# Delocalized SYZ mirrors and confronting top-down $S U(3)$-structure holographic meson masses at finite $g$ and $N_{c}$ with $\mathbf{P}($ article) $\mathbf{D}$ (ata) $\mathbf{G}$ (roup) values 

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#### Abstract

Meson spectroscopy at finite gauge coupling whereat any perturbative QCD computation would break down - and finite number of colors, from a top-down holographic string model, has thus far been entirely missing in the literature. This paper fills this gap. Using the delocalized type IIA SYZ mirror (with $S U(3)$ structure) of the holographic type IIB dual of large- $N$ thermal QCD of Mia et al. (Nucl Phys B 839:187. arXiv:0902.1540 [hep-th], 2010) as constructed in Dhuria and Misra (JHEP 1311:001. arXiv:1306.4339 [hep-th], 2013) at finite coupling and number of colors ( $N_{c}=$ number of $D 5(\overline{D 5})$-branes wrapping a vanishing two-cycle in the top-down holographic construct of Mia et al. (Nucl Phys B 839:187. arXiv:0902.1540 [hepth] , 2010) $=\mathcal{O}(1)$ in the IR in the MQGP limit of Dhuria and Misra (JHEP 1311:001. arXiv:1306.4339 [hep-th], 2013) at the end of a Seiberg-duality cascade), we obtain analytical (not just numerical) expressions for the vector and scalar meson spectra and compare our results with previous calculations of Sakai and Sugimoto (Prog Theor Phys 113:843. doi:10.1143/PTP.113.843. arXiv:hep-th/0412141, 2005) and Dasgupta et al. (JHEP 1507:122. doi:10.1007/ JHEP07(2015)122. arXiv:1409.0559 [hep-th], 2015), and we obtain a closer match with the Particle Data Group (PDG) results of Olive et al. (Particle Data Group) (Chin Phys C 38:090001, 2014). Through explicit computations, we verify that the vector and scalar meson spectra obtained by the gravity dual with a black hole for all temperatures (small and large) are nearly isospectral with the spectra obtained by a thermal gravity dual valid for only low temperatures; the isospectrality is much closer for vector mesons than scalar mesons. The black-hole gravity dual (with a horizon


[^0]radius smaller than the deconfinement scale) also provides the expected large- $N$ suppressed decrease in vector meson mass with increase of temperature.

## 1 Introduction

The AdS/CFT [6] correspondence and its non-conformal generalizations conjecture the equivalence between string theory on a ten-dimensional space-time and gauge theory living on the boundary of space-time. A generalization of the AdS/CFT correspondence is necessary to explore more realistic theories (less supersymmetric and non-conformal) such as QCD. The original AdS/CFT conjecture [6] proposed a duality between maximally supersymmetric $\mathcal{N}=$ $4 S U(N)$ SYM gauge theory and type IIB supergravity on $\mathrm{AdS}_{5} \times S^{5}$ in the low energy limit. Different generalized versions of the AdS/CFT have thus far been proposed to study non-supersymmetric field theories. One way of constructing gauge theories with less supersymmetry is to consider stacks of Dp branes at the singular tip of a Calabi-Yau cone. In this paper we use a large- $N$ top-down holographic dual of QCD [1] to obtain the meson spectrum from type IIA perspective. Embedding of additional $D$-branes (flavor branes) in the near-horizon limit gives rise to a modification of the original AdS/CFT correspondence which involves field theory degrees of freedom that transform in the fundamental representation of the gauge group. This is useful for describing field theories like QCD, where quark fields transform in the fundamental representation. Mesons operator or a gauge invariant bilinear operator corresponds to the bound state of anti-fundamental and fundamental field.

In the past decade, (glueballs and) mesons have been studied extensively to gain new insight into the non-perturbative regime of QCD. Various holographic set-ups such as soft-
wall model, hard-wall model, modified soft-wall model, etc. have been used to obtain the glueballs' and mesons' spectra and obtain interaction between them. In the following two paragraphs a brief summary of the work is given that has been done in past decades.

Most of existing literature on holographic meson spectroscopy is of the bottom-up variety based often on soft/hardwall AdS/QCD models. Here is a short summary of some of the relevant work. Soft-wall holographic QCD model was used in $[7,8]$ to obtain spectrum and decay constants for $1^{-+}$hybrid mesons and to study the scalar glueballs and scalar mesons at $T \neq 0$, respectively. In [7] no states with exotic quantum numbers were observed in the heavy quark sector. Comparison of the computed mass with the experimental mass of the $1^{-+}$candidates $\pi_{1}(1400), \pi_{1}(1600)$ and $\pi_{1}(2015)$, favored $\pi_{1}(1400)$ as the lightest hybrid state. In [9] an IR-improved soft-wall AdS/QCD model in good agreement with linear confinement and chiral symmetry breaking was constructed to study the mesonic spectrum. The model was constructed to rectify inconsistencies associated with both simple soft-wall and hard-wall models. The hardwall model gave a good realization for the chiral symmetry breaking, but the mass spectra obtained for the excited mesons did not match the experimental data well. The softwall model with a quadratic dilaton background field showed the Regge behavior for excited vector mesons but chiral symmetry breaking phenomena cannot be realized consistently in the simple soft-wall AdS/QCD model. A hard-wall holographic model of QCD was used in [10-12] to analyze the mesons. In [13] a two-flavor quenched dynamical holographic $\mathrm{QCD}(\mathrm{hQCD})$ model was constructed in the graviton-dilaton framework by adding two light flavors. In [3] the mesonic spectrum was obtained for a $D 4 / D 8(-\overline{D 8})$ brane configuration in type IIA string theory; in [14] massive excited states in the open string spectrum were used to obtain the spectrum for higher spin mesons $J \geq 2$. NLO terms were obtained by taking into account the effect of the curved background perturbatively, which led to corrections in formula $J=\alpha_{0}+\alpha^{\prime} M^{2}$. The results obtained for the meson spectrum were compared with the experimental data to identify $a_{2}(1320), b_{1}(1235), \pi(1300), a_{0}$ etc. first and second excited states. In [15] a holographic model was constructed with extremal $N_{c} D 4$-branes and $D 6$-flavor branes in the probe approximation. The model gave a good approximation for Regge behavior of glueballs but failed to explain mesonic spinning strings because the dual theory did not include quarks in the fundamental representations.

To our knowledge, the only top-down holographic dual of large- $N$ thermal QCD which is IR confining, UV conformal and UV-complete (e.g. the holographic Sakai-Sugimoto model [3] does not address the UV) with fundamental quarks is the one given in [1] involving $N$ D3-branes, $M D 5 /(\overline{D 5})$ branes wrapping a vanishing two-cycle and $N_{f} D 7(\overline{D 7})$ fla-
vor branes in a warped resolved conifold at finite temperature in the brane picture (and $M \overline{D 5}$-branes and $D 7(\overline{D 7})$-branes with a black hole and fluxes in a resolved warped deformed conifold gravitational dual). In [4], the authors (some also participating in [1]) obtained the vector and scalar mesonic spectra by taking a single T dual of the holographic type IIB background of [1]. Comparison of the (pseudo-)vector mesons with PDG results, provided a reasonable agreement. One of the main objectives of our work is to see if by taking a mirror of the type IIB background of [1] via delocalized Strominger-Yau-Zaslow's triple-T duality prescription - a new tool in this field - at finite gauge coupling and with finite number of colors - a new limit and one which is closest to realistic strongly coupled thermal QCD - one can obtain a better agreement between the mesonic spectra so obtained and PDG results than previously obtained in [3,4], and in the process gain new insights into a holographic understanding of thermal QCD.

In [16], we initiated top-down $G$-structure holographic large- $N$ thermal QCD phenomenology at finite gauge coupling and finite number of colors, in particular from the vantage point of the M-theory uplift of the delocalized SYZ type IIA mirror of the top-down UV-complete holographic dual of large- $N$ thermal QCD of [1], as constructed in [2]. We calculated up to $(\mathrm{N}) \mathrm{LO}$ in $N$, masses of $0^{++}, 0^{--}, 0^{-+}, 1^{++}$ and $2^{++}$glueballs, and we found very good agreement with some of the lattice results on the same. In this paper, we continue exploring top-down $G$-structure holographic large$N$ thermal QCD phenomenology at finite gauge coupling by evaluating the spectra of (pseudo-)vector and (pseudo)scalar mesons, and in particular comparing their ratios for both types with P (article) D (ata) G (roup) results.

The rest of the paper is organized as follows. In Sect. 2, via four sub-sections, we briefly review a UV-complete topdown type IIB holographic dual of large- $N$ thermal QCD (Sect. 2.1) as given in [1] and its M-theory uplift in the 'MQGP limit' as worked out in [2] (Sect. 2.2); Sect. 2.3 presents a discussion of the construction in [2] of the delocalized Strominger-Yau-Zaslow (SYZ) type IIA mirror of the aforementioned type IIB background of [1] and Sect. 2.4 has a brief review of $S U(3)$ and $G_{2}$ structures relevant to [1] (type IIB) and [2,17] (type IIA and M theory). Section 3 is on the construction of the embedding of $D 6$-branes via delocalized SYZ type IIA mirror of the embedding of D7branes of type IIB. Via five sub-sections, Sect. 4 is on obtaining the (pseudo-)vector meson spectra in the framework of [2] at finite coupling assuming a black-hole gravity dual for all temperatures, small and large. The (pseudo-)vector mesons correspond to gauge fluctuations about a background gauge field along the world volume of the $D 6$ branes. Unlike [4], the gravity dual involves a black hole $\left(r_{h} \neq 0\right)$ and consequently, while factorizing the gauge fluctuations along $\mathbb{R}^{3} \times \mathbb{S}^{1}$-radial direction into fluctuations along $\mathbb{R}^{3} \times \mathbb{S}^{1}$ and
eigenmode fluctuations along the radial direction, there are two types of eigenmodes along the radial direction - one denoted by $\alpha_{n}^{\{i\}}(Z)$, which is coupled to gauge fluctuations along the space-like $\mathbb{R}^{3}$, and the other denoted by $\alpha_{n}^{\{0\}}(Z)$, which is coupled to the compact time-like $\mathbb{S}^{1}$ (metric along which includes the black-hole function). After obtaining the EOMs for $\alpha_{n}^{\{i\}}$ and $\alpha_{n}^{\{0\}}$, the following is the outline of what is done in Sects. 4.1-4.5. First, in Sect. 4.1, assuming an IR-valued vector meson spectrum, the same is obtained by solving the EOMs near the horizon. Next, by converting the EOMs to a Schrödinger-like EOMs, the vector meson spectra are worked out for $\alpha_{n}^{\{i\}}$ eigenmodes in Sect. 4.2 (in the IR limit in Sect. 4.2.1 and the UV limit in Sect. 4.2.2) and $\alpha_{n}^{\{0\}}$ eigenmodes in Sect. 4.3 (in the IR limit in Sect. 4.3.1 and the UV limit in Sect. 4.3.2). Finally, using the WKB quantization prescription, the vector meson spectrum corresponding to the $\alpha_{n}^{\{i\}}$ eigenmodes is worked out (in the smalland large-mass limits) in Sect. 4.4, and that corresponding to the $\alpha_{n}^{\{0\}}$ eigenmodes (in the small- and large-mass limits) in Sect. 4.5. In Sect. 5, we obtain the scalar meson spectrum by considering fluctuations of the $D 6$-branes orthogonal to their world volume in the absence of any background gauge fields in a black-hole background for all temperatures, small and large. In the same vein as vector meson spectroscopy, after obtaining the EOM for the radial eigenfunction mode, the following is an outline of what is done in Sect. 5, devoted to scalar meson spectroscopy. First, in Sect. 5.1, assuming an IR-valued scalar meson spectrum, the same is obtained by solving the EOMs near the horizon. Next, by converting the EOMs to a Schrödinger-like EOMs, the scalar meson spectrum is worked out for in Sect. 5.2 (in the IR limit in Sect. 5.2.1 and the UV limit in Sect. 5.2.2). Finally, using the WKB quantization prescription, the scalar meson spectrum is worked out (in the small- and large-mass limits) in Sect. 5.3. In Sect. 6, we obtain the (pseudo-)vector meson spectrum in Sect. 6.1 (and the three sub-sub-sections therein) and (pseudo-)scalar meson spectrum in Sect. 6.2 using a thermal background, and hence verify that the mesonic spectra of Sects. 4 and 5 are nearly isospectral with Sect. 6. Section 7 presents a discussion of the new insights and results obtained in this work and some future directions. There are three supplementary appendices.

## 2 Background: a top-down type IIB holographic large- $N$ thermal QCD and its M-theory uplift in the 'MQGP' limit

Via four sub-sections, in this section, we will:

- provide a short review of the type IIB background of [1], a UV-complete holographic dual of large- $N$ thermal QCD, in Sect. 2.1,
- discuss the 'MQGP' limit of [2] and the motivation for considering this limit in Sect. 2.2,
- briefly review issues as discussed in [2,17-19], pertaining to construction of a delocalized S (trominger)-$\mathrm{Y}(\mathrm{au})-\mathrm{Z}$ (aslow) mirror and approximate supersymmetry, in Sect. 2.3,
- briefly review the new results of $[17,19]$ pertaining to construction of explicit $S U(3)$ and $G_{2}$ structures respectively of type IIB/IIA, and M-theory uplift.


### 2.1 Type IIB dual of large- $N$ thermal QCD

In this subsection, we will discuss a UV-complete holographic dual of large- $N$ thermal QCD as given in Dasgupta and Mia et al. [1]. In order to include fundamental quarks at non-zero temperature in the context of type IIB string theory, [1] considered at finite temperature, $N$ D3-branes, $M D 5$-branes wrapping a vanishing two-cycle and $M \overline{D 5}$ branes distributed along a resolved two-cycle and placed at the outer boundary of the IR-UV interpolating region/inner boundary of the UV region. The $D 5 / \overline{D 5}$ separation is given by $\mathcal{R}_{D 5 / \overline{D 5}}$. The radial space, in [1] is divided into the IR, the IR-UV interpolating region and the UV. $N_{f} D 7$-branes, via Ouyang embedding, are holomorphically embedded in the UV, the IR-UV interpolating region and dipping into the (confining) IR (up to a certain minimum value of $r$ corresponding to the lightest quark) and $N_{f} \overline{D 7}$-branes present in the UV and the UV-IR interpolating (not the confining IR). This is to ensure turning off of three-form fluxes. The resultant ten-dimensional geometry is given by a resolved warped deformed conifold. In the gravity dual D3-branes and the $D 5$-branes are replaced by fluxes in the IR. The finite temperature resolves ${ }^{1}$ and IR confinement deforms the conifold. Back-reactions are included in the warp factor and fluxes.

One has $S U(N+M) \times S U(N+M)$ color gauge group and $S U\left(N_{f}\right) \times S U\left(N_{f}\right)$ flavor gauge group, in the UV. It is expected that there will be a partial Higgsing of $S U(N+M) \times S U(N+M)$ to $S U(N+M) \times S U(N)$ at $r=\mathcal{R}_{D 5 / \overline{D 5}}$ [21]. The two gauge couplings, $g_{S U(N+M)}$ and $g_{S U(N)}$ flow logarithmically and oppositely in the IR: $4 \pi^{2}\left(\frac{1}{g_{S U(N+M)}^{2}}+\frac{1}{g_{S U(N)}^{2}}\right) e^{\phi} \sim \pi ; 4 \pi^{2}\left(\frac{1}{g_{S U(N+M)}^{2}}-\frac{1}{g_{S U(N)}^{2}}\right)$ $e^{\phi} \sim \frac{1}{2 \pi \alpha^{\prime}} \int_{S^{2}} B_{2}$. Had it not been for $\int_{S^{2}} B_{2}$, in the UV, one could have set $g_{S U(M+N)}^{2}=g_{S U(N)}^{2}=g_{Y M}^{2} \sim g_{s} \equiv$ constant (implying conformality), which is the reason for inclusion of $M \overline{D 5}$-branes at the common boundary of the UV-IR interpolating and the UV regions, to annul this contribution. In fact, the running also receives a contri-

[^1]bution from the $N_{f}$ flavor $D 7$-branes which needs to be annulled via $N_{f} \overline{D 7}$-branes. Under an NVSZ RG flow, the gauge coupling $g_{S U(N+M)}$ - having a larger rank - flows towards strong coupling and the $S U(N)$ gauge coupling flows towards weak coupling. Upon application of Seiberg duality, $S U(N+M)_{\text {strong }} \xrightarrow{\text { Seiberg Dual }} S U\left(N-\left(M-N_{f}\right)\right)_{\text {weak }}$ in the IR; assuming after duality cascade, $N$ decreases to 0 and there is a finite $M$, one will be left with $S U(M)$ gauge theory with $N_{f}$ flavors that confines in the IR - the finite temperature version of this was addressed by [1].

So, in the IR, at the end of the duality cascade, number of colors $N_{c}$ is identified with $M$, which in the 'MQGP limit' can be tuned to equal 3 . One can identify $N_{c}$ with $N_{\text {eff }}(r)+M_{\text {eff }}(r)$, where $N_{\text {eff }}(r)=$ $\int_{\text {Base of Resolved Warped Deformed Conifold }} F_{5}$ and $M_{\text {eff }}=\int_{S^{3}} \tilde{F}_{3}$ (the $S^{3}$ being dual to $e_{\psi} \wedge\left(\sin \theta_{1} \mathrm{~d} \theta_{1} \wedge \mathrm{~d} \phi_{1}-B_{1} \sin \theta_{2} \wedge\right.$ $\mathrm{d} \phi_{2}$ ), wherein $B_{1}$ is an asymmetry factor defined in [1], and $\left.e_{\psi} \equiv \mathrm{d} \psi+\cos \theta_{1} \mathrm{~d} \phi_{1}+\cos \theta_{2} \mathrm{~d} \phi_{2}\right)$ where $\tilde{F}_{3}(\equiv$ $\left.F_{3}-\tau H_{3}\right) \propto M(r) \equiv \frac{1}{1+e^{\alpha\left(r-\mathcal{R}_{D 5 / \overline{D 5}}\right)}}, \alpha \gg 1$ [22]. The number of colors $N_{c}$ varies between $M$ in the deep IR and a large value [even in the MQGP limit of (10) (for a large value of $N)$ ] in the UV. Hence, at very low energies, the number of colors $N_{c}$ can be approximated by $M$, which in the MQGP limit is taken to be finite and can hence be taken to be equal to three. In [1], the effective number of $D 3$-branes, $D 5$-branes wrapping the vanishing two-cycle and the flavor $D 7$-branes, denoted, respectively, by $N_{\text {eff }}(r), M_{\text {eff }}(r)$ and $N_{f}^{\text {eff }}(r)$, are given as
$N_{\mathrm{eff}}(r)=N\left[1+\frac{3 g_{s} M_{\mathrm{eff}}^{2}}{2 \pi N}\left(\log r+\frac{3 g_{s} N_{f}^{\mathrm{eff}}}{2 \pi}(\log r)^{2}\right)\right]$,
$M_{\mathrm{eff}}(r)=M+\frac{3 g_{s} N_{f} M}{2 \pi} \log r+\sum_{m \geq 1} \sum_{n \geq 1} N_{f}^{m} M^{n} f_{m n}(r)$,
$N_{f}^{\mathrm{eff}}(r)=N_{f}+\sum_{m \geq 1} \sum_{n \geq 0} N_{f}^{m} M^{n} g_{m n}(r)$.
It was argued in [17] that the length scale of the OKS-BH metric in the IR, after Seiberg duality cascading away almost the whole of $N_{\text {eff }}$, will be given by

$$
\begin{aligned}
& L_{\mathrm{OKS}-\mathrm{BH}} \sim \sqrt{M} N_{f}^{\frac{3}{4}} \sqrt{\left(\sum_{m \geq 0} \sum_{n \geq 0} N_{f}^{m} M^{n} f_{m n}(\Lambda)\right)} \\
& \quad \times\left(\sum_{l \geq 0} \sum_{p \geq 0} N_{f}^{l} M^{p} g_{l p}(\Lambda)\right)^{\frac{1}{4}} g_{s}^{\frac{1}{4}} \sqrt{\alpha^{\prime}} \\
& \equiv N_{f}^{\frac{3}{4}} \sqrt{\left(\sum_{m \geq 0} \sum_{n \geq 0} N_{f}^{m} M^{n} f_{m n}(\Lambda)\right)}
\end{aligned}
$$

$$
\begin{equation*}
\times\left.\left(\sum_{l \geq 0} \sum_{p \geq 0} N_{f}^{l} M^{p} g_{l p}(\Lambda)\right)^{\frac{1}{4}} L_{\mathrm{KS}}\right|_{\Lambda: \log \Lambda<\frac{2 \pi}{3 g s_{f}}} \tag{2}
\end{equation*}
$$

which implies that in the IR, relative to KS, there is a colorflavor enhancement of the length scale in the OKS-BH metric. Hence, in the IR, even for $N_{c}^{\mathrm{IR}}=M=3$ and $N_{f}=2$ (light flavors) upon inclusion of $n, m>1$ terms in $M_{\text {eff }}$ and $N_{f}^{\text {eff }}$ in (1), $L_{\mathrm{OKS}}$-BH $\gg L_{\mathrm{KS}}\left(\sim L_{\text {Planck }}\right)$ in the MQGP limit involving $g_{s} \stackrel{\sim}{<} 1$, implying that the stringy corrections are suppressed and one can trust supergravity calculations. Further, the global flavor group $S U\left(N_{f}\right) \times S U\left(N_{f}\right)$, is broken in the IR to $S U\left(N_{f}\right)$ as the IR has only $N_{f} D 7$-branes.

Hence, the type IIB model of [1] makes it an ideal holographic dual of thermal QCD because, it is UV conformal and IR confining with required chiral symmetry breaking in the IR. The quarks present in the theory transform in the fundamental representation, plus theory is defined for the full range of temperature both low and high.
(d) Supergravity solution on resolved warped deformed conifold

The metric in the gravity dual of the resolved warped deformed conifold with $g_{i}$ 's: $g_{1,2}\left(r, \theta_{1}, \theta_{2}\right)=1-\frac{r_{h}^{4}}{r^{4}}+$ $\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right)$ is given by

$$
\begin{align*}
\mathrm{d} s^{2}= & \frac{1}{\sqrt{h}}\left(-g_{1} \mathrm{~d} t^{2}+\mathrm{d} x_{1}^{2}+\mathrm{d} x_{2}^{2}+\mathrm{d} x_{3}^{2}\right) \\
& +\sqrt{h}\left[g_{2}^{-1} \mathrm{~d} r^{2}+r^{2} \mathrm{~d} \mathcal{M}_{5}^{2}\right] \tag{3}
\end{align*}
$$

The compact five-dimensional metric in (3) is given as

$$
\begin{align*}
& \mathrm{d} \mathcal{M}_{5}^{2}=h_{1}\left(\mathrm{~d} \psi+\cos \theta_{1} \mathrm{~d} \phi_{1}+\cos \theta_{2} \mathrm{~d} \phi_{2}\right)^{2} \\
& \quad+h_{2}\left(\mathrm{~d} \theta_{1}^{2}+\sin ^{2} \theta_{1} \mathrm{~d} \phi_{1}^{2}\right)+h_{4}\left(h_{3} \mathrm{~d} \theta_{2}^{2}+\sin ^{2} \theta_{2} \mathrm{~d} \phi_{2}^{2}\right) \\
& \quad+h_{5}\left[\cos \psi\left(\mathrm{~d} \theta_{1} \mathrm{~d} \theta_{2}-\sin \theta_{1} \sin \theta_{2} \mathrm{~d} \phi_{1} \mathrm{~d} \phi_{2}\right)\right. \\
& \left.\quad+\sin \psi\left(\sin \theta_{1} \mathrm{~d} \theta_{2} \mathrm{~d} \phi_{1}+\sin \theta_{2} \mathrm{~d} \theta_{1} \mathrm{~d} \phi_{2}\right)\right] \tag{4}
\end{align*}
$$

wherein we will assume $r \gg a, h_{5} \sim \frac{\text { (deformation parameter) }^{2}}{r^{3}} \ll$ 1 for $r \gg$ (deformation parameter) ${ }^{\frac{2}{3}}$. The $h_{i}$ appearing in the internal metric up to linear order depend on $g_{s}, M, N_{f}$ are given as
$h_{1}=\frac{1}{9}+\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right)$,
$h_{2}=\frac{1}{6}+\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right), \quad h_{4}=h_{2}+\frac{a^{2}}{r^{2}}$,
$h_{3}=1+\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right), \quad h_{5} \neq 0, \quad L=\left(4 \pi g_{s} N\right)^{\frac{1}{4}}$.
One sees from (4) and (5) that one has a non-extremal resolved warped deformed conifold involving an $S^{2}$-blowup (as $h_{4}-h_{2}=\frac{a^{2}}{r^{2}}$ ), an $S^{3}$-blowup (as $h_{5} \neq 0$ ) and squashing of
an $S^{2}$ (as $h_{3}$ is not strictly unity). The horizon (being at a finite $r=r_{h}$ ) is warped squashed $S^{2} \times S^{3}$. In the deep IR, in principle one ends up with a warped squashed $S^{2}(a) \times S^{3}(\epsilon), \epsilon$ being the deformation parameter. Assuming $\epsilon^{\frac{2}{3}}>a$ and given that $a=\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right) r_{h}$ [21], in the IR and in the MQGP limit, $N_{\text {eff }}(r \in \mathrm{IR})=\int_{\text {warped squashed } S^{2}(a) \times S^{3}(\epsilon)} F_{5}(r \in$ $\mathrm{IR}) \ll M=\int_{S^{3}(\epsilon)} F_{3}(r \in \mathrm{IR})$; we have a confining $S U(M)$ gauge theory in the IR.

The warp factor that includes the back-reaction in the IR is given as

$$
\begin{align*}
h= & \frac{L^{4}}{r^{4}}\left[1+\frac{3 g_{s} M_{\mathrm{eff}}^{2}}{2 \pi N} \log r\left\{1+\frac{3 g_{s} N_{f}^{\mathrm{eff}}}{2 \pi}\left(\log r+\frac{1}{2}\right)\right.\right. \\
& \left.\left.+\frac{g_{s} N_{f}^{\mathrm{eff}}}{4 \pi} \log \left(\sin \frac{\theta_{1}}{2} \sin \frac{\theta_{2}}{2}\right)\right\}\right] \tag{6}
\end{align*}
$$

where, in principle, $M_{\text {eff }} / N_{f}^{\text {eff }}$ are not necessarily the same as $M / N_{f}$; we, however, will assume that, up to $\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right)$, they are. Proper UV behavior requires [21]
$h=\frac{L^{4}}{r^{4}}\left[1+\sum_{i=1} \frac{\mathcal{H}_{i}\left(\phi_{1,2}, \theta_{1,2}, \psi\right)}{r^{i}}\right]$, large $r ;$
$h=\frac{L^{4}}{r^{4}}\left[1+\sum_{i, j ;(i, j) \neq(0,0)} \frac{h_{i j}\left(\phi_{1,2}, \theta_{1,2}, \psi\right) \log ^{i} r}{r^{j}}\right]$, small $r$.

In the IR, up to $\mathcal{O}\left(g_{s} N_{f}\right)$ and setting $h_{5}=0$, the threeforms are as given in [1]:

$$
\begin{aligned}
& \text { (a) } \widetilde{F}_{3}=2 M A_{1}\left(1+\frac{3 g_{s} N_{f}}{2 \pi} \log r\right) \\
& \times e_{\psi} \wedge \frac{1}{2}\left(\sin \theta_{1} \mathrm{~d} \theta_{1} \wedge \mathrm{~d} \phi_{1}-B_{1} \sin \theta_{2} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \phi_{2}\right) \\
& -\frac{3 g_{s} M N_{f}}{4 \pi} A_{2} \frac{\mathrm{~d} r}{r} \wedge e_{\psi} \wedge \\
& \times\left(\cot \frac{\theta_{2}}{2} \sin \theta_{2} \mathrm{~d} \phi_{2}-B_{2} \cot \frac{\theta_{1}}{2} \sin \theta_{1} \mathrm{~d} \phi_{1}\right) \\
& -\frac{3 g_{s} M N_{f}}{8 \pi} A_{3} \sin \theta_{1} \sin \theta_{2} \\
& \times\left(\cot \frac{\theta_{2}}{2} \mathrm{~d} \theta_{1}+B_{3} \cot \frac{\theta_{1}}{2} \mathrm{~d} \theta_{2}\right) \wedge \mathrm{d} \phi_{1} \wedge \mathrm{~d} \phi_{2} \\
& \text { (b) } H_{3}=6 g_{s} A_{4} M\left(1+\frac{9 g_{s} N_{f}}{4 \pi} \log r\right. \\
& \left.+\frac{g_{s} N_{f}}{2 \pi} \log \sin \frac{\theta_{1}}{2} \sin \frac{\theta_{2}}{2}\right) \frac{\mathrm{d} r}{r} \wedge \frac{1}{2} \\
& \times\left(\sin \theta_{1} \mathrm{~d} \theta_{1} \wedge \mathrm{~d} \phi_{1}-B_{4} \sin \theta_{2} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \phi_{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\frac{3 g_{s}^{2} M N_{f}}{8 \pi} A_{5}\left(\frac{\mathrm{~d} r}{r} \wedge e_{\psi}-\frac{1}{2} \mathrm{~d} e_{\psi}\right) \wedge \\
& \times\left(\cot \frac{\theta_{2}}{2} \mathrm{~d} \theta_{2}-B_{5} \cot \frac{\theta_{1}}{2} \mathrm{~d} \theta_{1}\right) \tag{8}
\end{align*}
$$

The asymmetry factors in (8) are given by $A_{i}=1+$ $\mathcal{O}\left(\frac{a^{2}}{r^{2}}\right.$ or $\frac{a^{2} \log r}{r}$ or $\left.\frac{a^{2} \log r}{r^{2}}\right)+\mathcal{O}\left(\frac{\text { deformation parameter }{ }^{2}}{r^{3}}\right), B_{i}=$ $1+\mathcal{O}\left(\frac{a^{2} \log r}{r}\right.$ or $\frac{a^{2} \log r}{r^{2}}$ or $\left.\frac{a^{2} \log r}{r^{3}}\right)+\mathcal{O}\left(\frac{(\text { deformation parameter })^{2}}{r^{3}}\right)$. As in the UV, $\frac{\left(\text { deformation parameter) }{ }^{2}\right.}{r^{3}} \ll \frac{(\text { resolution parameter) })^{2}}{r^{2}}$, we will assume the same three-form fluxes for $h_{5} \neq 0$. With $\mathcal{R}_{D 5 / \overline{D 5}}$ denoting the boundary common to the UVIR interpolating region and the UV region, $\tilde{F}_{l m n}, H_{l m n}=0$ for $r \geq \mathcal{R}_{D 5 / \overline{D 5}}$ is required to ensure conformality in the UV. Near the $\theta_{1}=\theta_{2}=0$-branch, assuming $\theta_{1,2} \rightarrow 0$ as $\epsilon^{\gamma_{\theta}>0}$ and $r \rightarrow \mathcal{R}_{\mathrm{UV}} \rightarrow \infty$ as $\epsilon^{-\gamma_{r}<0}, \lim _{r \rightarrow \infty} \tilde{F}_{\text {lmn }}=0$ and $\lim _{r \rightarrow \infty} H_{l m n}=0$ for all components except $H_{\theta_{1} \theta_{2} \phi_{1,2}}$; in the MQGP limit and near $\theta_{1,2}=\pi / 0$-branch, $H_{\theta_{1} \theta_{2} \phi_{1,2}}=$ $0 /\left.\frac{3 g_{s}^{2} M N_{f}}{8 \pi}\right|_{N_{f}=2, g_{s}=0.6, M=\left(\mathcal{O}(1) g_{s}\right)^{-\frac{3}{2}}} \ll 1$. So, the UV nature too is captured near $\theta_{1,2}=0$-branch in the MQGP limit. This mimics addition of $\overline{D 5}$-branes in [1] to ensure cancellation of $\tilde{F}_{3}$.

Further, to ensure UV conformality, it is important to ensure that the axion-dilaton modulus approaches a constant implying a vanishing beta function in the UV. This was discussed in detail in Appendix B of [17], wherein in particular, assuming the F-theory uplift involved, locally, an elliptically fibered $K 3$, it was shown that UV conformality and the Ouyang embedding are mutually consistent.

### 2.2 The 'MQGP Limit'

In [2], we had considered the following two limits:
(i) weak $\left(g_{s}\right)$ coupling - large 'tHooft coupling limit:

$$
g_{s} \ll 1, g_{s} N_{f} \ll 1, \frac{g_{s} M^{2}}{N} \ll 1, g_{s} M \gg 1, g_{s} N \gg 1
$$

effected by: $g_{s} \sim \epsilon^{d}, M \sim(\mathcal{O}(1) \epsilon)^{-\frac{3 d}{2}}$,

$$
\begin{equation*}
N \sim(\mathcal{O}(1) \epsilon)^{-19 d}, \quad \epsilon \ll 1, \quad d>0 \tag{9}
\end{equation*}
$$

(the limit in the first line though not its realization in the second line, considered in [1]);
(ii) MQGP limit: $\frac{g_{s} M^{2}}{N} \ll 1, g_{s} N \gg 1$, finite $g_{s}, M$
effected by: $g_{s} \sim \epsilon^{d}, M \sim(\mathcal{O}(1) \epsilon)^{-\frac{3 d}{2}}$,
$N \sim(\mathcal{O}(1) \epsilon)^{-39 d}, \quad \epsilon \lesssim 1, \quad d>0$.
The motivation for considering the MQGP limit which was discussed in detail in [17] is as follows:

1. Unlike the AdS/CFT limit wherein $g_{\mathrm{YM}} \rightarrow 0, N \rightarrow \infty$ such that $g_{\mathrm{YM}}^{2} N$ is large, for strongly coupled thermal systems like sQGP, what is relevant is $g_{\mathrm{YM}} \sim \mathcal{O}(1)$ and $N_{c}=3$. From the discussion in the previous paragraphs one sees that in the IR after the Seiberg duality cascade, effectively $N_{c}=M$, which in the MQGP limit of (10) can be tuned to 3 . Further, in the same limit, the string coupling $g_{s} \lesssim 1$. The finiteness of the string coupling necessitates addressing the same from an M-theory perspective. This is the reason for coining: 'MQGP limit'. In fact this is the reason why one is required to first construct a type IIA mirror, which was done in [2] à la delocalized Strominger-Yau-Zaslow mirror prescription, and then take its M-theory uplift.
2. The second set of reasons for looking at the MQGP limit of (10) is a calculational simplification in supergravity.

- In the UV-IR interpolating region and the UV, $\left(M_{\text {eff }}, N_{\text {eff }}, N_{f}^{\text {eff }}\right) \stackrel{\text { MQGP }}{\approx}\left(M, N, N_{f}\right)$.
- Asymmetry factors $A_{i}, B_{j}$ (in three-form fluxes) $\xrightarrow{\text { MQGP }}$ 1 in the UV-IR interpolating region and the UV.
- Simplification of ten-dimensional warp factor and non-extremality function in MQGP limit
2.3 Approximate supersymmetry, construction of the delocalized SYZ IIA mirror and its M-theory uplift in the MQGP limit

To implement the quantum mirror symmetry a la SYZ [23], one needs a special Lagrangian (sLag) $T^{3}$ fibered over a large base. Defining delocalized T duality coordinates, $\left(\phi_{1}, \phi_{2}, \psi\right) \rightarrow(x, y, z)$ valued in $T^{3}(x, y, z)$ [2]:
$x=\sqrt{h_{2}} h^{\frac{1}{4}} \sin \left\langle\theta_{1}\right\rangle\langle r\rangle \phi_{1}, \quad y=\sqrt{h_{4}} h^{\frac{1}{4}} \sin \left\langle\theta_{2}\right\rangle\langle r\rangle \phi_{2}$, $z=\sqrt{h_{1}}\langle r\rangle h^{\frac{1}{4}} \psi$,
using the results of [24] it was shown in $[18,19]$ that the following conditions are satisfied:
$\left.i^{*} J\right|_{\mathrm{RC} / \mathrm{DC}} \approx 0$,
$\left.\Im m\left(i^{*} \Omega\right)\right|_{\mathrm{RC} / \mathrm{DC}} \approx 0$,
$\left.\mathfrak{R e}\left(i^{*} \Omega\right)\right|_{\mathrm{RC} / \mathrm{DC}} \sim$ volume form $\left(T^{3}(x, y, z)\right)$,
for the $T^{2}$-invariant sLag of [24] for a deformed conifold $\sum_{i=1}^{4} z_{i}^{2}=1$ :
$K^{\prime}\left(r^{2}\right) \Im m\left(z_{1} \bar{z}_{2}\right)=c_{1}$,
$K^{\prime}\left(r^{2}\right) \Im m\left(z_{3} \bar{z}_{4}\right)=c_{2}, \quad \Im m\left(z_{1}^{2}+z_{2}^{2}\right)=c_{3}$,
and the $T^{2}$-invariant sLag of [24] of a resolved conifold:

$$
\begin{align*}
& \frac{K^{\prime}}{2}\left(|x|^{2}-|y|^{2}\right)+4 a^{2} \frac{\left|\lambda_{2}\right|^{2}}{\left|\lambda_{1}\right|^{2}+\left|\lambda_{2}\right|^{2}}=c_{1} \\
& \frac{K^{\prime}}{2}\left(|v|^{2}-|u|^{2}\right)+4 a^{2} \frac{\left|\lambda_{2}\right|^{2}}{\left|\lambda_{1}\right|^{2}+\left|\lambda_{2}\right|^{2}}=c_{2} \\
& \Im m(x y)=c_{3} \tag{14}
\end{align*}
$$

wherein one uses the following complex structure for a resolved conifold [25]:
$x=\left(9 a^{2} r^{4}+r^{6}\right)^{1 / 4} e^{i / 2\left(\psi-\phi_{1}-\phi_{2}\right)} \sin \frac{\theta_{1}}{2} \sin \frac{\theta_{2}}{2}$,
$y=\left(9 a^{2} r^{4}+r^{6}\right)^{1 / 4} e^{i / 2\left(\psi+\phi_{1}+\phi_{2}\right)} \cos \frac{\theta_{1}}{2} \cos \frac{\theta_{2}}{2}$,
$u=\left(9 a^{2} r^{4}+r^{6}\right)^{1 / 4} e^{i / 2\left(\psi+\phi_{1}-\phi_{2}\right)} \cos \frac{\theta_{1}}{2} \sin \frac{\theta_{2}}{2}$,
$v=\left(9 a^{2} r^{4}+r^{6}\right)^{1 / 4} e^{i / 2\left(\psi-\phi_{1}+\phi_{2}\right)} \sin \frac{\theta_{1}}{2} \cos \frac{\theta_{2}}{2}$

$$
\left(\begin{array}{l}
x  \tag{15}\\
y \\
u \\
v
\end{array}\right)=\frac{1}{\sqrt{2}}\left(\begin{array}{cccc}
1 & -i & 0 & 0 \\
1 & i & 0 & 0 \\
0 & 0 & -i & 1 \\
0 & 0 & -i & -1
\end{array}\right)
$$

In (14), $\left[\lambda_{1}: \lambda_{2}\right]$ are the homogeneous coordinates of the blown-up $\mathbb{C P}^{1}=S^{2} ; \frac{\lambda_{2}}{\lambda_{1}}=\frac{x}{-u}=\frac{v}{-y}=-e^{-i \phi_{1}} \tan \frac{\theta_{1}}{2}$. In (14), $\gamma\left(r^{2}\right) \equiv r^{2} K^{\prime}\left(r^{2}\right)=-2 a^{2}+4 a^{4} N^{-\frac{1}{3}}\left(r^{2}\right)+N^{\frac{1}{3}}\left(r^{2}\right)$, where $N\left(r^{2}\right) \equiv \frac{1}{2}\left(r^{4}-16 a^{6}+\sqrt{r^{8}-32 a^{6} r^{4}}\right)$. Hence, if the resolved warped deformed conifold is predominantly either resolved or deformed, the local $T^{3}$ of (11) is the required sLag to effect the SYZ mirror construction.

Interestingly, in the 'delocalized limit' [26] $\psi=\langle\psi\rangle$, under the coordinate transformation

$$
\binom{\sin \theta_{2} \mathrm{~d} \phi_{2}}{\mathrm{~d} \theta_{2}} \rightarrow\left(\begin{array}{cc}
\cos \langle\psi\rangle & \sin \langle\psi\rangle  \tag{16}\\
-\sin \langle\psi\rangle & \cos \langle\psi\rangle
\end{array}\right)\binom{\sin \theta_{2} \mathrm{~d} \phi_{2}}{\mathrm{~d} \theta_{2}}
$$

and $\psi \rightarrow \psi-\cos \left\langle\bar{\theta}_{2}\right\rangle \phi_{2}+\cos \left\langle\theta_{2}\right\rangle \phi_{2}-\tan \langle\psi\rangle \ln \sin \bar{\theta}_{2}$, the $h_{5}$ term becomes $h_{5}\left[\mathrm{~d} \theta_{1} \mathrm{~d} \theta_{2}-\sin \theta_{1} \sin \theta_{2} \mathrm{~d} \phi_{1} \mathrm{~d} \phi_{2}\right], e_{\psi} \rightarrow e_{\psi}$, i.e., one introduces a local (not global) isometry along $\psi$ in addition to the isometries along $\phi_{1,2}$.

To enable use of SYZ mirror duality via three T dualities, remembering that SYZ mirror symmetry is in fact a quantum mirror symmetry, one also needs to ensure a large base (implying large complex structures of the aforementioned two two-tori) of the $T^{3}(x, y, z)$ fibration, ensuring the disc instantons' contribution is very small [23]. This is effected via [27]:
$\mathrm{d} \psi \rightarrow \mathrm{d} \psi+f_{1}\left(\theta_{1}\right) \cos \theta_{1} \mathrm{~d} \theta_{1}+f_{2}\left(\theta_{2}\right) \cos \theta_{2} \mathrm{~d} \theta_{2}$,
$\mathrm{d} \phi_{1,2} \rightarrow \mathrm{~d} \phi_{1,2}-f_{1,2}\left(\theta_{1,2}\right) \mathrm{d} \theta_{1,2}$,
for appropriately chosen large values of $f_{1,2}\left(\theta_{1,2}\right)=$ $\pm \cot \theta_{1,2}$ [17]. The three-form fluxes remain invariant. The guiding principle behind choosing such large values of
$f_{1,2}\left(\theta_{1,2}\right)$, as given in [2], is that one requires the metric obtained after a SYZ mirror transformation applied to the non-Kähler resolved warped deformed conifold to be like a non-Kähler warped resolved conifold at least locally. For completeness, we summarize the Buscher triple-T duality rules $[2,28]$ in appendix A .

A single T duality along a direction orthogonal to the $D 3$-brane world volume, e.g., $z$ of (11), yields $D 4$ branes straddling a pair of $N S 5$-branes consisting of world-volume coordinates $\left(\theta_{1}, x\right)$ and $\left(\theta_{2}, y\right)$. Further, T dualizing along $x$ and then $y$ would yield a Taub-NUT space from each of the two $N S 5$-branes [29]. The $D 7$-branes yield $D 6$-branes which get uplifted to Kaluza-Klein monopoles in M-theory [30] which too involve Taub-NUT spaces. Globally, probably the 11-dimensional uplift would involve a seven-fold of $G_{2}$-structure, analogous to the uplift of $D 5$-branes wrapping a two-cycle in a resolved warped conifold [27]. We obtained a local $G_{2}$ structure in [17], which is summarized in Sect. 2.4.

## 2.4 $G$-structures

In this sub-section, we give a quick overview of $G=$ $S U(3), G_{2}$-structures and how the same appear in the holographic type IIB dual of [1], its delocalized type IIA SYZ mirror and its M-theory uplift constructed in [2].

Any metric-compatible connection can be written in terms of the Levi-Civita connection and the contorsion tensor $\kappa$ ([31] and references therein). Metric compatibility requires $\kappa \in \Lambda^{1} \otimes \Lambda^{2}, \Lambda^{n}$ being the space of $n$-forms. Alternatively, in $d$ complex dimensions, since $\Lambda^{2} \cong \operatorname{so}(d), \kappa$ also be thought of as $\Lambda^{1} \otimes \operatorname{so}(d)$. Given the existence of a $G$-structure, one can decompose $\operatorname{so}(d)$ into a part in the Lie algebra $g$ of $G \subset S O(d)$ and its orthogonal complement $g^{\perp}=s o(d) / g$. The contorsion $\kappa$ splits accordingly into $\kappa=\kappa^{0}+\kappa^{g}$, where $\kappa^{0}$ - the intrinsic torsion - is the part in $\Lambda^{1} \otimes g^{\perp}$. One can decompose $\kappa^{0}$ into irreducible $G$ representations providing a classification of $G$-structures in terms of which representations appear in the decomposition. Let us consider the decomposition of $T^{0}$ in the case of $S U(3)$-structure. The relevant representations are $\Lambda^{1} \sim 3 \oplus \overline{3}, g \sim 8, g^{\perp} \sim 1 \oplus 3 \oplus \overline{3}$. Thus the intrinsic torsion, an element of $\Lambda^{1} \oplus s u(3)^{\perp}$, can be decomposed into the following $S U(3)$ representations [31]:

$$
\begin{align*}
& \Lambda^{1} \otimes s u(3)^{\perp}=(3 \oplus \overline{3}) \otimes(1 \oplus 3 \oplus \overline{3}) \\
& \quad=(1 \oplus 1) \oplus(8 \oplus 8) \oplus(6 \oplus \overline{6}) \oplus(3 \oplus \overline{3}) \oplus(3 \oplus \overline{3})^{\prime} \\
& \quad \equiv W_{1} \oplus W_{2} \oplus W_{3} \oplus W_{4} \oplus W_{5} . \tag{18}
\end{align*}
$$

The $S U(3)$ structure torsion classes [32] can be defined in terms of $\mathrm{J}, \Omega, \mathrm{dJ}, \mathrm{d} \Omega$ and the contraction operator $\lrcorner: \Lambda^{k} T^{\star} \otimes$ $\Lambda^{n} T^{\star} \rightarrow \Lambda^{n-k} T^{\star}$. The torsion classes are then defined in the following way:

- $W_{1} \leftrightarrow[\mathrm{~d} J]^{(3,0)}$, given by real numbers $W_{1}=W_{1}^{+}+W_{1}^{-}$ with $\mathrm{d} \Omega_{+} \wedge J=\Omega_{+} \wedge \mathrm{d} J=W_{1}^{+} J \wedge J \wedge J$ and $\mathrm{d} \Omega_{-} \wedge J=$ $\Omega_{-} \wedge \mathrm{d} J=W_{1}^{-} J \wedge J \wedge J$.
- $W_{2} \leftrightarrow[\mathrm{~d} \Omega]_{0}^{(2,2)} ;\left(\mathrm{d} \Omega_{+}\right)^{(2,2)}=W_{1}^{+} J \wedge J+W_{2}^{+} \wedge J$ and $\left(\mathrm{d} \Omega_{-}\right)^{(2,2)}=W_{1}^{-} J \wedge J+W_{2}^{-} \wedge J$.
- $W_{3} \leftrightarrow[\mathrm{~d} J]_{0}^{(2,1)}$ is defined as $W_{3}=\mathrm{d} J^{(2,1)}-[J \wedge$ $\left.W_{4}\right]^{(2,1)}$.
- $\left.W_{4}=\frac{1}{2} J\right\lrcorner \mathrm{d} J$.
- $\left.W_{5}=\frac{1}{2} \Omega_{+}\right\lrcorner \mathrm{d} \Omega_{+}$(the subscript 0 indicative of the primitivity of the respective forms).

In [18], it was shown that the five $S U(3)$ structure torsion classes, in the MQGP limit, satisfied (schematically):

$$
\begin{align*}
& T_{S U(3)}^{\mathrm{IIB}} \in W_{1} \oplus W_{2} \oplus W_{3} \oplus W_{4} \oplus W_{5} \sim \frac{e^{-3 \tau}}{\sqrt{g_{s} N}} \\
& \quad \oplus\left(g_{s} N\right)^{\frac{1}{4}} e^{-3 \tau} \oplus \sqrt{g_{s} N} e^{-3 \tau} \oplus-\frac{2}{3} \oplus-\frac{1}{2} \tag{19}
\end{align*}
$$

( $r \sim e^{\frac{\tau}{3}}$ ), such that
$\frac{2}{3} W_{5}^{\overline{3}}=W_{4}^{\overline{3}}$
in the UV-IR interpolating region/UV, implying a Klebanov-Strassler-like supersymmetry [33]. Locally around $\theta_{1} \sim$ $\frac{1}{N^{\frac{1}{5}}}, \theta_{2} \sim \frac{1}{N^{\frac{3}{10}}}$, the type IIA torsion classes of the delocalized SYZ type IIA mirror metric were shown in [17] to be

$$
\begin{align*}
& T_{S U(3)}^{\mathrm{IIA}} \in W_{2} \oplus W_{3} \oplus W_{4} \oplus W_{5} \sim \gamma_{2} g_{s}^{-\frac{1}{4}} N^{\frac{3}{10}} \\
& \quad \oplus g_{s}^{-\frac{1}{4}} N^{-\frac{1}{20}} \oplus g_{s}^{-\frac{1}{4}} N^{\frac{3}{10}} \oplus g_{s}^{-\frac{1}{4}} N^{\frac{3}{10}} \approx \gamma W_{2} \oplus W_{4} \oplus W_{5} \\
& \text { fine tuning: } \gamma \approx 0 \approx W_{4} \oplus W_{5} . \tag{21}
\end{align*}
$$

Further,
$W_{4} \sim \mathfrak{R e} W_{5}$
indicative of supersymmetry after constructing the delocalized SYZ mirror.

The mirror type IIA metric after performing three T dualities, first along $x$, then along $y$ and finally along $z$, utilizing the results of [26] was worked out in [2]. The type IIA metric components were worked out in [2].

Apart from quantifying the departure from $S U$ (3) holonomy due to intrinsic contorsion supplied by the NS-NS threeform $H$, via the evaluation of the $S U(3)$ structure torsion classes, to our knowledge for the first time in the context of holographic thermal QCD at finite gauge coupling and for finite number of colors [in fact for $N_{c}=3$ in the IR] in [17]:
(i) The existence of approximate supersymmetry of the type IIB holographic dual of [1] in the MQGP limit near
the coordinate branch $\theta_{1}=\theta_{2}=0$ was demonstrated, which apart from the existence of a special Lagrangian three-cycle (as shown in $[17,18]$ ) is essential for construction of the local SYZ type IIA mirror.
(ii) It was demonstrated that the large- $N$ suppression of the deviation of the type IIB resolved warped deformed conifold from being a complex manifold, is lost on being duality-chased to type IIA - it was also shown that further fine tuning $\gamma_{2}=0$ in $W_{2}^{\mathrm{IIA}}$ can ensure that the local type IIA mirror is complex.
(iii) For the local type IIA $S U(3)$ mirror, the possibility of surviving approximate supersymmetry was demonstrated which is essential from the point of view of the end result of application of the SYZ mirror prescription.

We can get a one-form type IIA potential from the triple T dual (along $x, y, z$ ) of the type IIB $F_{1,3,5}$ in [2] and using which the following $D=11$ metric was obtained in [2] ( $u \equiv \frac{r_{h}}{r}$ ):

$$
\begin{align*}
\mathrm{d} s_{11}^{2}= & e^{-\frac{2 \phi^{I I A}}{3}}\left[g_{t t} \mathrm{~d} t^{2}+g_{\mathbb{R}^{3}}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}+\mathrm{d} Z^{2}\right)\right. \\
& \left.+g_{u u} d u^{2}+\mathrm{d} s_{I I A}^{2}\left(\theta_{1,2}, \phi_{1,2}, \psi\right)\right] \\
& +e^{\frac{4 \phi^{I I A}}{3}}\left(\mathrm{~d} x_{11}+A^{F_{1}}+A^{F_{3}}+A^{F_{5}}\right)^{2} \\
\equiv & \text { Black } M 3-\text { Brane }+\mathcal{O}\left(\left[\frac{g_{s} M^{2} \log N}{N}\right]\left(g_{s} N_{f}\right)\right) . \tag{23}
\end{align*}
$$

Let us now briefly discuss $G_{2}$ structure. We will be following [34-36]. If $V$ is a seven-dimensional real vector space, then a three-form $\varphi$ is said to be positive if it lies in the $G L(7, \mathbb{R})$ orbit of $\varphi_{0}$, where $\varphi_{0}$ is a three-form on $\mathbb{R}^{7}$ which is preserved by the $G_{2}$-subgroup of $G L(7, \mathbb{R})$. The pair $(\varphi, g)$ for a positive 3-form $\varphi$ and corresponding metric $g$ constitute a $G_{2}$-structure. The spaces of $p$-forms are known to decompose as the following irreps of $G_{2}$ [34]:
$\Lambda^{1}=\Lambda_{7}^{1}$,
$\Lambda^{2}=\Lambda_{7}^{2} \oplus \Lambda_{14}^{2}$,
$\Lambda^{3}=\Lambda_{1}^{3} \oplus \Lambda_{7}^{3} \oplus \Lambda_{27}^{3}$,
$\Lambda^{4}=\Lambda_{1}^{4} \oplus \Lambda_{7}^{4} \oplus \Lambda_{27}^{4}$,
$\Lambda^{5}=\Lambda_{7}^{5} \oplus \Lambda_{14}^{5}$,
$\Lambda^{6}=\Lambda_{7}^{6}$.
The subscripts denote the dimension of the representation and the components of the same representation/dimensionality, being isomorphic to each other. Let $M$ be a 7-manifold with a $G_{2}$-structure $(\varphi, g)$. Then the components of spaces of 2-, $3-, 4-$, and 5 -forms are given in $[34,36]$. The metric $g$ defines a reduction of the frame bundle F to a principal $S O$ (7)-subbundle of oriented orthonormal frames. Now, $g$ also defines
a Levi-Civita connection $\nabla$ on the tangent bundle $T M$, and hence on $F$. However, the $G_{2}$-invariant 3-form $\varphi$ reduces the orthonormal bundle further to a principal $G_{2}$-subbundle $Q$. The Levi-Civita connection can be pulled back to $Q$. On $Q$, $\nabla$ can be uniquely decomposed as
$\nabla=\bar{\nabla}+\mathcal{T}$
where $\bar{\nabla}$ is a $G_{2}$-compatible canonical connection, taking values in the sub-algebra $\mathfrak{g}_{2} \subset \mathfrak{s o}(7)$, while $\mathcal{T}$ is a 1-form taking values in $\mathfrak{g}_{2}^{\perp} \subset \mathfrak{s o}(7) ; \mathcal{T}$ is known as the intrinsic torsion of the $G_{2}$-structure - the obstruction to the LeviCivita connection being $G_{2}$-compatible. Now $\mathfrak{s o}$ (7) splits under $G_{2}$ as
$\mathfrak{s o}(7) \cong \Lambda^{2} V \cong \Lambda_{7}^{2} \oplus \Lambda_{14}^{2}$.

But $\Lambda_{14}^{2} \cong \mathfrak{g}_{2}$, so the orthogonal complement $\mathfrak{g}_{2}^{\perp} \cong \Lambda_{7}^{2} \cong V$. Hence $\mathcal{T}$ can be represented by a tensor $T_{a b}$ which lies in $W \cong V \otimes V$. Now, since $\varphi$ is $G_{2}$-invariant, it is $\bar{\nabla}$-parallel. So, the torsion is determined by $\nabla \varphi$. Now, from Lemma 2.24 of [35],
$\nabla \varphi \in \Lambda_{7}^{1} \otimes \Lambda_{7}^{3} \cong W$.
Due to the isomorphism between the $\Lambda_{7}^{a=1, \ldots, 5} \mathrm{~s}, \nabla \varphi$ lies in the same space as $T_{A B}$ and thus completely determines it. Equation (27) is equivalent to
$\nabla_{A} \varphi_{B C D}=T_{A}{ }^{E} \psi_{E B C D}$
where $T_{A B}$ is the full torsion tensor. Equation (28) can be inverted to yield
$T_{A}{ }^{M}=\frac{1}{24}\left(\nabla_{A} \varphi_{B C D}\right) \psi^{M B C D}$.

The tensor $T_{A}{ }^{M}$, like the space W, possesses 49 components and hence fully defines $\nabla \varphi$. In general $T_{A B}$ can be split into torsion components as
$\left.T=T_{1} g+T_{7}\right\lrcorner \varphi+T_{14}+T_{27}$
where $T_{1}$ is a function and gives the $\mathbf{1}$ component of $T$. We also have $T_{7}$, which is a 1 -form and hence gives the 7 component, and, $T_{14} \in \Lambda_{14}^{2}$ gives the $\mathbf{1 4}$ component. Further, $T_{27}$ is traceless symmetric, and gives the $\mathbf{2 7}$ component. Writing $T_{i}$ as $W_{i}$, we can split $W$ as
$W=W_{1} \oplus W_{7} \oplus W_{14} \oplus W_{27}$.

From [38], we see that a $G_{2}$ structure can be defined as
$\varphi_{0}=\frac{1}{3!} f_{A B C} e^{A B C}=e^{-\phi^{I I A}} f_{a b c} e^{a b c}+e^{-\frac{2 \phi^{I I A}}{3}} J \wedge e^{x_{10}}$,
where $A, B, C=1, \ldots, 6,10 ; a, b, c,=1, \ldots, 6$ and $f_{A B C}$ are the structure constants of the imaginary octonions. Using the same and [37]:
$\mathrm{d} \varphi_{0}=4 W_{1} *_{7} \varphi_{0}-3 W_{7} \wedge \varphi_{0}-*_{7} W_{27}$
$d *_{7} \varphi_{0}=-4 W_{7} \wedge *_{7} \varphi_{0}-2 *_{7} W_{14}$,
the $G_{2}$-structure torsion classes were worked out around $\theta_{1} \sim$ $\frac{1}{N^{\frac{1}{5}}}, \theta_{2} \sim \frac{1}{N^{\frac{3}{10}}}$ in [17] to
$T_{G_{2}} \in W_{2}^{14} \oplus W_{3}^{27} \sim \frac{1}{\left(g_{s} N\right)^{\frac{1}{4}}} \oplus \frac{1}{\left(g_{s} N\right)^{\frac{1}{4}}}$.
Hence, the approach of the seven-fold, locally, to having a $G_{2}$ holonomy ( $W_{1}^{G_{2}}=W_{2}^{G_{2}}=W_{3}^{G_{2}}=W_{4}^{G_{2}}=0$ ) is accelerated in the MQGP limit.

## 3 SYZ mirror of Ouyang embedding

To start off our study of meson spectroscopy, we first need to understand how the flavor D6-branes are embedded in the mirror (constructed in [2]) of the resolved warped deformed conifold of [1], i.e., the delocalized SYZ mirror of the Ouyang embedding of the flavor $D 7$-branes in [1].

One can show that the delocalized type IIA mirror metric of the resolved warped deformed conifold metric as worked out in [2], for fixed $\theta_{1}=\frac{\alpha_{\theta_{1}}}{N^{\frac{1}{5}}}$ in the $\left(\theta_{2}, T^{3}(x, y, z)\right)$ subspace near $\theta_{2}=\frac{\alpha_{\theta_{2}}}{N \frac{3}{10}}$ can be written as

$$
\begin{align*}
& \mathrm{d} s_{\text {IIA }}^{2}\left(\theta_{2}, T^{3}(x, y, z)\right) \\
& =\mathrm{d} \theta_{2}^{2} N^{\frac{7}{10}}\left(\xi_{\theta_{2} \theta_{2}} \frac{\alpha_{\theta_{1}}^{2}}{\alpha_{\theta_{2}}^{2}} \sqrt{g_{s}} \mathrm{~d} \theta_{2}+\xi_{\theta_{2} y} N^{-\frac{7}{20}} g_{s}^{\frac{1}{4}} \mathrm{~d} y\right. \\
& \left.\quad-\xi_{\theta_{2} z} \frac{\log r M N_{f}}{\alpha_{\theta_{2}}} N^{-\frac{13}{20}} g_{s}^{\frac{7}{4}} \mathrm{~d} z\right)+\mathrm{d} s^{2}\left(T^{3}(x, y, z)\right) \tag{35}
\end{align*}
$$

$\xrightarrow{N \gg 1} \xi_{\theta_{2} \theta_{2}} \frac{\alpha_{\theta_{1}}^{2}}{\alpha_{\theta_{2}}^{2}} \sqrt{g_{s}} \mathrm{~d} \theta_{2}^{2}+\mathrm{d} s^{2}\left(T^{3}(x, y, z)\right)$,
where the $T^{3}(x, y, z)$ metric is given by

$$
g_{i j}\left(T^{3}(x, y, z)\right)=\left(\begin{array}{ccc}
\frac{3^{2 / 3}\left(\alpha_{\theta_{1}}^{2}-\alpha_{\theta_{2}}^{2} \sqrt[5]{\frac{1}{N}}\right)}{\alpha_{\theta_{1}}^{2}} & \frac{2 \sqrt{2}\left(\alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}} \sqrt{N}-2 \alpha_{\theta_{2}}^{3} N^{3 / 10}\right)}{3 \sqrt[6]{3} \alpha_{\theta_{1}}^{6}} & \frac{2\left(9 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{1}} N^{4 / 5}-23^{2 / 3} N\right)}{27 \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2}}  \tag{36}\\
\frac{2 \sqrt{2}\left(\alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}} \sqrt{N}-2 \alpha_{\theta_{2}}^{3} N^{3 / 10}\right)}{3 \sqrt[6]{3} \alpha_{\theta_{1}}^{6}} & 3^{2 / 3} & \frac{\sqrt{2}\left(\alpha_{\theta_{2}}^{2}-3 N^{3 / 5}\right)}{3 \sqrt[6]{3} \alpha_{\theta_{2}} N^{3 / 10}} \\
\frac{2\left(9 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{1}} N^{4 / 5}-23^{2 / 3} N\right)}{27 \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2}} & \frac{\sqrt{2}\left(\alpha_{\theta_{2}}^{2}-3 N^{3 / 5}\right)}{3 \sqrt[6]{3} \alpha_{\theta_{2}} N^{3 / 10}} & \frac{2\left(\sqrt[5]{N} \alpha_{\theta_{1}}^{2}+\alpha_{\theta_{2}}^{2}\right) N^{2 / 5}}{3 \sqrt[3]{3} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2}}
\end{array}\right) .
$$

As stated earlier, the global uplift to M-theory of the type II $B$ background of [1] is expected to involve a seven-fold of $G_{2}$ structure (not $G_{2}$-holonomy due to non-zero $G_{4}$ ). It is hence extremely important to be able to see this, at least locally. It is in this sense that the results of [2] are of great significance as one explicitly sees, for the first time, in the context of holographic thermal QCD at finite gauge coupling, though locally, the aforementioned $G_{2}$ structure having worked out the non-trivial $G_{2}$-structure torsion classes.

Interestingly, one can diagonalize the local $T^{3}$ metric to

$$
\begin{align*}
\mathrm{d} s_{\text {IIA }}^{2}\left(T^{3}(x, y, z)\right)= & \frac{2 \mathrm{~d} \tilde{x}^{2}\left(9 \sqrt{2} \sqrt[6]{3} \alpha N^{4 / 5}-23^{2 / 3} N\right)}{27 \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{2}} \\
& +\frac{2 \mathrm{~d} \tilde{y}^{2}\left(23^{2 / 3} N-9 \sqrt{2} \sqrt[6]{3} \alpha N^{4 / 5}\right)}{27 \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{2}} \\
& +\frac{2 \mathrm{~d} \tilde{z}^{2}\left(3^{2 / 3} \alpha_{\theta_{1}}^{2} 2 N^{3 / 5}+3^{2 / 3} \alpha_{\theta_{2}}^{2} N^{2 / 5}\right)}{27 \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{2}} \tag{37}
\end{align*}
$$

where

$$
\begin{aligned}
\mathrm{d} \tilde{x} & =\frac{\mathrm{d} x\left(3 \alpha_{\theta_{1}}^{2}\left(\frac{1}{N}\right)^{2 / 5}+4\right)}{4 \sqrt{2}}+\frac{\mathrm{d} z\left(4-3 \alpha_{\theta_{1}}^{2}\left(\frac{1}{N}\right)^{2 / 5}\right)}{4 \sqrt{2}} \\
& +\frac{\mathrm{d} y \sqrt{\frac{1}{N}}\left(2 \alpha_{\theta_{2}}^{2} \sqrt[5]{\frac{1}{N}}\left(\left(54-3^{2 / 3}\right) \alpha_{\theta_{1}}^{6}-54 \sqrt{6} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}+66 \alpha_{\theta_{2}}^{4}\right)-\alpha_{\theta_{1}}^{2}\left(3^{2 / 3} \alpha_{\theta_{1}}^{6}+72 \alpha_{\theta_{2}}^{4}\right)\right)}{163^{5 / 6} \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}}
\end{aligned}
$$

$$
\begin{align*}
\mathrm{d} \tilde{y} & =\frac{\mathrm{d} x\left(3 \alpha_{\theta_{1}}^{2}\left(\frac{1}{N}\right)^{2 / 5}-4\right)}{4 \sqrt{2}}+\frac{\mathrm{d} z\left(3 \alpha_{\theta_{1}}^{2}\left(\frac{1}{N}\right)^{2 / 5}+4\right)}{4 \sqrt{2}} \\
& +\frac{\mathrm{d} y \sqrt{\frac{1}{N}}\left(\alpha_{\theta_{1}}^{2}\left(-\left(3^{2 / 3} \alpha_{\theta_{1}}^{6}+72 \alpha_{\theta_{2}}^{4}\right)\right)-2 \alpha_{\theta_{2}}^{2} \sqrt[5]{\frac{1}{N}}\left(\left(54+3^{2 / 3}\right) \alpha_{\theta_{1}}^{6}+54 \sqrt{6} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}-66 \alpha_{\theta_{2}}^{4}\right)\right)}{163^{5 / 6} \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}} \\
\mathrm{~d} \tilde{z} & =-\frac{9 \sqrt[6]{3} \sqrt{\alpha_{\theta_{2}}^{2}} \mathrm{~d} x\left(\frac{1}{N}\right)^{7 / 10}\left(\alpha_{\theta_{2}}^{2} \sqrt[5]{\frac{1}{N}}-2 \alpha_{\theta_{1}}^{2}\right)}{4 \sqrt{2}}+\frac{\mathrm{d} y\left(-3^{2 / 3} \alpha_{\theta_{1}}^{12}-768 \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}^{2} N+1728 \sqrt[3]{3} \alpha_{\theta_{2}}^{8}\right)}{768 \alpha_{\theta_{1}}^{4} \sqrt{\alpha_{\theta_{2}}^{2}} \alpha_{\theta_{2}} N} \\
& -\frac{3 \sqrt[6]{3} \alpha_{\theta_{2}}^{4} \mathrm{~d} z \sqrt{\frac{1}{N}}\left(3 \sqrt{3} \alpha_{\theta_{1}}^{3} \sqrt[5]{\frac{1}{N}}+\sqrt{2} \alpha_{\theta_{1}}^{2}-2 \sqrt{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{\frac{1}{N}}\right)}{2 \alpha_{\theta_{1}}^{4} \sqrt{\alpha_{\theta_{2}}^{2}}} . \tag{38}
\end{align*}
$$

Now, from the Buscher triple T duality results for the NS-NS $B$ as given in (A11), one sees that for small $\phi_{1,2}$ (which ensures decoupling of $M_{6}\left(\theta_{1,2}, \phi_{1,2}, \psi, x^{10}\right)$ from $M_{5}\left(\mathbb{R}^{1,3}, r\right)$ in the M-theory uplift in the MQGP limit)

$$
\left.\begin{array}{l}
B^{\mathrm{IIA}}\left(\theta_{1}=\frac{\alpha_{\theta_{1}}}{\left.N^{\frac{1}{5}}, \theta_{2} \sim \frac{\alpha_{\theta_{2}}}{N \frac{3}{10}}\right)}\right. \\
=\mathrm{d} \theta_{2}
\end{array}\right) \mathrm{d} x\left(-\frac{2 \sqrt{2} \sqrt[4]{\pi} \sqrt[4]{g_{s}} N^{3 / 4}\left(3 \sqrt{6} \alpha_{\theta_{1}}^{3}-2 \alpha_{\theta_{1}}^{2} \sqrt[5]{N}+2 \alpha_{\theta_{2}}^{2}\right)}{27 \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}}\right) .
$$

There is an important message we must take in from (39). As one realizes from (A11) and therefore (39), $B^{\mathrm{IIA}}$ is
independent of $M$ even up to NLO in $N$, even though $B^{\text {IIB }}$ is proportional to $M$. This will be important in obtaining the mesonic spectra in the subsequent sections and obtaining a good match with [5] without having to invoke $\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right)$ corrections which the authors of [4] had to use (and set to 0.5 - and yet consider the same as 'small' - to get a reasonable match with [5]).

The complete ten-dimensional type IIA metric in large- $N$ limit is given as

$$
\begin{align*}
& \mathrm{d} s_{\mathrm{IIA}}^{2} \approx G_{00}^{\mathrm{IIA}} \mathrm{~d} x_{0}^{2}+G_{11}^{\mathrm{IIA}} \mathrm{~d} x_{1}^{2}+G_{22}^{\mathrm{IIA}} \mathrm{~d} x_{2}^{2}+G_{33}^{\mathrm{IIA}} \mathrm{~d} x_{3}^{2} \\
& \quad+G_{r r}^{\mathrm{IIA}} d r^{2}+G_{\theta_{1} \theta_{1}}^{\mathrm{IIA}} \mathrm{~d} \theta_{1}^{2}+G_{\theta_{1} \tilde{x}}^{\mathrm{IIA}} \mathrm{~d} \theta_{1} \mathrm{~d} \tilde{x}+G_{\theta_{1} \tilde{y}}^{\mathrm{IIA}} \mathrm{~d} \theta_{1} \mathrm{~d} \tilde{y} \\
& \quad+G_{\theta_{1} \tilde{z}}^{\mathrm{IA}} \mathrm{~d} \theta_{1} \mathrm{~d} \tilde{z}+G_{\theta_{2} \theta_{2}}^{\mathrm{IA}} \mathrm{~d} \theta_{2}^{2}+\mathrm{d} s^{2}\left(T^{3}(\tilde{x}, \tilde{y}, \tilde{z})\right) . \tag{40}
\end{align*}
$$

To obtain the pull-back metric on the $D 6$ branes, we choose the first branch of the Ouyang embedding where $\left(\theta_{1}, \tilde{x}\right)=$ $(0,0)$ and we consider the $\tilde{z}$ coordinate as a function of r , i.e. $\tilde{z}(r)$. We then use the equation of motion of the field to find the explicit functional dependence. The coordinates for D6 brane are $x^{\mu}=x^{(0,1,2,3)}, r, \theta_{2}, \tilde{y}$. The pull-back of the metric is given by

$$
\begin{align*}
& G_{6 \mu \nu}^{\mathrm{IIA}} \mathrm{~d} x^{\mu} \mathrm{d} x^{\nu}=G_{00}^{\mathrm{IIA}} \mathrm{~d} x_{0}^{2}+G_{11}^{\mathrm{IIA}} \mathrm{~d} x_{1}^{2} \\
& \quad+G_{22}^{\mathrm{IIA}} \mathrm{~d} x_{2}^{2}+G_{33}^{\mathrm{IIA}} \mathrm{~d} x_{3}^{2}+\left(G_{r r}^{\mathrm{IIA}}+G_{\tilde{z} \tilde{z}}^{\mathrm{IIA}} \tilde{z}(r)^{2}\right) \mathrm{d} r^{2} \\
& G_{\theta_{2} \theta_{2}}^{\mathrm{IIA}} \mathrm{~d} \theta_{2}^{2}+G_{\tilde{y} \tilde{y}}^{\mathrm{IIA}} \mathrm{~d} \tilde{y}^{2} . \tag{41}
\end{align*}
$$

Near $\theta_{1}=\alpha_{\theta_{1}} N^{-1 / 5}$ and $\theta_{2}=\alpha_{\theta_{2}} N^{-3 / 10}$ the type IIA metric components up to NLO are given as follows:

$$
\begin{aligned}
& G_{00}^{\mathrm{IIA}}=-\frac{\left(r^{4}-r_{h}^{4}\right)\left(3 g_{s} M^{2} \log (r)\left(-2 g_{s} N_{f} \log \left(\alpha_{\theta 1} \alpha_{\theta 2}\right)+g_{s} N_{f} \log (N)-6 g_{s} N_{f}+g_{s} N_{f} \log (16)-8 \pi\right)-36 g_{s}^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)}{64 \pi^{5 / 2} \sqrt{g_{s}} N^{3 / 2} r^{2}} \\
& G_{11}^{\mathrm{IIA}}=\frac{r^{2}\left(3 g_{s} M^{2} \log (r)\left(-2 g_{s} N_{f} \log \left(\alpha_{\theta 1} \alpha_{\theta 2}\right)+g_{s} N_{f} \log N-6 g_{s} N_{f}+g_{s} N_{f} \log (16)-8 \pi\right)-36 g_{s}^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)}{64 \pi^{5 / 2} \sqrt{g_{s}} N^{3 / 2}} \\
& G_{22}^{\mathrm{IIA}}=\frac{r^{2}\left(3 g_{s} M^{2} \log (r)\left(-2 g_{s} N_{f} \log \left(\alpha_{\theta 1} \alpha_{\theta 2}\right)+g_{s} N_{f} \log N-6 g_{s} N_{f}+g_{s} N_{f} \log (16)-8 \pi\right)-36 g_{s}^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)}{64 \pi^{5 / 2} \sqrt{g_{s}} N^{3 / 2}}
\end{aligned}
$$

$$
\begin{align*}
& G_{33}^{\mathrm{IA}}=\frac{r^{2}\left(3 g_{s} M^{2} \log (r)\left(-2 g_{s} N_{f} \log \left(\alpha_{\theta 1} \alpha_{\theta 2}\right)+g_{s} N_{f} \log N-6 g_{s} N_{f}+g_{s} N_{f} \log (16)-8 \pi\right)-36 g_{s}{ }^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)}{64 \pi^{5 / 2} \sqrt{g_{s}} N^{3 / 2}} \\
& G_{r r}^{\mathrm{IA}}=\frac{\sqrt{g_{s}} r^{2}\left(6 a^{2}+r^{2}\right)\left(3 g_{s} M^{2} \log (r)\left(2 g_{s} N_{f} \log \left(\alpha_{\theta 1} \alpha_{\theta 2}\right)-g_{s} N_{f} \log N+6 g_{s} N_{f}-2 g_{s} N_{f} \log (4)+8 \pi\right)+36 g_{s}{ }^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)}{16 \pi^{3 / 2} \sqrt{N}\left(9 a^{2}+r^{2}\right)\left(r^{4}-r_{h}{ }^{4}\right)} \\
& G_{\tilde{y} \tilde{y}}^{\mathrm{II}}=-\frac{2\left(9 \sqrt{2} \sqrt[6]{3} \alpha N^{4 / 5}-23^{2 / 3} N\right)}{27 \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{2}} G_{\theta_{2} \theta_{2}}^{\mathrm{IIA}}=\frac{\sqrt{\pi} \sqrt{g_{s}} \sqrt{N}\left(\alpha_{\theta_{1}}^{2} 2 \sqrt[5]{N}+\alpha_{\theta_{2}}^{2}\right)}{\sqrt[3]{3} \alpha_{\theta_{2}}^{2}} . \tag{42}
\end{align*}
$$

One will assume that the embedding of the $D 6$-brane will be given by $i: \Sigma^{1,6}\left(\mathbb{R}^{1,3}, r, \theta_{2} \sim \frac{\alpha_{\theta_{2}}}{N \frac{3}{10}}, \tilde{y}\right) \hookrightarrow M^{1,9}$, effected by $\tilde{z}=\tilde{z}(r)$. The pull-back of the B-field along the directions of the D6-branes is given by

$$
\begin{align*}
& \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right) i^{*} B \\
& \quad=\delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right)\left[-B_{\theta_{2}} \tilde{z}^{\prime}(r) d r \wedge \mathrm{~d} \theta_{2}\right. \\
& \left.\quad+B_{\theta_{2} \tilde{y}} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \tilde{y}+B_{\theta_{2} \tilde{x}} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \tilde{x}\right], \tag{43}
\end{align*}
$$

where $B_{\theta_{2} \tilde{x}}, B_{\theta_{2} \tilde{y}}, B_{\theta_{2} \tilde{z}}$ can be read off from (117). Now, one can show that

Like [4], $\tilde{z}=$ constant, is a solution of (45). Alternatively, (45) is equivalent to $\frac{\tilde{z}^{\prime}(r)}{\sqrt{\Sigma_{0}\left(r ; g_{s}, N_{f}, N, M\right)+\Sigma_{1}\left(r ; g_{s}, N_{f}, N, M\right)\left(\tilde{z}^{\prime}\right)^{2}}}=$ $K$. Hence, the Euler-Lagrange equation for the $\tilde{z}(r)$ from the DBI action can be written in the following form:

$$
\begin{align*}
& z^{\prime}(r)^{2}-\frac{24461180928 \pi^{19 / 2} \alpha_{\theta_{1}}^{16} \alpha_{\theta_{2}}^{8} C_{1} g_{s}{ }^{4} K N^{49 / 5}}{C_{2}^{2}-24461180928 \pi^{19 / 2} \alpha_{\theta_{1}}^{16} \alpha_{\theta_{2}}^{8} C_{3} g_{s}{ }^{4} K N^{49 / 5}} \\
& \quad=0 \tag{46}
\end{align*}
$$

where $K$ is an arbitrary constant, while $C_{1}\left(r ; g_{s}, N_{f}, N\right)$, $C_{2}\left(r ; g_{s}, N_{f}, N\right)$ and $C_{3}\left(r ; g_{s}, N_{f}, N\right)$ up to NLO-in- $N$ after a large- $N$ expansion have the following forms:

$$
\begin{aligned}
& C_{1}\left(r ; g_{s}, N_{f}, N\right) \\
& =\frac{4194304 \pi^{17 / 2} \sqrt[5]{N} r^{6}\left(6 a^{2}+r^{2}\right)\left(27 \sqrt[3]{3} \alpha_{\theta_{1}}^{6}-12 \sqrt{6} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}+4 \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}-8 \alpha_{\theta_{2}}^{4}\right)}{\alpha_{\theta_{1}}^{6} \alpha_{\theta_{2}}^{4} g_{s}\left(9 a^{2}+r^{2}\right)},
\end{aligned}
$$

$C_{2}\left(r ; g_{s}, N_{f}, N\right)$

$$
=\frac{8388608 \pi^{8} \alpha_{\theta_{1}}^{2} \sqrt{g_{s}} N^{26 / 5} r^{4}\left(r^{4}-r_{h}^{4}\right)\left(81 \alpha_{\theta_{1}}^{6}-36 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}+43^{2 / 3} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}-43^{2 / 3} \alpha_{\theta_{2}}^{4}\right)}{27 \alpha_{\theta_{2}}^{2}},
$$

$$
\begin{align*}
& C_{3}\left(r ; g_{s}, N_{f}, N\right) \\
& =\frac{4194304 \pi^{8} N^{3 / 10} r^{4}\left(r^{4}-r_{h}^{4}\right)\left(81 \alpha_{\theta_{1}}^{6}-36 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}+43^{2 / 3} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}-43^{2 / 3} \alpha_{\theta_{2}}^{4}\right)}{27 \alpha_{\theta_{1}}^{6} \alpha_{\theta_{2}}^{6} g_{s}^{3 / 2}} . \tag{47}
\end{align*}
$$

$\operatorname{det}\left(i^{*}(g+B)\right)$

$$
\begin{equation*}
=\Sigma_{0}\left(r ; g_{s}, N_{f}, N, M\right)+\Sigma_{1}\left(r ; g_{s}, N_{f}, M, N\right)\left(\tilde{z}^{\prime}(r)\right)^{2} \tag{44}
\end{equation*}
$$

where the embedding functions $\Sigma_{0,1}\left(r ; g_{s}, N_{f}, M, N\right)$ are given in (B1).

Thus, the Euler-Lagrange equation of motion yields

$$
\begin{align*}
& \frac{d}{d r}\left(\frac{\tilde{z}^{\prime}(r)}{\sqrt{\Sigma_{0}\left(r ; g_{s}, N_{f}, N, M\right)+\Sigma_{1}\left(r ; g_{s}, N_{f}, N, M\right)\left(\tilde{z}^{\prime}\right)^{2}}}\right) \\
& \quad=0 . \tag{45}
\end{align*}
$$

Substituting the values of $C_{1}\left(r ; g_{s}, N_{f}, N\right), C_{2}\left(r ; g_{s}, N_{f}\right.$, $N)$ and $C_{3}\left(r ; g_{s}, N_{f}, N\right)$ in the differential equation presented above and keeping terms only up to NLO-in- $N$ after taking a large- $N$ expansion the differential equation acquires the following form:

$$
\begin{aligned}
& \tilde{z}^{\prime}(r)^{2}-\frac{590493^{2 / 3} \pi^{2} \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}^{6} g_{s}^{2} K\left(\frac{1}{N}\right)^{3 / 5}\left(6 a^{2}+r^{2}\right)}{2 r^{2}\left(9 a^{2}+r^{2}\right)\left(r^{4}-r_{h}^{4}\right)^{2}} \\
& -\frac{531441 \pi^{2} \alpha_{\theta_{1}}^{5} \alpha_{\theta_{2}}^{4} s_{s}^{2} K\left(\frac{1}{N}\right)^{4 / 5}\left(6 a^{2}+r^{2}\right)\left(4 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{2}}^{2}-9 \alpha_{\theta_{1}}^{3}\right)}{8 r^{2}\left(9 a^{2}+r^{2}\right)\left(r^{4}-r_{h}^{4}\right)^{2}} \\
& \quad=0 .
\end{aligned}
$$

Analogous to [4], from (45), one sees that $\tilde{z}=$ constant, is a valid solution and by choosing $\tilde{z}= \pm \mathcal{C} \frac{\pi}{2}$, one can choose the $D 6 / \overline{D 6}$-branes to be at "antipodal" points. Using a very similar computation for a thermal background with no black hole ( $r_{h}=0$ ), one can show that this constant embedding of $D 6$-branes, is still valid.

## 4 Vector meson spectroscopy in a black-hole background for all temperatures

Equipped with the embedding of the flavor D6-branes in the delocalized SYZ mirror of resolved warped deformed conifold of [1] from Sect. 3, we now proceed to obtaining the spectra as the Kaluza-Klein modes of the massless sector of open strings in type IIA at finite gauge coupling. In this and the next section, we do not worry about the issues like that the blackhole gravity dual is not considered at low temperatures where one must consider a thermal background. Happily, in Sect. 6, we will via an explicit computation verify that the mesonic spectra obtained in Sects. 4 (for [pseudo-]vector mesons) and 5 (for [pseudo-]scalar mesons), are nearly isospectral with one obtained by working with a thermal background without a black hole valid at only low temperatures.

We evaluate the masses of the (pseudo-)vector and (pesudo-)scalar mesons separately - the former by considering gauge fluctuations of a background gauge field along the world volume of the embedded flavor $D 6$-branes and the latter (without turning on a background gauge field) by looking at fluctuations of the embedding transverse to the world volume of the embedded $D 6$-branes.

As done in [4], let us redefine $(r, \tilde{z})$ in terms of new variables $(Y, Z)$ :
$r=r_{h} e^{\sqrt{Y^{2}+Z^{2}}}$,
$\tilde{z}=\mathcal{C} \arctan \frac{Z}{Y}$,
and the constant embedding of $D 6(\overline{D 6})$-branes can correspond to $\tilde{z}=\frac{\pi}{2}$ for $\mathcal{C}=1$ for $D 6$-branes and $\tilde{z}=-\frac{\pi}{2}$ for $\mathcal{C}=-1$ for $\overline{D 6}$-branes, both corresponding to $Y=0$. Now, consider turning on a gauge field fluctuation $\tilde{F} \frac{\sigma^{3}}{2}$ about a small background gauge field $F_{0} \frac{\sigma^{3}}{2}$ and the background $i^{*}(g+B)$. This implies

$$
\begin{aligned}
& \left.\operatorname{Str} \sqrt{\operatorname{det}_{\mathbb{R}^{1,3}, Z, \theta_{2}, \tilde{y}}\left(i^{*}(G+B)+\left(F_{0}+\tilde{F}\right) \frac{\sigma^{3}}{2}\right)}\right|_{Y=0} \\
& \quad \times \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right)=\sqrt{\operatorname{det}_{\theta_{2}, \tilde{y}}\left(i^{*}(g+B)\right)} \\
& \left.\operatorname{Str} \sqrt{\operatorname{det}_{\mathbb{R}^{1,3}, Z}\left(i^{*}(g+B)+\left(F_{0}+\tilde{F}\right) \frac{\sigma^{3}}{2}\right)}\right|_{Y=0}
\end{aligned}
$$

$$
\begin{align*}
& \times \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right) \\
= & \sqrt{\operatorname{det}_{\theta_{2}, \tilde{y}}\left(i^{*}(g+B)\right)} \sqrt{\operatorname{det}_{\mathbb{R}^{1,3}, Z}\left(i^{*} g\right)} \operatorname{Str} \\
& \times\left.\left(\mathbf{1}_{2}-\frac{1}{2}\left[\left(i^{*} g\right)^{-1}\left(\left(F_{0}+\tilde{F}\right) \frac{\sigma^{3}}{2}\right)\right]^{2}+\cdots\right)\right|_{Y=0} \\
& \times \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right) \tag{50}
\end{align*}
$$

Concentrating on the terms quadratic in $\tilde{F}$ :

$$
\begin{align*}
& S_{D 6}=-\frac{1}{2} \int \mathrm{~d}^{4} x \mathrm{~d} Z \mathrm{~d} \theta_{2} \mathrm{~d} \tilde{y} \\
& \quad \times \sqrt{\operatorname{det}_{\theta_{2}, \tilde{y}}\left(i^{*}(G+B)\right.} \sqrt{\operatorname{det}_{\mathbb{R}^{1,3}, Z}\left(i^{*} G\right)} \\
& \quad \times\left.\left[2 \sqrt{h} G^{Z Z} \tilde{G}^{\mu \nu} \tilde{F}_{\mu Z} \tilde{F}_{\nu Z}+h \tilde{G}^{\mu \mu_{1}} \tilde{G}^{\nu \nu_{1}} \tilde{F}_{\mu_{1} \nu} \tilde{F}_{\nu_{1} \mu}\right]\right|_{Y=0} \\
& \quad \times \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right) \tag{51}
\end{align*}
$$

where $\tilde{G}^{\mu \nu}$ are the unwarped $\mathbb{R}^{1,3}$ metric components. Substituting
$A_{\mu}\left(x^{\nu}, Z\right)=\sum_{n=1} B_{\mu}^{(n)}\left(x^{\nu}\right) \alpha_{n}^{\{\mu\}}(Z)$, no summation w.r.t. $\mu$,
$A_{Z}\left(x^{\nu}, Z\right)=\sum_{n=1} \phi^{(n)}\left(x^{\nu}\right) \beta_{n}(Z)$,
one obtains

$$
\begin{align*}
& \int \mathrm{d}^{4} x \mathrm{~d} Z\left(\mathcal{V}_{2}(Z) F_{\mu \nu}^{(n)} F_{(n)}^{\mu \nu} \alpha_{m}^{\{\mu\}}(Z) \alpha_{n}^{\{\mu\}}(Z)\right. \\
& \left.\quad+\mathcal{V}_{1}(Z) B_{\mu}^{(m)} B_{v}^{(n)} \dot{\alpha}_{m}^{\{\mu\}} \dot{\alpha}_{n}^{\{\mu\}}\right) \tag{53}
\end{align*}
$$

where $\mathcal{V}_{1,2}$ are given in (C1).
Now, $F_{\mu \nu}\left(x^{\rho},|Z|\right)=\sum_{n} \partial_{[\mu} B_{\nu]}^{(n)} \alpha_{n}(Z) \equiv F_{\mu \nu}^{(n)} \alpha_{n}(Z)$.
The EOM satisfied by $B_{\mu}\left(x^{\nu}\right)^{(n)}$ is $\partial_{\mu} \tilde{F}_{(n)}^{\mu \nu}+\partial_{\mu} \log$ $\sqrt{G_{\mathbb{R}^{1,3},|Z|}} \tilde{F}_{(n)}^{\mu \nu}=\partial_{\mu} \tilde{F}_{(n)}^{\mu \nu}=\mathcal{M}_{(n)}^{2} B_{(n)}^{\nu}$. After once integrating by parts, and utilizing the EOM for $B_{\mu}^{(n)}$, one rewrites (53) as

$$
\begin{align*}
& \int \mathrm{d}^{4} x \mathrm{~d} Z\left(-2 \mathcal{V}_{2}(Z) \mathcal{M}_{(m)}^{2} \alpha_{n}^{B_{\mu}} \alpha_{m}^{B_{\mu}}\right. \\
& \left.+\mathcal{V}_{1}(Z) \dot{\alpha}_{n}^{B_{\mu}} \dot{\alpha}_{m}^{B_{\mu}}\right) B^{\mu(n)} B_{\mu}^{(m)}, \tag{54}
\end{align*}
$$

which yields the following equations of motion:
$\alpha_{m}^{\{0\}}: \frac{d}{\mathrm{~d} Z}\left(\mathcal{V}_{1}(Z) \tilde{G}^{00}(Z) \dot{\alpha}_{m}^{\{0\}}\right)+2 \mathcal{V}_{2}(Z) \tilde{G}^{00} \mathcal{M}_{(m)}^{2} \alpha_{m}^{\{0\}}=0$,
$\alpha_{m}^{\{i\}}: \frac{d}{\mathrm{~d} Z}\left(\mathcal{V}_{1}(Z) \dot{\alpha}_{m}^{\{i\}}\right)+2 \mathcal{V}_{2}(Z) \mathcal{M}_{(m)}^{2} \alpha_{m}^{\{i\}}=0$.
Writing $a=r_{h}\left(0.6+4 \frac{g_{s} M^{2}}{N}\left(1+\log r_{h}\right)\right), m=\tilde{m}$ $\frac{r_{h}}{\sqrt{4 \pi g_{s} N}}$, one hence obtains the following EOMs:

$$
\begin{align*}
& \alpha_{n}^{\{i\} \prime \prime}(Z) \\
& +\alpha_{n}^{\{i\}}(Z)\left(\frac{-g_{s} N_{f}\left(e^{4|Z|}(-2 \log N+6|Z|+3)-2 \log N+6|Z|-3\right)-6 g_{s} N_{f}\left(e^{4|Z|}+1\right) \log \left(r_{h}\right)+8 \pi\left(e^{4|Z|}+1\right)}{\left(e^{4|Z|}-1\right)\left(g_{s} N_{f}(\log N-3|Z|)-3 g_{s} N_{f} \log \left(r_{h}\right)+4 \pi\right)}\right. \\
& -\frac{1}{N^{2}\left(g_{s} N_{f}(\log N-3 .|Z|)-3 . g_{s} N_{f} \log \left(r_{h}\right)+12.5664\right)^{2}}\left\{1.5 e^{-2|Z|}\left(4 . g_{s} M^{2} \log \left(r_{h}\right)+4 . g_{s} M^{2}+0.6 N\right)^{2}\right. \\
& \times\left[g_{s}{ }^{2} N_{f}{ }^{2}\left(2 . \log N^{2}-12 . \log N|Z|-6 . \log N+18 . Z^{2}+18 .|Z|+9 .\right)+18 . g_{s}{ }^{2} N_{f}{ }^{2} \log ^{2}\left(r_{h}\right)\right. \\
& +g_{s} N_{f} \log \left(r_{h}\right)\left(g_{s} N_{f}(-12 . \log N+36 .|Z|+18 .)-150.796\right)+g_{s} N_{f}(50.2655 \log N-150.796|Z|-75.3982) \\
& +315.827]\})+\alpha_{n}^{\{i\}}(Z) \frac{\tilde{m}^{2}\left(e^{2|Z|}-\frac{3 .\left(4 . g_{s} M^{2} \log \left(r_{h}\right)+4 . g_{s} M^{2}+0.6 N\right)^{2}}{N^{2}}\right)}{e^{4|Z|}-1}=0, \tag{56}
\end{align*}
$$

and

$$
\begin{align*}
& \alpha_{n}^{\{0\}}{ }^{\prime \prime}(Z)+\frac{\alpha_{n}^{\{0\}}{ }^{\prime}(Z)}{2\left(g_{s} N_{f} \log N-3 g_{s} N_{f} \log \left(r_{h}\right)-3 g_{s} N_{f}|Z|+4 \pi\right)^{2}} \\
& \quad \times\left\{\left(\frac { 1 } { N ^ { 2 } } \left\{e^{-2|Z|}\left(4 . g_{s} M^{2} \log \left(r_{h}\right)+4 . g_{s} M^{2}+0.6 N\right)^{2}\right.\right.\right. \\
& \quad \times\left(9 g _ { s } N _ { f } \left(-g_{s} N_{f} \log N+3 g_{s} N_{f} \log \left(r_{h}\right)+3 g_{s} N_{f}|Z|\right.\right. \\
& \\
& -4 \pi)+\left(2 g_{s} N_{f} \log (N)-6 g_{s} N_{f} \log \left(r_{h}\right)-6 g_{s} N_{f}|Z|\right. \\
& \\
& \left.-3 g_{s} N_{f}+8 \pi\right)\left(-3 g_{s} N_{f} \log N+9 g_{s} N_{f} \log \left(r_{h}\right)\right. \\
& \left.\left.\left.\quad+9 g_{s} N_{f}|Z|+9 g_{s} N_{f}-12 \pi\right)\right)\right\}+2\left(2 g_{s} N_{f} \log N\right. \\
& \left.\quad-6 g_{s} N_{f} \log \left(r_{h}\right)-6 g_{s} N_{f}|Z|-3 g_{s} N_{f}+8 \pi\right) \\
& \left.\left.\quad \times\left(g_{s} N_{f} \log N-3 g_{s} N_{f} \log \left(r_{h}\right)-3 g_{s} N_{f}|Z|+4 \pi\right)\right)\right\}  \tag{57}\\
& \quad+\alpha_{n}^{\{0\}}(Z) \frac{\tilde{m}^{2}\left(r_{h}^{2} e^{2|Z|}-\frac{3 . r_{h}^{2}\left(4 . g_{s} M^{2} \log \left(r_{h}\right)+4 . g_{s} M^{2}+0.6 N\right)^{2}}{N^{2}}\right)}{\left.r^{4|Z|}-1\right)} \\
& =0 .
\end{align*}
$$

We will now proceed to obtaining the (pseudo-)vector meson spectrum by three routes. The first will cater exclusively to an IR computation where we solve the $\alpha_{n}^{\{i\}}(Z)$ and $\alpha_{n}^{\{0\}}(Z)$ EOMs near the horizon. Imposing Neumann boundary condition at the horizon results in quantization of the (pseudo-)vector meson masses and via $N_{f}$ - and $M$ dependent contributions, we extract the temperature dependence of the (pseudo-)vector meson spectrum. We will see that up to LO in N , in the IR, there is a near isospectrality in the (pseudo-)vector meson spectrum obtained by solving the $\alpha_{n}^{\{i\}}(Z)$ and $\alpha_{n}^{\{0\}}(Z)$ EOMs. The second route will be to convert the $\alpha_{n}^{\{i\}}(Z)$ and $\alpha_{n}^{\{0\}}(Z)$ EOMs into Schr"odinger-like EOMs and to solve the same in the IR and UV separately and obtain (pseudo-)vector mass quantization by imposing Neumann boundary conditions at the horizon (IR)/asymptotic boundary (UV). It turns out the former yields a result, which up to LO in $N$, is of the same order as the IR results of route one. The UV-computations satisfy Neumann and/or Dirich-
let boundary conditions without any mass quantization condition. The third route catering to the IR-UV interpolating region and what gives us our main results that are directly compared with PDG results, is obtaining the (pseudo-)vector meson masses via WKB quantization condition. We also show that an IR WKB quantization (pseudo-)vector meson spectroscopy is nearly isospectral with the results of route one.
4.1 Vector meson spectrum from solution of EOMs near $r=r_{h}$

The $\alpha_{n}^{i}(Z)$ EOM, near the horizon, i.e., $Z=0(Y=0)$, is of the form

$$
\begin{equation*}
\alpha_{n}^{i \prime \prime}(Z)+\left(\frac{1}{|Z|}+\alpha_{1}\right) \alpha_{n}^{i \prime}(Z)+\left(\frac{\beta_{2}}{|Z|}+\alpha_{2}\right) \alpha_{n}^{i}(Z)=0 \tag{58}
\end{equation*}
$$

whose solution is given by

$$
\begin{align*}
& \left.\alpha_{n}^{\{i\}}(Z)=c_{1} e^{\frac{1}{2}|Z|\left(-\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}-\alpha_{1}\right.}\right) \\
& \quad \times U\left(-\frac{-\alpha_{1}+2 \beta_{2}-\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}, 1, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}|Z|\right) \\
& \quad+c_{2} e^{\frac{1}{2}|Z|\left(-\sqrt{\alpha_{1}^{2}-4 \alpha_{2}-\alpha_{1}}\right)} L_{\frac{-\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}-\alpha_{1}+2 \beta_{2}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}} \\
& \quad \times\left(|Z| \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}\right) \tag{59}
\end{align*}
$$

One sets $c_{2}=0$ as satisfying the Neumann boundary condition for the associate Laguerre function will not be feasible. From (55), one notes that the differential operator is even under $Z \rightarrow-Z$ - relevant to parity and charge conjugation [3] - and therefore one can think of solutions that are even or odd under $Z \rightarrow-Z$. As one knows from [3] that
$\alpha_{2 n}^{\{i\}}(-Z)=-\alpha_{2 n}^{\{i\}}(Z)$ and $\alpha_{2 n+1}^{\{i\}}(-Z)=\alpha_{2 n+1}^{\{i\}}(Z)$, (59) $\left(c_{2}=0\right)$ must be understood as

$$
\alpha_{n}^{\{i\}}(Z)=\left(\delta_{n, 2 \mathbb{Z}^{+}} \operatorname{Sign}(Z)+\delta_{n,\left(2 \mathbb{Z}^{+} \cup\{0\}\right)+1}\right)
$$

$$
\times e^{\frac{1}{2}|Z|\left(-\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}-\alpha_{1}\right)}
$$

$$
\times U\left(-\frac{-\alpha_{1}+2 \beta_{2}(n)-\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}}, 1,\right.
$$

$$
\begin{equation*}
\left.\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}|Z|\right) \tag{60}
\end{equation*}
$$

Setting $c_{2}=0$, one sees

$$
\begin{aligned}
& \alpha_{n}^{\{i\}}{ }^{\prime}(Z)=-\frac{1}{2} e^{-\frac{1}{2}|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}\right)} \\
& \quad \times\left[\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}\right)\right. \\
& \quad \times U\left(\frac{\alpha_{1}-2 \beta_{2}+\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}, 1, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}|Z|}\right) \\
& \quad+\left(\sqrt{\left.\alpha_{1}^{2}-4 \alpha_{2}+\alpha_{1}-2 \beta_{2}\right)}\right. \\
& \left.\quad \times U\left(\frac{\alpha_{1}-2 \beta_{2}+3 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}, 2, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}|Z|}\right)\right] \\
& = \\
& \quad-\frac{|Z| \Gamma\left(\frac{\alpha_{1}-2 \beta_{2}+\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}\right)}{1} \frac{2 \Gamma\left(\frac{\alpha_{1}-2 \beta_{2}+\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}\right)}{} \\
& \quad \beta_{2} \log \left(\alpha_{1}^{2}-4 \alpha_{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}\right) \psi^{(0)}\left(\frac{\alpha_{1}-2 \beta_{2}+\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}\right) \\
& -\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}-2 \beta_{2}\right) \psi^{(0)} \\
& \times\left(\frac{\alpha_{1}-2 \beta_{2}+3 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}\right) \\
& \left.+2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+2 \alpha_{1}+2 \beta_{2} \log |Z|-2 \beta_{2}+4 \gamma \beta_{2}\right\} \\
& +\mathcal{O}(|Z|) . \tag{61}
\end{align*}
$$

One therefore sees that one can impose the Neumann/ Dirichlet boundary condition $\alpha_{n}^{i \prime}\left(r=r_{h}\right)=0$ provided the following condition is imposed:

$$
\begin{equation*}
\frac{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}-2 \beta_{2}}{2 \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}=-n \in \mathbb{Z}^{-} \tag{62}
\end{equation*}
$$

One can show that (62) in the context of the EOM (56), for $a=0.6 r_{h}([16,19])$ :

$$
\begin{align*}
\alpha_{1} & =-1.08-\frac{9 . g_{s} M^{2}\left(4.8 \log \left(r_{h}\right)+4.8\right)}{\log N N} \\
& +\frac{1.5 M^{2}\left(4.8 \log \left(r_{h}\right)+4.8\right)\left(-18 . g_{s} N_{f} \log \left(r_{h}\right)+9 g_{s} N_{f}+75.398\right)}{N N_{f} \log ^{2}(N)} \\
& +\frac{3 . g_{s} M^{2}\left(4.8 \log \left(r_{h}\right)+4.8\right)}{N}+\frac{\left(-4.14 g_{s} N_{f}-3.016\right) \log \left(r_{h}\right)}{g_{s} N_{f} \log ^{2}(N)}+\frac{0.24}{\log N}, \\
\alpha_{2} & =\frac{g_{s} M^{2} \tilde{m}^{2}\left(7.2 \log \left(r_{h}\right)+7.2\right)}{N}+0.54 \tilde{m}^{2}, \\
\beta_{2} & =\frac{g_{s} M^{2} \tilde{m}^{2}\left(-3.6 \log \left(r_{h}\right)-3.6\right)}{N}-0.02 \tilde{m}^{2} . \tag{63}
\end{align*}
$$

Up to LO in $N$, NLO in $\log N$ (and assuming large $\left|\log r_{h}\right|$ ) LO in $\log r_{h}$, would yield the following meson spectrum:

$$
\begin{align*}
& \tilde{m}_{n}^{\alpha_{n}^{\langle i\}}}=0.5 \sqrt{-10800 . n^{2}-10800 . n+10800 \cdot \sqrt{(n+0.36)(n+0.5)(n+0.5)(n+0.64)}-2592 .} \\
& \quad+\frac{0.25\left(\frac{23.04(n+0.5)^{2}}{\sqrt{(n+0.36)(n+0.5)(n+0.5)(n+0.64)}}-24 .\right)}{\log N \sqrt{-10800 . n^{2}-10800 . n+10800 \cdot \sqrt{(n+0.36)(n+0.5)(n+0.5)(n+0.64)}-2592 .}} \\
& \quad+\frac{1}{g_{s}(\log N)^{2} \sqrt{-10800 . n^{2}-10800 . n+10800 \cdot \sqrt{(n+0.36)(n+0.5)(n+0.5)(n+0.64)}-2592 .} N_{f}} \\
& \quad \times\left\{0 . 2 5 \operatorname { l o g } ( r _ { h } ) \left(\frac{1}{((n+0.36)(n+0.5)(n+0.5)(n+0.64))^{3 / 2}}\right.\right. \\
& \quad \times\left\{n^{6}\left(-397.44 g_{s} N_{f}-289.529\right)+n^{5}\left(-1192.32 g_{s} N_{f}-868.588\right)+n^{4}\left(-1482.61 g_{s} N_{f}-1080.06\right)\right. \\
& \quad+n^{3}\left(-978.02 g_{s} N_{f}-712.473\right)+n^{2}\left(-360.915 g_{s} N_{f}-262.921\right) \\
& \left.\left.\left.\quad+n\left(-70.6251 g_{s} N_{f}-51.4493\right)-5.72314 g_{s} N_{f}-4.16922\right\}+414 . g_{s} N_{f}+301.593\right)\right\}+\mathcal{O}\left(\frac{1}{(\log N)^{3}}, \frac{1}{N}\right) . \tag{64}
\end{align*}
$$

Disregarding $n=0$ (as it yields a $\mathcal{O}\left(\frac{1}{N^{2}}\right)$-suppressed though imaginary value) one sees

$$
\begin{align*}
\tilde{m}_{n=1}^{\alpha_{0}^{(i)}}= & \frac{0.18\left(414 g_{s} N_{f}+0.089\left(-4487.65 g_{s} N_{f}-3269.19\right)+301.6\right) \log \left(r_{h}\right)}{g_{s} N_{f} \log ^{2}(N)} \\
& -\frac{0.15}{\log N}+0.69, \tilde{m}_{n=2}^{\alpha_{n}^{(i)}}=\frac{1}{g_{s} N_{f} \log ^{2}(N)}\left\{0 . 1 7 3 \operatorname { l o g } ( r _ { h } ) \left(0 . 0 0 4 \left(-5.723 g_{s} N_{f}\right.\right.\right. \\
& +16\left(-1482.61 g_{s} N_{f}-1080.06\right)+32\left(-1192.32 g_{s} N_{f}-868.588\right) \\
& +8\left(-978.02 g_{s} N_{f}-712.473\right)+4\left(-360.915 g_{s} N_{f}-262.92\right)+2\left(-70.625 g_{s} N_{f}-51.449\right) \\
& \left.\left.+64(-397.44 g s N f-289.529)-4.169)+414 . g_{s} N_{f}+904.77\right)\right\}-\frac{0.16}{\log N}+0.721 \tag{65}
\end{align*}
$$

Given that one is solving the EOM near the horizon, i.e., the
IR, one expects the masses to be small, something verified by (65).

Now, the EOM (57), near $r=r_{h}$, can be written as
$\alpha_{n}^{\{0\}}{ }^{\prime \prime}(Z)+\alpha_{1} \alpha_{n}^{\{0\}}{ }^{\prime}(Z)+\left(\frac{\beta_{2}}{|Z|}+\alpha_{2}\right) \alpha_{n}^{\{0\}}(Z)=0$,
whose solution is given by

$$
\begin{align*}
& \alpha_{n}^{\{0\}}(Z)=\left(\delta_{n, 2 \mathbb{Z}^{+}} \operatorname{Sign}(Z)+\delta_{n,\left(2 \mathbb{Z}^{+} \cup\{0\}\right)+1}\right) \\
& \left.\quad \times e^{-\frac{1}{2}|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}+\alpha_{1}\right.}\right) \\
& \quad \times U\left(1-\frac{\beta_{2}(n)}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}}, 2, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}|Z|\right) \tag{68}
\end{align*}
$$

Now

$$
\begin{align*}
& \left.\left.\alpha_{n}^{\{0\}}{ }^{\prime}(Z)\right|_{c_{2}=0}=-\frac{1}{2} c_{1} e^{-\frac{1}{2}|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}\right.}\right)\left[\left(|Z| \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}|Z|-2\right)\right. \\
& \left.\quad \times U\left(1-\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}, 2, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}|Z|\right)+2|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}-\beta_{2}\right) U\left(2-\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}, 3, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}}|Z|\right)\right] \\
& =\frac{c_{1}\left(\beta_{2} \log \left(\alpha_{1}^{2}-4 \alpha_{2}\right)+2 \beta_{2} \psi^{(0)}\left(1-\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}\right)+\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}+2 \beta_{2} \log |Z|+4 \gamma \beta_{2}\right)}{2 \beta_{2} \Gamma\left(-\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}\right)} \tag{69}
\end{align*}
$$

$$
\begin{align*}
& \left.\alpha_{n}^{\{0\}}(Z)=c_{1}|Z| e^{-\frac{1}{2}|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}+\alpha_{1}\right.}\right) \\
& \quad \times U\left(1-\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}}, 2, \sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)|Z|}\right) \\
& \left.\quad+c_{2}|Z| e^{-\frac{1}{2}|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}+\alpha_{1}\right.}\right) \\
& \quad \times F_{1}\left(1-\frac{\beta_{2}(n)}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)}} ; 2 ; \sqrt{\alpha_{1}^{2}-4 \alpha_{2}(n)|Z|}\right) \tag{67}
\end{align*}
$$

As $\frac{d}{\mathrm{~d} Z}\left[c_{2}|Z| e^{-\frac{1}{2}|Z|\left(\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}+\alpha_{1}\right.}\right){ }_{1} F_{1}\left(1-\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}} ;\right.$
2; $\left.\left.\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}|Z|\right)\right]$ vanishes at $|Z| \rightarrow 0^{+}$only for $c_{2}=0$, one sets $c_{2}=0$ at the very outset. Similar to (60):

Hence, one can successfully impose Neumann/Dirichlet boundary condition at the horizon: $\alpha_{n}^{\{0\}}{ }^{\prime}\left(r=r_{h}\right)=0$ by demanding:

$$
\begin{equation*}
\frac{\beta_{2}}{\sqrt{\alpha_{1}^{2}-4 \alpha_{2}}}=n \in \mathbb{Z}^{+} \tag{70}
\end{equation*}
$$

In the context of (57)

$$
\begin{aligned}
\alpha_{1}= & \frac{-\frac{3.016}{g_{s} N_{f}}+0.72 \log \left(r_{h}\right)-4.86}{\log N^{2}} \\
& +\frac{43.2 g_{s} M^{2} N_{f} \log \left(r_{h}\right)+43.2 g_{s} M^{2} N_{f}}{\log N N N_{f}} \\
& +\frac{g_{s} M^{2}\left(-14.4 \log \left(r_{h}\right)-14.4\right)}{N}+\frac{0.24}{\log N}+0.92
\end{aligned}
$$

$$
\begin{align*}
& \alpha_{2}=\frac{g_{s} M^{2} \tilde{m}^{2}\left(7.2 \log \left(r_{h}\right)+7.2\right)}{N}+0.54 \tilde{m}^{2} \\
& \beta_{2}=\frac{g_{s} M^{2} \tilde{m}^{2}\left(-3.6 \log \left(r_{h}\right)-3.6\right)}{N}-0.02 \tilde{m}^{2} . \tag{71}
\end{align*}
$$

One sees that (70) can be satisfied for only a single value of $\tilde{m}_{n}^{\alpha_{n}^{\{0\}}}$ - which we declare to be the ground state (largely due to the proximity of its value to (64)) - which satisfies the following condition:

$$
\begin{align*}
& \frac{1}{g_{s}{ }^{2} \log N^{4} N_{f}^{2}}\left\{\left(g_{s}{ }^{2} \log N M^{2} N_{f}(43.2-14.4 \log N)\right.\right. \\
& +g_{s} N_{f} \log \left(r_{h}\right)\left(g_{s} \log N M^{2}(43.2-14.4 \log N)\right. \\
& +0.72 N)+N\left(0.92 g_{s}(\log N-2.17166)(\log N\right. \\
& \left.\left.\left.+2.43253) N_{f}-3.016\right)\right)^{2}\right\}+\tilde{m}^{2} N\left(-28.8 g_{s} M^{2} \log \left(r_{h}\right)\right. \\
& \left.-28.8 g_{s} M^{2}-2.16 N\right)=0 \tag{72}
\end{align*}
$$

The solution when expanded in powers of $N$ and $\log N$

$$
\begin{align*}
& \tilde{m}_{n=0}^{\alpha_{n}^{\{0\}}}=\frac{0.163}{\log N}+0.626+\frac{-\frac{2.052}{g_{s} N_{f}}+0.49 \log \left(r_{h}\right)-3.307}{\log N^{2}} \\
& +\frac{g_{s} M^{2}\left(28.305 \log \left(r_{h}\right)+28.305\right)}{N \log N} \\
& +\frac{g_{s} M^{2}\left(-13.971 \log \left(r_{h}\right)-13.971\right)}{N}+\cdots \tag{73}
\end{align*}
$$

Now, from (65) and (73), disregarding $\mathcal{O}\left(\frac{\log r_{h}}{(\log N)^{2}}\right)$ terms, one sees that
$\tilde{m}_{n=1}^{\alpha^{(i)}}=\tilde{m}_{n=0}^{\alpha^{\{0\}}}, \quad$ for $N=105$.
Hence, from (65) and (73), one sees an IR isospectrality in the spectra of $\alpha_{n=1}^{i}$ and $\alpha_{n=0}^{0}$ mesons. Equation (73) beautifully captures the conformal $(N \rightarrow \infty)$, the non-conformal ( $N_{f}, M$-dependent) contributions as well as the temperature dependence via $\log r_{h}$ of vector mesons, and it does so explicitly. Also, from both (65) and (73), we see that the temperature dependence entering via $\log r_{h}$ does so at $\mathcal{O}\left(\frac{1}{N}\right)$.
4.2 Vector meson spectrum from conversion of $\alpha_{n}^{\{i\}}(Z)$ 's EOM to Schrödinger-like equations

The $\alpha_{n}^{\{i\}} \operatorname{EOM}$ (56), written as $\alpha_{n}^{\{i\}}{ }^{\prime \prime}(Z)+A(Z) \alpha_{n}^{\{i\}}{ }^{\prime}(Z)+$ $B(Z) \alpha_{n}^{\{i\}}(Z)=0$, with a field redefinition: $\psi_{n}^{i}(Z)=$ $\sqrt{\mathcal{C}_{1}(Z)} \alpha_{n}^{i}(Z)$, is converted to
$\psi_{n}^{\{i\}}{ }^{\prime \prime}(Z)+V\left(\alpha_{n}^{\{i\}}\right) \psi_{n}^{\{i\}}(Z)=0$,
where $V=\frac{\mathcal{C}_{1}^{\prime \prime}}{2 \mathcal{C}_{1}}-\frac{1}{4}\left(\frac{\mathcal{C}_{1}^{\prime}}{\mathcal{C}_{1}}\right)^{2}+B$. This potential for $\alpha_{n}^{\{i\}}(Z)$ can easily be worked out but due to the cumbersome nature of the
expression so obtained, we will not be giving its analytical expression.

For $\alpha_{n}^{\{i\}}$ vector mesons,

$$
\begin{align*}
\mathcal{C}_{1}= & -\frac{1}{2 r_{h}^{2}}\left\{e^{-4|Z|}\left(e^{4|Z|}-1\right)\right. \\
& \times\left(g_{s} N_{f} \log N\left(3 a^{2}+2{r_{h}}^{2} e^{2|Z|}\right)\right. \\
& -3 g_{s} N_{f} \log \left(r_{h}\right)\left(3 a^{2}+2{r_{h}}^{2} e^{2|Z|}\right) \\
& -9 a^{2} g_{s} N_{f}|Z|-9 a^{2} g_{s} N_{f}+12 \pi a^{2} \\
& \left.\left.-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|}|Z|+8 \pi r_{h}^{2} e^{2|Z|}\right)\right\} . \tag{76}
\end{align*}
$$

### 4.2.1 IR

The potential $V\left(\alpha_{n}^{\{i\}}\right)$, performing first a large- $N$ and then a small- $|Z|$ expansion, for $a=r_{h}\left(0.6+4 \frac{g_{s} M^{2}}{N}\left(1+\log r_{h}\right)\right)$ [19], is given by

$$
\begin{align*}
& V\left(\alpha_{n}^{\{i\}}\right)=\frac{1}{\left(e^{4|Z|}-1 .\right)^{2}}\left\{e ^ { - 2 | Z | } \left(e^{6|Z|}\left(6 .-1.08 \tilde{m}^{2}\right)\right.\right. \\
& \left.\left.\quad+e^{4|Z|}\left(2.16-\tilde{m}^{2}\right)+\tilde{m}^{2} e^{8|Z|}+\left(1.08 \tilde{m}^{2}-1\right) e^{2|Z|}-e^{10|Z|}-2.16\right)\right\} \\
& \quad+\frac{e^{-2|Z|}\left(g_{s}^{3} N_{f}^{3}\left(4.86-3 . e^{2|Z|}-3.24 e^{4|Z|}-1.62 e^{8|Z|}+3 . e^{10|Z|}\right)\right)}{g_{s}^{3} \log N N_{f}^{3}\left(e^{4|Z|}-1 .\right)^{2}} \\
& \quad+\mathcal{O}\left(\frac{1}{(\log N)^{2}}, \frac{g_{s} M^{2}}{N}\right) . \tag{77}
\end{align*}
$$

In the IR, (77) yields

$$
\begin{align*}
& V\left(\alpha_{n}^{\{i\}} ; I R\right)=\frac{-0.02 \tilde{m}^{2}-\frac{0.12}{\log N}+0.54}{|Z|} \\
& \quad+0.54 \tilde{m}^{2}+\frac{4.86}{\log N}+\frac{0.25}{Z^{2}}-3.49333 \\
& \quad+\mathcal{O}\left(|Z|, \frac{1}{(\log N)^{2}}, \frac{g_{s} M^{2}}{N}\right) \tag{78}
\end{align*}
$$

The solution to (77) is given in terms of Whittaker functions by

$$
\begin{align*}
& \psi_{n}^{\{i\}}(Z)=c_{1} M \frac{\log N\left(0.27-0.01 \tilde{m}^{2}\right)-0.06}{\sqrt{\log N} \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86}}, 0 \\
& \times\left(\frac{2 \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86}|Z|}{\sqrt{\log N}}\right) \\
& +c_{2} W \frac{\log N\left(0.27-0.01 \tilde{m}^{2}\right)-0.06}{\sqrt{\log N} \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86}}, 0 \\
& \times\left(\frac{2 \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86}|Z|}{\sqrt{\log N}}\right) . \tag{79}
\end{align*}
$$

One can show that

$$
\begin{equation*}
\left.\left.\frac{d}{\mathrm{~d} Z}\left(\frac{M \frac{\log N\left(0.27-0.01 \tilde{m}^{2}\right)-0.06}{\sqrt{\log N} \sqrt{-0.54 \log N \tilde{m}^{2}+3.4333 \log N-4.86}}, 0}{}\left(\frac{2 \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86|Z|}}{\sqrt{\log N}}\right)\right)\right|^{\sqrt{C_{1}}}\right)\left.\right|_{r=r_{h}}=0 \tag{80}
\end{equation*}
$$

implies
One can show that the Neumann boundary condition
$\tilde{m}=\left(2.543-\frac{1.769}{\log N}\right)+\mathcal{O}\left(\left(\frac{1}{\log N}\right)^{3 / 2}\right)$.
One can also show that

$$
\begin{equation*}
\left.\frac{d}{\mathrm{~d} Z}\left(\frac{W \frac{\log N\left(0.27-0.01 \tilde{m}^{2}\right)-0.06}{\sqrt{\log N \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86}}, 0}\left(\frac{2 \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86|Z|}}{\sqrt{\log N}}\right)}{\sqrt{C_{1}}}\right)\right|_{r=r_{h}}=0 \tag{82}
\end{equation*}
$$

implies
$\tilde{m}_{n}^{\alpha_{n}^{i}}=0.5 \sqrt{-10800 . n^{2}+10800 \cdot \sqrt{(n+0.376679)(n+0.623321)\left(n^{2}+n+0.25\right)}-10800 . n-2592}$.

The $n=0$ result of (83) -2.479 - is close to the LO result in (81). Once again, from considerations of parity and charge conjugation, similar to (60) and (68),
as well as the Dirichlet boundary conditions is identically satisfied. This hence does not yield values for $\tilde{m}$. Similarly, one can show that the Neumann boundary condition:
$\alpha_{n=0}^{\{i\}}(Z)=\operatorname{Sign}(Z) \frac{M \text { or } W \frac{\log N\left(0.27-0.01 \tilde{m}^{2}\right)-0.06}{\sqrt{\log N} \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86}}, 0}{\sqrt{\mathcal{C}_{1}(Z)}}\left(\frac{2 \sqrt{-0.54 \log N \tilde{m}^{2}+3.49333 \log N-4.86|Z|}}{\sqrt{\log N}}\right)$.

### 4.2.2 UV

Neglecting $N_{f}, M$-dependent terms in the potential in the

$$
\begin{equation*}
\lim _{Z \rightarrow \infty} \frac{d}{\mathrm{~d} Z}\left(\frac{c_{2} K_{1}\left(0.2 i e^{-|Z|} \sqrt{25 \cdot \tilde{m}^{2}+54 .}\right)}{\sqrt{C_{1}}}\right)=0 \tag{88}
\end{equation*}
$$ UV (as both become very small), one obtains

$$
\begin{align*}
& V\left(\alpha_{n}^{\{i\}} ; U V\right) \\
& =\frac{e^{-2|Z|}\left(e^{6|Z|}\left(6 .-1.08 \tilde{m}^{2}\right)+e^{4|Z|}\left(2.16-1 \tilde{m}^{2}\right)+\tilde{m}^{2} e^{8|Z|}+\left(1.08 \tilde{m}^{2}-1\right) e^{2|Z|}-e^{10|Z|}-2.16\right)}{\left(e^{4|Z|}-1\right)^{2}} \\
& =-1+\left(2.16+\tilde{m}^{2}\right) e^{-2|Z|}+\mathcal{O}\left(e^{-4|Z|}\right) . \tag{85}
\end{align*}
$$

The solution to the Schrödinger-like equation is

$$
\begin{align*}
& \psi_{n}^{\{i\}}(|Z| \in \mathrm{UV})=\left(\delta_{n, 2 \mathbb{Z}^{+}} \operatorname{Sign}(Z)+\delta_{n,\left(2 \mathbb{Z}^{+} \cup\{0\}\right)+1}\right) \\
& \quad \times\left[c_{1} I_{1}\left(0.2 i e^{-|Z|} \sqrt{25 \cdot \tilde{m}^{2}(n)+54 .}\right)\right. \\
& \left.\quad+c_{2} K_{1}\left(0.2 i e^{-|Z|} \sqrt{25 \tilde{m}^{2}(n)+54}\right)\right] \tag{86}
\end{align*}
$$

but not the Dirichlet boundary condition, is identically satisfied, therefore not providing values for $\tilde{m}$.
4.3 Vector meson spectrum from conversion of $\alpha_{n}^{\{0\}}(Z)$ 's EOM to Schrödinger-like equations

### 4.3.1 IR

One can show that

$$
\begin{align*}
& V\left(\alpha_{n}^{\{0\}} ; I R\right)=-1+|Z|\left(\frac{3.24}{\log N}-0.526667 \tilde{m}^{2}\right) \\
& -\frac{0.02 \tilde{m}^{2}}{|Z|}+0.54 \tilde{m}^{2}-\frac{3.24 Z^{2}}{\log (N)}+\frac{1.38}{\log N} \\
& \quad+\mathcal{O}\left(\frac{1}{(\log N)^{2}}, \frac{g_{s} M^{2}}{N}, Z^{3}\right) \tag{89}
\end{align*}
$$

The solution to
$\psi_{n}^{0 \prime \prime}(Z)+\left(\frac{a_{1}}{|Z|}+b_{1}\right) \psi_{n}^{0}(Z)=0$
is given by

$$
\begin{align*}
& \psi_{n}^{0}(Z)=c_{2}|Z| e^{-\sqrt{-b_{1}}|Z|}{ }_{1} F_{1}\left(\frac{-a_{1}}{2 \sqrt{-b_{1}}}+1 ; 2 ; 2 \sqrt{-b_{1}}|Z|\right) \\
& \quad+c_{1}|Z| e^{-\sqrt{-b_{1}}|Z|} U\left(\frac{-a_{1}}{2 \sqrt{-b_{1}}}+1,2,2 \sqrt{-b_{1}}|Z|\right) . \tag{91}
\end{align*}
$$

One can show that near $r=r_{h}$

$$
\begin{align*}
\frac{d}{\mathrm{~d} Z} & \left(\frac{c_{1}|Z| e^{-\sqrt{-b_{1}}|Z|} U\left(\frac{-a_{1}}{2 \sqrt{-b_{1}}}+1,2,2 \sqrt{-b_{1}}|Z|\right)}{\sqrt{C_{1}}(Z)}\right) \\
= & \frac{1}{2 a_{1} C_{1}(0)^{3 / 2}\left(a_{1} \sqrt{-b_{1}}+2 b_{1}\right) \Gamma\left(-\frac{a_{1}}{2 \sqrt{-b_{1}}}\right)} \\
& \times\left\{c _ { 1 } \left(2 a_{1}^{2} \sqrt{-b_{1}} C_{1}(0) \log |Z|+4 \gamma a_{1}^{2} \sqrt{-b_{1}} C_{1}(0)\right.\right. \\
& +a_{1} \sqrt{-b_{1}} C_{1}^{\prime}(0)+4 a_{1} b_{1} C_{1}(0) \log |Z| \\
& -2 a_{1} b_{1} C_{1}(0)+8 \gamma a_{1} b_{1} C_{1}(0) \\
& +2 a_{1} C_{1}(0)\left(a_{1} \sqrt{-b_{1}}+2 b_{1}\right) \log \left(2 \sqrt{-b_{1}}\right) \\
& +2 a_{1} C_{1}(0)\left(a_{1} \sqrt{-b_{1}}+2 b_{1}\right) \psi{ }^{(0)}\left(1-\frac{a_{1}}{2 \sqrt{-b_{1}}}\right) \\
& \left.\left.+2 b_{1} C_{1}^{\prime}(0)+4 \sqrt{-b_{1}} b_{1} C_{1}(0)\right)\right\}+\mathcal{O}(Z) \tag{92}
\end{align*}
$$

Hence by requiring
$\frac{a_{1}}{2 \sqrt{-b_{1}}}=n \in \mathbb{Z}^{+} \cup\{0\}$,
one can impose the Neumann boundary condition at the horizon, $r=r_{h}$. With
$a_{1}=-0.02 \tilde{m}^{2}$,
$b_{1}=-1+0.54 \tilde{m}^{2}+\frac{1.38}{\log N}$,
this yields

$$
\begin{align*}
m_{n}^{\alpha_{n}^{\{0\}}} & =0.5 \sqrt{10800 \cdot \sqrt{n^{4}+0.001 n^{2}}-10800 n^{2}} \\
& -\frac{2.56 n^{2}}{\log N \sqrt{n^{4}+0.001 n^{2}} \sqrt{10800 \cdot \sqrt{n^{4}+0.001 n^{2}}-10800 n^{2}}} \\
& +\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right) \\
= & \left\{1.36059-\frac{0.938489}{\log N}, 1.36077\right. \\
& \left.-\frac{0.93885}{\log N}, 1.3608-\frac{0.938917}{\log N}, 1.36081-\frac{0.938941}{\log N}, \ldots\right\} . \tag{95}
\end{align*}
$$

### 4.3.2 UV

In the UV disregarding the $M$ and $N_{f}$ (as there is no net $D 7-$ brane and $D 5$-brane charge in the UV in [1] and therefore in their mirror in [2])

$$
\begin{align*}
& V\left(\alpha^{\{0\}} ; U V\right)=\frac{e^{-2|Z|}\left(3 . e^{2|Z|}-1.62\right)}{\log N} \\
& \quad+\frac{\tilde{m}^{2} e^{2|Z|}}{e^{4|Z|}-1 .}-\frac{1.08 \tilde{m}^{2}}{e^{4|Z|}-1 .}-1 \\
& \quad=-1+e^{-2|Z|}\left(\tilde{m}^{2}-\frac{1.62}{\log N}\right)+\frac{3}{\log N}+\mathcal{O}\left(e^{-4|Z|}\right) \tag{96}
\end{align*}
$$

The solution to
$\psi_{n}^{\{0\} \prime \prime}(Z)+\left(A+B e^{-|Z|}\right) \psi_{n}^{\{0\}}(Z)=0$
is given by

$$
\begin{align*}
\psi_{n}^{\{0\}}(Z)= & c_{1} J_{-i \sqrt{A}}\left(\sqrt{B} \sqrt{e^{-2|Z|}}\right) \\
& +c_{2} J_{i \sqrt{A}}\left(\sqrt{B} \sqrt{e^{-2|Z|}}\right) . \tag{98}
\end{align*}
$$

One can show that $\psi_{n}^{0}(Z)$ does not satisfy the Dirichlet boundary condition in the UV but the Neumann boundary condition,
$\lim _{Z \rightarrow \infty} \frac{d}{\mathrm{~d} Z}\left(\frac{\psi_{n}^{\{0\}}(Z)}{\sqrt{\mathcal{C}_{1}(Z)}}\right)=0$,
is identically satisfied in the UV and hence one does not obtain any quantization condition on the masses $\tilde{m}$.
$4.4 \alpha_{n}^{\{i\}}(Z)$ meson spectroscopy from WKB quantization

The potential in the Schrödinger-like EOM having converted the $\alpha_{n}^{\{i\}}(Z)$-EOM to the same, is given by (75). To keep
the calculations tractable, we first perform a large- $N$ expansion of the potential and work up to LO in $N$, then expand $\sqrt{V\left(\alpha_{n}^{\{i\}}\right)}$ up to NLO in $\log N$.

Performing first a large- $N$ expansion, one obtains the following:

$$
\begin{align*}
& +\frac{1}{\tilde{m} N}\left(1.5 e^{-|Z|}+2.47 e^{-3|Z|}\right) \\
& +\mathcal{O}\left(\frac{1}{\tilde{m}^{2}}, \frac{1}{(\log N)^{2}}, e^{-5|Z|}\right) \tag{103}
\end{align*}
$$

$$
\begin{align*}
& \sqrt{V^{\alpha_{n}^{\{i\}}}(\tilde{m}, N)} \\
& =\sqrt{\frac{e^{-2|Z|}\left(e^{6|Z|}\left(6 .-1.08 \tilde{m}^{2}\right)+e^{4|Z|}\left(2.16-\tilde{m}^{2}\right)+\tilde{m}^{2} e^{8|Z|}+\left(1.08 \tilde{m}^{2}-1 .\right) e^{2|Z|}-e^{10|Z|}-2.16\right)}{\left(e^{4|Z|}-1 .\right)^{2}}} \\
& \quad-\frac{0.75\left(e^{4|Z|}-1\right)\left(2 . e^{2|Z|}-1.08 e^{4|Z|}+2 . e^{6|Z|}-3.24\right)}{\log N\left(-\tilde{m}^{2} e^{8|Z|}+\left(1-1.08 \tilde{m}^{2}\right) e^{2|Z|}+\left(\tilde{m}^{2}-2.16\right) e^{4|Z|}+\left(1.08 \tilde{m}^{2}-6 .\right) e^{6|Z|}+e^{10|Z|}+2.16\right)} \\
& \quad \times \sqrt{\frac{e^{-2|Z|}\left(e^{6|Z|}\left(6 .-1.08 \tilde{m}^{2}\right)+e^{4|Z|}\left(2.16-1 . \tilde{m}^{2}\right)+\tilde{m}^{2} e^{8|Z|}+\left(1.08 \tilde{m}^{2}-1 .\right) e^{2|Z|}-e^{10|Z|}-2.16\right)}{\left(e^{4|Z|}-1\right)^{2}}} \\
& \quad+\mathcal{O}\left(\left(\frac{1}{\log N}\right)^{2}\right) . \tag{100}
\end{align*}
$$

(a) Large- $\tilde{m}$ expansion: UV regime, i.e., $r>0.6 \sqrt{3} r_{h}$ or $|Z|>0.04$

One notes from (100) that $\sqrt{V} \in \mathbb{R}$ in the UV for large $\tilde{m}$ :

$$
\begin{align*}
& \sqrt{0.5 \tilde{m}^{2}-0.1 \sqrt{25 \cdot \tilde{m}^{4}-108 \cdot \tilde{m}^{2}}}<e^{|Z|} \\
& \quad<\sqrt{0.5 \tilde{m}^{2}+0.1 \sqrt{25 \cdot \tilde{m}^{4}-108 \cdot \tilde{m}^{2}}} \tag{101}
\end{align*}
$$

or

$$
\begin{align*}
& |Z| \in\left[\log \left(1.039+\frac{0.561}{\tilde{m}^{2}}+\mathcal{O}\left(\frac{1}{\tilde{m}^{3}}\right)\right)\right. \\
& \left.\quad \log \left(\tilde{m}-\frac{0.54}{\tilde{m}}+\mathcal{O}\left(\frac{1}{\tilde{m}^{3}}\right)\right)\right] \tag{102}
\end{align*}
$$

Thus, after performing a large- $N$ expansion, followed by a large- $\tilde{m}$ expansion and then a large- $|Z|$ expansion, one obtains

Finally, we have the WKB quantization condition
$\int_{\log \left(1.039+\frac{0.561}{\tilde{m}^{2}}\right)}^{\log \left(\tilde{m}-\frac{0.54}{\tilde{m}}\right)} \sqrt{V}=\left(n+\frac{1}{2}\right) \pi$
and up to $\mathcal{O}\left(\frac{1}{\log N}\right)$ obtains the results of Table 1.
(b) Small- $\tilde{m}$ expansion

We expand $\sqrt{V\left(\alpha_{n}^{\{i\}}\right)}$ up to $\mathcal{O}\left(\frac{1}{\log N}, \tilde{m}^{4}\right)$. One can show that $\sqrt{V} \in \mathbb{R}$ for $Z \in[0.01,0.47]$; given that $Z=0.0385$ corresponding to $r=\mathcal{R}_{D 5 / \overline{D 5}}$ - the $D 5-\overline{D 5}$ separation we put the lower limit by hand as 0.01 . One can show from the WKB quantization condition
$\int_{0.01}^{0.47} \mathrm{~d} Z \sqrt{V\left(\alpha_{n}^{\{i\}} ; Z\right)}=\left(n+\frac{1}{2}\right) \pi$,
that the following IR vector meson spectrum is generated:

$$
\begin{align*}
& m_{n}(I R) \\
& \quad=0.5 \sqrt{\frac{3.036-0.1136 \log N}{0.068 \log N+56.946}+2 \cdot \sqrt{\frac{(0.854513 n-0.0765252) \log ^{2}(N)+(715.605 n-67.1225) \log N-134.138}{(0.068 \log N+56.946)^{2}}}} . \tag{106}
\end{align*}
$$

$$
\begin{aligned}
& \sqrt{V^{\alpha_{n}^{\{i\}}}(\tilde{m}, N)}=\left(e^{-|Z|}-0.54 e^{-3|Z|}\right) \tilde{m} \\
& \quad+\frac{1}{\tilde{m}}\left(-0.5 e^{-|Z|}-0.27 e^{-|Z|}+2.03 e^{-3|Z|}\right)
\end{aligned}
$$

Happily, the ground state is non-zero and, for $N=6000$, is 0.81 - not that far off from the value $0.694-\frac{0.155}{\log N}$ in (65) obtained by solving the $\alpha_{n}^{\{i\}}(Z)$ equation of motion near $r=r_{h}$ or $Z=0-$ for $N=6000$ the same yields 0.677 .

Table 1 (Pseudo-)vector meson masses from WKB quantization applied to $V\left(\alpha_{n}^{\{i\}}\right)$

|  | (Pseudo-)vector <br> meson name | $J^{P C}$ | $m_{n>0}\left(\right.$ units of $\left.\frac{r_{h}}{\sqrt{4 \pi g_{s} N}}\right)$ | PDG mass <br> $(\mathrm{MeV})$ |
| :--- | :--- | :--- | :--- | :--- |
| $B_{\mu}^{(1)}$ | $\rho[770]$ | $1^{++}$ | $7.649-\frac{1.759}{\log N}$ | 775.49 |
| $B_{\mu}^{(2)}$ | $a_{1}[1260]$ | $1^{--}$ | $11.60-\frac{1.792}{\log N}$ | 1230 |
| $B_{\mu}^{(3)}$ | $\rho[1450]$ | $1^{++}$ | $15.535-\frac{1.81}{\log N}$ | 1465 |
| $B_{\mu}^{(4)}$ | $a_{1}[1640]$ | $1^{--}$ | $19.462-\frac{1.821}{\log N}$ | 1647 |

$4.5 \alpha_{n}^{\{0\}}$ spectroscopy from WKB quantization
Writing $m=\tilde{m} \frac{r_{h}}{\sqrt{4 \pi g_{s} N s}}, a=r_{h}\left(0.6+4 \frac{g_{s} M^{2}}{N}\left(1+\log r_{h}\right)\right)$, one can obtain the Schrödinger-like potential for $\alpha_{n}^{\{0\}}(Z)-$ due to its cumbersome form, we will not give the explicit form of its analytical expression.

After retaining terms up LO in $N$ in the potential, the square root of the Schrödinger-like potential for $\alpha_{n}^{\{0\}}(Z)$ after a large- $(\log ) N$ expansion yields

$$
\begin{align*}
& \sqrt{V^{\alpha_{n}^{\alpha 0\}}}(|Z|, N, \tilde{m})} \\
& =\sqrt{\frac{\tilde{m}^{2} e^{2|Z|}}{e^{4|Z|}-1}-\frac{1.08 \tilde{m}^{2}}{e^{4|Z|}-1 .}+0 . e^{-2|Z|}-1 .} \\
& \quad+\frac{e^{-2|Z|}\left(1.5 e^{2|Z|}-0.81\right)}{\log N \sqrt{\frac{\tilde{m}^{2} e^{2|Z|}}{e^{4|Z|}-1 .}-\frac{1.08 \tilde{m}^{2}}{e^{4|Z|}-1}}} \\
& \quad+\mathcal{O}\left(\left(\frac{1}{\log N}\right)^{2}\right) \tag{107}
\end{align*}
$$

### 4.5.1 Large- $\tilde{m}$ expansion

One can show that $V\left(\alpha_{n}^{\{0\}}\right) \in \mathbb{R}$ provided

$$
\begin{align*}
& 0.5 \log \left(0.1\left(5 \tilde{m}^{2}-\sqrt{25 \cdot \tilde{m}^{4}-108 \tilde{m}^{2}+100}\right)\right)<|Z| \\
& \quad<0.5 \log \left(0.1\left(5 \tilde{m}^{2}+\sqrt{25 \tilde{m}^{4}-108 \tilde{m}^{2}+100}\right)\right) \tag{108}
\end{align*}
$$

or

$$
\begin{align*}
& |Z| \in\left[0.0385+\mathcal{O}\left(\frac{1}{\tilde{m}^{2}}\right), \log \tilde{m}-\frac{0.54}{\tilde{m}^{2}}+\mathcal{O}\left(\frac{1}{\tilde{m}^{3}}\right)\right] \\
& \quad \approx[0.0385, \log \tilde{m}] \tag{109}
\end{align*}
$$

which will be the turning points for the WKB quantization condition implementation.

One obtains the following large- $\tilde{m}$ expansion from (107):

$$
\begin{aligned}
& \sqrt{V^{\alpha_{n}^{\{0\}}}(|Z|, N, \tilde{m})}=\tilde{m}\left(e^{-|Z|}-0.54 e^{-3|Z|}\right) \\
& +\frac{1}{\tilde{m}}\left(-\frac{e^{-|Z|}}{2}-0.27 e^{-|Z|}+0.03 e^{-3|Z|}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{\tilde{m} \log N}\left(1.5 e^{|Z|}-0.531 e^{-3|Z|}\right) \\
& +\mathcal{O}\left(\frac{1}{(\log N)^{2}}, \frac{1}{\tilde{m}^{2}}, e^{-5|Z|}\right) \tag{110}
\end{align*}
$$

The WKB quantization condition,
$\int_{0.0385}^{\log \tilde{m}} \mathrm{~d} Z \sqrt{V^{\alpha_{n}^{[0]}}(Z)}=\left(n+\frac{1}{2}\right) \pi$
yields a cubic of the form: $a+b \tilde{m}+\frac{c}{\tilde{m}}+\frac{d}{\tilde{m}^{2}}=g$ where
$a=-1.5+\frac{1.5}{\log N}$,
$b=0.802$,
$c=0.269-\frac{1.717}{\log N}$,
$d=0.27$,
$g=\left(n+\frac{1}{2}\right) \pi$.
The only real root up to $\mathcal{O}\left(\frac{1}{\log N}\right)$ yields the vector meson spectrum of Table 2 (disregarding $n=0$ as it does not satisfy the large- $\tilde{m}$ assumption).

One hence notes a near isospectrality between the (pseudo)vector meson spectra from Tables 1 and 2, and as will be seen in Table 4, upon comparison with PDG, it is the results of Table 2 that are slightly closer to the PDG values than those of Table 1 (Table 3).

The WKB quantization does not work for $\alpha_{n}^{\{0\}}(Z)$ for small $\tilde{m}$; as can easily be shown in the large- $N$ limit there are no turning points of the potential.

## 5 Scalar meson spectroscopy using a black-hole background for all temperatures

Unlike vector meson spectroscopy, the scalar meson spectrum will be obtained by considering a fluctuation of the $D 6$-brane world volume along $Y$ by switching off any $D 6$ brane world-volume fluxes as in [4]. Now, $Y \neq 0$ and the D6-brane metric (41), using (49) and the embedding
$Y=Y\left(x^{\mu}, Z\right)$,

Table 2 (Pseudo-)vector meson masses from WKB quantization applied to $V\left(\alpha_{n}^{\{0\}}\right)$

|  | (Pseudo-)vector meson name | $J^{P C}$ | $m_{n>0}\left(\right.$ units of $\left.\frac{r_{h}}{\sqrt{4 \pi g_{s} N}}\right)$ | PDG mass $(\mathrm{MeV})$ |
| :--- | :--- | :--- | :--- | :--- |
| $B_{\mu}^{(1)}$ | $\rho[770]$ | $1^{++}$ | $7.698-\frac{1.604}{\log N}$ | 775.49 |
| $B_{\mu}^{(2)}$ | $a_{1}[1260]$ | $1^{--}$ | $11.634-\frac{1.692}{\log N}$ | 1230 |
| $B_{\mu}^{(3)}$ | $\rho[1450]$ | $1^{++}$ | $15.56-\frac{1.736}{\log N}$ | 1465 |
| $B_{\mu}^{(4)}$ | $a_{1}[1640]$ | $1^{--}$ | $19.483-\frac{1.762}{\log N}$ | 1647 |

Table 3 The lightest sector meson masses

| $m_{n=1}$ | $1.331-\frac{0.167}{\log N}$ |
| :--- | :--- |
| $m_{n=2}$ | $1.958-\frac{0.226}{\log N}$ |

is therefore

$$
\begin{align*}
& G_{6 \mu \nu}^{\mathrm{IIA}} \mathrm{~d} x^{\mu} \mathrm{d} x^{\nu}=G_{\mu \nu}^{\mathrm{IIA}}\left(1+\mathcal{C}_{1}\left(x^{\kappa}, Z\right) G_{\mathrm{IIA}}^{\rho \lambda} \partial_{\rho} Y \partial_{\lambda} Y\right) \\
& \quad \times \mathrm{d} x^{\mu} \mathrm{d} x^{\nu}+\mathcal{C}_{2}\left(x^{\kappa}, Z, \dot{Y}\right) \mathrm{d} Z^{2}+\mathcal{C}_{3}\left(x^{\kappa}, Z, \dot{Y}\right) \mathrm{d} x^{\mu} \mathrm{d} Z \partial_{\mu} Y \\
& G_{\theta_{2} \theta_{2}}^{\mathrm{II}} \mathrm{~d} \theta_{2}^{2}+G_{\tilde{y} \tilde{y}}^{\mathrm{IA}} \mathrm{~d} \tilde{y}^{2}, \tag{114}
\end{align*}
$$

where

$$
\begin{align*}
& \mathcal{C}_{1}\left(x^{\kappa}, Z\right)=\mathcal{A} Y^{2}+\mathcal{B} Z^{2} \\
& \mathcal{C}_{2}\left(x^{\kappa}, Z, \dot{Y}\right)=\left(\mathcal{A} Y^{2}+\mathcal{B} Z^{2}\right) \dot{Y}^{2} \\
& \quad+\left(\mathcal{A} Y^{2}+\mathcal{B} Z^{2}\right)+2 Y Z(\mathcal{A}-\mathcal{B}) \dot{Y}, \\
& \mathcal{C}_{3}\left(x^{\kappa}, Z, \dot{Y}\right)=2\left(\mathcal{A} Y^{2}+\mathcal{B} Z^{2}\right) \dot{Y}+2 Y Z(\mathcal{A}-\mathcal{B}) \tag{115}
\end{align*}
$$

where
$\mathcal{A}=\frac{G_{r r}^{\mathrm{IA}} r_{h}^{2} e^{2 \sqrt{Y^{2}+Z^{2}}}}{\left(Y^{2}+Z^{2}\right)}$,
$\mathcal{B}=\frac{G_{\tilde{Z} \tilde{z}}^{\mathrm{IIA}}}{\left(Y^{2}+Z^{2}\right)^{2}}$.
$B_{\text {NS }-\mathrm{NS}}^{\text {IA }}$ [2] in diagonal basis $\left(\theta_{2}, \tilde{x}, \tilde{y}, \tilde{z}\right)$ is given by
$B^{I I A}=B_{\theta_{2} \tilde{y}} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \tilde{y}+B_{\theta_{2} \tilde{z}} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \tilde{z}+B_{\theta_{2} \tilde{x}} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \tilde{x}$.

Thus, its pull-back on $D_{6}$ is given by

$$
\begin{align*}
i^{*} B^{\mathrm{IIA}}= & B_{\theta_{2} \tilde{y}} \mathrm{~d} \theta_{2} \wedge \mathrm{~d} \tilde{y}+\mathcal{C}_{4}\left(x^{\kappa}, Z, \dot{Y}\right) \mathrm{d} Z \wedge \mathrm{~d} \theta_{2} \\
& +\mathcal{C}_{5}\left(x^{\kappa}, Z\right) \partial_{\mu} Y \mathrm{~d} x^{\mu} \wedge \mathrm{d} \theta_{2} \tag{118}
\end{align*}
$$

where
$\mathcal{C}_{4}\left(x^{\kappa}, Z, \dot{Y}\right)=\left(\frac{B_{\theta_{2} \tilde{z}}^{\mathrm{IIA}}}{Y^{2}+Z^{2}}\right)(\dot{Y} Z-Y)$,
$\mathcal{C}_{5}\left(x^{\kappa}, Z\right)=\left(\frac{B_{\theta_{2} \tilde{z}}^{\mathrm{II}}}{Y^{2}+Z^{2}}\right) Z$.

Now, $B_{\theta_{2} \tilde{z}}$ and $B_{\theta_{2} \tilde{y}}$ are as given in (39).
Therefore
$i^{*}(G+B)^{\mathrm{IA}}=\left(\begin{array}{ll}\mathbb{A}_{4 \times 4} & \mathbb{B}_{4 \times 3} \\ \mathbb{C}_{3 \times 4} & \mathbb{D}_{3 \times 3}\end{array}\right)$,
where
$\mathbb{A}=\left(\begin{array}{cccc}G_{00}^{\mathrm{IIA}} \mathcal{T} & 0 & 0 & 0 \\ 0 & G_{x^{1 / 1} 1_{1}} \mathcal{T} & 0 & 0 \\ 0 & 0 & G_{x^{2} x^{2} \mathcal{I}} \mathcal{I} & 0 \\ 0 & 0 & 0 & G_{x^{3} x^{3}}^{\mathrm{II}} \mathcal{T}\end{array}\right)$,
$\mathcal{T}=\left(1+\mathcal{C}_{1} G_{I I A}^{\rho \lambda} \partial_{\rho} Y \partial_{\lambda} Y\right)$,
$\mathbb{B}_{4 \times 3}=\left(\begin{array}{ccc}i^{*} G_{x^{0} Z}^{\mathrm{IIA}} & i^{*} B_{x^{\mathrm{IA}} \theta_{2}}^{\mathrm{II}} & 0 \\ i^{*} G_{x^{I A} Z}{ }^{\mathrm{IA}} & i^{*} B_{x^{\mathrm{IA}} \theta_{2}} & 0 \\ i^{*} G_{x^{2} Z} \mathrm{IA} & i^{*} B_{x^{2} \theta_{2}}^{\mathrm{IA}} & 0 \\ i^{*} G_{x^{3} Z}{ }^{\mathrm{IA}} & i^{*} B_{x^{3} \theta_{2}}^{\mathrm{IA}} & 0\end{array}\right)$,
$\mathbb{D}_{3 \times 3}=\left(\begin{array}{ccc}i^{*} G_{Z Z}^{\mathrm{II}} & i^{*} B_{Z \theta_{2}}^{\mathrm{II}} & 0 \\ -i^{*} B_{Z \theta_{2}}^{\mathrm{IA}} & i^{*} G_{\theta_{2} \theta_{2}}^{\mathrm{IA}} & i^{*} B_{\theta_{2} \tilde{\tilde{y}}}^{\mathrm{IIA}} \\ 0 & -i^{*} B_{\theta_{2} \tilde{y}}^{\mathrm{IA}} i^{*} G_{\bar{y} \tilde{y}}^{\mathrm{II}}\end{array}\right)$.
Now, $\operatorname{det}\left(i^{*}(G+B)^{\text {IIA }}\right)=\operatorname{det} \mathbb{A} \operatorname{det}\left(\mathbb{D}-\mathbb{C} \mathbb{A}^{-1} \mathbb{B}\right)$, and retaining terms in the following up to $\mathcal{O}\left(Y^{2}, \dot{Y}^{2}, \partial_{\mu} Y \partial_{\nu} Y\right)$ (indicated by a tilde below), one obtains

$$
\begin{align*}
& \sqrt{\operatorname{det} \mathbb{A}} \sim \sqrt{-G^{\mathrm{IA}}} \mathbb{R}^{1,3} \\
& \operatorname{det}\left(1+\frac{\mathcal{C}_{1}(Y=0)}{2} G_{\mathrm{IIA}}^{\mu \nu} \partial_{\mu} Y \mathbb{C A}^{-1} \mathbb{B}\right) \sim \dot{Y}^{2} \Omega_{1}+Y^{2} \Omega_{2} \\
& +\Omega_{3} G_{\mathrm{II}}^{\mu \nu} \partial_{\mu} Y \partial_{\nu} Y+\Omega_{4}, \\
& \text { implying: } \\
& \sqrt{\operatorname{det}\left(\mathbb{D}-\mathbb{C A}^{-1} \mathbb{B}\right)} \\
& \sim \sqrt{\Omega_{4}}\left(1+\frac{\Omega_{1}}{\Omega_{4}} \frac{\dot{Y}^{2}}{2}+\frac{\Omega_{2}}{\Omega_{4}} \frac{\dot{Y}^{2}}{2}+\frac{\Omega_{3}}{\Omega_{4}} \frac{G_{\mathrm{IIA}}^{\mu \nu}}{2} \partial_{\mu} Y \partial_{\nu} Y\right) ; \\
& \quad \cdot \equiv \frac{d}{\mathrm{dZ}} . \tag{121}
\end{align*}
$$

Finally, one thus obtains the following DBI action for $N_{f}$ $D 6$-branes (setting the tension to unity):

$$
\begin{align*}
S_{D 6} & =N_{f} \int d^{4} x \mathrm{~d} Z \mathrm{~d} \theta_{2} \mathrm{~d} \tilde{y} \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right) \\
& \times e^{-\phi^{\mathrm{IIA}}} \sqrt{\Omega_{4}} \sqrt{-G^{\mathrm{IIA}}} \mathbb{R}^{1,3}\left[1+\frac{\mathcal{C}_{1}+\frac{\Omega_{3}}{\Omega_{4}}}{2} G_{\text {IIA }}^{\mu \nu} \partial_{\mu} Y \partial_{\nu} Y\right. \\
& \left.+\frac{\Omega_{1}}{\Omega_{4}} \frac{\dot{Y}^{2}}{2}+\frac{\Omega_{2}}{\Omega_{4}} \frac{\dot{Y}^{2}}{2}\right]\left.\right|_{\theta_{1}=\frac{\alpha_{\theta_{1}}}{N^{\frac{1}{5}}}, \tilde{x}=0} \\
= & N_{f} \int \mathrm{~d}^{4} x \mathrm{~d} Z \mathrm{~d} \theta_{2} \mathrm{~d} \tilde{y} \delta\left(\theta_{2}-\frac{\alpha_{\theta_{2}}}{N^{\frac{3}{10}}}\right) \\
& \times\left[\mathcal{S}_{1}(Z) G_{\text {IIA }}^{\mu \nu} \partial_{\mu} Y \partial_{\nu} Y+\mathcal{S}_{2}(Z) \dot{Y}^{2}\right. \\
& \left.+\mathcal{S}_{3}(Z) Y^{2}\right]\left.\right|_{\theta_{1}=\frac{\alpha_{\theta_{1}}}{N^{\frac{1}{5}}}, \tilde{x}=0}, \tag{122}
\end{align*}
$$

where $\mathcal{S}_{1,2,3}$ are defined in (C2).
Now, similar to [4], we make the KK ansatz:
$Y\left(x^{\mu}, Z\right)=\sum_{n=1} \mathcal{Y}^{(n)}\left(x^{\mu}\right) \mathcal{Z}_{n}(Z)$,
together with the following identifications, normalization and EOM:

$$
\begin{align*}
& \int \mathrm{d} Z\left(\mathcal{S}_{2}(Z) \dot{\mathcal{Z}}_{m}(Z) \dot{\mathcal{Z}}_{n}(Z)+\mathcal{S}_{3}(Z) \mathcal{Z}_{m}(Z) \mathcal{Z}_{n}(Z)\right) \\
& =\frac{m_{n}^{2}}{2} \delta_{m n}, \int \mathrm{~d} Z \mathcal{S}_{1}(Z) \mathcal{Z}_{m}(Z) \mathcal{Z}_{n}(Z)=\frac{1}{2} \delta_{m n} \\
& \quad-\partial_{Z}\left(\mathcal{S}_{2}(Z) \partial_{Z} \mathcal{Z}_{n}\right)+\mathcal{S}_{3}(Z) \mathcal{Z}_{n}(Z)=\mathcal{S}_{1}(Z) m_{n}^{2} \mathcal{Z}_{n} \tag{124}
\end{align*}
$$

Making a field redefinition: $\mathcal{Z}_{n}(Z)=|Z| \mathcal{G}_{n}(Z)$, one obtains the following EOM for $\mathcal{G}(Z)$ :
the (pseudo-)scalar meson masses. The second route will be to convert the $\mathcal{G}_{n}(Z)$ EOM into Schrödinger-like EOM and to solve the same in the IR and UV separately and obtain (pseudo-)scalar mass quantization by imposing Neumann boundary conditions at the horizon (IR)/asymptotic boundary (UV). It turns out the former yields a result which, up to LO in $N$, is of the same order as the IR results of route one. The UV computations satisfy Neumann and/or Dirichlet boundary conditions without any mass quantization condition. The third route catering to the IR-UV interpolating region and giving us our main results that are directly compared with PDG results, is obtaining the (pseudo-)scalar meson masses via the WKB quantization condition.
5.1 Scalar meson spectrum from solution to EOM near $r=r_{h}$

Analogous to (58)-(62), one can rewrite (125) and solve the same near $r=r_{h}$, impose Neumann boundary condition at $r=r_{h}$ with the following identifications:

$$
\begin{align*}
\alpha_{1}= & 0.92+\frac{0.24}{\log N} \\
\alpha_{2}= & \frac{0.02 \alpha_{\theta_{2}}^{2} \tilde{m}^{2}}{\alpha_{\theta_{1}}^{2} 2 \sqrt[5]{N}} \\
& +\frac{g_{s} M^{2} \tilde{m}^{2}\left(-3.6 \log \left(r_{h}\right)-3.6\right)}{N}-0.02 \tilde{m}^{2}, \\
\beta_{2}= & -\frac{0.54 \alpha_{\theta_{2}}^{2} \tilde{m}^{2}}{\alpha_{\theta_{1}}^{2} 2 \sqrt[5]{N}}+\frac{g_{s} M^{2} \tilde{m}^{2}\left(7.2 \log \left(r_{h}\right)+7.2\right)}{N} \\
& +0.54 \tilde{m}^{2} . \tag{126}
\end{align*}
$$

$$
\begin{align*}
& \mathcal{G}_{n}^{\prime \prime}(Z)+\mathcal{G}_{n}^{\prime}(Z) \frac{1}{\left(2 g_{s} N_{f} \log N-6 g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right)+8 \pi\right)^{2}} \\
& \times\left\{\left(\frac{2\left(2 g_{s} N_{f} \log N-6 g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right)+8 \pi\right)\left(4 g_{s} N_{f} e^{4|Z|} \log N-12 g_{s} N_{f} e^{4|Z|}\left(\log \left(r_{h}\right)+|Z|\right)-3 g_{s} N_{f} e^{4|Z|}+3 g_{s} N_{f}+16 \pi e^{4|Z|}\right)}{e^{4|Z|}-1}\right.\right. \\
& -3 e^{-2|Z|}\left(\frac{4 g_{s} M^{2} \log \left(r_{h}\right)}{N}+\frac{4 g_{s} M^{2}}{N}+0.6\right)^{2}\left[-24 g_{s}{ }^{2} N_{f}{ }^{2} \log N\left(\log \left(r_{h}\right)+|Z|\right)+4 g_{s}{ }^{2} N_{f}{ }^{2} \log ^{2}(N)-12 g_{s}{ }^{2} N_{f}{ }^{2} \log N\right. \\
& +36 g_{s}{ }^{2} N_{f}{ }^{2}\left(\log \left(r_{h}\right)+|Z|\right)^{2}+36 g_{s}{ }^{2} N_{f}{ }^{2}\left(\log \left(r_{h}\right)+|Z|\right)+18 g_{s}{ }^{2} N_{f}{ }^{2}+32 \pi g_{s} N_{f} \log N-96 \pi g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right) \\
& \left.\left.\left.-48 \pi g_{s} N_{f}+64 \pi^{2}\right]\right)\right\}+\mathcal{G}_{n}(Z) \frac{\tilde{m}^{2}\left(\alpha_{\theta_{1}}^{2} 2 \sqrt[5]{N}-\alpha_{\theta_{2}}^{2}\right)\left(e^{2|Z|}-\frac{3 \cdot\left(4 \cdot g s M^{2} \log \left(r_{h}\right)+4 . g_{s} M^{2}+0.6 N\right)^{2}}{N^{2}}\right)}{\alpha_{\theta_{1}}^{2} 2 \sqrt[5]{N}\left(e^{4|Z|}-1\right)}=0 . \tag{125}
\end{align*}
$$

Analogous to obtaining the (pseudo-)vector meson spectrum in Sect. 4, we will now proceed to obtaining the (pseudo-)scalar meson spectrum by three routes. The first will cater exclusively to an IR computation where we solve the $\mathcal{G}_{n}(Z)$ EOM near the horizon. Imposing the Neumann boundary condition at the horizon results in quantization of

The analog of (62) for scalar mesons up to $\mathcal{O}\left(\frac{1}{\log N}\right)$ yields

$$
\begin{align*}
\tilde{m}_{n} & =0.5 \sqrt{0.548697 n^{2}+0.548697 \sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+166.926\right)}+0.548697 n+3.54458} \\
& +\frac{0.25\left(\frac{22.9689(n+0.5)^{2}}{\sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+166.926\right)}}+0.888889\right)}{\log N \sqrt{0.548697 n^{2}+0.548697 \sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+166.926\right)}+0.548697 n+3.54458}} \\
& +\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right) .
\end{align*}
$$

The lightest scalar meson masses are in Table 3.
Our result implies that $\frac{m_{n=1}^{2}}{m_{n=0}^{2}}=2.16$ if one disregards the $\mathcal{O}\left(\frac{1}{N}\right)$ corrections. On comparison with the PDG table for scalar meson masses, if one assumes that the lightest scalar mesons are $f 0[980] / a 0$ [980], $f 0$ [1370] then their masssquared ratio is 1.95 - not too far from our result.
5.2 Scalar mass spectrum from solution of the Schrödinger-like equation

### 5.2.1 IR

In the IR, one can show that the potential in a Schrödingerlike form simplifies to

Now

$$
\begin{aligned}
& \mathcal{C}_{1}(Z)=-\frac{1}{2 r_{h}^{2}}\left\{e^{-2|Z|}\left(e^{4|Z|}-1\right)\right. \\
& \quad \times\left(2 g _ { s } N _ { f } \operatorname { l o g } ( N ) \left(3 r _ { h } ^ { 2 } \left(\frac{4 g_{s} M^{2} \log \left(r_{h}\right)}{N}\right.\right.\right. \\
& \left.\left.\quad+\frac{4 g_{s} M^{2}}{N}+0.6\right)^{2}+2 r_{h}^{2} e^{2|Z|}\right) \\
& \quad-6 g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right)\left(3 r _ { h } ^ { 2 } \left(\frac{4 g_{s} M^{2} \log \left(r_{h}\right)}{N}\right.\right. \\
& \left.\left.\quad+\frac{4 g_{s} M^{2}}{N}+0.6\right)^{2}+2 r_{h}^{2} e^{2|Z|}\right)
\end{aligned}
$$

$$
\begin{align*}
V(\mathrm{IR})= & \frac{\frac{-0.36 g_{s}{ }^{2} N_{f}{ }^{2} \log \left(r_{h}\right)+2.43 g_{s}{ }^{2} N_{f}{ }^{2}+1.50796 g_{s} N_{f}}{g_{s} \log N^{2} N_{f}{ }^{2}}-\frac{0.12}{\log N}-0.02 \tilde{m}^{2}-0.46}{|Z|} \\
& +\frac{13.86 g_{s}{ }^{4} N_{f}{ }^{4} \log \left(r_{h}\right)+g_{s}{ }^{2} N_{f}{ }^{2}\left(-2.97 g_{s}{ }^{2} N_{f}{ }^{2}-58.0566 g_{s} N_{f}\right)}{g_{s}{ }^{4} \log N^{2} N_{f}{ }^{4}}+0.54 \tilde{m}^{2}+\frac{0.25}{Z^{2}}-3.413 . \tag{128}
\end{align*}
$$

The solution to the Schrödinger-like equation $\Phi_{n}^{\prime \prime}(Z)+$ $V(\operatorname{IR})(Z) \Phi_{n}(Z)=0$, where $\Phi_{n}(Z)=\sqrt{\mathcal{C}_{1}} \mathcal{G}_{n}(Z)$, and $V(\mathrm{IR})(Z)=\frac{c_{1}}{Z^{2}}+\frac{a_{1}}{|Z|}+b_{1}$ with
$c_{1}=0.25$,
$a_{1}=\frac{-0.36 g_{s}{ }^{2} N_{f}{ }^{2} \log \left(r_{h}\right)+2.43 g_{s}{ }^{2} N_{f}^{2}+1.50796 g_{s} N_{f}}{g_{s}{ }^{2} N_{f}{ }^{2} \log ^{2}(N)}$ $-0.02 \tilde{m}^{2}-\frac{0.12}{\log (N)}-0.46$,
$b_{1}=0.54 \tilde{m}^{2}-3.413$,
is given by

$$
\begin{align*}
& +2\left(r _ { h } ^ { 2 } ( 1 2 \pi - 9 g _ { s } N _ { f } ) \left(\frac{4 g_{s} M^{2} \log \left(r_{h}\right)}{N}\right.\right. \\
& \left.\left.\left.\left.+\frac{4 g_{s} M^{2}}{N}+0.6\right)^{2}+8 \pi r_{h}^{2} e^{2|Z|}\right)\right)\right\} \tag{131}
\end{align*}
$$

One can show that the Neumann boundary condition is identically satisfied by $\mathcal{Z}_{n}(Z)=|Z| \mathcal{G}_{n}(Z)=|Z| \frac{\Phi_{n}(Z)}{\sqrt{\mathcal{C}_{1}(Z)}}$ and is hence uninteresting, and it will therefore not be implemented. We will implement the Neumann boundary condition on $\mathcal{G}_{n}(Z)=\frac{\Phi_{n}(Z)}{\sqrt{\mathcal{C}_{1}(Z)}}$. One sees that

$$
\left.\begin{array}{l}
\Phi_{n}(Z)=c_{1} M_{\frac{g_{s}\left(\left(-0.01 \tilde{m}^{2}(n)-0.23\right) \log N^{2}-0.06 \log N+1.215\right) N_{f}-0.18 g_{s} \log \left(r_{h}\right) N_{f}+0.753982}{}}^{g_{s} \log N^{2} \sqrt{3.41333-0.54 \tilde{m}^{2}(n)} N_{f}}, 0 \\
\quad+c_{2} W_{\frac{g_{s}\left(\left(-0.01 \tilde{m}^{2}(n)-0.23\right) \log N^{2}-0.06 \log N+1.215\right) N_{f}-0.18 g s}{} \log \left(r_{h}\right) N_{f}+0.753982}^{g_{s} \log N^{2} \sqrt{3.41333-0.54 \tilde{m}^{2}(n)} N_{f}}, 0 \tag{130}
\end{array}\left(2 \sqrt{3.41333-0.54 \tilde{m}^{2}(n)}|Z|\right)\right) .
$$

$$
\begin{align*}
& \frac{d}{\mathrm{~d} Z}\left(\frac{M_{g s\left(\left(-0.01 \tilde{m}^{2}-0.23\right) \log N^{2}-0.06 \log N+1.215\right) N_{f}-0.188_{s} \log \left(r_{h}\right) N_{f}+0.753982}, 0}{g_{s} \log N^{2} \sqrt{3.4133-0.54 \tilde{m}^{2} N_{f}}} \sqrt{\sqrt{\mathcal{C}_{1}(Z)}}\left(2 \sqrt{3.41333-0.54 \tilde{m}^{2}}|Z|\right)\right. \\
& \quad=\frac{1}{N^{2}}\left(0.54 \tilde{m}^{2}-3.41333\right)^{\frac{1}{4}}\left(\omega_{1}\left(g_{s}, M, N_{f}\right)+\omega_{2}\left(g_{s}, M, N_{f} ; \log r_{h}\right) \tilde{m}^{2}\right)+\mathcal{O}(Z) \tag{132}
\end{align*}
$$

Hence, either
$0.54 \tilde{m}^{2}-3.41333=0$, implying $\tilde{m}=2.514$,
or
$\omega_{1}\left(g_{s}, M, N_{f}\right)+\omega_{2}\left(g_{s}, M, N_{f} ; \log r_{h}\right) \tilde{m}^{2}=0$, implying
$\tilde{m}=3.07694+\frac{\frac{95.6605}{g_{s} N_{f}}-22.8373 \log \left(r_{h}\right)+3.34479}{\log ^{2}(N)}$

$$
\begin{equation*}
-\frac{7.61242}{\log (N)}+\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right) \tag{134}
\end{equation*}
$$

One can show that

$$
\begin{align*}
& \frac{\left.W_{-\frac{i a_{1}}{2 \sqrt{b_{1}}}, 0}\left(2 i \sqrt{b_{1}}|Z|\right)\right|_{|Z| \sim 0}}{\sqrt{\mathcal{C}_{1}(Z)}}=\frac{\frac{2 i \sqrt{b_{1}}\left(\psi^{(0)}\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}-\frac{1}{2}\right)+\log \left(2 i \sqrt{b_{1}}\right)+\log |Z|+2 \gamma\right)}{\Gamma\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}-\frac{1}{2}\right)}+\frac{a_{1}\left(\psi^{(0)}\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}+\frac{1}{2}\right)+\log \left(2 i \sqrt{b_{1}}\right)+\log |Z|+2 \gamma\right)}{\Gamma\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}+\frac{1}{2}\right)}}{\sqrt{2} \sqrt{i \sqrt{b_{1}}} \sqrt{\mathcal{C}_{1}(0)} \sqrt{|Z|}} \\
& \quad \sqrt{|Z|}\left(\frac{2 b_{1} C 1^{\prime}(0) \log |Z|}{\Gamma\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}-\frac{1}{2}\right)}-\frac{2 b_{1} C 1^{\prime}(0) \log |Z|}{\Gamma\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}+\frac{1}{2}\right)}-\frac{i a_{1} \sqrt{b_{1}} C 1^{\prime}(0) \log |Z|}{\Gamma\left(\frac{i a_{1}}{2 \sqrt{b_{1}}}+\frac{1}{2}\right)}\right)  \tag{135}\\
& 2 \sqrt{2}\left(i \sqrt{b_{1}}\right)^{3 / 2} \mathcal{C}_{1}(0)^{3 / 2}
\end{align*} O\left(Z^{3 / 2}\right) . \quad .
$$

One therefore can satisfy the Neumann boundary condition
at $r=r_{h}$ if
$\frac{1}{2}-\frac{i a_{1}}{2 \sqrt{b_{1}}}=-n \in \mathbb{Z}^{-} \cup\{0\}$,
which yields the following quantization condition on $\tilde{m}$ :

$$
\begin{align*}
m_{n} & =0.5 \sqrt{-10800 . n^{2}+10800 \cdot \sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+0.271719\right)}-10800 . n-2792} \\
& +\frac{6 . n^{2}-6 \cdot \sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+0.271719\right)}+6 . n+1.5}{\log N \sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+0.271719\right)} \sqrt{-10800 . n^{2}+10800 \cdot \sqrt{\left(n^{2}+n+0.25\right)\left(n^{2}+n+0.271719\right)}-10800 . n-2792 .}} \\
& +\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right) . \tag{137}
\end{align*}
$$

$$
\begin{align*}
V(U V)= & \frac{e^{-2|Z|}\left(e^{6|Z|}\left(8-1.08 \tilde{m}^{2}\right)+e^{4|Z|}\left(-\tilde{m}^{2}\right)+1.08 \tilde{m}^{2} e^{2|Z|}+\left(\tilde{m}^{2}+1.08\right) e^{8|Z|}-4 e^{10|Z|}-1.08\right)}{\left(e^{4|Z|}-1\right)^{2}} \\
& +\frac{\alpha_{\theta_{2}}^{2} \tilde{m}^{2}\left(e^{2|Z|}+1.08 e^{4|Z|}-e^{6|Z|}-1.08\right)}{\alpha_{\theta_{1}}^{2} \sqrt[5]{N}\left(e^{4|Z|}-1\right)^{2}} . \tag{138}
\end{align*}
$$

The solution to the Schrödinger-like equation $\Phi^{\prime \prime}(Z)+$ $V(Z) \Phi(Z)=0$ is given by
$\Phi(Z)=c_{1} J_{-i \sqrt{x_{2}}}\left(e^{|Z|} \sqrt{x_{1}}\right)+c_{2} J_{i \sqrt{x_{2}}}\left(e^{|Z|} \sqrt{x_{1}}\right)$,
where
$x_{1}=1.08+\tilde{m}^{2}$,
$x_{2}=8-1.08 \tilde{m}^{2}$.
One can show that in the UV

One sees that $\sqrt{V} \in \mathbb{R}$ for $|Z| \in[0.0385,0.5 \log (0.54$ $\left.\left.+\frac{1.013 \times 10^{9}}{\tilde{m}^{2}}\right)\right]$, which we will approximate by $|Z| \in$ $\left[0.0385,0.5 \log \left(1.013 \times 10^{9}\right) \approx 10.368\right]$. The WKB quantization condition,
$\int_{0.0385}^{10.368} \mathrm{~d} Z \sqrt{V(Z)}=\left(n+\frac{1}{2}\right) \pi$,
yields the following scalar meson spectrum:

$$
\begin{equation*}
\tilde{m}_{n}=\frac{\alpha_{\theta_{1}}^{2}(157.08 n+78.5398)+0.5 \sqrt{\alpha_{\theta_{1}}^{4}\left(98696 n^{2}+98696 n+2.08747 \times 10^{9}\right)-1.04372 \times 10^{9} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}}}{82 \alpha_{\theta_{1}}^{2}-41 \alpha_{\theta_{2}}^{2} \sqrt[5]{N}} \tag{144}
\end{equation*}
$$

$$
\begin{align*}
& \frac{d}{\mathrm{~d} Z}\left(\frac{J_{ \pm i \sqrt{x_{2}}}\left(e^{|Z|} \sqrt{x_{1}}\right)}{\sqrt{\mathcal{C}_{1}(Z)}}\right) \\
& \quad \sim \cos \left(e^{|Z|} \sqrt{x_{1}}\right) \times \mathcal{O}\left(e^{-\frac{3|Z|}{2}}\right), \tag{141}
\end{align*}
$$

which tells us that the Neumann boundary condition is identically satisfied and one does not obtain any mass quantization condition in the UV from this approach.

### 5.3 Scalar mass spectrum via WKB quantization condition

For the purpose of simplification of this calculation, we will be disregarding $N_{f}$ and $M$ because one can show that the WKB quantization condition integral fails to converge for IR-valued $\tilde{m}$ and in the UV, $M$ and $N_{f}$ are very small. One will hence work in the large- $\tilde{m} / \mathrm{UV}$ limit. From (138), one thus obtains

$$
\begin{align*}
& \sqrt{V}=\tilde{m} \sqrt{\frac{1.08-e^{2|Z|}-1.08 e^{4|Z|}+e^{6|Z|}}{1 .-2 e^{4|Z|}+e^{8|Z|}}} \\
& +\frac{0.5\left(8 e^{6|Z|}+1.08 e^{8|Z|}-4 e^{10|Z|}-1.08\right) \sqrt{\frac{1.08-e^{2|Z|}-1.08 e^{4|Z|}+e^{6|Z|}}{1 .-2 e^{4|Z|}+e^{8|Z|}}}}{\tilde{m}\left(1.08 e^{2|Z|}-e^{4|Z|}-1.08 e^{6|Z|}+e^{8|Z|}\right)} \\
& +\frac{0.5 \alpha_{\theta_{2}}^{2} \tilde{m}\left(e^{2|Z|}+1.08 e^{4|Z|}-e^{6|Z|}-1.08\right)}{\alpha_{\theta_{1}}^{2} \sqrt[5]{N}\left(e^{4|Z|}-1 .\right)^{2} \sqrt{\frac{1.08-e^{2|Z|}-1.08 e^{4|Z|}+e^{6|Z|}}{1 .-2 e^{4|Z|}+e^{8|Z|}}}} \\
& +\mathcal{O}\left(\frac{1}{\tilde{m} N^{\frac{1}{5}}}, \frac{1}{\tilde{m}^{2}}\right) . \tag{142}
\end{align*}
$$

One can argue that $Y\left(x^{\mu}, Z\right)$ is even under parity: $\left(x^{1,2,3}, Z\right)$ $\rightarrow\left(-x^{1,2,3},-Z\right)$. The idea is the following. The type IIB set-up of [1] includes $D 3$-branes with world-volume coordinates $x^{0,1,2,3}$ and $D 7$-branes with world-volume coordinates $\left(x^{0,1,2,3}, r, \tilde{x}, \theta_{1}, \tilde{z}\right){ }^{2}$ which after three T dualities along $\tilde{x}, \tilde{y}, \tilde{z}$ yield two sets of $D 6$-branes, one set with worldvolume coordinates $\left(x^{0,1,2,3}, \tilde{x}, \tilde{y}, \tilde{z}\right)$ (obtained from a triple T dual of the $D 3$-branes) and the other set with world-volume coordinates $\left(x^{0,1,2,3}, r, \theta_{2}, \tilde{y}\right)$ (obtained from a triple T dual of the $D 7$-branes). One hence sees that the two sets of $D 6$ branes are separated in $r$ or $Y$. In the type IIB set-up of [1], the flavor $D 7$-branes never touch the D3-branes, which in the SYZ or triple T dual picture implies that the two sets of $D 6$-branes never touch each other. This, like [3,4], implies one can construct a $C_{5} \sim Y \mathrm{~d} x^{0} \wedge \mathrm{~d} x^{1} \wedge \mathrm{~d} x^{2} \wedge \mathrm{~d} x^{3} \wedge \mathrm{~d} Z$ which vanishes precisely when the two sets of $D 6$-branes touch. From this $C_{5}$, one can construct a Chern-Simons action: $\int_{w . v . o f D 6} F_{2} \wedge C_{5}$ where $F_{2}=d A_{1}$ corresponds to a gauge field on the $D 6$-brane world volume. If one demands the Chern-Simons action be invariant under parity - which includes $Z \rightarrow-Z$ - given that $F_{2}$ is even,

[^2]one sees that $Y$ is even under parity. Similarly, under charge conjugation - which includes $Z \rightarrow-Z$ - noting that $F_{2}$ is charge-conjugation odd implies that $Y$ is chargeconjugation even. From (123), under 5D parity, $\mathcal{Z}_{n}(-Z)=$ $(-)^{n+1} \mathcal{Z}_{n}(Z), \mathcal{Y}^{(n)}\left(-x^{\mu}\right)=(-)^{n+1} \mathcal{Y}^{(n)}\left(x^{\mu}\right), n \in \mathbb{Z}^{+}[3]$.

We assume that the three lightest scalar mesons from the PDG are $f 0[980] / a 0[980], f 0[1370]$ and $f 0[1450]$. We could choose $\alpha_{\theta_{1}}$ and $\alpha_{\theta_{2}}$ to match $\frac{m_{n=3}}{m_{n=1}}$ with PDG exactly (this is not normalizing our $\frac{m_{n=3}}{m_{n=1}}$ result to match PDG values)! This is effected by imposing the following condition on $\alpha_{\theta_{1}}, \alpha_{\theta_{2}}$ :
$\sqrt{3} a \neq 0)$, one can show that the $\alpha_{N}(Z)$ EOM - there is no need to attach a superscript to $\alpha_{N}$ anymore as $r_{h}=0$ - near the horizon can be written in the form
$\alpha_{n}^{\prime \prime}(Z)+\left(a_{1}+b_{1}|Z|\right) \alpha_{n}^{\prime}(Z)+\left(a_{2}+b_{2}|Z|\right) \alpha_{n}(Z)=0$,
where up to $\mathcal{O}\left(\frac{1}{\log N}\right)$ :
$\frac{m_{n=3}}{m_{n=1}}=\frac{549.7787 \alpha_{\theta_{1}}^{2}+0.5 \sqrt{2.08767 \times 10^{9} \alpha_{\theta_{1}}^{4}-1.04372 \times 10^{9} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}}}{235.6194 \alpha^{2}+0.5 \sqrt{2.08747 \times 10^{9} \alpha_{\theta_{1}}^{4}-1.04372 \times 10^{9} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}}}=\frac{1350}{980}$,
which is satisfied by $\alpha_{\theta_{1}}=0.70765 N^{\frac{1}{10}} \alpha_{\theta_{2}}$. Having done so, the ratio $\frac{m_{n=5}}{m_{n=1}}$ is not too far off of the PDG value - see Table 8 in Sect. 7! The (pseudo-)scalar meson $\left(0^{--}\right) 0^{++}$ masses are listed in Table 4. (The entries against $0^{--}$are blanks as there are, as of now, no known candidates with this $J, P, C$ assignment.)

## 6 Meson spectroscopy in a thermal background and near isospectrality with black-hole background

In this section we show an interesting near isospectrality of the (pseudo-) vector meson spectrum (in Sect. 6.1) and (pseudo-)scalar meson spectrum (in Sect. 6.2) obtained using a thermal background which is valid for low temperatures, with the corresponding results of Sects. 4 and 5 obtained using a black-hole background (expected to be valid/stable at high temperatures) for all temperatures.

As the techniques are similar to and in fact simpler than the ones used in Sects. 4 and 5, we will only present the main results to substantiate our claim.
6.1 Vector meson spectroscopy in a thermal background

### 6.1.1 Solving the EOM near an IR cut-off $r=r_{0}$

Writing $r=r_{0} e^{\sqrt{Y^{2}+Z^{2}}}-r_{0}$ being an IR cut-off ${ }^{3}-$ and defining $m=\tilde{m} \frac{r_{0}}{\sqrt{4 \pi g_{s} N}}$, setting $r_{h}=0$ and introducing a bare resolution parameter $a=\gamma r_{0}$ (to ensure that $\mathcal{R}_{D 5 / \overline{D 5}}=$

[^3]$a_{1}=2-3 \gamma^{2}-\frac{3}{\log N}+\frac{9 \gamma^{2}}{\log N}$,
$b_{1}=6 \gamma^{2}-18 \frac{\gamma^{2}}{\log N}$,
$a_{2}=\left(1-3 \gamma^{2}\right) \tilde{m}^{2}$,
whose solution is given by
\[

$$
\begin{align*}
& \alpha_{n}(Z)=e^{-a_{1}|Z|+\frac{b_{2}|Z|}{b_{1}}-\frac{b_{1} Z^{2}}{2}} \\
& \quad \times\left(c _ { 2 } \left(\frac{b_{1}^{3}-a_{2} b_{1}^{2}+a_{1} b_{2} b_{1}-b_{2}^{2}}{2 b_{1}^{3}} ; \frac{1}{2} ;\right.\right. \\
& \left.\quad \frac{\left(|Z| b_{1}^{2}+a_{1} b_{1}-2 b_{2}\right)^{2}}{2 b_{1}^{3}}\right) \\
& \left.\quad+c_{1} H_{\frac{-a_{1} b_{1} b_{2}+a_{2} b_{1}^{2}-b_{1}^{3}+b_{2}^{2}}{b_{1}^{3}}}\left(\frac{a_{1} b_{1}+b_{1}^{2}|Z|-2 b_{2}}{\sqrt{2} b_{1}^{3 / 2}}\right)\right) . \tag{148}
\end{align*}
$$
\]

By imposing the Neumann boundary condition at $r=r_{0}$ by assuming $c_{2}=0$, numerically, e.g., for $N=6000, \gamma=0.6$ (similar to $a\left(r_{h} \neq 0\right)=\left(0.6+\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right)\right) r_{h}$ ), one obtains as a root - corresponding to the lightest vector meson $-\tilde{m} \approx$ 1.04 - of the same order as the LO value in (65). One gets only the root $\tilde{m}=0$ if $c_{1}=0$.

### 6.1.2 Schrödinger-like EOM

One can rewrite the EOM as a Schrödinger-like equation in terms of $\psi_{n}(Z)=\sqrt{\mathcal{C}_{1}(Z)} \alpha_{n}(Z)$ where $\psi_{n}(Z)$ satisfies $\psi_{n}^{\prime \prime}(Z)+V(Z) \psi_{n}(Z)=0$, where

Table 4 (Pseudo-)scalar meson masses from WKB quantization

|  | (Pseudo-)scalar meson name | $J^{P C}$ | $m_{n>0}\left(\right.$ units of $\left.\frac{r_{h}}{\sqrt{4 \pi g_{s} N}}\right)$ | PDG mean mass (MeV) |
| :--- | :--- | :--- | :--- | :--- |
| $\mathcal{Y}^{(1)}$ | $f 0[980] / a 0[980$ | $0^{++}$ | 9207.44 | 980 |
| $\mathcal{Y}^{(2)}$ | - | $0^{--}$ | $10,861.9$ | - |
| $\mathcal{Y}^{(3)}$ | $f 0[1370]$ | $0^{++}$ | $12,683.7$ | 1350 |
| $\mathcal{Y}^{(4)}$ | - | $0^{--}$ | $14,640.8$ | - |
| $\mathcal{Y}^{(5)}$ | $f 0[1450]$ | $0^{++}$ | 16,704 | 1474 |

$$
\begin{align*}
\mathcal{C}_{1}(Z)= & \frac{1}{2}\left[3 \gamma^{2}\left(g_{s} N_{f}(\log (N)-3|Z|-3)+4 \pi\right)\right. \\
& -3 g_{s} N_{f}\left(3 \gamma^{2}+2 e^{2|Z|}\right) \log \left(r_{0}\right) \\
& \left.+2 e^{2|Z|}\left(g_{s} N_{f}(\log (N)-3|Z|)+4 \pi\right)\right] \tag{149}
\end{align*}
$$

and

$$
\begin{align*}
V(Z)= & -1+e^{-2|Z|} \tilde{m}^{2}-3 e^{-4|Z|} \gamma^{2} \tilde{m}^{2} \\
& +\frac{3-\frac{9}{2} e^{-2|Z|} \gamma^{2}}{\log N}+\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right) . \tag{150}
\end{align*}
$$

Near $r=r_{0}$ - the IR - the EOM can written as $\psi_{n}^{\prime \prime}(Z)+$ $(a+b|Z|) \psi_{n}(Z)$, where
$a=-1+\frac{3}{\log N}-\frac{9 \gamma^{2}}{2 \log N}+\tilde{m}^{2}\left(1-3 \gamma^{2}\right)$,
$b=\frac{9 \gamma^{2}}{\log N}+\tilde{m}^{2}\left(12 \gamma^{2}-2\right)$,
the solution to which is given by

$$
\begin{equation*}
\psi(Z)=c_{1} A i\left(-\frac{a+b|Z|}{(-b)^{2 / 3}}\right)+c_{2} B i\left(-\frac{a+b|Z|}{(-b)^{2 / 3}}\right) . \tag{152}
\end{equation*}
$$

In the large- $\left|\log r_{0}\right|$ limit and setting $\gamma=0.6$, one can show that $\tilde{m}=0.36$.

In the $\mathrm{UV}, V(Z)=-1+e^{-2|Z|} \tilde{m}^{2}+\mathcal{O}\left(e^{-4|Z|}\right)$, and the solution to the EOM is given by $J_{1}\left(e^{-|Z|} \tilde{m}\right)$ and $Y_{1}\left(e^{-|Z|} \tilde{m}\right)$ which satisfy the Neumann/Dirichlet boundary conditions, identically, in the UV and do not provide $\tilde{m}$ quantization.

### 6.1.3 WKB quantization condition

In the UV, one can show that

$$
\begin{align*}
& \sqrt{V\left(\alpha_{n}\right)}=\frac{3-\frac{9}{2} \gamma^{2} e^{-2|Z|}}{2 \log (N) \sqrt{-3 \gamma^{2} \tilde{m}^{2} e^{-4|Z|}+\tilde{m}^{2} e^{-2|Z|}-1}} \\
& \quad+\sqrt{-3 \gamma^{2} \tilde{m}^{2} e^{-4|Z|}+\tilde{m}^{2} e^{-2|Z|}-1} \\
& \quad+\mathcal{O}\left(\left(\frac{1}{\log N}\right)^{2}\right) . \tag{153}
\end{align*}
$$

One can see that $\sqrt{V} \in \mathbb{R}$ for $|Z| \in\left[\log \left(\sqrt{3} \gamma+\mathcal{O}\left(\frac{1}{\tilde{m}^{2}}\right)\right)\right.$, $\left.\log \left(\tilde{m}-\frac{3}{2 \tilde{m}}+\mathcal{O}\left(\frac{1}{\tilde{m}^{2}}\right)\right)\right]$. One can then show that
$\int_{\log (\sqrt{3} \gamma)}^{\log \left(\tilde{m}-\frac{3}{2 \tilde{m}}\right)} \sqrt{V}=\left(n+\frac{1}{2}\right) \pi$
yields

$$
\begin{align*}
& m_{n}^{\alpha_{n}}=\frac{3}{20} \sqrt{3}\left(\sqrt{2} \sqrt{\gamma^{2}\left(2(2 \pi n+\pi)^{2}+12 \pi(2 n+1)+13\right)}\right. \\
& +2 \gamma(2 \pi n+\pi+3)) \\
& \quad \times \frac{\sqrt{3}\left(\frac{\gamma^{2}(7-36 \pi(2 n+1))}{\sqrt{2} \sqrt{\gamma^{2}\left(2(2 \pi n+\pi)^{2}+12 \pi(2 n+1)+13\right)}}-18 \gamma\right)}{20 \log (N)} \\
& \quad+\mathcal{O}\left(\left(\frac{1}{\log N}\right)^{2}\right) . \tag{155}
\end{align*}
$$

Hence, disregarding $n=0$, the spectrum of Table 5, nearly isospectral with Table 1 gotten using a black-hole background, is obtained.
6.2 Scalar meson spectroscopy in a thermal background

### 6.2.1 Solving the EOM near an IR cut-off $r=r_{0}$

The EOM for $\mathcal{G}_{n}(Z)$, near the horizon, is again of the form (146) wherein
$a_{1}=4-3 \gamma^{2}+\frac{\left(9 \gamma^{2}-3\right)}{\log N}+\frac{\left(27 \gamma^{2}-9\right) \log r_{0}}{(\log N)^{2}}$,
$b_{1}=6 \gamma^{2}-\frac{18 \gamma^{2}}{\log N}-\frac{54 \gamma^{2} \log r_{0}}{(\log N)^{2}}$,
$a_{2}=\tilde{m}^{2}\left(1-3 \gamma^{2}\right)$,
$b_{2}=\tilde{m}^{2}\left(12 \gamma^{2}-2\right)$.
Quite interestingly, this IR computation is able to resolve $f 0[980]\left(m_{f 0[980]}=990 \mathrm{MeV}\right)$ and $a 0[980]\left(m_{a 0[980]}=\right.$ 980 MeV ) because, for $\gamma=0.6$, numerically one can show that the two smallest roots of the equation obtained by imposing Neumann boundary condition on $|Z| \mathcal{G}_{n}\left(r=r_{0}\right)$ by set-

Table 5 (Pseudo-)vector meson masses from WKB quantization applied to $V\left(\alpha_{n}^{\{0\}}\right)$

|  | (Pseudo-)vector meson name | $J^{P C}$ | $m_{n>0}\left(\right.$ units of $\left.\frac{r_{0}}{\sqrt{4 \pi g_{s} N}}\right)$ | PDG mass $(\mathrm{MeV})$ |
| :--- | :--- | :--- | :--- | :--- |
| $B_{\mu}^{(1)}$ | $\rho[770]$ | $1^{++}$ | $7.716-\frac{1.636}{\log N}$ | 775.49 |
| $B_{\mu}^{(2)}$ | $a_{1}[1260]$ | $1^{--}$ | $11.644-\frac{1.714}{\log N}$ | 1230 |
| $B_{\mu}^{(3)}$ | $\rho[1450]$ | $1^{++}$ | $15.567-\frac{1.753}{\log N}$ | 1465 |
| $B_{\mu}^{(4)}$ | $a_{1}[1640]$ | $1^{--}$ | $19.488-\frac{1.776}{\log N}$ | 1647 |

ting $c_{2}=0$ are: 1.83 and 1.94 - the second in particular not far off of the results of Table 1 gotten using a black-hole gravity dual - and $\frac{1.94}{1.83}=1.06$ and $\frac{m_{f 0[980]}}{m_{a 0[980]}}=1.01$ - very close indeed! A black-hole computation could not do so.

### 6.2.2 Schrödinger-like EOM

With

$$
\begin{align*}
& \mathcal{C}_{1}(Z)=e^{2|Z|}\left(3 \gamma^{2}\left(g_{s} N_{f}(\log (N)-3|Z|-3)+4 \pi\right)\right. \\
& \quad-3 g_{s} N_{f}\left(3 \gamma^{2}+2 e^{2|Z|}\right) \log \left(r_{0}\right) \\
& \left.\quad+2 e^{2|Z|}\left(g_{s} N_{f}(\log (N)-3|Z|)+4 \pi\right)\right) \tag{157}
\end{align*}
$$

and the potential in the Schrödinger-like EOM (analogous to (150)) given by

$$
\begin{align*}
V\left(\mathcal{G}_{n}\right)= & -4-3 e^{-4|Z|} \gamma^{2} \tilde{m}^{2}+e^{-2|Z|}\left(3 \gamma^{2}+\tilde{m}^{2}\right) \\
& +\frac{6-\frac{27}{2} e^{-2|Z|} \gamma^{2}}{\log N}+\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right) \tag{158}
\end{align*}
$$

the $a, b$, analogous to (151), are given by
$a=-4+3 \gamma^{2}+\tilde{m}^{2}\left(1-3 \gamma^{2}\right)+\frac{\left(6-27 \gamma^{2}\right)}{\log N}$,
$b=-6 \gamma^{2}+\tilde{m}^{2}\left(12 \gamma^{2}-2\right)+\frac{27 \gamma^{2}}{\log N}$.
One gets a solution analogous to (152); only for $c_{2}=0$ we can show numerically that, for $\gamma=0.6, \tilde{m} \approx 0.85$ - not too far off of the smallest root in Sect. 6.2.1 and about the same order as the results of Sect. 5.2.1 - one can approximately satisfy the Neumann boundary condition at $r=r_{0}$.

### 6.2.3 WKB quantization condition

Once again, as was assumed for the black-hole background computation, given that scalars are typically more massive than vector mesons, implying that we address the UV-IR interpolating/UV region in which $M, N_{f}$ are very small, one sees that

$$
\begin{align*}
\sqrt{V\left(\mathcal{G}_{n}\right)}= & \sqrt{-4-3 e^{-2|Z|} \gamma^{2} \tilde{m}^{2}+e^{-2|Z|}\left(3 \gamma^{2}+\tilde{m}^{2}\right)} \\
& +\mathcal{O}\left(\frac{1}{N^{\frac{1}{5}}}\right) \tag{160}
\end{align*}
$$

One sees that $|Z| \in\left[\log (\sqrt{3} \gamma), \log \left(\frac{\tilde{m}}{2}-\frac{9 \gamma^{2}}{4 \tilde{m}^{2}}\right)\right], \sqrt{V} \in$ $\mathbb{R}$ and the WKB quantization condition:

$$
\begin{equation*}
\int_{\log (\sqrt{3} \gamma)}^{\log \left(\frac{\tilde{m}}{2}-\frac{9 \gamma^{2}}{4 m^{2}}\right)} \sqrt{V\left(\mathcal{G}_{n}(Z)\right)}=\left(n+\frac{1}{2}\right) \pi \tag{161}
\end{equation*}
$$

yields

$$
\begin{align*}
& m_{n}=\frac{1}{20} \sqrt{3} \gamma(6(2 \pi n+\pi+6) \\
& \left.\quad+\sqrt{2} \sqrt{18(2 \pi n+\pi)^{2}+216 \pi(2 n+1)+349}\right) \tag{162}
\end{align*}
$$

Hence, disregarding $n=0$, the spectrum of Table 6 is generated.

For a low-temperature thermal gravity dual, we do not trust values $n>3$ and hence have not quoted these.

## 7 Summary, new insights into thermal QCD and future directions

A top-down finite-gauge-coupling finite-number-of-colors holographic thermal QCD calculation pertaining to meson spectroscopy, ${ }^{4}$ has thus far been missing in the literature. This paper fills this gap. We should keep in mind that even though lattice QCD is a good tool to deal with IR physics it is hard to include fundamental fermions in the same. However, incorporation of fermions is easily taken care of in the top-down type IIB construct of [1] and its type IIA mirror in [2]. In this paper, we have calculated (pseudo-)vector and (pseudo-)scalar meson spectra from the delocalized type IIA SYZ mirror (constructed in [2]) of the UV-complete topdown type IIB holographic dual of large- $N$ thermal QCD (constructed in [1]), at finite coupling and with finite number of colors (part of the 'MQGP' limit), and we compared our

[^4]Table 6 (Pseudo-)scalar meson masses from WKB quantization

|  | (Pseudo-)scalar meson name | $J^{P C}$ | $m_{n>0}\left(\right.$ units of $\left.\frac{r_{0}}{\sqrt{4 \pi g_{s} N}}\right)$ | PDG mean mass (MeV) |
| :--- | :--- | :--- | :--- | :--- |
| $\mathcal{Y}^{(1)}$ | $f 0[980] / a 0[980]$ | $0^{++}$ | 15.745 | 980 |
| $\mathcal{Y}^{(2)}$ | - | $0^{--}$ | 22.359 | - |
| $\mathcal{Y}^{(3)}$ | $f 0[1370]$ | $0^{++}$ | 28.934 | 1350 |

results with [3-5]. We first do a computation with a blackhole background assuming the same to be valid for all temperatures, low and high (similar in spirit to the computations in [8]). We then repeat the computation in a thermal background with no black hole which is valid for low temperatures. What we learn about QCD is that the mirror of [1] when considered in the 'MQGP limit' - involving finite gauge/string coupling and finite $N_{c}=M$ (at the end of a Seiberg duality cascade) and not just a large 't Hooft coupling - can, almost without any fine tuning, generate the low-lying vector and scalar meson spectra from the massless string sector. An analytical finite-gauge-coupling computation in perturbative (thermal) QCD is very hard if not unfeasible. This, however, is easily done as a classical supergravity computation in our set-up.

- Summary of new results obtained (points 1.-6.) and the new insights gained into thermal QCD (point 7.)

1. In Tables 1 and 2 , even if we drop the $\mathcal{O}\left(\frac{1}{\log N}\right)$ terms in the vector meson masses ( $\mathrm{BH} /$ thermal background) obtained by a WKB quantization condition, and assume $n=1,2,3,4$ to correspond, respectively, to $\rho$ [770], $a_{1}$ [1260], $\rho$ [1450], $a_{1}$ [1640], then Table 7 compares mass ratios from our results at LO in $N$ (obtained from a WKB quantization condition) with those from [3,4] (up to first order in $\delta=\frac{g_{s} M^{2}}{N}<1$ ) and [5].

The authors of [4] obtain a variety of values by adjusting the values of and working up to first order in $\delta$, as well as a constant appearing in a 'squashing factor' in the metric. Their best values for (pseudo-)vector meson mass ratios are quoted in Table 5, column 5. But they need to do a lot of fine tuning, incorporate contributions to the results from the $\mathcal{O}(\delta)$
terms, and choose $\delta=0.5$, which in fact cannot justify disregarding terms of higher powers of $\delta$ as $\delta=0.5$ is not very small to warrant the same. Our results, specially coming from the WKB quantization condition applied to $V\left(\alpha_{n}^{\{0\}}\right)$ for the BH gravitational dual or $V\left(\alpha_{n}\right)$ for the thermal gravitational background, working even up to LO in $N$ without having to explicitly numerically compute the $\mathcal{O}(\delta)(\delta \sim 0.001$ for our calculations and thereby justifying dropping higher powers of $\delta$ ) contribution, display the following features (Table 6):

- Our $m_{a_{1}[1260]}^{2} / m_{\rho[770]}^{2}$ is close to [3,4], and not too far off of the PDG value.
- Our $m_{\rho[1450]}^{2} / m_{\rho[770]}^{2}$ is the same as (for BH background)/very close to (for thermal background) [4] (but without any fine tuning and already at LO in $N$ ) - within $\approx 15 \%$ of the PDG value.
- Our $m_{a_{1}[1640]}^{2} / m_{\rho[770]}^{2}$ is closer to the PDG value than [3].

2. There is a near isospectrality between the lightest (pseudo-)vector meson masses calculated by a BH and thermal backgrounds.
3. The thermal background, to the order permissible by our analytical/numerical computations, does not provide a temperature dependence of $\tilde{m}$ at low temperatures - in agreement with one's expectations [8]. Encouraged by the aforementioned isospectrality, by solving the EOMs for the gauge field fluctuations along the $D 6$ world volume in a BH gravitational dual, close to the horizon, we are able to capture the explicit temperature dependence of the lowest lying vector meson mass with the temperature dependence appearing at $\mathcal{O}\left(\frac{1}{(\log N)^{2}}, \frac{g_{s} M^{2}}{N}\right)$. The temperature-dependent meson mass $\tilde{m}$ will have the following form (Table 7):

Table 7 Comparison of mesons masses ratio

|  | Ratio $\left(\alpha_{n}^{\{i\}}\right)(\mathrm{BH})$ | Ratio $\left(\alpha_{n}^{\{0\}}\right)(\mathrm{BH})$ | Ratio (thermal) | Sakai-Sugimoto (as <br> given in [4]) | Best value in [4]: $\delta=0.5$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Exp. value PDG <br> (as given in [4]) |  |  |  |  |  |
| $m_{a_{1}[1260]}^{2}$ <br> $m_{\rho[770]}^{2}$ | 2.30 | 2.28 | 2.28 | 2.32 | 2.31 |
| $\frac{m_{\rho[1450]}^{2}}{m_{\rho[770]}^{2}}$ | 4.12 | 4.09 | 4.07 | 4.22 | 4.09 |
| $\frac{m_{a_{1}[1640]}^{2}}{m_{\rho[770]}^{2}}$ | 6.47 | 6.41 | 6.38 | 6.62 | 5.57 |

$$
\begin{align*}
& \tilde{m}_{\text {lightest }}=\alpha+\frac{\beta}{\log N}+\frac{\left(\delta_{1}+\delta_{2} \log r_{h}\right)}{(\log N)^{2}} \\
& \quad+\frac{g_{s} M^{2}\left(\kappa_{1}+\kappa_{2} \log r_{h}\right)}{N}+\mathcal{O}\left(\frac{g_{s} M^{2}}{N} \frac{\log r_{h}}{\log N}\right) \tag{163}
\end{align*}
$$

$\delta_{2}>0$. The temperature, assuming the resolution to be larger than the deformation in the resolved warped deformed conifold in the type IIB background of [1] in the MQGP limit, and utilizing the IR-valued warp factor $h\left(r, \theta_{1} \sim N^{-\frac{1}{5}}, \theta_{2} \sim N^{-\frac{3}{10}}\right)$, is [19]

$$
\begin{align*}
T & =\frac{\partial_{r} G_{00}^{\mathcal{M}}}{4 \pi \sqrt{G_{00}^{\mathcal{M}} G_{r r}^{\mathcal{M}}}}=r_{h}\left[f r a c 12 \pi^{3 / 2} \sqrt{g_{s} N}\right. \\
& \left.-\frac{3 g_{s}^{\frac{3}{2}} M^{2} N_{f} \log \left(r_{h}\right)\left(-\log N+12 \log \left(r_{h}\right)+\frac{8 \pi}{g_{s} N_{f}}+6-\log (16)\right)}{64 \pi^{7 / 2} N^{3 / 2}}\right] \\
& +a^{2}\left(\frac{3}{4 \pi^{3 / 2} \sqrt{g_{s}} \sqrt{N} r_{h}}\right. \\
& \left.-\frac{9 g_{s}^{3 / 2} M^{2} N_{f} \log \left(r_{h}\right)\left(\frac{8 \pi}{g_{s} N_{f}}-\log (N)+12 \log \left(r_{h}\right)+6-2 \log (4)\right)}{128 \pi^{7 / 2} N^{3 / 2} r_{h}}\right) . \tag{164}
\end{align*}
$$

Using (164) and the arguments of [19], one can invert (164) and express $r_{h}$ in terms of $T$ [41] in the MQGP limit. Assuming $\log r_{h}$ in (163) to be in fact $\log \left(\frac{r_{h}}{\Lambda}\right), \Lambda>$ $r_{h}$ being the scale at which confinement occurs, one sees that, as per expectations, the vector meson masses decrease with temperature [8] with the same being large$N$ suppressed [42] (and references therein).
4. On comparing scalar meson mass ratios obtained from (144) using a black-hole gravitational dual WKB quantization and PDG values, we obtain Table 8.

The agreement with the PDG values for the lightest three scalar meson candidates (if assumed to be $f 0$ [980], $f 0[1370], f 0[1450])$ is quite nice. We do not expect the agreement for more massive scalar mesons. This is for the following reason. As discussed in $[4,14]^{5}$ massive open string excitations can contribute to the meson (specially scalar) masses (as scalar mesons are typically heavier than (pseudo)vector mesons). We do not attempt to estimate the same as open string quantization in a curved RR background is a hard problem, and in this paper, like [4], we have assumed that the mesons arise from the massive KK modes of the massless open string sector. The difference between our results and the PDG results for the mass ratios of vector and scalar

[^5]Table 8 The lightest scalar meson mass ratios

| Our results | PDG values |
| :---: | :---: |
| $m_{n=3}$ | $m_{\text {f0[1370] }}$ |
| $m_{n=1}$ | $m_{\text {f0 } 0980]}$ |
| 1.38 | 1.38 |
| $\frac{m_{n=5}}{m_{n=1}}$ | $\frac{m_{f o[1450]}}{m_{\text {f }}}$ |
| $m_{n=1}$ | $m_{\text {f0[ } 980]}$ |
| 1.81 | 1.50 |

mesons, for heavier mesons, could be attributable to the contributions coming to meson masses from the massive open string sector (which we have not calculated) in addition to the ones coming from the massless open string sector (which we have calculated in this paper).
5. Using a thermal background, though on the one hand, it appears to be possible to in fact resolve $a 0$ [980] (average mass of 980 MeV ) and $f 0$ [980] (average mass 990 MeV ), on the other hand assuming $f 0$ [980]/a0[980], $f 0$ [1370], $f 0[1450]$ to be the lightest scalar mesons, and a WKB quantization condition yields a mass ratio of the first two as 1.83 , the mass ratio of $f 0[1370$ ] and $a 0$ [980] being 1.38 ; as $f 0[1370]$ is expected to have a mass range of $1200-1500 \mathrm{MeV}$ [5], with 1500 MeV the ratio - as per PDG values - increases to 1.53 .
The thermal background is not able to correctly account for $f 0$ [1470]. The black-hole background, as is evident from Table 8, does a good job though.
6. The $0^{--}$pseudo-scalar mesons in Table 4 do not, thus far, have corresponding entries in the PDG tables. This is one point of difference between our results and PDG tables.
7. There are three main insights we gain in thermal QCD by evaluating mesonic (vector/scalar) spectra and comparing with PDG values.
(a) The first is the realization that, from a type IIA perspective, meson spectroscopy in the mirror of top-down holographic type IIB duals of large- $N$ thermal QCD at finite coupling and number of colors ${ }^{6}$ (closer to a realistic QCD calculation) which are UV-complete - we know of only [1] that falls in this category for which the mirror was worked out in [2] - will give results closer to PDG values rather than a single T dual of the same. Even though obtaining the mirror requires a lot of work, once performed, one can obtain very good agreement with PDG tables already at $\mathcal{O}\left(\left(\frac{g_{s} M^{2}}{N}\right)^{0}\right)$ (for vector mesons, without any fine tuning). This is a major lesson we learn from

[^6]our computation. There are two major reasons for the same.

- As noted in Sect. 3, the mirror of [1] picks up subdominant terms in $N$ of $\mathcal{O}\left(N^{0}\right)$ in $B^{\text {IIA }}$ which are therefore bigger than the $\mathcal{O}\left(\frac{g_{s} M^{2}}{N}\right)$ contributions, and which were missed in [4]. This is the reason why the authors of [4] had to set $\frac{g_{s} M^{2}}{N}=0.5-$ not a small enough fraction to warrant disregarding $\mathcal{O}\left(\left(\frac{g_{s} M^{2}}{N}\right)^{2}\right)$ contributions which they did - to obtain a reasonable match with [5].
- In the context of [1], this is expected to be related to the following. ${ }^{7}$ The brane construct of [1] involves $N$ D3branes, $M D 5$-branes wrapping the vanishing $S^{2}, M \overline{D 5}$ branes wrapping the same $S^{2}$ but placed at the antipodal points of the resolved $S^{2}(a)$ relative to the $D 3, D 5$ branes, $N_{f}$ flavor $D 7$-branes wrapping an $S^{3}$ and radially extending all the way into the IR starting from the UV and $N_{f} \overline{D 7}$-branes wrapping the same $S^{3}$ but going only up to the IR-UV interpolating region. The mirror of this results in $D 6$ branes in a deformed conifold. Now, the gravity dual of the above picture - which is what we work with - involves a resolved warped deformed conifold with a black hole and $\overline{D 5}, D 7$ branes and $\overline{D 7}$ branes (plus fluxes) on the type IIB side and the delocalized mirror yields a non-Kähler warped resolved conifold with a black hole and $D 6, \overline{D 6}$ branes (plus fluxes) on the type IIA side. The latter (warped resolved conifolds) are easier to deal with computationally than the former (resolved warped deformed conifolds).
(b) This is related to (a) above. There is an intimate connection between Strominger-Yau-Zaslow mirror of resolved warped deformed conifolds and thermal QCD at strong coupling and finite number of colors; hence, delocalized Strominger-Yau-Zaslow mirror construction is an entirely new technique used for hadron spectroscopy.
(c) The third is that a BH gravitational dual and a thermal gravity dual yield nearly isospectral spectra for the light vector mesons; the same is partially true for the lightest scalar mesons too. Explicit computations reveal that a BH type IIA gravity dual obtained by delocalized SYZ mirror transform of the type IIB holographic dual of [1] is not only able to provide a good match with PDG values for the lightest vector and scalar mesons, it is also able to thereby obtain an explicit temperature dependence of the (pseudo-)vector masses as a bonus and realize the $\log r_{h}$ dependence in the same appears at the sub-dominant

[^7]$\mathcal{O}\left(\frac{1}{(\log N)^{2}}\right)$ with the desired feature of a small large- $N$ suppressed decrease with increase of temperature.

- Future directions: glueball decays into mesons

It will be interesting to look at the various glueball-tomeson decay modes. To that end, Performing a KaluzaKlein reduction similar to [39]: $A_{Z}=\phi(Z) \pi\left(x^{\mu}\right), A_{\mu}=$ $\psi(Z) \rho_{\mu}\left(x^{\nu}\right)$, and similar to [40], we can look at the following M-theory metric perturbations $h_{M N}(M, N=$ $0, \ldots, 10 ; \mu=t, a, a=1,2,3)$ :
$h_{t t}\left(r, x^{\mu}\right)=q_{1}(r) G\left(x^{\mu}\right) G_{t t}^{\mathcal{M}}$,
$h_{r r}\left(r, x^{\mu}\right)=q_{2}(r) G\left(x^{\mu}\right) G_{r r}^{\mathcal{M}}$,
$h_{r a}\left(r, x^{\mu}\right)=g_{3}(r) \partial_{a} G\left(x^{\mu}\right) G_{a a}^{\mathcal{M}}$,
$h_{a b}\left(r, x^{\mu}\right)=G_{a b}^{\mathcal{M}}\left(q_{4}(r)+q_{5} \frac{\partial_{a} \partial_{b}}{m^{2}}\right)$
$G\left(x^{\mu}\right)$ no summation,
$h_{10{ }_{10}}\left(r, x^{\mu}\right)=q_{6}(r) G\left(x^{\mu}\right) G_{1010}^{\mathcal{M}}$.

Using Witten's prescription of going from type IIA to M-theory we could work back and using the aforementioned M-theory metric perturbations, work out the type IIA metric perturbations which hence would yield (in the following $\tilde{G}_{\alpha \beta}^{\text {IIA }}=G_{\alpha \beta}^{\text {IIA }}+h_{\alpha \beta} ; \alpha, \beta=0,1 \ldots, 9$ and $h_{\alpha \beta}$ being type IIA metric perturbations):

$$
\begin{aligned}
& e^{\frac{4 \phi^{\mathrm{IA}}}{3}}=G_{1010}^{\mathcal{M}}+h_{1010}, \\
& \frac{\tilde{G}_{r r, t t}^{\mathrm{IAA}}}{\sqrt{G_{1010}^{\mathcal{M}}+h_{1010}}}=G_{r r, t t}^{\mathcal{M}}+h_{r r, t t}, \\
& \frac{\tilde{G}_{r a}^{\mathrm{IIA}}}{\sqrt{G_{1010}^{\mathcal{M}}+h_{1010}}}=h_{r a}, \\
& \frac{\tilde{G}_{a b}^{\mathrm{IIA}}}{\sqrt{G_{1010}^{\mathcal{M}}+h_{1010}}}=G_{a b}^{\mathcal{M}}+h_{a b} .
\end{aligned}
$$

Solving (with a slight abuse of notation) the first order perturbation of the M-theory Einstein's EOM (assuming the flux term providing a cosmological constant): $R_{M N}^{(1)} \sim \frac{G_{4} \wedge * G_{4}}{\sqrt{G}} h_{M N}$, for $q_{1, \ldots 6}$, one can obtain the glueball-meson interaction Lagrangian density (metric perturbation corresponding to glueballs and gauge field fluctuations corresponding to mesons), using which one can work out glueball decays into mesons.

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## Appendix A: Triple-T duality rules

In this section, we summarize the Buscher triple T duality rules for T dualizing first along $x$, then along $y$ followed by dualizing along $z$. The starting metric in the type IIB theory has the following components:

$$
\begin{align*}
\mathrm{d} s_{\text {IIB }}^{2}= & g_{\mu \nu}^{\mathrm{IIB}} \mathrm{~d} x^{\mu} \mathrm{d} x^{\nu}+g_{x \mu}^{\mathrm{IIB}} \mathrm{~d} x \mathrm{~d} x^{\mu} \\
& +g_{y \mu}^{\mathrm{IIB}} \mathrm{~d} y \mathrm{~d} x^{\mu}+g_{z \mu}^{\mathrm{IIB}} \mathrm{~d} z \mathrm{~d} x^{\mu}+g_{x y}^{\mathrm{IIB}} \mathrm{~d} x \mathrm{~d} y \\
& +g_{x z}^{\mathrm{IB}} \mathrm{~d} x \mathrm{~d} z+g_{z y}^{\mathrm{IIB}} \mathrm{~d} z \mathrm{~d} y \\
& +g_{x x}^{\mathrm{IB}} \mathrm{~d} x^{2}+g_{y y}^{\mathrm{IIB}} \mathrm{~d} y^{2}+g_{z z}^{\mathrm{IIB}} \mathrm{~d} z^{2}, \tag{A1}
\end{align*}
$$

where $\mu, v \neq x, y, z$. As shown in [28], the form of the metric of the mirror manifold after performing three T dualities, first along $x$, then along $y$ and finally along $z$ :

$$
\begin{align*}
\mathrm{d} s^{2} & =\left(G_{\mu \nu}-\frac{G_{z \mu} G_{z \nu}-\mathcal{B}_{z \mu} \mathcal{B}_{z \nu}}{G_{z z}}\right) \mathrm{d} x^{\mu} \mathrm{d} x^{\nu} \\
& +2\left(G_{x \nu}-\frac{G_{z x} G_{z \nu}-\mathcal{B}_{z x} \mathcal{B}_{z v}}{G_{z z}}\right) \mathrm{d} x d x^{\mu} \\
& +2\left(G_{y \nu}-\frac{G_{z y} G_{z v}-\mathcal{B}_{z y} \mathcal{B}_{z v}}{G_{z z}}\right) \mathrm{d} y \mathrm{~d} x^{\nu} \\
& +2\left(G_{x y}-\frac{G_{z x} G_{z y}-\mathcal{B}_{z x} \mathcal{B}_{z y}}{G_{z z}}\right) \mathrm{d} x \mathrm{~d} y \\
& +\frac{\mathrm{d} z^{2}}{G_{z z}}+2 \frac{\mathcal{B}_{\mu z}}{G_{z z}} \mathrm{~d} x^{\mu} \mathrm{d} z+2 \frac{\mathcal{B}_{x z}}{G_{z z}} \mathrm{~d} x \mathrm{~d} z+2 \frac{\mathcal{B}_{y z}}{G_{z z}} \mathrm{~d} y \mathrm{~d} z \\
& +\left(G_{x x}-\frac{G_{z x}^{2}-\mathcal{B}_{z x}^{2}}{G_{z z}}\right) \mathrm{d} x^{2}+\left(G_{y y}-\frac{G_{z y}^{2}-\mathcal{B}_{z y}^{2}}{G_{z z}}\right) \mathrm{d} y^{2} . \tag{A2}
\end{align*}
$$

The various components of the metric after three successive T dualities along $x, y$ and $z$, respectively, can be written as [28]

$$
\begin{aligned}
& G_{\mu \nu}=\frac{g_{\mu \nu}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x \mu}^{\mathrm{IB}} g_{x \nu}^{\mathrm{IIB}}+b_{x \mu}^{\mathrm{IB}} b_{x \nu}^{\mathrm{IB}}}{g_{x x}^{\mathrm{IX}}} \\
& \frac{-\left(g_{y \mu}^{\mathrm{IIB}} g_{x x}^{\mathrm{IB}}-g_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IX}}+b_{x y}^{\mathrm{IB}} b_{x \mu}^{\mathrm{IIB}}\right)\left(g_{y v}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-g_{x y}^{\mathrm{IB}} g_{x \nu}^{\mathrm{IB}}+b_{x y}^{\mathrm{IIB}} b_{x \nu}^{\mathrm{IB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IB} 2}\right)}
\end{aligned}
$$

$$
\frac{+\left(b_{y \mu}^{\mathrm{IB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IB}}+b_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IIB}}\right)\left(b_{y \nu}^{\mathrm{IBB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} b_{x \nu}^{\mathrm{IB}}+b_{x y}^{\mathrm{IBB}} g_{x \nu}^{\mathrm{IB}}\right)}{g_{x x}^{\mathrm{IIP}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IBP}} 2\right),}
$$

$$
\begin{align*}
G_{\mu z}= & \frac{g_{\mu z}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-g_{x \mu}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}+b_{x \mu}^{\mathrm{IIB}} b_{x z}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IBB}}}  \tag{A3}\\
& -\frac{\left(g_{y \mu}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-g_{x y}^{\mathrm{IB}} g_{x \mu}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} b_{x \mu}^{\mathrm{IB}}\right)\left(g_{y z}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IBB}} b_{x z}^{\mathrm{IIB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB} 2}\right)} \\
& +\frac{\left(b_{y \mu}^{\mathrm{IBB}} g_{x x}^{\mathrm{IBB}}-g_{x y}^{\mathrm{IIB}} b_{x \mu}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IBB}} g_{x \mu}^{\mathrm{IBB}}\right)\left(b_{y z}^{\mathrm{IB}} g_{x x}^{\mathrm{IBB}}-g_{x y}^{\mathrm{IIB}} b_{x z}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IBB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IBB} 2}+b_{x y}^{\mathrm{IIB} 2}\right)}
\end{align*}
$$

$$
G_{y y}=\frac{g_{x x}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB} 2}+b_{x y}^{\mathrm{IIB} 2}}
$$

$$
G_{x x}=\frac{g_{y y}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2}
$$

$$
\begin{equation*}
G_{x y}=\frac{-g_{x y}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2} \tag{A7}
\end{equation*}
$$

$$
G_{\mu x}=\frac{b_{\mu x}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IIB}}}+\frac{\left(g_{\mu y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} b_{x \mu}^{\mathrm{IIB}}\right) b_{x y}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2\right)}
$$

$$
\begin{equation*}
+\frac{\left(b_{y \mu}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} b_{x \mu}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IIB}}\right) g_{x y}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IB}}\left(g_{y y}^{\mathrm{IB}} g_{x x}^{\mathrm{IXB}}-g_{x y}^{\mathrm{IB}} 2+b_{x y}^{\mathrm{IIB}}{ }^{2}\right)} \tag{A8}
\end{equation*}
$$

$$
G_{z x}=\frac{b_{z x}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IIB}}}+\frac{\left(g_{z y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} b_{x z}^{\mathrm{IIB}}\right) b_{x y}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2\right)}
$$

$$
\begin{equation*}
+\frac{\left(b_{y z}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} b_{x z}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}\right) g_{x y}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2\right)} \tag{A9}
\end{equation*}
$$

In the above formulas we have denoted the type IIB $B$ fields by $b_{m n}^{\mathrm{IIB}}$. For the generic case we will switch on all the components of the $B$ field:

$$
\begin{align*}
B^{\mathrm{IIB}}= & b_{\mu \nu}^{\mathrm{IIB}} \mathrm{~d} x^{\mu} \wedge \mathrm{d} x^{\nu}+b_{x \mu}^{\mathrm{IB}} \mathrm{~d} x \wedge \mathrm{~d} x^{\mu}+b_{y \mu}^{\mathrm{IB}} \mathrm{~d} y \wedge \mathrm{~d} x^{\mu}+b_{z \mu}^{\mathrm{IBB}} \mathrm{~d} z \\
& \wedge \mathrm{~d} x^{\mu}+b_{x y}^{\mathrm{IIB}} \mathrm{~d} x \wedge \mathrm{~d} y+b_{x z}^{\mathrm{IB}} \mathrm{~d} x \wedge \mathrm{~d} z+b_{z y}^{\mathrm{IIB}} \mathrm{~d} z \wedge \mathrm{~d} y \tag{A10}
\end{align*}
$$

$$
\begin{align*}
& G_{z z}=\frac{g_{z z}^{\mathrm{IIB}} g_{x x}^{\mathrm{II}}-j_{x z}^{2}+b_{x z}^{2}}{g_{x x}^{\mathrm{II}}}  \tag{A4}\\
& -\frac{\left(g_{y z}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IBB}}+b_{x y}^{\mathrm{IB}} b_{x z}^{\mathrm{IBB}}\right)^{2}}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IB}} 2\right)}+\left(b_{y z}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-g_{x y}^{\mathrm{IIB}} b_{x z}^{\mathrm{IBB}}\right. \\
& \frac{\left.+b_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}\right)^{2}}{\left.g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB} 2}+b_{x y}^{\mathrm{IIB} \mathrm{2}}\right),}  \tag{A5}\\
& G_{y \mu}=-\frac{b_{y \mu}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-b_{x \mu}^{\mathrm{II}} g_{x y}^{\mathrm{II}}+b_{x y}^{\mathrm{IIB}} g_{\mu x}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IB}} 2+b_{x y}^{\mathrm{IIB}} 2}, \\
& G_{y z}=-\frac{b_{y z}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-b_{x z}^{\mathrm{II}} g_{x y}^{\mathrm{IIB}}+b_{x y}^{\mathrm{II}} g_{z x}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IB} 2}+b_{x y}^{\mathrm{II} 2}}, \tag{A6}
\end{align*}
$$

After applying again the T dualities, the type IIA NS-NS $B$ field in the mirror set-up will take the form

$$
\begin{align*}
B^{I I A}= & \left(\mathcal{B}_{\mu \nu}+\frac{2 \mathcal{B}_{z[\mu} G_{\nu] z}}{G_{z z}}\right) \mathrm{d} x^{\mu} \wedge \mathrm{d} x^{\nu} \\
& +\left(\mathcal{B}_{\mu x}+\frac{2 \mathcal{B}_{z[\mu} G_{x] z}}{G_{z z}}\right) \mathrm{d} x^{\mu} \wedge \mathrm{d} x \\
& \times\left(\mathcal{B}_{\mu y}+\frac{2 \mathcal{B}_{z[\mu} G_{y] z}}{G_{z z}}\right) \mathrm{d} x^{\mu} \wedge \mathrm{d} y \\
& +\left(\mathcal{B}_{x y}+\frac{2 \mathcal{B}_{z[x} G_{y] z}}{G_{z z}}\right) \mathrm{d} x \wedge \mathrm{~d} y \\
& +\frac{G_{z \mu}}{G_{z z}} \mathrm{~d} x^{\mu} \wedge \mathrm{d} z+\frac{G_{z x}}{G_{z z}} \mathrm{~d} x \wedge \mathrm{~d} z+\frac{G_{z y}}{G_{z z}} \mathrm{~d} y \wedge \mathrm{~d} z \tag{A11}
\end{align*}
$$

Here the $G_{m n}$ components have been given above, and the various $\mathcal{B}$ components can be written as

$$
\begin{aligned}
& \mathcal{B}_{\mu \nu}=\frac{b_{\mu \nu}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}+b_{x \mu}^{\mathrm{IB}} g_{v x}^{\mathrm{IIB}}-b_{x \nu}^{\mathrm{IB}} g_{\mu x}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IB}}}
\end{aligned}
$$

$$
\begin{align*}
& \mathcal{B}_{\mu z}=\frac{b_{\mu z}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}+b_{x \mu}^{\mathrm{II}} g_{z x}^{\mathrm{IB}}-b_{x z}^{\mathrm{IB}} g_{\mu x}^{\mathrm{II}}}{g_{x x}^{\mathrm{IB}}}  \tag{A12}\\
& +\frac{2\left(g_{y[\mu}^{\mathrm{IB}} g_{x x}^{\mathrm{IB}}-g_{x y}^{\mathrm{II}} g_{x[\mu}^{\mathrm{IB}}+b_{x y}^{\mathrm{IIB}} b_{x[\mu}^{\mathrm{IB}}\right)\left(b_{z] \mid \mathrm{IB}} g_{x x}^{\mathrm{IB}}-b_{z] x}^{\mathrm{IB}} g_{x y}^{\mathrm{IB}}-b_{x y}^{\mathrm{II}} g_{z] x}^{\mathrm{IB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IB} 2}\right)},  \tag{A13}\\
& \mathcal{B}_{\mu y}=\frac{g_{\mu y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IB}}+b_{x y}^{\mathrm{IIB}} b_{x \mu}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB} 2}+b_{x y}^{\mathrm{IIB}} 2}, \\
& \mathcal{B}_{z y}=\frac{g_{z y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} b_{x z}^{\mathrm{IB}}}{g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{II}} 2},  \tag{A14}\\
& \mathcal{B}_{\mu x}=\frac{g_{\mu x}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IB}}}-\frac{g_{x y}^{\mathrm{IIB}}\left(g_{\mu y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} g_{x \mu}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} b_{x \mu}^{\mathrm{IIB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2\right)} \\
& +\frac{b_{x y}^{\mathrm{IIB}}\left(b_{x \mu}^{\mathrm{IIB}} g_{x y}^{\mathrm{IIB}}-b_{y \mu}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-b_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2\right)},  \tag{A15}\\
& \mathcal{B}_{z x}=\frac{g_{z x}^{\mathrm{IIB}}}{g_{x x}^{\mathrm{IIB}}}-\frac{g_{x y}^{\mathrm{IIB}}\left(g_{z y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IB}} g_{x z}^{\mathrm{IIB}}+b_{x y}^{\mathrm{IIB}} b_{x z}^{\mathrm{IB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB} 2} 2\right)} \\
& +\frac{b_{x y}^{\mathrm{IIB}}\left(b_{x z}^{\mathrm{IIB}} g_{x y}^{\mathrm{IIB}}-b_{y z}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-b_{x y}^{\mathrm{IIB}} g_{x z}^{\mathrm{IIB}}\right)}{g_{x x}^{\mathrm{IIB}}\left(g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{IIB}} 2\right)},  \tag{A16}\\
& \mathcal{B}_{x y}=\frac{-b_{x y}^{\mathrm{IIB}}}{g_{y y}^{\mathrm{IIB}} g_{x x}^{\mathrm{IIB}}-g_{x y}^{\mathrm{IIB}} 2+b_{x y}^{\mathrm{III}} 2} . \tag{A17}
\end{align*}
$$

## Appendix B: Vector meson embedding functions appearing in the DBI action for D6-branes

In this appendix we give the embedding functions $\Sigma_{0,1}\left(r ; g_{s}\right.$, $\left.N_{f}, M, N\right)$ relevant to the embedding of $D 6$-branes in the delocalized SYZ type IIA mirror of the type IIB construct of [1] that appears in (44) in Sect. 3. The same are given by

$$
\begin{aligned}
& \Sigma_{0}\left(r ; g_{s}, N_{f}, M, N\right) \\
& \equiv-\frac{1}{97844723712 \pi^{11} \alpha_{\theta_{1}}^{8} \alpha_{\theta_{2}}^{4} g_{s} N^{26 / 5}\left(9 a^{2}+r^{2}\right)}\left\{r^{6}\left(6 a^{2}+r^{2}\right)\right. \\
& \times\left[81 \sqrt{2} 3^{5 / 6} \alpha_{\theta_{1}}^{9} \sqrt[5]{N}-54 \sqrt[3]{3} \alpha_{\theta_{1}}^{8} N^{2 / 5}+81 \sqrt{2} 3^{5 / 6} \alpha_{\theta_{1}}^{7} \alpha_{\theta_{2}}^{2}\right. \\
&-54(2+\sqrt[3]{3}) \alpha_{\theta_{1}}^{6} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}+24 \sqrt{6} \alpha_{\theta_{1}}^{5} \alpha_{\theta_{2}}^{2} N^{2 / 5}-8 \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}^{2} N^{3 / 5} \\
&\left.-24 \sqrt{6} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{4} \sqrt[5]{N}+16 \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{4} N^{2 / 5}-8 \alpha_{\theta_{2}}^{6} \sqrt[5]{N}\right] \\
& \times\left(3 g _ { s } M ^ { 2 } \operatorname { l o g } ( r ) \left(-2 g_{s} N_{f} \log \left(\alpha_{\theta_{1}} \alpha_{\theta_{2}}\right)+g_{s} N_{f} \log N\right.\right. \\
&\left.\left.-6 g_{s} N_{f}+g_{s} N_{f} \log (16)-8 \pi\right)-36 g_{s}^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)^{4} \\
& \times\left(3 g _ { s } M ^ { 2 } \operatorname { l o g } ( r ) \left(2 g_{s} N_{f} \log \left(\alpha \alpha_{\theta_{2}}\right)-g_{s} N_{f} \log N+6 g_{s} N_{f}\right.\right. \\
&\left.\left.\left.-2 g_{s} N_{f} \log (4)+8 \pi\right)+36 g_{s}^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)\right\}
\end{aligned}
$$

$\Sigma_{1}\left(r ; g_{s}, N_{f}, M, N\right)$

$$
\begin{align*}
\equiv & \frac{1}{165112971264 \pi^{19 / 2} \alpha_{\theta_{1}}^{10} \alpha_{\theta_{2}}^{6} g_{s}^{3 / 2} N^{49 / 10}}\left\{r^{4}\left(r^{4}-r_{h}^{4}\right)\right. \\
& \times\left[-486 \sqrt{6} \alpha_{\theta_{1}}^{11} N+324 \alpha_{\theta_{1}}^{10} N^{6 / 5}\right. \\
& +972 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{1}}^{9} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}\left(\left(-9-3 \sqrt[3]{3}+63^{2 / 3}\right) \alpha_{\theta_{2}}^{2}-\sqrt[3]{3} N^{3 / 5}\right) \\
& +108 \alpha_{\theta_{1}}^{8}\left(27 \sqrt{2} \sqrt[6]{3}\left(3^{2 / 3}-3\right) \alpha_{\theta_{2}}^{5} \sqrt[10]{N}+18(\sqrt[3]{3}-1)^{2} \alpha_{\theta_{2}}^{4} N^{2 / 5}\right. \\
& \left.+2\left(3+3^{2 / 3}\right) \alpha_{\theta_{2}}^{2} N\right)-9 \alpha_{\theta_{1}}^{7}\left(243 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{2}}^{6}-216 \sqrt[3]{3}(\sqrt[3]{3}-1)\right. \\
& \left.\alpha_{\theta_{2}}^{5} N^{3 / 10}+54 \sqrt{6} \alpha_{\theta_{2}}^{4} N^{3 / 5}+16 \sqrt{2} \sqrt[6]{3} \alpha_{\theta_{2}}^{2} N^{6 / 5}\right) \\
& +2 \alpha_{\theta_{1}}^{6}\left(2433^{2 / 3} \alpha_{\theta_{2}}^{6} \sqrt[5]{N}+54\left(3+23^{2 / 3}\right) \alpha_{\theta_{2}}^{4} N^{4 / 5}\right. \\
& \left.+83^{2 / 3} \alpha_{\theta_{2}}^{2} N^{7 / 5}\right)-163^{2 / 3} \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}^{4} N^{6 / 5}+144 \sqrt{2} \sqrt[6]{3} \alpha^{3} \alpha_{\theta_{2}}^{6} N^{4 / 5} \\
& \left.-163^{2 / 3} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}}^{6} N+163^{2 / 3} \alpha_{\theta_{2}}^{8} N^{4 / 5}\right]\left(3 g _ { s } M ^ { 2 } \operatorname { l o g } ( r ) \left(-2 g_{s} N_{f}\right.\right. \\
& \left.\log \left(\alpha \alpha_{\theta_{2}}\right)+g_{s} N_{f} \log N-6 g_{s} N_{f}+g_{s} N_{f} \log (16)-8 \pi\right) \\
& \left.\left.-36 g_{s}^{2} M^{2} N_{f} \log ^{2}(r)+32 \pi^{2} N\right)^{4}\right\} . \tag{B1}
\end{align*}
$$

## Appendix C: Functions appearing in the vector and scalar meson actions

Appendix C.1: Vector meson action functions
The functions $\mathcal{V}_{1,2}$ appearing in Eq. (53) in Sect. 4 in the context of the vector meson action obtained by substituting
a KK ansatz (52) into the DBI action for $N_{f}=2$ D6-branes of (51) are given by

$$
\begin{aligned}
\mathcal{V}_{1}(Z) & =e^{-\Phi^{I I A}} \sqrt{h} G^{Z Z} \sqrt{\operatorname{det}_{\theta_{2}, \tilde{y}}\left(i^{*}(G+B)\right)} \sqrt{\operatorname{det}_{\mathbb{R}^{1,3}, Z}\left(i^{*} G\right)}, \\
= & \frac{1}{36 \sqrt{2} \pi^{3 / 2} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}} g_{s}{ }^{3 / 2}}\left\{\sqrt[5]{N} e^{-4|Z|}\left(e^{4|Z|}-1\right)\right. \\
& \times\left(3 a^{2} g_{s} N_{f} \log N-9 a^{2} g_{s} N_{f} \log \left(r_{h}\right)\right. \\
& -9 a^{2} g_{s} N_{f}|Z|-9 a^{2} g_{s} N_{f}+12 \pi a^{2}+2 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log N \\
& \left.\left.-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|}|Z|-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log \left(r_{h}\right)+8 \pi r_{h}^{2} e^{2|Z|}\right)\right\} \\
& +\frac{1}{576 \pi^{3 / 2} \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}^{3} g_{s}^{3 / 2}}\left\{e ^ { - 4 | Z | } ( e ^ { 4 | Z | } - 1 ) \left(27 \sqrt{2} \sqrt[3]{3} \alpha_{\theta_{1}}^{6}\right.\right. \\
& \left.-24 \sqrt{3} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}-8 \sqrt{2} \alpha_{\theta_{2}}^{4}\right) \\
& \times\left(-3 a^{2} g_{s} N_{f} \log \left(\frac{1}{N}\right)-9 a^{2} g_{s} N_{f} \log \left(r_{h}\right)-9 a^{2} g_{s} N_{f}|Z|\right. \\
& -9 a^{2} g_{s} N_{f}+12 \pi a^{2}-2 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log \left(\frac{1}{N}\right) \\
& \left.\left.-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|}|Z|-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log \left(r_{h}\right)+8 \pi r_{h}^{2} e^{2|Z|}\right)\right\} \\
& +\mathcal{O}\left(\frac{1}{N^{6 / 5}}\right),
\end{aligned}
$$

$$
\mathcal{V}_{2}(Z)=e^{-\Phi^{I I A}} \frac{h}{2} \sqrt{\operatorname{det}_{\theta_{2}, \tilde{y}}\left(i^{*}(G+B)\right)} \sqrt{\operatorname{det}_{\mathbb{R}^{1,3},|Z|}\left(i^{*} G\right)}
$$

$$
=\frac{1}{18 \sqrt{2 \pi} \alpha_{\theta_{1}}^{2} \alpha_{\theta_{2}} \sqrt{g_{s}} r_{h}^{2}}\left\{N ^ { 6 / 5 } e ^ { - 2 | Z | } \left(3 a^{2} g_{s} N_{f} \log \left(\frac{1}{N}\right)\right.\right.
$$

$$
+9 a^{2} g_{s} N_{f} \log \left(r_{h}\right)+9 a^{2} g_{s} N_{f}|Z|
$$

$$
-9 a^{2} g_{s} N_{f}-12 \pi a^{2}+2 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log N-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|}|Z|
$$

$$
\left.\left.-6 g_{s} N_{f} r_{h}{ }^{2} e^{2|Z|} \log \left(r_{h}\right)+8 \pi r_{h}^{2} e^{2|Z|}\right)\right\}
$$

$$
+\frac{1}{288 \sqrt{\pi} \alpha_{\theta_{1}}^{4} \alpha_{\theta_{2}}^{3} \sqrt{g_{s} r_{h}{ }^{2}}}
$$

$$
\times\left\{N e^{-2|Z|}\left(27 \sqrt{2} \sqrt[3]{3} \alpha_{\theta_{1}}^{6}-24 \sqrt{3} \alpha_{\theta_{1}}^{3} \alpha_{\theta_{2}}^{2}-8 \sqrt{2} \alpha_{\theta_{2}}^{4}\right)\right.
$$

$$
\times\left(-3 a^{2} g_{s} N_{f} \log N+9 a^{2} g_{s} N_{f} \log \left(r_{h}\right)\right.
$$

$$
+9 a^{2} g_{s} N_{f}|Z|-9 a^{2} g_{s} N_{f}-12 \pi a^{2}+2 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log N
$$

$$
\left.\left.-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|}|Z|-6 g_{s} N_{f} r_{h}^{2} e^{2|Z|} \log \left(r_{h}\right)+8 \pi r_{h}^{2} e^{2|Z|}\right)\right\}
$$

$$
\begin{equation*}
+\mathcal{O}\left(\frac{1}{N^{6 / 5}}\right) \tag{C1}
\end{equation*}
$$

Appendix C.2: Scalar meson action functions

The scalar meson functions $\mathcal{S}_{1,2,3}$ appearing in the DBI action (122) for $N_{f} D 6$-branes and (124) are given by

$$
\begin{aligned}
\mathcal{S}_{1} & =\frac{1}{1296 \sqrt{2} \sqrt[3]{3} \pi \alpha_{\theta_{1}}^{2} 4 \alpha_{\theta_{2}}^{5} g_{s} Z^{2}}\left\{\mathcal { C } ^ { 2 } N ^ { 1 1 / 1 0 } r _ { h } { } ^ { 2 } \left(27 \sqrt[3]{3} \alpha_{\theta_{1}}^{2} 6\right.\right. \\
& \left.-12 \sqrt{6} \alpha_{\theta_{1}}^{2} 3 \alpha_{\theta_{2}}^{2}+8 \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{2} \sqrt[5]{N}-8 \alpha_{\theta_{2}}^{4}\right) \\
& \times\left[-g_{s} N_{f} \log (N)\left(3\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2}-2 e^{2|Z|}\right)\right.
\end{aligned}
$$

$$
\begin{align*}
& +3 g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right)\left(3\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2}-2 e^{2|Z|}\right) \\
& \left.\left.-3\left(3 g_{s} N_{f}+4 \pi\right)\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2}+8 \pi e^{2|Z|}\right]\right\}, \\
& \mathcal{S}_{2}=\frac{1}{15552 \sqrt{2} \pi^{2} \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{5} g_{s}{ }^{2} Z^{2}}\left\{\mathcal{C}^{2} \sqrt[10]{N} r_{h}{ }^{4} e^{-2|Z|}\left(e^{4|Z|}-1\right)\right. \\
& \times\left(81 \alpha_{\theta_{1}}^{2} 4-36 \sqrt{2} \sqrt[6]{3} \alpha \alpha_{\theta_{2}}^{2}+83^{2 / 3} \alpha_{\theta_{2}}^{2} \sqrt[5]{N}\right) \\
& \times\left[g_{s} N_{f} \log (N)\left(3\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2}+2 e^{2|Z|}\right)\right. \\
& -3 g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right)\left(3\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2}+2 e^{2|Z|}\right) \\
& \left.\left.+\left(12 \pi-9 g_{s} N_{f}\right)\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2}+8 \pi e^{2|Z|}\right]\right\} \\
& \mathcal{S}_{3}=\frac{1}{1944 \sqrt{2} \pi^{2} \alpha_{\theta_{1}}^{2} 2 \alpha_{\theta_{2}}^{3} g_{s}{ }^{2}\left(e^{4|Z|}-1\right) Z^{4}}\left\{\sqrt[5]{N} r_{h}{ }^{4} e^{-2|Z|}\right. \\
& \times\left(g _ { s } N _ { f } \operatorname { l o g } ( N ) \left(3 \left(3^{2 / 3} \mathcal{C}^{2} \sqrt[10]{N}\left(e^{4|Z|}-1\right)^{2}\right.\right.\right. \\
& \left.-27 \sqrt{\pi} \alpha_{\theta_{2}}^{2} \sqrt{g_{s}} e^{4|Z|} Z^{2}\left(-2|Z|+e^{4|Z|}(6|Z|+1)-1\right)\right) \\
& \times\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2} \\
& +2 e^{2|Z|}\left(27 \sqrt{\pi} \alpha_{\theta_{2}}^{2} \sqrt{g_{s}} e^{4|Z|} Z^{2}\left(e^{4|Z|}(4|Z|+1)-1\right)\right. \\
& \left.\left.+3^{2 / 3} \mathcal{C}^{2} \sqrt[10]{N}\left(e^{4|Z|}-1\right)^{2}\right)\right) \\
& -3 g_{s} N_{f}\left(\log \left(r_{h}\right)+|Z|\right)\left[3 \left(3^{2 / 3} \mathcal{C}^{2} \sqrt[10]{N}\left(e^{4|Z|}-1\right)^{2}\right.\right. \\
& \left.-27 \sqrt{\pi} \alpha_{\theta_{2}}^{2} \sqrt{g_{s}} e^{4|Z|} Z^{2}\left(-2|Z|+e^{4|Z|}(6|Z|+1)-1\right)\right) \\
& \times\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2} \\
& +2 e^{2|Z|}\left(27 \sqrt{\pi} \alpha_{\theta_{2}}^{2} \sqrt{g_{s}} e^{4|Z|} Z^{2}\left(e^{4|Z|}(4|Z|+1)-1\right)\right. \\
& \left.\left.+3^{2 / 3} \mathcal{C}^{2} \sqrt[10]{N}\left(e^{4|Z|}-1\right)^{2}\right)\right] \\
& +3\left(\frac{4 g_{s} M^{2}\left(\log \left(r_{h}\right)+1\right)}{N}+0.6\right)^{2} \\
& \times\left[\left(3^{2 / 3} \mathcal{C}^{2} \sqrt[10]{N}\left(e^{4|Z|}-1\right)^{2}\left(4 \pi-3 g_{s} N_{f}\right)\right.\right. \\
& -27 \sqrt{\pi} \alpha_{\theta_{2}}^{2} \sqrt{g_{s}} e^{4|Z|}|Z| \\
& \times\left(3 g_{s} N_{f}|Z|\left(3|Z|+e^{4|Z|}(|Z|+1)-1\right)\right. \\
& \left.\left.+4 \pi|Z|\left(-2|Z|+e^{4|Z|}(6|Z|+1)-1\right)\right)\right] \\
& +2 \sqrt{\pi} e^{2|Z|}\left(27 \alpha_{\theta_{2}}^{2} \sqrt{g_{s}} e^{4|Z|} Z^{2}\right. \\
& \times\left(4 \pi\left(e^{4|Z|}(4|Z|+1)-1\right)-3 g_{s} N_{f}\left(e^{4|Z|}-1\right)|Z|\right) \\
& \left.\left.\left.+43^{2 / 3} \sqrt{\pi} \mathcal{C}^{2} \sqrt[10]{N}\left(e^{4|Z|}-1\right)^{2}\right)\right)\right\} \text {. } \tag{C2}
\end{align*}
$$

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[^1]:    ${ }^{1}$ The non-zero resolution parameter ' $a$ ' is also there to introduce a separation $\mathcal{R}_{D 5 / \overline{D 5}}$ between the $D 5$ and $\overline{D 5}$ branes, which as in [19] we assume is $\sqrt{3} a$ (as $r>\sqrt{3} a$ lies in the large- $r$ region in a resolved conifold [20]).

[^2]:    ${ }^{2}$ There are also $D 5$-branes with world-volume coordinates $\left(x^{0,1,2,3}, \theta_{1}, \tilde{x}\right)$ and $\overline{D 5}$-branes with world-volume coordinates $\left(x^{0,1,2,3}, \theta_{1}, \tilde{x}\right)$ which, relative to the $D 5$-branes are at the antipodal point of the resolved $S_{a}^{2}\left(\theta_{2}, \phi_{2}\right)$; their bound state, however, is equivalent to producing a net $D 3$-brane charge provided a certain topological condition is satisfied (see [19] and references therein).

[^3]:    ${ }^{3}$ This is not actually a parameter put in by hand. In the spirit of a topdown approach, one can show that a Hawking-Page transition occurs at a temperature at an $r_{0}$ given in terms of $r_{h}$ and an $\mathcal{O}(1)$ constant of proportionality relating the modulus of the Ouyang embedding parameter corresponding to the holomorphic embedding of type IIB flavor branes to $r_{0}-$ see [19].

[^4]:    ${ }^{4}$ For glueball spectroscopy, see [16].

[^5]:    $\overline{5}$ One of us (AM) thanks K. Dasgupta for emphasizing this point to him.

[^6]:    ${ }^{6}$ In the IR, as explained in Sect. $2.1, N_{c}=M$ which can be $\mathcal{O}(1)$ in the MQGP limit of [2] - taken to be three in this paper - and not $N$.

[^7]:    ${ }^{7}$ One of us (AM) thanks K. Dasgupta for discussion of this point.

