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# Two particle entanglement and its geometric duals

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**Abstract** We show that for a system of two entangled particles, there is a dual description to the particle equations in terms of classical theory of conformally stretched spacetime. We also connect these entangled particle equations with Finsler geometry. We show that this duality translates strongly coupled quantum equations in the pilot-wave limit to weakly coupled geometric equations.

## 1 Introduction

The forces of nature are described by fields which one defines on the spacetime except gravity, which is defined by the spacetime itself as explained by general relativity. In general relativity, physical effects are described elegantly in terms of differential geometry of curved spacetime. All the interactions in GR are completely described by purely geometric equations. In short, GR says that gravity is nothing but a curvature in the spacetime fabric caused by heavy physical objects placed on it. Thus GR, as it stands today, is one of the most successful theories of modern physics.

Quantum mechanics, however, is concerned with a probability interpretation of objects and is a phenomenon which is not very clear and intuitive as compared to the mathematically beautiful GR. Thus it naturally led many physicists to the attempt to reformulate quantum mechanics in a geometric language, like GR. References [1–19] are a few such efforts towards the geometrical rewriting of quantum laws.

It is sometimes useful to develop different mathematical theories describing the same physics. Such an equivalent description of different theories is very helpful at times to avoid certain difficulties; for instance, the well-known AdS/CFT duality [20] provides an equivalence between a specific gravitational theory and a lower dimensional non-gravitational theory, and it is helpful in answering some difficult questions arising on one side of the correspondence by manipulations on the other side. Specifically, the duality is the following:

$$\text{String Theory on } AdS_5 \times S^5 \sim \mathcal{N} = 4\ SU(N) \text{ gauge theory in } 4D.$$

There are also other dualities in string theory which interconnect various string theories and untangle difficulties related to either side.

Concerning a similar approach, various results have been reported on supersymmetric quantum mechanical models and their topological aspects, a few of which are [21–25], since the introduction of a topological index by Witten [26].

In order to reformulate quantum mechanics in a geometric fashion, one needs to associate physical reality to objects and to define the background space. Quantum correlation rests almost entirely on the consideration of non-locality between spatially separated particles. To write quantum mechanics in a geometric way, it is thus important to include determinism in the quantum description. When the wave function is factored using the ansatz [27,28] $\psi = Pe^{\frac{iS}{\hbar}}$ into an amplitude and a phase, the particle momentum can be expressed in terms of the guiding equation $m_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = \nabla_i S$ and writing the Schrödinger equation using this ansatz leads to the following equation of motion:

$$m_i \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\nabla_i (V_i + Q). \tag{1}$$

$P$ and $S$ represent the amplitude and dimensionless phase of the pilot wave, respectively. The added term is the "quantum potential",

$$Q = \sum_{i=1}^{n} \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 |\psi|}{|\psi|}, \tag{2}$$

which includes the position of all the constituent particles. Keeping (2) in mind, one can see from (1) that the dynamics of single particle is specified by the entire system and thus non-locality is inherent in the system. It is actually this term which associates physical reality with the particles.

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In [16] it was shown following the formalism of [27,28] how the relativistic Klein–Gordon equation can be rephrased in a geometric way and how the particle trajectories can be represented by geodesics on a conformally curved $4n$ dimensional configuration space. A similar approach [29] dealt with the non-relativistic limit of the above.

The curious and bizarre phenomenon of quantum entanglement has attracted physicists since long ago. Its principal importance is in its implications for quantum information processing. As mentioned above, there is some work reporting a correspondence between geometry and physics, in view of which it seems plausible that one may find geometric alternatives for quantum entanglement. The purpose of this paper is to present an alternate (geometric) language for quantum entanglement in the pilot wave limit. We will examine the simple case of two particle entanglement in terms of physical trajectories and write the purely physical equations of quantum entanglement by using the factored form of the wave function described above and then translate these physical equations to geometry in two ways: (1) in terms of $1+6$ dimensional conformally rescaled configuration space geometry, and (2) in terms of Finsler geometry. As a reference model, we will consider a system of two identical particles (with spin) in the vicinity of external magnetic field described by the spin incorporated Schrödinger equation. Nonlocality is present in the system through the quantum potential. The choice of the particular model is well suited to envision how an external magnetic field assists a particle's spin to be entangled.

The paper is organized as follows: Sect. 2 is devoted to the model description and derivation of the physics side of four entanglement equations in the pilot-wave limit. In Sect. 3, we present the (configuration space) geometric interpretation of quantum entanglement equations from Sect. 2. In Sect. 4, we extend the Einstein–Hilbert action for Finsler spacetime and show that particle equations from Sect. 2 can also be written in the language of Finsler geometry. The results are briefly summarized in Sect. 5.

## 2 Two particle entanglement

Two particles are said to be entangled if the quantum states of two particles could not be described independently, even if the two particles are at cosmic spatial distance. The state of two entangled particles cannot be separated as product states

$\psi(x_1, x_2, t) \neq \psi(x_1, t)\psi(x_2, t)$.

If one of the particle is in a particular state of spin 'up', the other particle must have spin 'down'. Now we need to incorporate the spin of the particles; the spin incorporated in the Schrödinger equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m_j}\nabla_j^2 - \frac{\hbar^2}{2m_j}\frac{e_j^2}{c^2\hbar^2}A_j^2(x_j) - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right)\psi \tag{3}$$

where $e_j$ is the charge of the $j$th particle, $\mu_j$ is the magnetic moment due to magnetic affects produced by the spin of the particle, $B(x_j)$ is the magnetic field, $A_j(x_j)$ is the magnetic vector potential due to magnetic field $B(x_j)$, $\mathbf{S}_j$ is the spin operator and $S_j$ its eigenvalue: the spin of the particle. The index $j$ represents which one of the two particles is affected. There is no spin–spin coupling because the particles may be spatially separated by large distances and there will be no significant interaction between the magnetic fields generated by the two particles. But the magnetic field of the two particles does have interaction with the applied external magnetic field and they will experience a torque. For this purpose, we need to consider the last term which accounts for the potential energy $\mu.B(x_j)$ of the magnetic moment in the external magnetic field. In double index notation one can rewrite Eq. (3) as

$$\sum_j\left[\frac{\hbar^2}{2m_j}\partial_j^K\partial_{jK} + \frac{e_j^2}{2m_jc^2}A_j^{K2}(x_j) + \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + i\hbar\partial_0\right]\psi = 0 \tag{4}$$

where $j = 1, 2$ and $K$ is the space index in 3 dimensions.

### 2.1 First equation

Following [27,28], the wave function can be factored into an amplitude and a phase,

$$\psi(x_1, x_2, t) = P(x_1, x_2, t)\exp\left(\frac{iS(x_1, x_2, t)}{\hbar}\right). \tag{5}$$

Using (5) in (4) leads to

$$\sum_j\left(\frac{\hbar^2}{2m_j}\partial_j^K\partial_{jK} + \frac{e_j^2}{2m_j}A_j^{K2}(x_j) + \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + i\hbar\partial_0\right) \times P(x_1, x_2, t)\exp\left(\frac{iS(x_1, x_2, t)}{\hbar}\right) = 0. \tag{6}$$

We work in the limit of absolute "time", designated as $t$. In this limit, space and time are not on an equal footing, so the spatial and temporal derivatives will be treated differently. Moreover, $\partial P/\partial t = 0$, as for large $t$ the amplitude on the average is zero. We get (see Appendix A)

$$2m_jQ(x_1, x_2, t) = \sum_j\left[(\partial_j^K S)(\partial_{jK}S) - e_j^2A_j^{K2}(x_j) - 2m_j\frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + 2m_j\partial_0 S\right]. \tag{7}$$

On the L.H.S. of Eq. (7) above, we have used the definition $Q = \sum_j \frac{\hbar^2}{2m_j} \frac{\partial_j^K \partial_{jK} P(t,\vec{x_j})}{P(t,\vec{x_j})}$.

This is the first equation of two particle entanglement. Here $Q(x_1, x_2, t)$ is the quantum potential and $P(x_1, x_2, t)$ is the associated pilot wave. If entanglement is lost, the quantum potential will be the sum of two terms with each term depending on the position of one particle only. Since the quantum potential locks the position of two particles, if one of the two particles has its magnetic moment aligned to the external magnetic field, the other particle will have an anti-aligned magnetic moment. The two particles will then each experience a torque; these torques are in opposite directions (one clockwise and the other counterclockwise). Therefore, we need to consider this specific form of the Schrödinger equation to understand how the spins of the two particles are entangled.

### 2.2 Second equation

The wave function, as defined above, permits one to construct the conserved current as

$$\partial_0(\psi^*\psi) - \sum_{j=1}^{n} \partial_j^m \left( \frac{i\hbar}{2m_j} (\psi^* \overleftrightarrow{\partial}_{jm} \psi) \right) = 0. \tag{8}$$

Using the factored form of the wavefunction from Eq. (5) in Eq. (8) leads to

$$\partial_0(P^2) + \sum_j \partial_{jK} \left( \frac{P^2}{m_j} (\partial_j^K S) \right) = 0. \tag{9}$$

Equation (9) is the second of the entanglement equations, representing the conserved current.

### 2.3 Third equation

In order to characterize the particle trajectories in terms of the pilot wave, one needs to define a guiding equation as follows:

$$\frac{\mathrm{d}x_j^K}{\mathrm{d}t} = \frac{\hbar}{m_j} \operatorname{Im} \left( \frac{\psi, D_j^K \psi}{\psi, \psi} \right) (x_1, x_2) \tag{10}$$

with the following definition of the covariant derivative:

$$D_j^K \psi = \nabla_j P e^{\frac{iS}{\hbar}} - \frac{ie_j}{c\hbar} A_j^K(x_j) P e^{\frac{iS}{\hbar}} \tag{11}$$

and using $\psi$ from Eq. (5) in Eq. (10), we get

$$\frac{\mathrm{d}x_j^K}{\mathrm{d}t} = \frac{(\partial_j^K S) - ie_j A_j^K(x_j)}{m_j}. \tag{12}$$

Momentum is given by

$$\pi_j^K = (\partial_j^K S) - ie_j A_j^K(x_j). \tag{13}$$

Each of the two particles follows a trajectory specified by the other particle (described by the same wave function). In fact, the wave function can geographically stretch over the entire universe. This mysterious interdependence of the wave function interlocks the two particles into a single physical reality.

### 2.4 Fourth equation

The equation of motion for two entangled particles is given by (see Appendix B)

$$m_j \frac{\mathrm{d}^2 x_j^K}{\mathrm{d}t^2} = \partial_{lN} \left[ 2Q + \frac{2\mu_j.B(x_j)\mathbf{S}_j}{S_j} - \frac{ieSF^*}{m_j} - 2\partial_0 S \right] \tag{14}$$

where

$$F^* = \partial_j^K A_l^N + \partial_l^N A_j^K. \tag{15}$$

Therefore, the particle trajectory is specified by the quantum potential $Q(x_1, x_2, t)$ and the combined electric and magnetic effects $[\frac{2\mu_j.B(x_j)\mathbf{S}_j}{S_j} - \frac{ieSF^*}{m_j}]$ and is guided by the pilot wave.

This equation confirms that the two particles are entangled because the motion of each particle is effected by the quantum potential which depends on the position of both particles. This makes entanglement in this approach even more perceptible, because due to nonlocality, the particles are so coupled that the state of one particle is entirely specified by the other particle.

In Eqs. (7), (9), (13) and (14) $P$, $S$ and $Q$ of the two particles depend on $1+6$ dimensions, 6 space dimensions with a single time coordinate. One can define

$$x^L = (t, \vec{x_1}^1, \vec{x_1}^2, \vec{x_1}^3, \vec{x_2}^1, \vec{x_2}^2, \vec{x_2}^3) \tag{16}$$

such that $\partial_j^K, \partial_l^N \to \partial^L$ and $\partial_{jK}, \partial_{lN} \to \partial_L$, with

$$Q = \frac{\hbar^2}{2m_j} \frac{\partial^L \partial_L P(t, \vec{x_j})}{P(t, \vec{x_j})}.$$

All four entanglement equations obtained above can now be written as

$$2m_j Q(x_1, x_2, t) = \Bigg[ (\partial^L S)(\partial_L S) - e_j^2 A^{L2}(x_j) - 2m_j \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + 2m_j \partial_0 S \Bigg], \tag{17}$$

$$\partial_L \left( \frac{P^2(\partial^L S)}{m_j} \right) + \partial_0(P^2) = 0, \tag{18}$$

$$\pi^L = (\partial^L S) - ie_j A^L(x_j), \tag{19}$$

$$m_j \frac{\mathrm{d}^2 x^L}{\mathrm{d}t^2} = \partial_L \left[ 2Q + \frac{2\mu_j.B(x_j)\mathbf{S}_j}{S_j} - 2\partial_0 S - \frac{ieSF^*}{m_j} \right]. \tag{20}$$

Note that in contrast to [16], this model is concerned with spin-1/2 particles and is non-relativistic. In Ref. [16] (concerned with a relativistic model), the geometric theory was developed in $4n$ dimensions, but in our case it is necessary to work with 6 dimensions of space and one of time to picture two particle entanglement. In the non-relativistic limit, $\psi^*\psi$ can be interpreted as a probability density, but it is not possible to provide an interpretation for the probability of the Klein–Gordon equation by $\psi^*\psi$. The probability interpretation of the Klein–Gordon equation is given in terms of the Klein–Gordon current, which is conserved with respect to time. The effects of the external field interaction are absorbed into the field strength tensor defined as

$$F = \partial_j^K A_l^N - \partial_l^N A_j^K$$

but here $\partial_j^K$ is not a four gradient as in [16] and the tensor defines the electromagnetic field in 3 dimensional space. In the following section, we will show that this set of four equations can be written in a geometric way.

## 3 Entanglement equations in terms of $1 + 6$ dimensional configuration space

We consider a $1+6$ dimensional configuration space of two particles with a single time coordinate. The coordinates are defined as

$$\hat{x}^\Lambda = \left(\hat{t}, \hat{\hat{x}}_1^1, \hat{\hat{x}}_1^2, \hat{\hat{x}}_1^3, \hat{\hat{x}}_2^1, \hat{\hat{x}}_2^2, \hat{\hat{x}}_2^3\right). \tag{21}$$

The scalar equation specifying curvature in this setup employs $1+6$ dimensions and is given by

$$P_s(\hat{R} + k\hat{L}_M) = \hat{R} + k\hat{L}_M. \tag{22}$$

Here $P_s$ is symmetrization operator between different particles $x_i^\lambda$ and $x_j^\lambda$, $\hat{R}$ is the Ricci scalar, $\hat{L}_M$ is the matter Lagrangian and $k$ is the coupling constant, which accounts for the matter–field interaction and now with this symmetrization condition, the particle action reads

$$S(\hat{g}_{\Lambda\Delta}) = \int \mathrm{d}t \int \mathrm{d}x^6 \sqrt{|\hat{g}|}(\hat{R} + k\hat{L}_M). \tag{23}$$

The metric $\hat{g}$ is factorized into a conformal function $\phi(\vec{x}_j, t)$ and a flat part $\eta$ [16,29] to describe the local conformal part of the theory. The conformal transformation here is given by

$$\hat{g}_{\Lambda\Gamma} = \phi^{\frac{4}{5}} \eta_{LG}. \tag{24}$$

This rescaling, however, does not change the physics. The inverse of the metric is given by

$$\hat{g}^{\Lambda\Gamma} = \phi^{\frac{-4}{5}} \eta^{LG}. \tag{25}$$

The lower Greek and lower Roman index are identified as $\hat{\partial}_\Lambda = \partial_L$ so that the adjoint derivatives are different in each notation, i.e.,

$$\hat{\partial}^\Lambda = g^{\Lambda\Sigma}\hat{\partial}_\Sigma = \phi^{\frac{-4}{5}}\eta^{LS}\partial_S,$$
$$\hat{\partial}^\Lambda = \phi^{\frac{-4}{5}}\partial^L,$$
$$\hat{\partial}_\Lambda = \phi^{\frac{4}{5}}\partial_L.$$

### 3.1 Geometric dual to the first equation

The particle action in terms of $\phi$ and $g_{LD}$ is

$$S(\phi, g_{LD}) = -\int \mathrm{d}t \int \mathrm{d}x^6 \sqrt{|g|}\left[\frac{24}{5}(\partial^L\phi)(\partial_L\phi) + \phi^2\left(R + k\left(L_M - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right)\right)\right].$$

In this model we are concerned with a flat Minkowski background space so $g_{LG} = \eta_{LG}$ and $|g| = 1$ and $R = 0$. So the action simplifies to

$$S(\phi) = -\int \mathrm{d}t \int \mathrm{d}x^6 \left[\frac{24}{5}(\partial^L\phi)(\partial_L\phi) + \phi^2 k\left(L_M - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right)\right].$$

The equation of motion is

$$-\frac{24}{5}\frac{\partial^L\partial_L\phi}{\phi} = k\left(L_M - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right). \tag{26}$$

The matter Lagrangian $L_M$ is given by

$$L_M = \frac{2(\hat{\partial}^\Lambda S_H - ie_j\hat{A}^\Lambda(x_j))(\hat{\partial}_\Lambda S_H - ie_j\hat{A}_\Lambda(x_j))}{2\hat{M}_G} + \frac{\partial S_H}{\partial t}.$$

The equation of motion with this Lagrangian then gives

$$-2\hat{M}_G\frac{24}{k(5)}\frac{\partial^L\partial_L\phi}{\phi} = 2(\hat{\partial}^\Lambda S_H)(\hat{\partial}_\Lambda S_H) - 2e_j^2\hat{A}^{\Lambda 2}(x_j) - 2\hat{M}_G\frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + 2\hat{M}_G\dot{S}_H. \tag{27}$$

With the matching conditions

$$k = -\frac{24}{5}.\frac{2\hat{M}_G}{\hbar^2},$$
$$\phi(\vec{x}_j, t) = P(\vec{x}_j, t),$$
$$S_H(\vec{x}_j, t) = S(\vec{x}_j, t),$$
$$m_j = \hat{M}_G,$$

we find that Eq. (27) is identical to Eq. (17).

### 3.2 Geometric dual to the second equation

Using conservation of the energy-momentum tensor,

$$\nabla_\Lambda T^{\Lambda\Delta} = 0, \tag{28}$$

we find (see Appendix C)

$$\frac{(\hat{\partial}_\Lambda S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + \nabla_\Lambda(\partial_0 S_H) = 0. \tag{29}$$

The Levi-Civita connection is given by

$$\Gamma^\Sigma_{\Lambda\Delta} = \frac{1}{2} g^{\Sigma\Xi}(\partial_\Lambda g_{\Delta\Xi} + \partial_\Delta g_{\Xi\Lambda} - \partial_\Xi g_{\Lambda\Delta}). \tag{30}$$

Making use of the conformal rescaling of the metric defined above, Eq. (61) leads to

$$\Gamma^\Sigma_{\Lambda\Delta} = \frac{1}{2}\phi^{\frac{-4}{5}}\left[\left(\partial_L \phi^{\frac{4}{5}}\right)\delta^S_D + \left(\partial_D \phi^{\frac{4}{5}}\right)\delta^S_L - \left(\partial^S \phi^{\frac{4}{5}}\right)\eta_{LD}\right]. \tag{31}$$

With this result, Eq. (29) reads as follows.
For the first term in (29)

$$\nabla_\Lambda(\hat{\partial}^\Lambda S_H) = \phi^{\frac{-14}{5}}\partial_L(\phi^2(\partial^L S_H)) = 0. \tag{32}$$

For the second term in (29)

$$\nabla_\Lambda(\partial_0 S_H) = \partial_0(\partial_L S_H). \tag{33}$$

From (29), with the first term and second term as above, we get

$$\left(\frac{\partial_L\left[\phi^2(\partial^L S_H)\right]}{\hat{M}_G} + \partial_0(\phi^2)\right) = 0. \tag{34}$$

With the matching conditions defined above, Eq. (34) is identical to Eq. (18). Note that $\phi(x,t)$ enters the theory in two ways: $\phi^2(x,t)$ gives the probability interpretation and $\phi(x,t)$ accounts for the interaction of matter and field.

### 3.3 Geometric dual to the third equation

The momentum is defined by the derivative of $S_H$ (Hamilton principal function) as suggested by the Hamilton–Jacobi formalism,

$$\hat{\pi}^\Lambda = (\hat{\partial}^\Lambda S_H - i e_j \hat{A}^\Lambda(x_j)). \tag{35}$$

This is identical to the third equation of Eq. (19).

### 3.4 Geometric dual to equation of motion

The geometric dual to the trajectory equation of motion Eq. (20), is the following (see Appendix D):

$$\hat{M}_G \frac{d^2\hat{x}^\Lambda}{d\hat{s}^2} = \hat{\partial}_\Delta\left[-\frac{24}{k(5)}Q + \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} - \frac{ieS_H\hat{F}^*_\Lambda}{\hat{M}_G} - \dot{S}_H\right] \tag{36}$$

where

$$\hat{F}^*_\Lambda = (\hat{\partial}_\Delta \hat{A}_\Lambda + \hat{\partial}_\Lambda \hat{A}_\Delta)$$

where $s = t$. We can see that Eq. (36), with the defined matching conditions, is identical to Eq. (20).
To conclude the above two sections, one must note the following:

- Equations (17)–(20) of two entangled particles have been rephrased in terms of a nonlocal theory. Though the developed geometric theory is intimately related to the local theory of general relativity, the non-locality here is attributed to the extra dimensions. The symmetrization condition too accounts for nonlocal interactions in the geometric theory. This requires the coordinates of each particle to be identical so in order to change one particle's coordinates the changes need to be made simultaneously for the other particle. Let us note that, although the geometric translation of quantum equations follows from the Einstein–Hilbert action, it is still different from GR in the context where space and time are not treated on the same footing and also gravity works in $4D$ spacetime not in $1+6$ dimensional configuration space.
- The dual to the equations of two particle entanglement in geometric theory works with $1+6$ dimensions only. Each particle resides in its own reference frame with 3 dimensions of space and a common temporal dimension.
- Equation (27) and Eqs. (34)–(36) are merely the translation of quantum equations, Eqs. (17)–(20), in geometric language. A set of matching conditions is defined to connect the quantum equations with their geometrical counterparts. These conditions are chosen so as to give the best connecting link between the two theories. These conditions connect the quantum phase $S$ with the Hamilton principal function $S_H$, the amplitude of pilot wave $P$ with the conformal function of the metric $\phi$ and the mass $m_j$ with the mass $\hat{M}_G$. The coupling constant of the geometric theory is

$$k = -\frac{24}{5}.\frac{2\hat{M}_G}{\hbar^2}.$$

Further, note that the particles are strongly coupled to the applied external magnetic field that makes them experience a torque, but the coupling on the geometric side of this duality is weak ($k < 1$). Hence, by means of the matching conditions (to switch between geometry and physics of the two particle entanglement), we arrive at a strong–weak duality—strong on the physics side and weak on the geometric side.

## 4 Entanglement equations in terms of Finsler geometry

Finsler geometry gives insight into a novel approach to discussing the dynamics and geometry of matter–fields. GR works with a geometric background furnished with a $4D$ Lorentzian manifold to put field theories into causal geometrical interpretation. One can, however, expand this geometric background to a non-metric, general length measure background Finsler spacetime [30–32]. The dynamics thus described is compatible with GR.

The Einstein–Hilbert action including the matter–field interaction is given by

$$S[g, \phi_i] = \int_M d^4x\sqrt{|g|}\left(R + kL_M[g, \phi_i]\right). \tag{37}$$

In the limit when we replace

$$\sqrt{|g|} = \sqrt{|g|}\sqrt{|h|}, \tag{38}$$

$$R = R_{ab}, \tag{39}$$

the Einstein–Hilbert action could then be written for Finsler space in terms of a general length measure $F$ (Finsler function) over a tangent bundle $TM$. We consider the sphere $S_P$ to be fibered over each point of the $4D$ spacetime manifold $M$, in the tangent space $T_PM$,

$$S_P = \left[y \in T_PM \mid \sqrt{F_{|P}(y, y)} = 1\right].$$

The Einstein–Hilbert action can then be written as an action on the sphere bundle $\sum$ which is a subset of tangent bundle $TM$ obtained by union over all points as follows:

$$S_P \subset T_PM.$$

Introducing the notion

$$(x^a, \theta^\alpha), \qquad a = 0, 1, 2, 3, \quad \alpha = 1, 2, 3.$$

The resulting Einstein–Hilbert action in Finsler space for the sphere bundle becomes [31]

$$S[F, \phi_i] = \int_\Sigma d^4x d^3\theta\sqrt{|g||h|}\left(\hat{R}_{ab} + k\hat{L}_M[g, \phi_i]\right) \tag{40}$$

where $F$ is the Finsler function. Also note that the coupling constant arising in Eq. (40) is different from that of the GR coupling constant.

Using

$$R_{ab} = g^{ab}R \tag{41}$$

and the conformal transformation,

$$\hat{g}_{\Lambda\Gamma} = \phi^{4/5}(x_j, \Theta)\eta_{LG}, \tag{42}$$

the inverse of the metric is given by

$$\hat{g}^{\Lambda\Gamma} = \phi^{-4/5}(x_j, \Theta)\eta^{LG}. \tag{43}$$

The particle action in terms of $\phi$ and Finsler function $F$ becomes

$$\begin{aligned} S[F, \phi_i] = \int_\Sigma & d^4x d^3\theta\sqrt{|g||h|}\Big[\hat{g}^{ab}\phi^{-23/5} \\ & \times\left(\frac{-24}{5}\partial^L\phi\partial_L\phi + R\phi^2\right) \\ & + k\phi^2\phi^{-14/5}\left(L_M - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right)\Big]; \end{aligned} \tag{44}$$

when we choose $R = 0$,

$$\begin{aligned} S[F, \phi_i] = & \int_\Sigma d^4x d^3\theta\sqrt{|g||h|}\phi^{-14/5} \\ & \times\Big[-\phi^{-13/5}\frac{24}{5}\partial^L\phi\partial_L\phi \\ & + k\phi^2\left(L_M - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right)\Big] \end{aligned} \tag{45}$$

$$\Rightarrow -\phi^{-13/5}\frac{24}{5}\frac{\partial^L\partial_L\phi}{\phi} = k\left(L_M - \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}\right). \tag{46}$$

The matter Lagrangian $L_M$ is given by

$$\begin{aligned} L_M = & \frac{2\left(\hat{\partial}^\Lambda S_H - ie_j\hat{A}^\Lambda(x_j)\right)\left(\hat{\partial}_\Lambda S_H - ie_j\hat{A}_\Lambda(x_j)\right)}{2\hat{M}_G} \\ & + \frac{\partial S_H}{\partial\Theta}. \end{aligned} \tag{47}$$

The equation of motion with this Lagrangian then gives

$$\begin{aligned} & -2\hat{M}_G\phi^{-13/5}\frac{24}{k(5)}\frac{\partial^L\partial_L\phi}{\phi} \\ & = 2(\hat{\partial}^\Lambda S_H)(\hat{\partial}_\Lambda S_H) - 2e_j^2\hat{A}^{\Lambda 2}(x_j) \\ & \quad - 2\hat{M}_G\frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + 2\hat{M}_G\frac{\partial S_H}{\partial\Theta}. \end{aligned} \tag{48}$$

With the matching conditions

$$k=-\phi^{-13/5}\frac{24}{5}.\frac{2\hat{M}_G}{\hbar^2}, \tag{49}$$

$$\phi(\vec{x}_j,\Theta)=P(\vec{x}_j,t), \tag{50}$$

$$S_H(\vec{x}_j,\Theta)=S(\vec{x}_j,t), \tag{51}$$

$$m_j=\hat{M}_G, \tag{52}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\Theta}, \tag{53}$$

where

$$\partial\Theta=\partial\theta_\alpha,\quad \alpha=1,2,3, \tag{54}$$

(48) is identical to (17).

### 4.1 Geometric dual to the second equation

As before, using conservation of the energy-momentum tensor and following the same procedure as for the previous dual (see Appendix C), we obtain

$$\frac{(\hat{\partial}_\Lambda S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G}+\nabla_\Lambda\frac{\partial S_H}{\partial\Theta}=0. \tag{55}$$

Using the definition of Levi-Civita connection (61) in (62), we obtain

$$\frac{(\hat{\partial}_\Lambda S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G}=\phi^{\frac{-14}{5}}\partial_L(\phi^2\partial^L S_H)=0 \tag{56}$$

and

$$\nabla_\Lambda\frac{\partial S_H}{\partial\Theta}=\frac{\partial}{\partial\Theta}(\partial_L S_H). \tag{57}$$

One obtains from (56) and (57)

$$\left(\frac{\partial_L\left[\phi^2(\partial^L S_H)\right]}{\hat{M}_G}+\frac{\partial\phi^2}{\partial\Theta}\right)=0. \tag{58}$$

Equation (58) with the defined matching conditions is identical to Eq. (18)

### 4.2 Geometric dual to the third equation

The particle trajectories are governed by

$$\hat{\pi}^\Lambda=\left(\hat{\partial}^\Lambda S_H-ie_j\hat{A}^\Lambda(x_j)\right). \tag{59}$$

Equation (59) with the defined matching conditions is identical to Eq. (19).

### 4.3 Geometric dual to the equation of motion

The Finsler geometric dual to the trajectory equation of motion Eq. (20) is obtained in a similar way to the earlier case of configuration space geometry (see Appendix D); one finds

$$\hat{M}_G\frac{\mathrm{d}^2\hat{x}^\Lambda}{\mathrm{d}\hat{\Theta}^2} =\hat{\partial}^\Lambda\left[Q+\frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j}-\frac{ieS_H\hat{F}^*_\Lambda}{\hat{M}_G}-\frac{\partial S_H}{\partial\Theta}\right]. \tag{60}$$

This equation describes the dynamics of particle in Finsler spacetime and is identical to Eq. (20) with given matching conditions.

To conclude this section, let us note the following:

- For geometry developed over configuration space, nonlocality is encoded into the theory by means of a quantum potential, a symmetrization condition and extra dimensions. In the Finslerian model, however, the quantum potential is responsible for nonlocality. This potential affects the particles in such a way that it is not possible to isolate one particle from the other, asserting that the two particles are entangled.
- Finsler geometry is developed in 7 dimensions over a Finslerian manifold which is fibered over by a unit sphere at each point. We must note that we could restore gravity from Finsler spacetime if the general length measure (Finsler function $F$) is identical to the metric length or by means of Eqs. (38) and (39).
- The Finsler geometry is connected with the physical equations of two particle entanglement by defining an appropriate set of matching conditions. These conditions connect the quantum phase $S$ with the Hamilton principal function $S_H$, the amplitude of the pilot wave $P$ with the conformal function of the metric $\phi$, the mass $m_j$ with the mass $\hat{M}_G$ and the time coordinate $t$ with polar angle $\Theta$. The coupling required to match the physics side of the duality with (Finsler) geometric side is given by

  $$k=-\phi^{-13/5}\frac{24}{5}.\frac{2\hat{M}_G}{\hbar^2}. \tag{61}$$

  The coupling constant arising in the (Finsler) action in Eq. (40) is the following:

  $$k=\frac{4\pi G}{c^4}.\frac{1}{y^a y^b}. \tag{62}$$

  Note that the coupling constant in Eq. (61) includes $\phi^{-13/5}$, This function accounts for the matter–field interaction and is found to be $\phi^{-13/5}=-\frac{8\pi G}{c^4}.\frac{1}{y^a y^b}\frac{5}{24}.\frac{\hbar^2}{4\hat{M}_G}$. The term $\frac{8\pi G}{c^4}$ is the Einstein constant. Thus, the particles being very light (e.g., electrons) are weakly affected by the gravitational field. Therefore, one can conclude that the strongly coupled quantum equations have a dual geometric description in terms of weakly coupled equations in a Finsler framework.

## 5 Summary

We studied the geometric duality of the equations of two entangled particles with configuration space and Finsler space, respectively. The physics side of the duality constitutes a set of four quantum equations for two entangled particles with spin, in the pilot-wave limit. We presented two types of geometric dualities for this set of four equations. The first geometric description follows from the action (23) where the particles move along geodesics over a $1+6$ dimensional configuration space. The second geometric description follows from the Einstein–Hilbert extended Finsler action (40) over a 7 dimensional manifold.

The constant $\kappa$ specifying the matter–field interaction in both geometric theories is different and that is because the Finsler gravity action includes the Ricci tensor in contrast to the configuration space one. This makes the Hamilton–Jacobi equation (7) slightly different in the two geometric formulations while the other duals appears to be merely a reformulation of (18)–(20) in two different geometries. In each case a suitable set of matching conditions is defined to connect the geometric side (configuration space and Finsler space, respectively) with the equations of quantum entanglement.

The duality presented in this paper is such that one could translate from a strongly coupled quantum theory in the pilot-wave limit, to weakly coupled geometric theories (either a $(1+6)D$ configuration space or Finsler space). The (two entangled) particles have a strong interaction with the external magnetic field, which causes the particles to experience torque. One can deduce from the two (geometric) couplings,

$$k = -\frac{24}{5} \cdot \frac{2\hat{M}_G}{\hbar^2} \tag{63}$$

and

$$k = -\phi^{-13/5}\frac{24}{5} \cdot \frac{2\hat{M}_G}{\hbar^2}, \tag{64}$$

that the strongly coupled quantum equations have a correspondence with weakly coupled configuration space and Finsler space geometric theories.

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## Appendix A: First equation

Starting from Eq. (6)

$$\sum_j \left( \frac{\hbar^2}{2m_j} \partial_j^K \partial_{jK} + \frac{e_j^2}{2m_j} A_j^{K2}(x_j) + \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + i\hbar\partial_0 \right) \times P(x_1, x_2, t) \exp\left( \frac{iS(x_1, x_2, t)}{\hbar} \right) = 0 \tag{65}$$

leads to the following:

$$\sum_j \left[ \hbar^2 \partial_j^K \partial_{jK} P + \frac{i\hbar^2}{\hbar} P \partial_j^K (\partial_{jK} S) \frac{iS}{\hbar} + e_j^2 A_j^{K2}(x_j) P + 2m_j \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} P - 2m_j P \partial_0 S \right] = 0. \tag{66}$$

On average for large $t$, $\partial P/\partial t = 0$. Taking the real part of the above equation after using a Taylor series, we get

$$2m_j \sum_j \frac{\hbar^2}{2m_j} \frac{\partial_j^K \partial_{jK} P}{P} = \sum_j \left[ \left( \partial_j^K S \right) (\partial_{jK} S) - e_j^2 A_j^{K2}(x_j) - 2m_j \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + 2m_j \partial_0 S \right]. \tag{67}$$

Note that the term $\sum_j \frac{\hbar^2}{2m_j} \frac{\partial_j^K \partial_{jK} P}{P}$ on the left hand side of (67) is the formal definition of the quantum potential $Q$. Therefore,

$$2m_j Q(x_1, x_2, t) = \sum_j \left[ \left( \partial_j^K S \right) (\partial_{jK} S) - e_j^2 A_j^{K2}(x_j) - 2m_j \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} + 2m_j \partial_0 S \right]. \tag{68}$$

## Appendix B: Trajectory equation of motion

Using the definition of momenta

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\left( \partial_j^K S \right) - ie_j A_j^K(x_j)}{m_j} \tag{69}$$

and using the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_l \frac{\mathrm{d}}{\mathrm{d}x_l^N} \frac{\mathrm{d}x_l^N}{\mathrm{d}t} \tag{70}$$

one could derive the equation of motion for two entangled particles:

$$\frac{\mathrm{d}^2 x_j^K}{\mathrm{d}t^2} = \sum_l \frac{\mathrm{d}}{\mathrm{d}x_l^N} \frac{\mathrm{d}x_l^N}{\mathrm{d}t} \frac{\left(\partial_j^K S\right) - ie_j A_j^K(x_j)}{m_j}. \tag{71}$$

After some algebra, one obtains

$$m_j^2 \frac{\mathrm{d}^2 x_j^K}{\mathrm{d}t^2} = \sum_l \Big(\partial_{lN} \Big[\left(\partial_l^N S\right)\left(\partial_j^K S\right) - ieSF^* + ie_l A_l^N(x_l) ie_j A_j^K(x_j)\Big]\Big) \tag{72}$$

where the field strength tensor is given by

$$F^* = \partial_j^K A_l^N + \partial_l^N A_j^K. \tag{73}$$

Using single index notation where $\partial_j^K, \partial_l^N \to \partial^L$ we have

$$m_j^2 \frac{\mathrm{d}^2 x^L}{\mathrm{d}t^2} = \left(\partial_L[(\partial^L S)(\partial_L S) - ieSF^* - e^2 A^{L2}(x_l)]\right). \tag{74}$$

Since

$$2m_j \left[Q + \frac{\mu_j . B(x_j)\mathbf{S}_j}{S_j} - \partial_0 S\right] = (\partial^L S)(\partial_L S) - e_j^2 A^{L2}(x_j), \tag{75}$$

the trajectory equation of motion becomes

$$m_j \frac{\mathrm{d}^2 x^L}{\mathrm{d}t^2} = \partial_L \left[2Q + \frac{2\mu_j . B(x_j)\mathbf{S}_j}{S_j} - \frac{ieSF^*}{m_j} - 2\partial_0 S\right]. \tag{76}$$

## Appendix C: Geometric dual to second equation

The stress-energy tensor is given by

$$T^{\Lambda\Delta} = 2\frac{\delta L_M}{\delta g_{\Lambda\Delta}} + g^{\Lambda\Delta} L_M. \tag{77}$$

Substituting for the matter Lagrangian one finds

$$T^{\Lambda\Delta} = \frac{2(\hat{\partial}^\Lambda S_H)(\hat{\partial}^\Delta S_H)}{\hat{M}_G} + g^{\Lambda\Delta}\left(\frac{(\hat{\partial}^\Lambda S_H)(\hat{\partial}_\Lambda S_H)}{\hat{M}_G} + \partial_0 S_H - \frac{e_j^2 \hat{A}^{\Lambda 2}}{\hat{M}_G}\right) \tag{78}$$

where $\partial_0 = \partial/\partial t$, and since the stress-energy tensor is covariantly conserved $\nabla_\Lambda T^{\Lambda\Delta} = 0$, we have

$$\frac{2\nabla_\Lambda(\hat{\partial}^\Lambda S_H)(\hat{\partial}^\Delta S_H)}{\hat{M}_G} + \frac{2(\hat{\partial}^\Lambda S_H) g^{\Lambda\Delta}\nabla_\Lambda(\hat{\partial}_\Lambda S_H)}{\hat{M}_G} + \frac{\nabla_\Lambda g^{\Lambda\Delta}(\hat{\partial}_\Lambda S_H)(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + \frac{g^{\Lambda\Delta}\nabla_\Lambda(\hat{\partial}_\Lambda S_H)(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + \frac{g^{\Lambda\Delta}(\hat{\partial}_\Lambda S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + \nabla_\Lambda g^{\Lambda\Delta}(\partial_0 S_H) + g^{\Lambda\Delta}\nabla_\Lambda(\partial_0 S_H) = 0. \tag{79}$$

The covariant derivative of the metric is taken to be zero, $\nabla_\Lambda g^{\Lambda\Delta} = 0$. Thus, one finds

$$\frac{2(\hat{\partial}^\Delta S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + \frac{2(\hat{\partial}^\Lambda S_H)\nabla^\Delta(\hat{\partial}_\Lambda S_H)}{\hat{M}_G} + \frac{(\hat{\partial}^\Lambda S_H)\nabla^\Delta(\hat{\partial}_\Lambda S_H)}{\hat{M}_G} + \frac{g^{\Lambda\Delta}(\hat{\partial}_\Lambda S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + g^{\Lambda\Delta}\nabla_\Lambda(\partial_0 S_H) = 0. \tag{80}$$

From this, we can write

$$\frac{(\hat{\partial}^\Delta S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} = 0 \tag{81}$$

$$\frac{(\hat{\partial}^\Lambda S_H)\nabla^\Delta(\hat{\partial}_\Lambda S_H)}{\hat{M}_G} = 0 \tag{82}$$

$$\frac{(\hat{\partial}_\Lambda S_H)\nabla_\Lambda(\hat{\partial}^\Lambda S_H)}{\hat{M}_G} + \nabla_\Lambda(\partial_0 S_H) = 0. \tag{83}$$

## Appendix D: Dual to trajectory equation of motion

The total derivative is

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} = \frac{\mathrm{d}\hat{x}^\Lambda}{\mathrm{d}\hat{s}}\hat{\partial}_\Lambda. \tag{84}$$

Using a conformal transformation, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} = \phi^{\frac{4}{5}} \frac{\mathrm{d}x^L}{\mathrm{d}s}\partial, \tag{85}$$

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} = \phi^{\frac{4}{5}} \frac{\mathrm{d}}{\mathrm{d}s}. \tag{86}$$

Applying this relation to the momenta

$$\frac{\mathrm{d}^2\hat{x}^\Lambda}{\mathrm{d}\hat{s}^2} = \frac{\mathrm{d}}{\mathrm{d}\hat{s}} \frac{\left(\hat{\partial}^\Lambda S_H - ie_j\hat{A}^\Lambda(x_j)\right)}{\hat{M}_G} \tag{87}$$

and using the identity

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} = \frac{\mathrm{d}}{\mathrm{d}\hat{x}^\Lambda}\frac{\mathrm{d}\hat{x}^\Lambda}{\mathrm{d}\hat{s}} \tag{88}$$

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} = \hat{\partial}_\Lambda \frac{\mathrm{d}\hat{x}^\Lambda}{\mathrm{d}\hat{s}} \tag{89}$$

one could derive the trajectory equation of motion:

$$\frac{\mathrm{d}^2\hat{x}^\Lambda}{\mathrm{d}\hat{s}^2} = \hat{\partial}_\Delta \frac{\mathrm{d}\hat{x}^\Delta}{\mathrm{d}\hat{s}} \frac{\left(\hat{\partial}^\Lambda S_H - ie_j \hat{A}^\Lambda(x_j)\right)}{\hat{M}_G}. \quad (90)$$

Using the definition of the momentum and after doing some algebra, (90) yields

$$\hat{M}_G^2 \frac{\mathrm{d}^2\hat{x}^\Lambda}{\mathrm{d}\hat{s}^2} = \hat{\partial}_\Delta\big[(\hat{\partial}^\Delta S_H)(\hat{\partial}^\Lambda S_H) - ie_j S_H \hat{F}^{\Delta *} + ie_j \hat{A}^\Delta(x_j) ie_j \hat{A}^\Lambda(x_j)\big] \quad (91)$$

with the field strength tensor given by

$$\hat{F}^{\Delta *} = (\hat{\partial}^\Delta \hat{A}^\Lambda + \hat{\partial}^\Lambda \hat{A}^\Delta). \quad (92)$$

Changing the index $\Delta$ in (91) inside the parentheses using the metric gives

$$\hat{M}_G^2 \frac{\mathrm{d}^2\hat{x}^\Lambda}{\mathrm{d}\hat{s}^2} = \hat{\partial}^\Lambda\big[(\hat{\partial}_\Lambda S_H)(\hat{\partial}^\Lambda S_H) - ie_j S_H \hat{F}^*_\Lambda - e_j^2 \hat{A}_\Lambda \hat{A}^\Lambda(x_j)\big]. \quad (93)$$

Substituting

$$(\hat{\partial}_\Lambda S_H)(\hat{\partial}^\Lambda S_H) - e_j^2 \hat{A}^{\Lambda 2}(x_j) = M_G\left[-\frac{24}{k(5)}\frac{\partial^L\partial_L\phi}{\phi} + \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} - \dot{S}_H\right] \quad (94)$$

in (93) gives the equation of motion,

$$\hat{M}_G^2 \frac{\mathrm{d}^2\hat{x}^\Lambda}{\mathrm{d}\hat{s}^2} = \hat{\partial}^\Lambda\left[M_G\left[-\frac{24}{k(5)}Q + \frac{\mu_j.B(x_j)\mathbf{S}_j}{S_j} - \dot{S}_H\right] - ieS_H\hat{F}^*_\Lambda\right]. \quad (95)$$

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