Big|_{r=r_c} \\ &= \frac{\pi r_c^2}{2G} \left[1 - \sqrt{1 - \frac{8GM}{r_c^2}} \right] \\ &= \frac{2}{\mu} \left[1 - \sqrt{1 - 2\pi\mu M} \right]. \end{aligned} \quad (59)$$

where $L = \int d\phi \sqrt{g_{\phi\phi}}|_{r=r_c}$ is the proper size on the boundary, and the non-vanishing components are

$$\mu^t = \frac{1}{f(r)}, \quad T_{tt} = \frac{1}{4G} \left(f(r)^2 - \frac{1}{r} f(r)^3 \right). \quad (60)$$

This energy completely agrees with the energy spectrum of $T\bar{T}$ deformed CFT_2 , given in [4,40].

3 The dual geometry of the highly excited states in CFT_2

For a highly excited state of CFT_2 ,

$$|\psi_h\rangle = \lim_{z \rightarrow 0} z^{\Delta_h} \mathcal{O}_h(z) |0\rangle, \quad (61)$$

where $\Delta_h \sim \mathcal{O}(c)$ is the conformal dimension of a heavy primary operator \mathcal{O}_h , usually it is very difficult to calculate the entanglement entropy. Using the replica trick, the entanglement entropy can be calculated by a four point function, which consists of two heavy operators exciting the states and two light twist operators. This four point function can be expanded in conformal blocks with OPE. The problem is that generically the closed form of the conformal blocks is unknown. However, it is believed that, in order to have a gravity dual, a CFT should have a large central charge and a sparse spectrum of light operators. Remarkably, for such CFTs, the conformal block expansion is well approximated by the identity block and the entanglement entropy of a finite size system is [23–25]

$$S_{EE} = \frac{c}{3} \log \left(\frac{2\beta_\Psi}{a} \sinh \left(\frac{\Delta x}{2\beta_\Psi} \right) \right), \quad (62)$$

with $\beta_\Psi = \left(\sqrt{\frac{12\Delta_h}{c}} - 1 \right)^{-1}$. This entanglement entropy is identical to that of a finite temperature system with $T = \beta_\Psi^{-1}$. As three dimensional gravity has no propagation, the excited states of the dual CFT_2 only leads to local defects and global topologies in the bulk. This is why the entanglement entropies of finite size system, finite temperature system and their excited states take the same form, and their dual metrics are connected by simple transformations. It is then easy to understand that we really only need to consider a single representative of CFTs with distinct topologies.

So, we can safely use the same procedure which we have used in [21] to fix the leading behaviors of the dual geometries of CFT_2 at finite temperature to get the geodesic length:

$$\begin{aligned} \cosh \left(\frac{L_{\text{bulk}}}{R} \right) &= \left[\cosh \left(\frac{\Delta x}{\beta_\Psi} \right) - \sqrt{\left(1 - \left(\frac{y}{\beta_\Psi} \right)^2 \right) \left(1 - \left(\frac{y'}{\beta_\Psi} \right)^2 \right)} \right. \\ &\quad \times \left. \left(1 + \left(\frac{\Delta y}{\beta_\Psi} \right)^2 \cdot \mathcal{O} \left(\frac{y}{\beta_\Psi} \right) \right) \cosh \left(\frac{\Delta t}{\beta_\Psi} \right) \right], \end{aligned} \quad (63)$$

and the metric

$$\begin{aligned} ds^2 &= \frac{R_{AdS}^2}{y^2} \left(- \left(1 - \frac{y^2}{\beta_\Psi^2} \right) dt^2 + \left(1 - \frac{y^2}{\beta_\Psi^2} \right)^{-1} \right. \\ &\quad \times \left. \left(1 + \mathcal{O} \left(\frac{y}{\beta_\Psi} \right) \right) dy^2 + dx^2 \right) \end{aligned} \quad (64)$$

Setting $R_{AdS} = 1$ for simplicity, the metric in globe coordinate system is

$$\begin{aligned} ds^2 &= - \left(r^2 + 1 - \frac{12\Delta_h}{c} \right) dt^2 + \frac{dr^2}{\left(r^2 + 1 - \frac{12\Delta_h}{c} \right)} (1 + \dots) \\ &\quad + r^2 d\phi^2. \end{aligned} \quad (65)$$

When $\Delta_h < \frac{c}{12}$, β_Ψ is imaginary, it is a finite size system with rescaled length. The dual geometry describes a conical defect placed in the center of the global AdS. The defect is caused by the back-reaction of a massive particle. When $\Delta_h > \frac{c}{12}$, the primary state $|\psi_h\rangle$ approximates to a thermal state and the dual excited state in AdS is heavy enough to form a black hole. These conclusions are in consistent with the holographic calculations [24,25].

4 Conclusion

In this paper, we fixed the leading behaviors of the deformed BTZ black hole metric from the entanglement entropy of $T\bar{T}$ deformed CFT_2 . The metric shows explicitly that the dual region $\frac{6R_{AdS}^4}{c\mu} \geq r^2 \geq r_H^2$ is finite in the bulk. We

used the deformed BTZ metric to calculate the propagation speed and energy spectrum. Both results match perfectly with those calculated in the $T\bar{T}$ deformed CFT_2 , and no identification is necessary. We then showed how to get the dual geometry of highly excited states of CFT_2 . The metric describes a conical defect located in the center of the global AdS for $\Delta_h < \frac{c}{12}$, or covers a BTZ black hole at temperature $T = \beta_\Psi^{-1} = \sqrt{\frac{12\Delta_h}{c} - 1}$ as $\Delta_h > \frac{c}{12}$.

Finally, we wish to clarify two significant reasons that why our derivations may look heavy:

1. The advantages of our method is to also cover the time-like direction, i.e. the full spacetime metric, naturally. It is very difficult because we only know the information of the lower dimensional theories.
2. Our purpose is not only to figure out the linear order but also the singularity and event horizon of BTZ spacetime. The behaviors of black hole's singularity and event horizon can not be extracted by the leading term of the spacetime metric directly. This is why we use more results of entanglement entropies, and aim to fix more accurate leading behaviors of bulk geometries.

Moreover, in this paper, our aim is not to explain how bulk geometry is emerged from entanglement entropies, or to derive the bulk dynamics (Einstein's equation) from the boundary theory, but only to show that the entanglement entropies of CFT_2 are enough to fix the leading behaviors of the bulk spacetime geometries.

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