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Higher spin currents in the orthogonal coset theory

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Abstract In the coset model $(D_N^{(1)} \oplus D_N^{(1)}, D_N^{(1)})$ at levels (k_1, k_2), the higher spin 4 current that contains the quartic WZW currents contracted with a completely symmetric $SO(2N)$ invariant d tensor of rank 4 is obtained. The three-point functions with two scalars are obtained for any finite N and k_2 with $k_1 = 1$. They are determined also in the large N 't Hooft limit. When one of the levels is the dual Coxeter number of $SO(2N)$, $k_1 = 2N - 2$, the higher spin $\frac{7}{2}$ current, which contains the septic adjoint fermions contracted with the above d tensor and the triple product of structure constants, is obtained from the operator product expansion (OPE) between the spin $\frac{3}{2}$ current living in the $\mathcal{N} = 1$ superconformal algebra and the above higher spin 4 current. The OPEs between the higher spin $\frac{7}{2}, 4$ currents are described. For $k_1 = k_2 = 2N - 2$ where both levels are equal to the dual Coxeter number of $SO(2N)$, the higher spin 3 current of $U(1)$ charge $\frac{4}{3}$, which contains the six products of spin $\frac{1}{2}$ (two) adjoint fermions contracted with the product of the d tensor and two structure constants, is obtained. The corresponding $\mathcal{N} = 2$ higher spin multiplet is determined by calculating the remaining higher spin $\frac{7}{2}, \frac{7}{2}, 4$ currents with the help of two spin $\frac{3}{2}$ currents in the $\mathcal{N} = 2$ superconformal algebra. The other $\mathcal{N} = 2$ higher spin multiplet, whose $U(1)$ charge is opposite to the one of the above $\mathcal{N} = 2$ higher spin multiplet, is obtained. The OPE between these two $\mathcal{N} = 2$ higher spin multiplets is also discussed.

1 Introduction

The proposal by Gaberdiel and Gopakumar [1], the duality between the higher spin gauge theory on AdS_3 space [2] and the large N 't Hooft limit of a family of $W_N (\equiv WA_{N-1})$ minimal models is the natural analog of the Klebanov and Polyakov duality [3] relating the $O(N)$ vector model in three-dimensions to a higher spin theory on AdS_4 space. Then the obvious generalization of [1] is to consider the Klebanov and Polyakov duality in one dimension lower. By replacing the $SU(N)$ group by $SO(2N)$, the relevant most general coset model is described as [4,5]

$$\frac{\hat{SO}(2N)_{k_1} \oplus \hat{SO}(2N)_{k_2}}{\hat{SO}(2N)_{k_1+k_2}}. \tag{1.1}$$

One can also consider the case where the $SU(N)$ group by $SO(2N+1)$ but this is not described in this paper. It is well known that the conformal weight (or spin) of the primary state is equal to the quadratic Casimir eigenvalue divided by the sum of the level and the dual Coxeter number of the finite Lie algebra [6,7]. For example, for $SO(2N)$, the quadratic Casimir eigenvalue for the adjoint representation is given by $2N - 2$, while the dual Coxeter number is $2N - 2$. Then we are left with the adjoint fermion of spin $\frac{1}{2}$ at the critical level which is equal to the dual Coxeter number. One can apply this critical behavior to the two numerator factors in (1.1) simultaneously. In the description of these adjoint free fermions, the central charge grows like N^2 in the large N 't Hooft limit: the so-called stringy coset model [8]. See also the relevant work in [9–13].

Although some constructions on the higher spin currents in [14] have been done, there are two unknown coefficients in the expression of higher spin 4 current. Moreover, the spin 1 currents in the numerators of (1.1) are described with the double index notation. Each index is a vector representation of $SO(2N)$ and because of the antisymmetry property of these spin 1 currents, the number of independent fields is given by $\frac{1}{2}[(2N)^2 - 2N] = N(2N-1)$. In order to obtain the description of the above free adjoint fermions, one should write down the spin 1 currents with a single adjoint index. It is well known that the real free fermions transforming in the adjoint representation of $SO(2N)$ realize the affine Kac–Moody algebra for the critical level. It is equivalent to the theory of $\frac{1}{2}2N(2N-1) = N(2N-1)$ free fermions [7].

Before one considers the adjoint free fermion description, one should obtain the higher spin 4 current from the spin

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1 currents living in the numerator factors of (1.1) and having a single adjoint index. The higher spin 4 current is the $SO(2N)$ singlet field [6]. Then one should have a quantity contracted with the quartic terms in the above spin 1 currents. This is known as the d symbol; it is a completely symmetric $SO(2N)$ invariant tensor of rank 4. In the calculation of any OPE between the higher spin currents, one should use various contraction identities between the above d symbol and the structure constant f. Recall that in the defining OPE between the spin 1 currents, the structure constant f symbol appears. As far as I know, there are no known identities between f symbol and d symbol except of the ff contraction in the literature. This is one of the reasons why the double index notation in [14] is used.

In this paper, one starts with the definition of the d symbol which is given by one half times the trace over six quartic terms in the $SO(2N)$ generators. When one meets the relevant contraction identities in the calculation of any OPE, one can try to obtain the tensorial structure in the right-hand sides of these identities. Of course in each term, there should be present N dependence coefficients explicitly. The tensorial structure in terms of multiple product of f symbol, d symbol and the symmetric $SO(2N)$ invariant tensor δ of rank 2 occurs naturally during the explicit calculation of OPE. As one applies for $N = 2, 3, 4$ and 5 cases in the $SO(2N)$ generators, one can determine the N dependence coefficients explicitly.

It turns out that the higher spin 4 current is obtained completely except of overall normalization factor. The eigenvalue equations of zero mode of the higher spin 4 current acting on several primary states can be determined explicitly. The corresponding three-point functions can be obtained. By choosing the overall factor correctly, one observes the standard three-point functions in the large N 't Hooft limit from the asymptotic symmetry algebra in the AdS_3 bulk theory.

According to the observations in [8,9], the Gaberdiel and Gopakumar proposal in the unitary case is still valid for arbitrary N and k_2. One of the main novelties of this paper is the fact that the (scalar–scalar-higher spin current) three-point functions for finite N and k_2 are obtained. They do depend on these finite values and in the large N 't Hooft limit they contain the 't Hooft coupling constant which is the ratio of these two quantities as usual. Of course, the standard three-point functions depend on the three complex coordinates appearing in the above three quantities via two-point function between the two scalars and some factors which can be determined by the conformal symmetry. See also [15]. There are also, in general, spin dependent pieces in the three-point functions. In our case, the higher spin is fixed by $s = 4$. In the large N 't Hooft limit one can also analyze the next leading order behavior (for example, $\frac{1}{N}$) of the three-point functions. It is interesting to describe the asymptotic symmetry algebra in the AdS_3 bulk theory and see whether the above finite N and k_2 behavior in the corresponding eigenvalues or in the corresponding three-point functions can be reproduced.

From the description of adjoint fermions living in the first factor in the numerator of (1.1), one obtains the well known $\mathcal{N} = 1$ superconformal algebra generated by the spin 2 stress energy tensor and its superpartner, spin $\frac{3}{2}$ current. It turns out that the higher spin $\frac{7}{2}$ current consists of septic, quintic, cubic and linear terms in the adjoint fermions with appropriate derivative terms. The $\mathcal{N} = 2$ superconformal algebra is realized by two adjoint fermions living in the two numerator factors in (1.1). In this case, the higher spin 3 current with $U(1)$ charge $\frac{4}{3}$ is given by the multiple product of two fermions contracted with dff or ff tensors without any derivative terms. Moreover, its three partners, higher spin $\frac{7}{2}$, $\frac{7}{2}$, and 4 currents, are determined.

In Sect. 2, the higher spin 4 current is obtained, the three-point functions are given and the OPE between the higher spin 4 current and itself is described under some constraints.

In Sect. 3, the higher spin $\frac{7}{2}$ current is obtained, and the three OPEs between this higher spin $\frac{7}{2}$ current and the higher spin 4 current are described using the Jacobi identities.

In Sect. 4, the lowest higher spin 3 current is obtained, and its three other higher spin $\frac{7}{2}$, $\frac{7}{2}$ and 4 currents are obtained which can be denoted as $\mathcal{N} = 2$ lowest higher spin multiplet with definite $U(1)$ charge $\frac{4}{3}$. Furthermore, another $\mathcal{N} = 2$ lowest higher spin multiplet with definite $U(1)$ charge $-\frac{4}{3}$ is obtained. The OPEs between these higher spin multiplets in $\mathcal{N} = 2$ superspace are given using the Jacobi identities.

In Sect. 5, we list some future directions related to this work.

In Appendices A–L, which appear in the arXiv version only (arXiv:1701.02410), the technical details appearing in Sects. 2–4 are given.

2 The coset model with arbitrary two levels (k_1, k_2)

From the spin 1 currents of the coset model, one constructs the spin 2 stress energy tensor. By generalizing the Sugawara construction, the higher spin 4 current is obtained from the quartic terms in the spin 1 currents with the $SO(2N)$ invariant tensors of ranks 4, 2. The corresponding three-point functions of zero mode of the higher spin 4 current with two scalars are described. The OPE between the higher spin 4 current and itself for particular k_1 and N is obtained.

2.1 Spin 2 current and Virasoro algebra

The standard stress energy tensor satisfies the following OPE [6]:

$$T(z)\,T(w) = \frac{1}{(z-w)^4}\,\frac{c}{2} + \frac{1}{(z-w)^2}\,2T(w) + \frac{1}{(z-w)}\,\partial T(w) + \cdots. \tag{2.1}$$

For the coset model in (1.1), the above stress energy tensor can be obtained by the usual Sugawara construction [6],

$$T(z) = -\frac{1}{2(k_1+2N-2)} J^a J^a(z) - \frac{1}{2(k_2+2N-2)} K^a K^a(z) + \frac{1}{2(k_1+k_2+2N-2)} (J^a+K^a)(J^a+K^a)(z). \tag{2.2}$$

The affine Kac–Moody algebra $\hat{SO}(2N)_{k_1} \oplus \hat{SO}(2N)_{k_2}$ in (1.1) is described by the following OPEs [6]:

$$J^a(z)\,J^b(w) = -\frac{1}{(z-w)^2}\,k_1\delta^{ab} + \frac{1}{(z-w)}\,f^{abc}J^c(w) + \cdots,$$
$$K^a(z)\,K^b(w) = -\frac{1}{(z-w)^2}\,k_2\delta^{ab} + \frac{1}{(z-w)}\,f^{abc}K^c(w) + \cdots. \tag{2.3}$$

The adjoint indices $a, b, \ldots$ corresponding to $SO(2N)$ group run over $a, b = 1, 2, \ldots, \frac{1}{2}2N(2N-1)$. The Kronecker delta δ^{ab} appearing in (2.3) is the second rank $SO(2N)$ symmetric invariant tensor. The structure constant f^{abc} is antisymmetric as usual. The diagonal affine Kac–Moody algebra $\hat{SO}(2N)_{k_1+k_2}$ in (1.1) can be obtained by adding the above two spin 1 currents, $J^a(z)$ and $K^a(z)$. Of course, we have $J^a(z)\,K^b(w) = +\cdots$.

The central charge appearing in the above OPE (2.1) is given by [6]

$$c(k_1,k_2,N) = \frac{1}{2}2N(2N-1)\left[\frac{k_1}{(k_1+2N-2)} + \frac{k_2}{(k_2+2N-2)} - \frac{(k_1+k_2)}{(k_1+k_2+2N-2)}\right]. \tag{2.4}$$

Note that the dual Coxeter number of $SO(2N)$ is equal to $(2N-2)$ and the dimension of $SO(2N)$ is given by $\frac{1}{2}2N(2N-1)$.

Then the Virasoro algebra realized in the coset model (1.1) [5,16] is summarized by (2.1) together with (2.2) and (2.4).

2.2 Higher spin 4 current

The 28 $SO(8)$ generators T^a are given in Appendix A. Then the structure constant introduced in the above is given by

$$f^{abc} = -\frac{i}{2}\mathrm{Tr}[T^cT^aT^b - T^cT^bT^a]. \tag{2.5}$$

Then one obtains $[T^a, T^b] = i f^{abc}T^c$.

The totally symmetric $SO(2N)$ invariant tensor of rank 4 is defined as [17,18]

$$T^aT^bT^c + T^aT^cT^b + T^cT^aT^b + T^bT^aT^c + T^bT^cT^a + T^cT^bT^a = d^{abcd}\,T^d. \tag{2.6}$$

That is, one can express the d tensor as[1]

$$d^{abcd} = \frac{1}{2}\mathrm{Tr}[T^dT^aT^bT^c + T^dT^aT^cT^b + T^dT^cT^aT^b + T^dT^bT^aT^c + T^dT^bT^cT^a + T^dT^cT^bT^a]. \tag{2.7}$$

Note that one uses $\mathrm{Tr}(T^aT^b) = 2\delta^{ab}$.

One obtains the product of the structure constants

$$f^{abc}f^{abd} = 2(2N-2)\delta^{cd}, \tag{2.8}$$

and the triple product leads to

$$f^{adb}f^{bec}f^{cfa} = -(2N-2)f^{def}. \tag{2.9}$$

Furthermore, one obtains the following non-trivial triple product between d tensor (2.7) and f tensor (2.5):

$$d^{adeb}f^{bfc}f^{cga} = -\frac{4}{3}(N-1)d^{defg} + 4\delta^{df}\delta^{eg} + 4\delta^{dg}\delta^{ef} - 8\delta^{de}\delta^{fg} - \frac{1}{3}(2N-5)f^{dfh}f^{heg} - \frac{1}{3}(2N-5)f^{dgh}f^{hef}. \tag{2.10}$$

By multiplying f^{dfh} into (2.10) and rearranging the indices, one obtains with (2.9)

$$d^{abcf}f^{agd}f^{bde}f^{che} = 2(2N^2-7N+11)f^{fgh}. \tag{2.11}$$

For the index condition $f = d$ in (2.10) together with the identity (2.8), one obtains

$$d^{aabc} = 2(4N-1)\delta^{bc}. \tag{2.12}$$

Note that this behavior is different from the one of the unitary case where the trivial result $d^{aabc} = 0$ arises [19]. One also has

$$d^{abcd}d^{abce} = 12[N(2N-1)+2]\delta^{de}. \tag{2.13}$$

Let us describe how one can obtain the higher spin current with the help of the d tensor we introduced. For the second

[1] One can consider the rank 3 tensor as $d^{abc} = \frac{1}{2}\mathrm{Tr}[T^aT^bT^c + T^bT^aT^c]$, which is identically zero.

rank $SO(2N)$ invariant symmetric tensor δ^{ab}, one describes the stress energy tensor in (2.2). According to the observation of footnote 1, there is no non-trivial third rank $SO(2N)$ invariant symmetric tensor d^{abc}. Then the next non-trivial higher spin current can be constructed from the fourth rank $SO(2N)$ invariant symmetric tensor d^{abcd} (2.6).

Let us consider the following higher spin 4 current, along the line of [6,19,20]:

$$\begin{aligned}
W^{(4)}(z) = d^{abcd}[& A_1\, J^a J^b J^c J^d + A_2\, J^a J^b J^c K^d \\
&+A_3\, J^a J^b K^c K^d \\
&+A_4\, J^a K^b K^c K^d + A_5\, K^a K^b K^c K^d](z) \\
&+[A_6\, \partial J^a \partial J^a + A_7\, \partial^2 J^a J^a \\
&+A_8\, \partial K^a \partial K^a + A_9\, \partial^2 K^a K^a \\
&+A_{10}\, \partial J^a \partial K^a + A_{11}\, \partial^2 J^a K^a \\
&+A_{12}\, J^a \partial^2 K^a + A_{13}\, f^{abc} J^a \partial J^b K^c \\
&+A_{14}\, f^{abc} J^a K^b \partial K^c + A_{15}\, J^a J^a J^b J^b \\
&+A_{16}\, K^a K^a K^b K^b + A_{17}\, J^a J^a K^b K^b \\
&+A_{18}\, J^a J^a J^b K^b + A_{19}\, J^a K^a K^b K^b \\
&+A_{20}\, J^a J^b K^a K^b](z).
\end{aligned} \tag{2.14}$$

One should obtain the 20 relative (k_1, k_2, N)-dependent coefficients. The first five quartic terms in (2.14) can easily be understood in the sense that they are the only possible terms from each spin 1 current, $J^a(z)$ and $K^a(z)$ using the d^{abcd} tensor. The next seven derivative terms in (2.14) can be found from the second derivative of stress energy tensor $\partial^2 T(z)$. The remaining eight terms can arise in $TT(z)$.

First of all, the higher spin 4 current should have the regular terms with the diagonal spin 1 current in the coset model as follows [6,19,20]:

$$J'^a(z)\, W^{(4)}(w) = + \cdots, \quad J'^a(z) \equiv (J^a + K^a)(z). \tag{2.15}$$

Let us calculate the OPEs between the diagonal spin 1 current and the 20 terms in (2.14) in order to use the condition (2.15). One can perform the various OPEs by following the procedures done in the unitary case [21]. Let us focus on the A_1 term in (2.14) which has the regular OPE with $K^a(z)$. Then Eqs. (2.22), (2.23) and (2.24) of [21] can be used. For example, Eq. (2.24) of [21] provides the information of the OPE between the $J'^a(z)$ and the above A_1 term. Using Eqs. (2.11) and (2.8), one can simplify the fourth-order pole in (2.24) of [21] which was given by $f^{abf} f^{fci} d^{bcde} f^{geh} f^{idg} J^h(w)$.

It turns out that we are left with $J^a(w)$ with an N-dependent $SO(2N)$ group theoretical factor. The third-order pole,

$$\begin{aligned}
& f^{abf} d^{bcde} (f^{hdg} f^{fch} J^g J^e + f^{heg} f^{fch} J^d J^g \\
& + f^{heg} f^{fdh} J^c J^g)(w) + f^{acf} d^{bcde} f^{geh} f^{fdg} J^b J^h(w),
\end{aligned} \tag{2.16}$$

can be simplified with the help of (2.11). We are left with $f^{abc} J^b J^c(w)$ in (2.16) with N dependent coefficient factor which is proportional to $\partial J^a(w)$. Finally the second-order pole,

$$\begin{aligned}
& -4k_1 d^{abcd} J^b J^c J^d(w) \\
& \quad + d^{bcde} (f^{abf} f^{fcg} J^g J^d J^e + f^{abf} f^{fdg} J^c J^g J^e \\
& \quad + f^{abf} f^{feg} J^c J^d J^g \\
& \quad + f^{acf} f^{fdg} J^b J^g J^e + f^{acf} f^{feg} J^b J^d J^g \\
& \quad + f^{adf} f^{feg} J^b J^c J^g)(w),
\end{aligned} \tag{2.17}$$

can be simplified further together with (2.10).

Then we obtain the final OPE as follows:

$$\begin{aligned}
& J'^a(z)\, d^{bcde} J^b J^c J^d J^e(w) \\
& \quad = \frac{1}{(z-w)^4}\, 2(2N-2)(4N^2-14N+22) J^a(w) \\
& \quad - \frac{1}{(z-w)^3}\, 2(4N^2-14N+22) f^{abc} J^b J^c(w) \\
& \quad + \frac{1}{(z-w)^2} [-(4k_1+8(N-1)) d^{abcd} J^b J^c J^d \\
& \quad - (12+(2N-2)(2N-5)) f^{abc} \partial J^b J^c \\
& \quad + (12+(2N-2)(2N-5)) f^{abc} J^b \partial J^c](w) + \cdots.
\end{aligned} \tag{2.18}$$

There is no first-order pole in Eq. (2.18). One can check the second-order pole in (2.18) from (2.17).

Let us consider the A_2 term in (2.14) where there exists a $K^d(z)$ dependence. Starting from Eqs. (2.23) and (2.19) of [21] with Eqs. (2.11) and (2.10), one can simplify the third-order pole, $f^{acf} f d^{bcde} f^{geh} f^{fdg} J^h K^b(w)$, as $f^{abc} J^b K^c(w)$ with an N dependent factor. Similarly, the second-order pole,

$$\begin{aligned}
& -3k_1 d^{abcd} J^c J^d K^b(w) - k_2 d^{abcd} J^b J^c J^d(w) \\
& \quad + d^{bcde} (f^{acf} f^{fdg} J^g J^e K^b + f^{acf} f^{feg} J^d J^g K^b \\
& \quad + f^{adf} f^{feg} J^c J^g K^b)(w),
\end{aligned} \tag{2.19}$$

can be simplified in terms of several independent terms. It turns out that in this case also there are no first-order poles.

Therefore, one obtains the following OPE corresponding to A_2 term:

$$\begin{aligned}
& J'^a(z)\, d^{bcde} J^b J^c J^d K^e(w) \\
& \quad = \frac{1}{(z-w)^3} (4N^2-14N+22) f^{abc} J^b K^c(w) \\
& \quad + \frac{1}{(z-w)^2} [-(3k_1+4(N-1)) d^{abcd} J^b J^c K^d \\
& \quad - k_2 d^{abcd} J^b J^c J^d + 12 J^b J^b K^a
\end{aligned}$$

$$-12J^aJ^bK^b - 12f^{abc}\partial J^bK^c$$
$$+(2N-5)f^{abc}f^{cde}J^bJ^eK^d](w) + \cdots . \quad (2.20)$$

One can see the second-order pole in (2.20) from (2.19).

Let us consider the A_3 term in (2.14). From Eq. (2.22) of [21], one has the relevant OPEs. For example, the second-order pole,

$$(-2k_1d^{abcd}J^dK^bK^c + f^{adf}f^{feg}d^{bcde}J^gK^bK^c$$
$$-2k_2d^{abcd}K^dJ^bJ^c + f^{adf}f^{feg}d^{bcde}K^gJ^bJ^c)(w), \quad (2.21)$$

can be reexpressed in terms of various independent terms with the help of the identity (2.10). It turns out that the relevant OPE coming from (2.21) can be summarized as

$$J'^a(z)\,d^{bcde}J^bJ^cK^dK^e(w) = \frac{1}{(z-w)^2}$$
$$\times\Big[-\left(2k_1+\frac{4}{3}(N-1)\right)d^{abcd}J^bK^cK^d$$
$$-\left(2k_2+\frac{4}{3}(N-1)\right)d^{abcd}J^bJ^cK^d$$
$$+8J^bK^aK^b + 4f^{abc}\partial J^bK^c$$
$$-8J^aK^bK^b - 4f^{abc}J^b\partial K^c$$
$$-\frac{1}{3}(2N-5)f^{abc}f^{cde}J^dK^eK^b$$
$$-\frac{1}{3}(2N-5)f^{abc}f^{cde}J^dK^bK^e$$
$$-8J^bJ^bK^a + 8J^aJ^bK^b$$
$$-\frac{1}{3}(2N-5)f^{abc}f^{cde}J^eJ^bK^d$$
$$-\frac{1}{3}(2N-5)f^{abc}f^{cde}J^bJ^eK^d\Big](w) + \cdots . \quad (2.22)$$

It is useful to realize that this OPE remains the same after the exchange of $J^a(w)$ and $K^a(w)$ together with $k_1 \leftrightarrow k_2$. The left-hand side is invariant under this transformation because the d tensor is totally symmetric. The 12 terms in the second-order pole can be divided into two groups and each of them has their own counterpart.

It is straightforward to complete this calculation step by step. We summarize the remaining 17 OPEs in Appendix B. Then we have the complete expressions in (2.18), (2.20), (2.22), and Appendix B.

The higher spin 4 current should transform as a primary field under the stress energy tensor (2.2). According to the previous regular condition (2.15), the diagonal spin 1 current $J'^a(z)$ does not have any singular terms in the OPE with the higher spin 4 current $W^{(4)}(w)$ after we use the results of Appendix B. Then there are no singular terms in the OPE between the stress energy tensor in the denominator of the coset model (1.1) and the higher spin 4 current because the former is given by $J'^aJ'^a(z)$. The singular terms can arise from the OPE between the stress energy tensor in the numerator of the coset model and the higher spin 4 current. Therefore, one should have the following condition [6,19,20]:

$$\hat{T}(z)\,W^{(4)}(w)\Big|_{\frac{1}{(z-w)^n},\,n=3,4,5,6} = 0. \quad (2.23)$$

Here the stress energy tensor in the numerator is described by

$$\hat{T}(z) \equiv -\frac{1}{2(k_1+2N-2)}J^aJ^a(z)$$
$$-\frac{1}{2(k_2+2N-2)}K^aK^a(z). \quad (2.24)$$

Of course, the higher spin 4 current has the standard OPE (the second- and first-order poles) with stress energy tensor (2.2) as usual.

Let us calculate the OPE between the stress energy tensor (2.24) and the A_1 term in (2.14). First of all, because the A_1 term does not contain the $K^a(w)$ spin 1 current, one can consider the OPE between the first term of (2.24) and the A_1 term. It is well known that the spin 1 current $J^a(w)$ transforms as a primary field under the first term of (2.24) (i.e., the stress energy tensor in the first factor of the numerator). Then one should obtain the OPE $J^b(z)$ $d^{bcde}J^cJ^dJ^e(w)$ and it turns out that there exists a nontrivial second-order pole given by $-3k_1(8N-2)J^cJ^c(w)$, where the identity (2.12) is used. Note that the structure constant term vanishes due to the presence of d^{bcde}. Furthermore, one should calculate the OPE between the above stress energy tensor and the previous expression $d^{bcde}J^cJ^dJ^e(w)$ where the order of the singular terms is greater than 2. Then we are left with $-3k_1(8N-2)J^cJ^c(w)$ by combing the contribution $-2k_1(8N-2)J^cJ^c(w)$ from the contraction between the stress energy tensor and $J^c(w)$ and the contribution $-k_1(8N-2)J^cJ^c(w)$ from the OPE between the stress energy tensor and $d^{bcde}J^dJ^e(w)$. Therefore, the final total contribution is summarized by $-6k_1(8N-2)J^cJ^c(w)$ and we present this OPE as follows:

$$\hat{T}(z)\,d^{bcde}J^bJ^cJ^dJ^e(w) = -\frac{1}{(z-w)^4}$$
$$\times 12k_1(4N-1)J^aJ^a(w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad (2.25)$$

This result in (2.25) shows a behavior different from the corresponding OPE in the unitary case, because in the latter there is no contribution from the fourth-order pole because the above d^{aabc} tensor for the $SU(N)$ group vanishes [19].

Let us move on the A_2 term in (2.14). In this case, the spin 1 current $K^d(w)$ is present. However, the contribution in the higher singular terms of the stress energy tensor coming from the second term of (2.24) vanishes. Then one can calculate the OPE between the stress energy tensor in the first factor of the numerator and the A_2 term. By using the previous procedure one can obtain the contribution $-2k_1(8N-2)J^cK^c(w)$ from the contraction with $J^b(w)$ current and the contribution $-k_1(8N-2)J^cK^c(w)$ from the contraction with other remaining factor $d^{bcde}J^cJ^dK^e(w)$. By adding these two, one obtains the following OPE:

$$\hat{T}(z)\,d^{bcde}J^bJ^cJ^dK^e(w) = -\frac{1}{(z-w)^4} \times 6k_1(4N-1)J^aK^a(w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad (2.26)$$

Now let us describe the contribution from the A_3 term in (2.14) where the quadratic $K^cK^d(w)$ appears. In this case, one should also calculate the contribution from the stress energy tensor in the second term in (2.24). As done before, the contribution from the contraction with the $J^b(w)$ spin 1 current is given by $-k_1(8N-2)K^cK^c(w)$. Similarly the contribution from the contraction with the remaining factor is given by $-k_2(8N-2)J^cJ^c(w)$. Then we are left with

$$\hat{T}(z)\,d^{bcde}J^bJ^cK^dK^e(w) = -\frac{1}{(z-w)^4} \times [2k_2(4N-1)J^aJ^a + 2k_1(4N-1)K^aK^a](w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad (2.27)$$

One also sees the symmetry under the transformation $J^a(z) \leftrightarrow K^a(z)$ and $k_1 \leftrightarrow k_2$.

It is straightforward to perform the other remaining calculations step by step. We summarize the remaining 17 OPEs in Appendix C. Then we are left with (2.25)–(2.27), and Appendix C.

Now one can determine the undetermined coefficient functions $A_1, A_2, \ldots, A_{20}$ appearing in the higher spin 4 current in (2.14). The 23 linear equations are given in Appendix B explicitly. The eight linear equations are given in Appendix C. By solving them, one obtains the final expressions in Appendix D. They depend on k_1, k_2 and N. The corresponding coefficients for $k_1 = 1$ are presented in Appendix E. Appendix F corresponds to the case where $k_1 = 2N-2$.

2.3 Three-point functions [22] with two scalars where $k_1 = 1$

The zero modes of the current satisfy the commutation relations of the underlying finite dimensional Lie algebra $SO(2N)$. For the state $|(v;0)\rangle$, T^a corresponds to iK_0^a and for the state $|(0;v)\rangle$, T^a corresponds to iJ_0^a as follows:

$$|(v;0)\rangle: \quad T^a \leftrightarrow iK_0^a, \quad |(0;v)\rangle: \quad T^a \leftrightarrow iJ_0^a. \quad (2.28)$$

Note that from the defining equation of the OPEs (2.3), one obtains

$$[J_m^a, J_n^b] = -k_1 m\delta^{ab}\delta_{m+n,0} + f^{abc}J_{m+n}^c, \; [K_m^a, K_n^b] = -k_2 m\delta^{ab}\delta_{m+n,0} + f^{abc}K_{m+n}^c. \quad (2.29)$$

In (2.29), the central terms for the zero modes vanish. Recall that our generators for the $SO(2N)$ satisfy $[T^a, T^b] = if^{abc}T^c$ [6].

The large N 't Hooft limit is described as [23,24]

$$N, k_2 \to \infty, \quad \lambda \equiv \frac{2N}{2N-2+k_2} \quad \text{fixed}. \quad (2.30)$$

The presence of the numerical value -2 in the denominator of (2.30) is not important in the large N 't Hooft limit [25].

Compared to the large $\mathcal{N}=4$ holography in [10,26,27] where one can obtain the eigenvalue equations from the several low N values inside the package of [28], one should analyze both the coefficients and the zero modes of the 20 terms in higher spin 4 current in order to obtain the corresponding eigenvalue equations.

2.3.1 Eigenvalue equation of the zero mode of the higher spin 4 current acting on the state $|(0;v)\rangle$

Let us consider the eigenvalue equation of the zero mode of the A_1 term of the higher spin 4 current in (2.14) acting on the primary state $(0;v)$

$$d^{abcd}(J^aJ^bJ^cJ^d)_0|(0;v)\rangle. \quad (2.31)$$

Using the fact that the zero mode is nothing but the product of each zero mode but the ordering is reversed [19,20], Eq. (2.31) becomes

$$d^{abcd}(J^dJ^cJ^bJ^a)_0|(0;v)\rangle. \quad (2.32)$$

Note that the ground state transforms as a vector representation with respect to J_0^a, while the zero mode K_0^a has vanishing eigenvalue equation [22],

$$K_0^a|(0;v)\rangle = 0. \quad (2.33)$$

Equation (2.32) becomes

$$\frac{1}{2N}d^{abcd}(-i)^4\mathrm{Tr}(T^dT^cT^bT^a)|(0;v)\rangle. \quad (2.34)$$

In order to use the previous identity in (2.6), one can express the above A_1 term as follows:

$$\frac{1}{6}d^{abcd}(J^aJ^bJ^cJ^d + J^bJ^cJ^aJ^d + J^bJ^aJ^cJ^d + J^cJ^aJ^bJ^d + J^aJ^cJ^bJ^d + J^cJ^bJ^aJ^d), \quad (2.35)$$

due to the symmetric property of the d tensor. Then the equivalent expression corresponding to (2.34) with (2.35) can be written in terms of

$$\frac{1}{2N}\frac{1}{6}d^{abcd}\mathrm{Tr}(T^dT^cT^bT^a+T^dT^aT^cT^b+T^dT^cT^aT^b$$
$$+T^dT^bT^aT^c+T^dT^bT^cT^a+T^dT^aT^bT^c)$$
$$\times|(0;v)\rangle. \quad (2.36)$$

The reason why the extra $\frac{1}{2N}$ exists is that one should have the eigenvalue, not the trace. Using the identity (2.6), one can reexpress (2.36) as

$$\frac{1}{2N}\frac{1}{3}d^{abcd}d^{cbad}=\frac{1}{2N}\frac{1}{3}12[N(2N-1)+2]\delta^{aa}$$
$$=\frac{1}{2N}\frac{1}{3}12[N(2N-1)+2]\frac{1}{2}2N(2N-1)$$
$$\rightarrow 8N^3. \quad (2.37)$$

Here the identity (2.13) is used and we take the large N limit at the last result in (2.37).

One can analyze the other 19 terms in (2.14). Among them, the 16 terms which have the $K^a(z)$ spin 1 current do not contribute to the eigenvalue equation, because one can take the zero mode and change the ordering of the zero modes as in (2.32). Then one can move the rightmost zero mode K_0^a to the right and use the previous condition (2.33). On the other hand, the remaining A_6, A_7 and A_{15} terms can contribute to the eigenvalue equation.

The zero mode of the A_6 term of the higher spin 4 current acting on the primary state $(0;v)$ is

$$(\partial J^a\partial J^a)_0|(0;v)\rangle=(\partial J^a)_0(\partial J^a)_0|(0;v)\rangle$$
$$=(-J_0^a)(-J_0^a)|(0;v)\rangle=J_0^aJ_0^a|(0;v)\rangle, \quad (2.38)$$

where the zero mode of ∂J^a in (2.38) can be obtained from the usual mode expansion and is given by the zero mode of $-J^a$. Now using the correspondence (2.28), the above expression leads to

$$\frac{1}{2N}\mathrm{Tr}(iT^aiT^a)|(0;v)\rangle=-\frac{1}{2N}2\delta^{aa}$$
$$=-\frac{1}{2N}2\frac{1}{2}2N(2N-1)\rightarrow -2N, \quad (2.39)$$

where the extra factor $\frac{1}{2N}$ is considered as in (2.36) and the large N limit is taken.

Now the final contribution from the zero mode of the A_7 term of the higher spin 4 current acting on the primary state $(0;v)$ is given by

$$(\partial^2J^aJ^a)_0|(0;v)\rangle=J_0^a(\partial^2J^a)_0|(0;v)\rangle=J_0^a2J_0^a|(0;v)\rangle, \quad (2.40)$$

where the zero mode of ∂^2J^a in (2.40) can be obtained from the usual mode expansion also and is given by the zero mode of $2J^a$. Therefore, one can follow the previous description. It turns out that

$$2\frac{1}{2N}\mathrm{Tr}(iT^aiT^a)|(0;v)\rangle$$
$$=-2\frac{1}{2N}2\frac{1}{2}2N(2N-1)\rightarrow -4N. \quad (2.41)$$

For the A_{15} term, one has the eigenvalue equation

$$(J^aJ^aJ^bJ^b)_0|(0;v)\rangle=\delta^{ab}\delta^{cd}(J^aJ^bJ^cJ^d)_0|(0;v)\rangle. \quad (2.42)$$

By following the procedure in the A_1 term, one sees that the above (2.42) can be written as

$$\frac{1}{2N}\frac{1}{3}\delta^{ab}\delta^{cd}d^{cbad}=\frac{1}{2N}\frac{1}{3}2(4N-1)\frac{1}{2}2N(2N-1)$$
$$\rightarrow\frac{8}{3}N^2, \quad (2.43)$$

where the identity (2.12) is used in (2.43). Furthermore, the A_{15} term itself behaves as N^0 in Appendix E. Then there is no contribution at the leading order approximation.

By combining (2.37), (2.39) and (2.41) with the corresponding coefficients in the large N limit of Appendix E, the zero mode eigenvalue equation leads to

$$W_0^{(4)}|(0;v)\rangle=\left[8N^3A_1+(-2N)\left(N^2\frac{12(2\lambda-9)}{5(2\lambda-3)}A_1\right)\right.$$
$$\left.+(-4N)\left(-N^2\frac{8(2\lambda-9)}{5(2\lambda-3)}A_1\right)\right]$$
$$\times|(0;v)\rangle=N^3\left[\frac{96(\lambda-2)}{5(2\lambda-3)}\right]A_1|(0;v)\rangle. \quad (2.44)$$

One can also calculate the same eigenvalue equation at finite N and k_2 corresponding to (2.44), which will appear later.

2.3.2 *Eigenvalue equation of the zero mode of the higher spin 4 current acting on the state $|(v;0)\rangle$*

Let us describe the eigenvalue equation of the zero mode of the A_1 term of the higher spin 4 current in (2.14) acting on the primary state $(v;0)$,

$$d^{abcd}(J^aJ^bJ^cJ^d)_0|(v;0)\rangle=d^{abcd}J_0^dJ_0^cJ_0^bJ_0^a|(v;0)\rangle. \quad (2.45)$$

Note that the ground state transforms as a vector representation with respect to K_0^a and the singlet condition for the primary state $(v;0)$ can be described as [22]

$$(J_0^a+K_0^a)|(v;0)\rangle=0. \quad (2.46)$$

Then Eq. (2.45) is equivalent to

$$-d^{abcd}J_0^dJ_0^cJ_0^bK_0^a|(v;0)\rangle=-d^{abcd}K_0^aJ_0^dJ_0^cJ_0^b|(v;0)\rangle, \quad (2.47)$$

where Eq. (2.46) is used and the zero mode K_0^a is moved to the left. Now the singlet condition is applied to the rightmost

J_0^b and we are left with

$$d^{abcd} K_0^a J_0^d J_0^c K_0^b |(v;0)\rangle = d^{abcd} K_0^a K_0^b J_0^d J_0^c |(v;0)\rangle. \tag{2.48}$$

One can further take the previous steps and obtains

$$d^{abcd} K_0^a K_0^b K_0^c K_0^d |(v;0)\rangle. \tag{2.49}$$

Then using the correspondence (2.28), Eq. (2.49) becomes

$$\frac{1}{2N} d^{abcd} (-i)^4 \mathrm{Tr}(T^a T^b T^c T^d) |(0;v)\rangle, \tag{2.50}$$

which leads to the previous eigenvalue in (2.37).[2]

What happens for the A_5 term of the higher spin 4 current in (2.14)? According to the large N behavior of the coefficient A_5, this coefficient behaves as $\frac{1}{N}$ in Appendix E and, moreover, the analysis of eigenvalue equation leads to N^3 behavior. Therefore, the total power for the large N behavior is given by N^2 and can be ignored in this approximation.

Let us move on the A_6 term. The eigenvalue equation leads to

$$(\partial J^a \partial J^a)_0 |(v;0)\rangle = J_0^a J_0^a |(v;0)\rangle \\ = -J_0^a K_0^a |(v;0)\rangle = K_0^a K_0^a |(v;0)\rangle, \tag{2.54}$$

where the singlet condition (2.46) is used. After using the correspondence (2.28), this becomes the previous result in (2.39).

Similarly, the A_7 term eigenvalue equation gives

$$(\partial^2 J^a J^a)_0 |(v;0)\rangle = J_0^a 2 J_0^a |(v;0)\rangle \\ = -2 J_0^a K_0^a |(v;0)\rangle = 2 K_0^a K_0^a |(v;0)\rangle, \tag{2.55}$$

which leads to (2.41).

For the A_8 and A_9 terms of the higher spin 4 current, these coefficients behave as N from Appendix E in the large N limit and the corresponding eigenvalues behave as N. Then the total power of the large N behavior is given by 2 and these terms can be ignored at the leading order calculation.[3]

Let us consider the A_{13} term of the higher spin 4 current in (2.14). One can easily see that

$$f^{abc} J^a \partial J^b K^c(z) = J^a J^b J^a K^b(z) - J^a J^a J^b K^b(z), \tag{2.59}$$

by writing the derivative term as the commutator of normal ordered product. Then the zero mode of Eq. (2.59) is given by

$$(K_0^b J_0^a J_0^b J_0^a - K_0^b J_0^b J_0^a J_0^a) |(v;0)\rangle \\ = -(K_0^b K_0^a K_0^b K_0^a - K_0^b K_0^a K_0^a K_0^b) |(v;0)\rangle. \tag{2.60}$$

Then Eq. (2.60) becomes

$$-\frac{1}{2N} (-i)^4 \mathrm{Tr}(T^b T^a T^b T^a - T^b T^a T^a T^b) |(v;0)\rangle. \tag{2.61}$$

Furthermore, Eq. (2.61) will reduce to

$$-\frac{1}{2N} (-i)^4 i f^{bac} \mathrm{Tr}(T^b T^a T^c) |(v;0)\rangle \\ = -\frac{1}{2N} (-i)^4 i f^{bac} \frac{1}{2} \mathrm{Tr}(T^b T^a T^c - T^b T^c T^a) |(v;0)\rangle. \tag{2.62}$$

One can use the identity (2.5) and obtains, together with (2.8),

$$-\frac{1}{2N} (-i)^4 i f^{bac} \frac{1}{2} 2 i f^{bac} \\ = \frac{1}{2N} 2(2N-2) \frac{1}{2} 2N(2N-1) \rightarrow 4N^2. \tag{2.63}$$

Let us focus on the A_{14} term. One has the relation

$$f^{abc} J^a K^b \partial K^c(z) = J^a K^b K^a K^b(z) - J^a K^b K^b K^a(z). \tag{2.64}$$

---

[2] The eigenvalue equation of the zero mode of the A_2 term of the higher spin 4 current in (2.14) acting on the primary state $(v;0)$ can be written as

$$d^{abcd} (J^a J^b J^c K^d)_0 |(v;0)\rangle = d^{abcd} K_0^d J_0^c J_0^b J_0^a |(v;0)\rangle, \tag{2.51}$$

which is equivalent to (2.47) with an extra minus sign due to the symmetric property of the d symbol. Then we are left with the fact that Eq. (2.51) is equal to the previous result (2.50) with minus sign.

The eigenvalue equation of the zero mode of the A_3 term of the higher spin 4 current in (2.14) acting on the primary state $(v;0)$ leads to

$$d^{abcd} (J^a J^b K^c K^d)_0 |(v;0)\rangle = d^{abcd} K_0^d K_0^c J_0^b J_0^a |(v;0)\rangle, \tag{2.52}$$

which is equal to (2.48) and thus (2.52) becomes Eq. (2.50).

Similarly, the eigenvalue equation of the zero mode of the A_4 term of the higher spin 4 current in (2.14) acting on the primary state $(v;0)$ can be described as

$$d^{abcd} (J^a K^b K^c K^d)_0 |(v;0)\rangle = d^{abcd} K_0^d K_0^c K_0^b J_0^a |(v;0)\rangle \\ = -d^{abcd} K_0^d K_0^c K_0^b K_0^a |(v;0)\rangle, \tag{2.53}$$

where the singlet condition is used and the above expression (2.53) leads to (2.50) with an extra minus sign.

[3] Let us describe the next A_{10} term of the higher spin 4 current in (2.14). One obtains

$$(\partial J^a \partial K^a)_0 |(v;0)\rangle = K_0^a J_0^a |(v;0)\rangle = -K_0^a K_0^a |(v;0)\rangle, \tag{2.56}$$

where Eq. (2.56) is equivalent to Eq. (2.54) with an extra minus sign. We can also calculate the eigenvalue equation for the A_{11} term,

$$(\partial^2 J^a K^a)_0 |(v;0)\rangle = K_0^a 2 J_0^a |(v;0)\rangle = -2 K_0^a K_0^a |(v;0)\rangle. \tag{2.57}$$

Equation (2.57) is equivalent to (2.55) with an extra minus sign.
One can continue to calculate the eigenvalue equation corresponding to the A_{12} term as follows:

$$(J^a \partial^2 K^a)_0 |(v;0)\rangle = 2 K_0^a J_0^a |(v;0)\rangle = -2 K_0^a K_0^a |(v;0)\rangle. \tag{2.58}$$

Then Eq. (2.58) is the same contribution from A_{11} term.

The zero mode of (2.64) can be described as

$$(K_0^b K_0^a K_0^b J_0^a - K_0^a K_0^b K_0^b J_0^a)|(v;0)\rangle$$
$$= -(K_0^b K_0^a K_0^b K_0^a - K_0^a K_0^b K_0^b K_0^a)|(v;0)\rangle. \quad (2.65)$$

Then Eq. (2.65) becomes

$$-\frac{1}{2N}(-i)^4 \mathrm{Tr}(T^b T^a T^b T^a - T^a T^b T^b T^a)|(v;0)\rangle. \quad (2.66)$$

Furthermore, Eq. (2.66) will reduce to

$$-\frac{1}{2N}(-i)^4 i f^{bac} \mathrm{Tr}(T^c T^b T^a)|(v;0)\rangle, \quad (2.67)$$

by combining the first two generators. Equation (2.67) is equivalent to (2.62) and (2.63).

Are there any contributions from the A_{15}–A_{20} terms? These coefficients behave as N^0, $\frac{1}{N^2}$, $\frac{1}{N}$, N^0, $\frac{1}{N}$ and N^0, respectively, from Appendix E. There are no contributions. Then one obtains the final eigenvalue equation as follows:

$$\begin{aligned}
W_0^{(4)}|(v;0)\rangle &= \Bigg[8N^3 A_1 - 8N^3\left(\frac{4\lambda}{(\lambda-1)}\right)A_1 \\
&+8N^3\left(\frac{12\lambda^2}{(\lambda-1)(2\lambda-3)}\right)A_1 \\
&-8N^3\left(\frac{8\lambda^3}{(\lambda-3)(\lambda-1)(2\lambda-3)}\right)A_1 \\
&-2N\left(N^2\frac{12(2\lambda-9)}{5(2\lambda-3)}\right)A_1 - 4N\left(-N^2\frac{8(2\lambda-9)}{5(2\lambda-3)}\right)A_1 \\
&+2N\left(\frac{N^2 48(\lambda-2)\lambda}{5(\lambda-1)(2\lambda-3)}\right)A_1 \\
&+4N\left(-N^2\frac{16(\lambda-12)\lambda}{5(\lambda-1)(2\lambda-3)}\right)A_1 \\
&+4N\left(-N^2\frac{16\lambda\left(\lambda^2+15\lambda+6\right)}{5(\lambda-3)(\lambda-1)(2\lambda-3)}\right)A_1 \\
&+4N^2\left(-N\frac{24\lambda}{(\lambda-1)(2\lambda-3)}\right)A_1 \\
&+4N^2\left(N\frac{48\lambda^2}{(\lambda-3)(\lambda-1)(2\lambda-3)}\right)A_1\Bigg] \\
&\times|(v;0)\rangle = -N^3\left[\frac{96(\lambda+1)(\lambda+2)(\lambda+3)}{5(\lambda-3)(\lambda-1)(2\lambda-3)}\right]A_1|(v;0)\rangle.
\end{aligned} \quad (2.68)$$

The eigenvalue has a simple factorized form.

With the following normalization:

$$A_1 = -\frac{5}{96N^3}(\lambda-3)(\lambda-1)(2\lambda-3), \quad (2.69)$$

the two eigenvalue equations (2.44) and (2.68) lead to

$$W_0^{(4)}|(v;0)\rangle = (1+\lambda)(2+\lambda)(3+\lambda)|(v;0)\rangle,$$
$$W_0^{(4)}|(0;v)\rangle = (1-\lambda)(2-\lambda)(3-\lambda)|(0;v)\rangle. \quad (2.70)$$

If one takes the overall normalization factor for the $W^{(4)}(z)$ as A_4 rather than A_1 as in (2.69), then A_4 becomes $A_4 = -\frac{5}{12N^3}\lambda^3$. In principle, one can calculate the OPE between $W^{(4)}(z)$ and $W^{(4)}(w)$ from the explicit 20 terms in (2.14), although the complete computation of the eighth-order singular terms is rather involved for general (k_2, N) manually. Then one expects that the central term, the eighth-order pole of the above OPE, is given by $A_4^2\, f(\lambda, N)$ where $f(\lambda, N)$ is a (fractional) function of λ and N (after the large N limit is taken). That is, our normalization is given by the central term of the OPE between the higher spin 4 current and itself which behaves as $\frac{25}{144N^6}\lambda^6\, f(\lambda, N)$ where $f(\lambda, N)$ is not known at the moment.

The above eigenvalues are also observed in [24] by following the descriptions in [29] where the unitary case is analyzed.

One of the primaries is given by $(v;0)\otimes(v;0)$ and the other primary is given by $(0;v)\otimes(0;v)$ by pairing up identical representations on the holomorphic and antiholomorphic sectors in the context of diagonal modular invariant [15]. Let us denote them as follows:

$$\mathcal{O}_+ = (v;0)\otimes(v;0), \quad \mathcal{O}_- = (0;v)\otimes(0;v). \quad (2.71)$$

The ratio of the three-point functions, from (2.70), is given by

$$\frac{\langle\mathcal{O}_+\mathcal{O}_+W^{(4)}\rangle}{\langle\mathcal{O}_-\mathcal{O}_-W^{(4)}\rangle} = \frac{(1+\lambda)(2+\lambda)(3+\lambda)}{(1-\lambda)(2-\lambda)(3-\lambda)}, \quad (2.72)$$

in the notation of (2.71). This is the same form as for the unitary case [21,30]. In the corresponding unitary bulk calculation of [15], for $\lambda = \frac{1}{2}$, this ratio for generic spin is given by $(-1)^s(2s-1)$ with spin s. One expects that the orthogonal bulk computation will give rise to the behavior of (2.72).

2.3.3 *Eigenvalue equation of the zero mode of the higher spin 4 current acting on the state* $|(v;v)\rangle$

For the primary $(v;v)$ with the condition $J_0^a|(v;v)\rangle = 0$, one can calculate the eigenvalue equation [8]. The non-trivial contributions arise from the A_5, A_8, and A_9 terms. It turns out that

$$W_0^{(4)}|(v;v)\rangle = -N^2\frac{48\lambda^2(\lambda^2+1)}{5(\lambda-3)(\lambda-1)(2\lambda-3)}A_1|(v;v)\rangle. \quad (2.73)$$

In (2.73), Appendix E is used.

2.3.4 *Further eigenvalue equations*

We also present the eigenvalue equations [30] at finite N and k_2, by using Appendix E, as follows:

$$
\begin{aligned}
W_0^{(4)}|(0;v)\rangle = & \frac{6A_1}{(3k_2+2N-2)d(1,k_2,N)} \\
& \times(32k_2^3N^4 - 64k_2^3N^2 + 96k_2^3N - 55k_2^3 \\
& +160k_2^2N^5 - 168k_2^2N^4 - 268k_2^2N^3 \\
& +690k_2^2N^2 - 617k_2^2N + 203k_2^2 \\
& +128k_2N^6 - 64k_2N^5 - 680k_2N^4 \\
& +1548k_2N^3 - 1726k_2N^2 + 987k_2N \\
& -220k_2 + 256N^6 - 1024N^5 \\
& +1584N^4 - 1384N^3 + 900N^2 \\
& -382N + 68)|(0;v)\rangle, \\
W_0^{(4)}|(v;0)\rangle = & \frac{6(k_2+2N-1)(k_2+4N-3)(3k_2+8N-5)A_1}{k_2(k_2+2N-2)(3k_2+2N-2)(3k_2+4N-4)d(1,k_2,N)} \\
& \times(32k_2^3N^4 - 64k_2^3N^2 + 96k_2^3N - 55k_2^3 \\
& +224k_2^2N^5 - 120k_2^2N^4 - 500k_2^2N^3 \\
& +1038k_2^2N^2 - 907k_2^2N + 292k_2^2 \\
& +384k_2N^6 - 64k_2N^5 - 1752k_2N^4 \\
& +3636k_2N^3 - 3930k_2N^2 + 2213k_2N \\
& -487k_2 + 1024N^6 - 3328N^5 \\
& +5184N^4 - 5576N^3 + 3976N^2 \\
& -1566N + 250)|(v;0)\rangle. \qquad (2.74)
\end{aligned}
$$

Of course, the eigenvalue equations (2.74) become (2.44) and (2.68), respectively, in the large N 't Hooft limit. Compared to the unitary case in [30], the above eigenvalues do not have a simple factorized form. This is because of the fact that the identities between f and d symbols contain rather complicated functions of N.

For convenience, we also present the eigenvalue equations for the spin 2 stress energy tensor (2.2) with $k_1 = 1$,

$$
\begin{aligned}
T_0|(0;v)\rangle &= \frac{k_2}{2(k_2+2N-1)}|(0;v)\rangle \rightarrow \frac{(1-\lambda)}{2}|(0;v)\rangle, \\
T_0|(v;0)\rangle &= \frac{(k_2+4N-3)}{2(k_2+2N-2)}|(v;0)\rangle \rightarrow \frac{(1+\lambda)}{2}|(v;0)\rangle. \qquad (2.75)
\end{aligned}
$$

Note that the conformal dimension of $(0;v)$ can be obtained from the formula [4,8,23,31,32]

$$
\begin{aligned}
h(0;v) &= \frac{1}{2}(2N-1)\left[\frac{1}{1+(2N-2)} - \frac{1}{1+k_2+(2N-2)}\right] \\
&= \frac{k_2}{2(k_2+2N-1)}, \qquad (2.76)
\end{aligned}
$$

where the overall factor $\frac{1}{2}(2N-1)$ is the quadratic Casimir eigenvalue of the $SO(2N)$ vector representation. Similarly, the conformal dimension of $(v;0)$ can be obtained,

$$
\begin{aligned}
h(v;0) &= \frac{1}{2}(2N-1)\left[\frac{1}{1+(2N-2)} + \frac{1}{k_2+(2N-2)}\right] \\
&= \frac{(k_2+4N-3)}{2(k_2+2N-2)}. \qquad (2.77)
\end{aligned}
$$

Then the two results (2.76) and (2.77) are coincident with the ones in (2.75).[4]

2.4 The OPE between the higher spin 4 current and itself where $k_1 = 1$, $N = 4$ and k_2 is arbitrary

Let us describe the OPE between the higher spin 4 current and itself. Because it is rather involved to calculate this OPE manually, one fixes the value of N and then one can compute this OPE inside the package of [28]. For fixed $N = 4$, which is the lowest value one can consider non-trivially, one obtains the fourth-order pole of this OPE, by realizing that the right structure constants should behave according to the well-known results [33], as follows:

$$
\begin{aligned}
W^{(4)}(z)\, W^{(4)}(w)\Big|_{\frac{1}{(z-w)^4}} &= \frac{3}{10}\partial^2 T(w) + \frac{42}{(5c+22)} \\
&\times\left(T^2 - \frac{3}{10}\partial^2 T\right)(w) + \sqrt{\frac{18(c+24)}{(5c+22)}}\, W^{(4)}(w) \\
&+W^{(4')}(w). \qquad (2.81)
\end{aligned}
$$

Here the central charge reduces to

$$
c(k_1=1, k_2, N=4) = \frac{4k_2(k_2+13)}{(k_2+6)(k_2+7)}, \qquad (2.82)
$$

which can be obtained from (2.4) by substituting the two values of $k_1 = 1$ and $N = 4$. The overall factor can be fixed as

[4] Furthermore, one can write down the eigenvalue equation for the state $|(v;v)\rangle$

$$
T_0|(v;v)\rangle = \frac{(2N-1)}{2(k_2+2N-2)(k_2+2N-1)}|(v;v)\rangle \rightarrow \frac{\lambda^2}{4N}|(v;v)\rangle. \qquad (2.78)
$$

Note that in the large N 't Hooft limit the eigenvalue (2.78) reduces to zero.

The conformal dimension of $(v;v)$ can be obtained as follows:

$$
\begin{aligned}
h(v;v) &= \frac{1}{2}(2N-1)\left[\frac{1}{k_2+(2N-2)} - \frac{1}{1+k_2+(2N-2)}\right] \qquad (2.79) \\
&= \frac{(2N-1)}{2(k_2+2N-2)(k_2+2N-1)}.
\end{aligned}
$$

This looks similar to the unitary case [29]: the overall factor is again the quadratic Casimir eigenvalue of $SO(2N)$ in the vector representation. In the denominator one has (k_2+2N-2) and this quantity plus one. There exists a relation together with (2.76), (2.77) and (2.79),

$$
h(v;v) = h(0;v) + h(v;0) - 1, \qquad (2.80)
$$

which was also observed in [23]. The identity in (2.80) is checked from (2.75) and (2.78).

$$A_1(k_1=1,k_2,N=4)=\frac{k_2}{2520(k_2+7)}\times\sqrt{\frac{(k_2+2)(k_2+4)}{3(k_2+9)(k_2+11)}}, \quad (2.83)$$

by comparing the coefficient of the first term in the right-hand side of (2.81).

Let us emphasize that there exists a new primary field in (2.81) which is given by

$$\begin{aligned}W^{(4')}(z)\Big|_{k_1=1,k_2,N=4} &= \frac{1}{(k_2+2)(k_2+11)}\\ &\times\Big[-\frac{1}{9}d^{abcd}J^aJ^bK^cK^d+\frac{2}{35}(k_2-1)k_2\,\partial J^a\partial J^a\\ &-\frac{4}{105}(k_2-1)k_2\,\partial^2J^aJ^a+\frac{28}{15}\partial J^a\partial K^a\\ &+\frac{64}{105}(k_2-1)\,\partial^2J^aK^a\\ &-\frac{4}{35}(k_2-1)\,f^{abc}J^a\partial J^bK^c-\frac{2}{735}(k_2-1)k_2\,J^aJ^aJ^bJ^b\\ &-\frac{68}{315}J^aJ^aK^bK^b\\ &+\frac{8}{105}(k_2-1)\,J^aJ^aJ^bK^b-\frac{28}{45}J^aJ^bK^aK^b\\ &+\frac{1}{90}d^{abef}d^{efcd}J^aJ^bK^cK^d\Big](z). \quad (2.84)\end{aligned}$$

In other words, there exists a nonzero expression by combining the fourth-order pole with the first line of (2.81) with minus sign. Furthermore, one can express the various nonzero terms as the one in (2.84). One can easily see that the ten operators except the last operator appear in the previous higher spin 4 current in (2.14). It is straightforward to analyze the description appearing in Appendices B and C for the last operator in (2.84).

Let us further restrict to the simplest case where one can see the full structure of the corresponding OPE without losing any terms in the right-hand side. In other words, in the particular limit where $k_2\to\infty$ corresponding to $c=4$, the structure constants do not vanish. That is, there is no $(c-4)$ factor in the right-hand side of the OPE.

Then the higher spin 4 current can be written in terms of

$$\begin{aligned}W^{(4)}(z)\Big|_{k_1=1,k_2\to\infty,N=4} &= \frac{1}{2520\sqrt{3}}(d^{abcd}J^aJ^bJ^cJ^d\\ &+18\partial J^a\partial J^a-12\,\partial^2J^aJ^a\\ &-3\,J^aJ^aJ^bJ^b)(z), \quad (2.85)\end{aligned}$$

by substituting $N=4$ and $k_2\to\infty$ limit in Appendix E. The normalization factor is consistent with the general form in (2.83). The field contents in (2.85) are given in terms of the numerator spin 1 current (having the level $k_1=1$) of the coset model. Of course, the stress energy tensor contains only the first term with $k_1=1$ in (2.2) in this limit.

Then one can obtain the corresponding higher spin $4'$ current from (2.84) by taking the $k_2\to\infty$ limit and it turns out that

$$\begin{aligned}W^{(4')}(z)\Big|_{k_1=1,k_2\to\infty,N=4} &= \frac{2}{35}\Big(\partial J^a\partial J^a-\frac{2}{3}\partial^2J^aJ^a\\ &-\frac{1}{21}J^aJ^aJ^bJ^b\Big)(z). \quad (2.86)\end{aligned}$$

In (2.86), there is no d symbol.

Now we can calculate the OPE between the higher spin 4 current (2.85) and itself as follows:

$$\begin{aligned}W^{(4)}(z)\,W^{(4)}(w)\Big|_{k_1=1,k_2\to\infty,N=4} &= \frac{1}{(z-w)^8}\frac{c}{4}\\ &+\frac{1}{(z-w)^6}2\,T(w)+\frac{1}{(z-w)^5}\frac{1}{2}2\,\partial T(w)\\ &+\frac{1}{(z-w)^4}\Big[\frac{3}{20}2\,\partial^2T+\frac{42}{(5c+22)}\\ &\times\Big(T^2-\frac{3}{10}\partial^2T\Big)+C_{44}^4\,W^{(4)}\Big](w)\\ &+\frac{1}{(z-w)^3}\Big[\frac{1}{30}2\,\partial^3T+\frac{1}{2}\frac{42}{(5c+22)}\\ &\times\partial\Big(T^2-\frac{3}{10}\partial^2T\Big)+\frac{1}{2}C_{44}^4\,\partial W^{(4)}\Big](w)\\ &+\frac{1}{(z-w)^2}\Big[\frac{1}{168}2\,\partial^4T+\frac{5}{36}\frac{42}{(5c+22)}\\ &\times\partial^2\Big(T^2-\frac{3}{10}\partial^2T\Big)+\frac{5}{36}C_{44}^4\,\partial^2W^{(4)}\\ &+\frac{24(72c+13)}{(5c+22)(2c-1)(7c+68)}\Big(T(T^2-\frac{3}{10}\partial^2T)\\ &-\frac{3}{5}\partial^2TT+\frac{1}{70}\partial^4T\Big)\\ &-\frac{(95c^2+1254c-10904)}{6(5c+22)(2c-1)(7c+68)}\Big(\frac{1}{2}\partial^2(T^2-\frac{3}{10}\partial^2T)\\ &-\frac{9}{5}\partial^2TT+\frac{3}{70}\partial^4T\Big)\\ &+\frac{28}{3(c+24)}C_{44}^4\Big(TW^{(4)}-\frac{1}{6}\partial^2W^{(4)}\Big)+C_{44}^6\,W^{(6)}\Big](w)\\ &+\frac{1}{(z-w)}\Big[\frac{1}{1120}2\,\partial^5T+\frac{1}{36}\frac{42}{(5c+22)}\\ &\times\partial^3\Big(T^2-\frac{3}{10}\partial^2T\Big)+\frac{1}{36}C_{44}^4\,\partial^3W^{(4)}\\ &+\frac{1}{2}\frac{24(72c+13)}{(5c+22)(2c-1)(7c+68)}\\ &\times\partial\Big(T(T^2-\frac{3}{10}\partial^2T)-\frac{3}{5}\partial^2TT+\frac{1}{70}\partial^4T\Big)\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\,\frac{(95c^2+1254c-10904)}{6(5c+22)(2c-1)(7c+68)} \\
& \times \partial \left(\frac{1}{2}\partial^2(T^2-\frac{3}{10}\partial^2 T)-\frac{9}{5}\partial^2 T T+\frac{3}{70}\partial^4 T\right) \\
& +\frac{1}{2}\,\frac{28}{3(c+24)}\,C_{44}^{4}\,\partial\left(T W^{(4)}-\frac{1}{6}\partial^2 W^{(4)}\right) \\
& +\frac{1}{2}\,C_{44}^{6}\,\partial W^{(6)}\Big](w) \\
& +\frac{1}{(z-w)^4}\,W^{(4')}(w)+\frac{1}{(z-w)^3}\,\frac{1}{2}\partial W^{(4')}(w) \\
& +\frac{1}{(z-w)^2}\,\frac{1}{3}\left(T W^{(4')}-\frac{1}{6}\partial^2 W^{(4')}\right)(w) \\
& +\frac{1}{(z-w)}\,\frac{1}{2}\frac{1}{3}\partial\left(T W^{(4')}-\frac{1}{6}\partial^2 W^{(4')}\right)(w)+\cdots .
\end{aligned} \tag{2.87}
$$

Here the central charge coming from (2.82) is given by

$$
c(k_1=1,k_2\to\infty,N=4)=4, \tag{2.88}
$$

from (2.4) by substituting the right numbers. Moreover, the two structure constants are given by

$$
C_{44}^{4}=\sqrt{\frac{18(c+24)}{(5c+22)}},\quad C_{44}^{6}=\sqrt{\frac{12(c-1)(11c+656)}{(2c-1)(7c+68)}}, \tag{2.89}
$$

together with (2.88). Note that there are two extra last lines in (2.87) associated with the new primary higher spin $4'$ current, compared to the previous result in [33]. Equation (2.89) already appeared in [33–35].

2.5 Next higher spin currents

In the second-order pole of (2.87), there exists a primary higher spin 6 current. One can imagine the six products of spin 1 current with correct contractions of $SO(2N)$ indices. Let us consider the higher spin 4 current $W^{(4)}(z)$ which contains $d^{abcd} J^a J^b J^c J^d(z)$ and the same higher spin 4 current which contains $d^{d'efg} J^{d'} J^e J^f J^g(z)$. Then one has the second-order pole of this OPE, $d^{abcd} d^{d'efg} \delta^{dd'} J^a J^b J^c J^e J^f J^g(w)$, by considering the singular term between $J^d(z)$ and $J^{d'}(w)$. This gives rise to the term of $d^{abcd} d^{defg} J^a J^b J^c J^e J^f J^g(w)$. Then one expects that the higher spin 6 current contains this term and is given by $W^{(6)}(z) = d^{abcd} d^{defg} J^a J^b J^c J^e J^f J^g(z) + \cdots$. According to the description of [17,18], the tensorial structure of $SO(2N)$ symmetric invariant tensor of rank 6 can be determined by the product of two rank 4 d symbols. Therefore the above expression can be rewritten in terms of d tensor of rank 6 and one should have $W^{(6)}(z) = d^{abcefg} J^a J^b J^c J^e J^f J^g(z) + \cdots$. It would be interesting to observe the full expression for the higher spin 6 current.

3 Higher spin currents with $\mathcal{N}=1$ supersymmetry in the stringy coset model with two levels $(2N-2, k_2)$

In the presence of adjoint fermions coming from the equality of one of the levels and the dual Coxeter number of $SO(2N)$, one can construct the higher spin $\frac{7}{2}$ current which is the superpartner of the previous higher spin 4 current. In doing this, the role of the spin $\frac{3}{2}$ current living in the $\mathcal{N}=1$ superconformal algebra is crucial. The OPE between this $\mathcal{N}=1$ lowest higher spin multiplet, denoted by $(\frac{7}{2},4)$, is described using the Jacobi identities.

3.1 Spin $\frac{3}{2}$, 2 currents and $\mathcal{N}=1$ superconformal algebra

The spin $\frac{3}{2}$ current can be obtained from the spin $\frac{1}{2}$ current and spin 1 current as follows [6,16]:

$$
\begin{aligned}
G(z) &= \sqrt{\frac{4(N-1)}{(2N-2+k_2)(4N-4+k_2)}} \\
&\quad \times \left(\frac{k_2}{6(N-1)}\,\psi^a J^a-\psi^a K^a\right)(z).
\end{aligned} \tag{3.1}
$$

Here one has

$$
\psi^a(z)\,\psi^b(w)=-\frac{1}{(z-w)}\,\frac{1}{2}\delta^{ab}+\cdots . \tag{3.2}
$$

Furthermore, we can express the spin 1 current from the above spin $\frac{1}{2}$ current satisfying (3.2) as

$$
J^a(z)\equiv f^{abc}\psi^b\psi^c(z). \tag{3.3}
$$

It is easy to check that this spin 1 current satisfies the first equation of (2.3) with $k_1=(2N-2)$.

Then it is easy to see that there are only two terms in (3.1) and the relative coefficients can be fixed by using the above spin $\frac{3}{2}$ current, which should transform as a primary field under the stress energy tensor (2.2) with $k_1=(2N-2)$ as follows:

$$
T(z)\,G(w)=\frac{1}{(z-w)^2}\,\frac{3}{2}G(w)+\frac{1}{(z-w)}\,\partial G(w)+\cdots . \tag{3.4}
$$

In other words, the condition (3.4) determines the relative coefficients of (3.1).

The overall factor in (3.1) can be determined by the following OPE between the spin $\frac{3}{2}$ current and itself:

$$
G(z)\,G(w)=\frac{1}{(z-w)^3}\,\frac{2c}{3}+\frac{1}{(z-w)}\,2T(w)+\cdots . \tag{3.5}
$$

Here the central charge in (3.5) is given by (2.4) with the condition $k_1=(2N-2)$.

It is useful to write down the following OPEs which will be used in later calculations:

$$\begin{aligned}
\hat{G}(z)\,\psi^a(w) &= \frac{1}{(z-w)}\frac{1}{2}\left[-\frac{k_2}{2(N-1)}J^a+K^a\right](w)+\cdots,\\
\hat{G}(z)\,J^a(w) &= -\frac{1}{(z-w)^2}\,k_2\psi^a(w)\\
&\quad+\frac{1}{(z-w)}\left[-k_2\partial\psi^a-f^{abc}\psi^b K^c\right](w)+\cdots,\\
\hat{G}(z)\,K^a(w) &= \frac{1}{(z-w)^2}\,k_2\psi^a(w)\\
&\quad+\frac{1}{(z-w)}\left[k_2\partial\psi^a+f^{abc}\psi^b K^c\right](w)+\cdots,
\end{aligned} \tag{3.6}$$

where we introduce the following quantity:

$$\hat{G}(z) \equiv \left(\frac{k_2}{6(N-1)}\,\psi^a J^a-\psi^a K^a\right)(z). \tag{3.7}$$

Compared to the unitary case [36–38], the behavior of relative coefficient, which is equal to one over three times the level divided by the dual Coxeter number, occurs in (3.7). See also [6,16].

3.2 Eigenvalue equation of the zero mode of the higher spin 4 current

We also present the eigenvalue equations for the spin 2 stress energy tensor (2.2) with $k_1=(2N-2)$,

$$\begin{aligned}
T_0|(0;v)\rangle &= \frac{k_2(2N-1)}{8(N-1)(k_2+4N-4)}|(0;v)\rangle\\
&\rightarrow \frac{(1-\lambda)}{4(1+\lambda)}|(0;v)\rangle,\\
T_0|(v;0)\rangle &= \frac{(2N-1)(k_2+6N-6)}{8(N-1)(k_2+2N-2)}|(v;0)\rangle\\
&\rightarrow \frac{(1+2\lambda)}{4}|(v;0)\rangle.
\end{aligned} \tag{3.8}$$

In (3.8), the large N 't Hooft limit is taken at the final stage. Note that the conformal dimension of $(0;v)$ can be obtained from the formula

$$\begin{aligned}
h(0;v) &= \frac{1}{2}(2N-1)\left[\frac{1}{(2N-2)+(2N-2)}\right.\\
&\quad\left.-\frac{1}{(2N-2)+k_2+(2N-2)}\right]\\
&= \frac{k_2(2N-1)}{8(N-1)(k_2+4N-4)},
\end{aligned} \tag{3.9}$$

where the overall factor $\frac{1}{2}(2N-1)$ is the quadratic Casimir eigenvalue of the $SO(2N)$ vector representation. Similarly, the conformal dimension of $(v;0)$ can be obtained:[5]

$$\begin{aligned}
h(v;0) &= \frac{1}{2}(2N-1)\left[\frac{1}{(2N-2)+(2N-2)}\right.\\
&\quad\left.+\frac{1}{k_2+(2N-2)}\right]\\
&= \frac{(2N-1)(k_2+6N-6)}{8(N-1)(k_2+2N-2)}.
\end{aligned} \tag{3.13}$$

As done in Sect. 2, one obtains the following eigenvalue equations:

$$\begin{aligned}
W_0^{(4)}|(0;v)\rangle &= -N^3\left[\frac{48}{(2\lambda-3)}\right]A_1|(0;v)\rangle,\\
W_0^{(4)}|(v;0)\rangle &= -N^3\left[\frac{48(\lambda+1)^2(2\lambda+1)(4\lambda+3)}{(\lambda-3)(\lambda-1)(2\lambda-3)}\right]\\
&\quad A_1|(v;0)\rangle,\\
W_0^{(4)}|(v;v)\rangle &= -N^3\left[\frac{96\lambda^2\left(4\lambda^2+\lambda+1\right)}{(\lambda-3)(\lambda-1)(2\lambda-3)}\right]\\
&\quad A_1|(v;v)\rangle.
\end{aligned} \tag{3.14}$$

Using Eqs. (3.14), one can obtain the several three-point functions. The relevant coefficients are given in Appendix F.

3.3 Higher spin $\frac{7}{2}$, 4 currents

One way to determine the higher spin $\frac{7}{2}$ current is to use the OPE between the spin $\frac{3}{2}$ current and the higher spin 4 current in previous section. Note that the corresponding coefficients at the critical level $k_1=(2N-2)$ are given in Appendix F. In other words, from the $\mathcal{N}=1$ super primary condition [36,37], one should have

[5] Moreover, the eigenvalue equation for the state $|(v;v)\rangle$ can be obtained as follows:

$$T_0|(v;v)\rangle = \frac{(N-1)(2N-1)}{(k_2+2N-2)(k_2+4N-4)}|(v;v)\rangle \rightarrow \frac{\lambda^2}{2(\lambda+1)}|(v;v)\rangle. \tag{3.10}$$

The conformal dimension of $(v;v)$ in (3.10) can also be obtained as follows:

$$\begin{aligned}
h(v;v) &= \frac{1}{2}(2N-1)\left[\frac{1}{k_2+(2N-2)}-\frac{1}{(2N-2)+k_2+(2N-2)}\right]\\
&= \frac{(N-1)(2N-1)}{(k_2+2N-2)(k_2+4N-4)}.
\end{aligned} \tag{3.11}$$

There exists a relation together with (3.9), (3.13) and (3.11),

$$h(v;v) = h(0;v)+h(v;0)-\frac{(2N-1)}{4(N-1)}. \tag{3.12}$$

Here the last term in (3.12) is the ratio of quadratic Casimir eigenvalue for the vector representation and the dual Coxeter number of $SO(2N)$.

$$\hat{G}(z)\, W^{(4)}(w)\Big|_{\frac{1}{(z-w)^2}} = \frac{1}{(z-w)^2} 7 \times \sqrt{\frac{(2N-2+k_2)(4N-4+k_2)}{4(N-1)}}\, W^{(\frac{7}{2})}(w) + \mathcal{O}\left(\frac{1}{(z-w)}\right). \tag{3.15}$$

In order to calculate the second-order pole of (3.15), one can use the three OPEs in (3.6). The explicit results are given in Appendix G. Of course, this will give us the final higher spin $\frac{7}{2}$ current but it is rather non-trivial to simplify in simple form. Therefore, after we identify the correct field contents for fixed $N = 4$, we introduce the undetermined coefficients and fix them using the previous methods we used in previous section. That is, the higher spin $\frac{7}{2}$ current should not have any singular terms with the diagonal spin 1 current and transform as a primary higher spin current under the stress energy tensor.

Then one can express the higher spin $\frac{7}{2}$ current as follows [6,19,20]:

$$\begin{aligned} W^{(\frac{7}{2})}(z) = & B_1\, d^{abcd} \psi^a J^b J^c J^d(z) \\ & + B_2\, d^{abcd} f^{aef} f^{beg} J^c J^d \psi^f K^g(z) \\ & + B_3\, d^{abcd} J^a K^b \psi^c K^d(z) \\ & + B_4\, d^{abcd} f^{aef} f^{beg} K^c K^d \psi^f K^g(z) \\ & + B_5\, J^a \psi^a J^b J^b(z) + B_6\, K^a K^a \psi^b K^b(z) \\ & + B_7\, J^a J^a \psi^b K^b(z) + B_8\, J^a J^a \psi^b J^b(z) \\ & + B_9\, \psi^a K^a K^b K^b(z) \\ & + B_{10}\, f^{abc} f^{cde} K^a K^e \psi^b K^d(z) \\ & + B_{11}\, J^a \psi^a K^b K^b(z) + B_{12}\, \psi^a J^b K^a K^b(z) \\ & + B_{13}\, J^a J^b \psi^a K^b(z) \\ & + B_{14}\, f^{abc} f^{cde} J^a J^e \psi^b K^d(z) \\ & + B_{15}\, J^a J^b K^a \psi^b(z) \\ & + X(k_2, N)(GT - \frac{1}{8}\partial^2 G)(z). \end{aligned} \tag{3.16}$$

The B_7 term can be written as $(\psi^a J^b J^b K^a - 2 f^{abc} \psi^a \partial J^b K^c - (2N-2)\partial^2 \psi^a K^a)(z)$ by moving the field ψ^b to the left. Similarly, the B_8 term can be described as $(\psi^a J^a J^b J^b + (2N-2)\partial^2 \psi^a J^a - (2N-2)\psi^a \partial^2 J^a + 2(2N-2)\partial \psi^a \partial J^a)(z)$. For the B_{13} term one obtains $(\psi^a J^b J^a K^b + (2N-2)\partial^2 \psi^a K^a)(z)$. For the B_{14} term one can write down $(3(2N-2) f^{abc} \psi^a \partial J^b K^c + (N-2)^2 \partial^2 \psi^a K^a)(z)$. For the B_{15} term, one has $(\psi^a J^b J^a K^b + f^{abc} \psi^a \partial J^b K^c)(z)$ by moving ψ^b to the left. Furthermore, the B_2 term and the B_4 term can be simplified using the identity (2.10). For the remaining other terms, the fermion ψ^a can be moved to the leftmost position without any extra terms because of the properties of the f and d symbols. The B_5, B_6, B_7, B_{11}, B_{12}, and B_{13} terms can be seen from $GT(z)$. The B_8, B_9, and B_{15} terms are written in terms of B_5, B_6, and B_{13} terms plus derivative terms, respectively.

Note that the last term in (3.16) is a quasiprimary field in the sense that the OPE between the stress energy tensor and this field does not contain the third-order pole. We realize that this term does not appear for the particular $N = 4$ case.

We would like to determine the undetermined coefficients B_1–B_{15} and $X(k_2, N)$ in (3.16). As in (2.15), one should have the regular condition as follows:

$$J'^a(z)\, W^{(\frac{7}{2})}(w) = + \cdots. \tag{3.17}$$

In Appendix H, we present the OPEs between the diagonal spin 1 current and the 15 fields in (3.16). Moreover, the higher spin $\frac{7}{2}$ current transforms as a primary field under the stress energy tensor. In other words, one has

$$\hat{T}(z)\, W^{(\frac{7}{2})}(w)\Big|_{\frac{1}{(z-w)^n},\, n=3,4,5} = 0, \tag{3.18}$$

as in (2.23). One obtains the corresponding OPEs in Appendix I.

By solving the various linear equations on the coefficients satisfying the above requirements (3.17) and (3.18), one obtains the final coefficients in Appendix J.

For consistency check, one can calculate the OPE between $\hat{G}(z)$ and $W^{(\frac{7}{2})}(w)$. The first-order pole should behave as $\sqrt{\frac{(2N-2+k_2)(4N-4+k_2)}{4(N-1)}}\, W^{(4)}(w)$. In doing this, the OPEs in (3.6) are crucial. In order to see the presence of higher spin 4, the rearrangement of the normal ordered product should be taken because the above first-order pole terms contain unwanted terms. Of course, we do not have to worry about the extra contractions in the OPEs because we are interested in the first-order pole as described above.

3.4 The OPEs between the higher spin $\frac{7}{2}$, 4 currents

It is natural to ask how the OPEs between the higher spin $\frac{7}{2}$ current and the higher spin 4 current arise. They have rather long expressions for the $N = 4$ case.

Therefore, one tries to obtain the corresponding OPEs from the Jacobi identities for the above higher spin currents and other relevant higher spin currents. We will consider only the three OPEs, $W^{(\frac{7}{2})}(z)\, W^{(\frac{7}{2})}(w)$, $W^{(\frac{7}{2})}(z)\, W^{(4)}(w)$ and $W^{(4)}(z)\, W^{(4)}(w)$. What kind of new primary higher spin currents are present in the right-hand side of OPEs? From the OPEs of $W^{(\frac{7}{2})}(z)\, W^{(4)}(w)$ or $W^{(4)}(z)\, W^{(\frac{7}{2})}(w)$, one can think of the presence of a new higher spin $\frac{13}{2}$ current at the first order pole. Furthermore, from the OPE $W^{(4)}(z)\, W^{(4)}(w)$, the new higher spin 6 current can appear in the second-order pole of this OPE. Note that there is no new higher spin 7 current in the first-order pole. The reason is as follows. One can calculate the OPE $W^{(4)}(w)\, W^{(4)}(z)$

in the presence of the new higher spin 7 current at the first-order pole, use the symmetry $z \leftrightarrow w$ and end up with the OPE $W^{(4)}(z)\, W^{(4)}(w)$. By focusing on the new higher spin 7 current, one realizes that there exists an extra minus sign. Therefore, the new higher spin 7 current should vanish.

Then one can assign the above two higher spin currents as one single $\mathcal{N}=1$ higher spin current, denoted by $(6', \frac{13}{2})$ where the numbers stand for each spin. From the OPE in $W^{(4)}(z)\, W^{(4)}(w)$, the second-order pole provides a new higher spin 6 current. Then one can turn to the $\mathcal{N}=1$ higher spin current denoted by $(\frac{11}{2}, 6)$. Furthermore, from the bosonic higher spin $4'$ current in the previous section, one can introduce its superpartner whose spin is given by $\frac{9}{2}$. The corresponding $\mathcal{N}=1$ higher spin current is characterized by $(4', \frac{9}{2})$ using the above notation.

For $N=4$, one can calculate the OPE in $W^{(\frac{7}{2})}(z)\, W^{(\frac{7}{2})}(w)$. By requiring that the seventh-order pole should be equal to $\frac{2c}{7}$, one can determine the coefficient A_1 as

$$A_1(k_1=6, k_2, N=4) = \frac{k_2}{5040(k_2+12)} \times \sqrt{\frac{(k_2+2)(k_2+4)}{6(k_2+6)(k_2+12)(k_2+14)(k_2+16)}}. \tag{3.19}$$

The fifth-order pole gives $2T(w)$ and the fourth-order pole gives $\partial T(w)$. Similar behaviors arise in (2.87). Let us describe the third-order pole. One can easily check that the following quantity together with (3.19):

$$\frac{1}{(4c+21)(10c-7)}\Big[8(37c+3)TT + 3(2c-117)\partial G G - \frac{3}{10}(302c-327)\partial^2 T\Big](w) \tag{3.20}$$

is a quasiprimary field. The third-order pole subtracted by both (3.20) and $\frac{3}{10}\partial^2 T(w)$ (which is a descendant field) is a primary field. However, this is not written in terms of the previous higher spin 4 current. This implies that there exists a new primary higher spin $4'$ current. The structure constants appearing in (3.20) are obtained from the Jacobi identities. Because we are dealing with the extensions of the $\mathcal{N}=1$ superconformal algebra, the $\partial G G(w)$ term appears in addition to $TT(w)$ and $\partial^2 T(w)$.

By assuming that the $\mathcal{N}=1$ OPE between the $\mathcal{N}=1$ higher spin $\frac{7}{2}$ multiplet contains the $\mathcal{N}=1$ higher spin $4', \frac{11}{2}, 6'$ multiplets, one obtains the complete structure of these OPEs in a component approach (and $\mathcal{N}=1$ superspace). They are given in Appendix K in terms of the central charge and some undetermined structure constants. It would be interesting to see whether there exist other additional higher spin currents or not. See also the work in [39] where the Jacobi identities are used.

3.5 The OPE in the $\mathcal{N}=1$ superspace

From the three OPEs in a component approaches described in Appendix K, one summarizes its $\mathcal{N}=1$ superspace in simple notation as follows:

$$\Big[\mathbf{W}^{(\frac{7}{2})} \cdot \mathbf{W}^{(\frac{7}{2})}\Big] = [\mathbf{I}] + \Big[\mathbf{W}^{(\frac{7}{2})}\Big] + \Big[\mathbf{W}^{(4')}\Big] + \Big[\mathbf{W}^{(\frac{11}{2})}\Big] + \Big[\mathbf{W}^{(6')}\Big], \tag{3.21}$$

where $[\mathbf{I}]$ appearing in (3.21) is the $\mathcal{N}=1$ superconformal family of the identity operator. According to the field contents in [40], where k_2 is fixed as $k_2=1$, the above OPE should not contain the $\mathcal{N}=1$ higher spin integer multiplets. See also [41]. The right-hand side should contain the first, the second and the fourth terms. It would be interesting to observe this behavior explicitly. First of all, the single higher spin 4 current should exist by combining the previous two kinds of higher spin 4 currents under the constraint $k_2=1$.

4 Higher spin currents with $\mathcal{N}=2$ supersymmetry in the stringy coset model with two levels $(2N-2, 2N-2)$

The additional adjoint fermions allow us to construct the spin $1, \frac{3}{2}$ currents in the $\mathcal{N}=2$ superconformal algebra. Furthermore, the additional higher spin $3, \frac{7}{2}$ currents can be found explicitly along the line of [42]. The lowest higher spin 3 current of $U(1)$ charge $\frac{4}{3}$ is obtained and it can be written in terms of two adjoint fermions. There exists another $\mathcal{N}=2$ higher spin multiplet, which consists of the above same spin contents, $(3, \frac{7}{2}, \frac{7}{2}, 4)$ with different $U(1)$ charges. Finally, the OPE between these two $\mathcal{N}=2$ higher spin multiplets is described.

4.1 Spin $1, \frac{3}{2}, \frac{3}{2}, 2$ currents and $\mathcal{N}=2$ superconformal algebra

Let us introduce the second adjoint fermions which satisfy the following OPE:

$$\chi^a(z)\,\chi^b(w) = -\frac{1}{(z-w)}\,\frac{1}{2}\delta^{ab} + \cdots. \tag{4.1}$$

It is easy to see that one can express the spin 1 current from the above spin $\frac{1}{2}$ current with (4.1) as

$$K^a(z) \equiv f^{abc}\chi^b\chi^c(z). \tag{4.2}$$

This spin 1 current satisfies the second equation of (2.3) with $k_2=(2N-2)$.

Then it is straightforward to construct the four generating currents, denoted by $(1, \frac{3}{2}, \frac{3}{2}, 2)$, corresponding to the $\mathcal{N}=2$

superconformal algebra as follows [43]:

$$\begin{aligned}
J(z) &= \frac{2}{3}\, i\, \psi^a \chi^a(z), \\
G^+(z) &= -\frac{1}{6\sqrt{3(2N-2)}} \\
&\quad \times \left[\psi^a J^a - 3\,\psi^a K^a - i\,\chi^a K^a + 3\,i\,\chi^a J^a\right](z), \\
G^-(z) &= -\frac{1}{6\sqrt{3(2N-2)}} \\
&\quad \times \left[\psi^a J^a - 3\,\psi^a K^a + i\,\chi^a K^a - 3\,i\,\chi^a J^a\right](z), \\
T(z) &= -\frac{1}{4(2N-2)}\, J^a J^a(z) - \frac{1}{4(2N-2)}\, K^a K^a(z) \\
&\quad + \frac{1}{6(2N-2)} (J^a + K^a)(J^a + K^a)(z).
\end{aligned} \tag{4.3}$$

By realizing that the difference between $G^+(z)$ and $G^-(z)$ occurs in the third and fourth terms, under $\chi^a(z) \to -\chi^a(z)$, one sees the relation $G^+(z) \leftrightarrow G^-(z)$.

Let us introduce the following spin 1 current by taking the product of two adjoint fermions:

$$L^a \equiv f^{abc} \psi^b \chi^c. \tag{4.4}$$

The central charge can be reduced to

$$c = \frac{1}{3} N(2N-1), \tag{4.5}$$

which can be obtained from (2.4) by substituting the corresponding two levels. In order to construct the higher spin currents, let us introduce the following intermediate spin 2 current:

$$\begin{aligned}
M_1^a &\equiv d^{abcd} \psi^b \chi^c J^d, \\
M_2^a &\equiv d^{abcd} \psi^b \chi^c K^d, \\
M_3^a &\equiv d^{abcd} \psi^b \chi^c L^d,
\end{aligned} \tag{4.6}$$

together with (3.3), (4.2) and (4.4). Compared to the unitary case in [44], the contracted indices appear in the two different adjoint fermions (because of the symmetric d tensor) as well as the spin 1 currents.

4.2 Higher spin 3, $\frac{7}{2}$, $\frac{7}{2}$, 4 currents

From the experience of Sects. 2 and 3, there exist the higher spin 4 current and the $\mathcal{N}=1$ higher spin $\frac{7}{2}$ current denoted by $(\frac{7}{2}, 4)$; then there are two choices where the above $\mathcal{N}=1$ higher spin $\frac{7}{2}$ multiplet can arise from the lower two component currents or higher two component currents. Let us try to find the higher spin currents by taking the second choice.

By writing the possible candidate terms for the higher spin 3 current, one can think of the product of spin 1 currents (3.3), (4.2) or (4.4) and the intermediate spin 2 currents in (4.6). Furthermore, one can think of the product of each component field in the spin $\frac{3}{2}$ currents living in the $\mathcal{N}=2$ superconformal algebra. Of course, one should consider the possible derivative terms. Therefore, one can consider the following higher spin 3 current [6]:

$$\begin{aligned}
W_{\frac{4}{3}}^{(3)}(z) &= a_1\, J^a M_1^a(z) + a_2\, K^a M_1^a(z) + a_3\, L^a M_1^a(z) \\
&\quad + a_4\, J^a M_2^a(z) + a_5\, K^a M_2^a(z) + a_6\, L^a M_2^a(z) \\
&\quad + a_7\, J^a M_3^a(z) + a_8\, K^a M_3^a(z) + a_9\, L^a M_3^a(z) \\
&\quad + a_{10}\, J^a \partial J^a(z) + a_{11}\, J^a \partial K^a(z) \\
&\quad + a_{12}\, J^a \partial L^a(z) + a_{13}\, \partial J^a K^a(z) + a_{14}\, K^a \partial K^a(z) \\
&\quad + a_{15}\, K^a \partial L^a(z) + a_{16}\, \partial J^a L^a(z) \\
&\quad + a_{17}\, \partial K^a L^a(z) + a_{18}\, L^a \partial L^a(z) \\
&\quad + a_{19}\, (\psi^a J^a)(\psi^b J^b)(z) + a_{20}\, (\psi^a J^a)(\psi^b K^b)(z) \\
&\quad + a_{21}\, (\psi^a J^a)(\chi^b J^b)(z) + a_{22}\, (\psi^a J^a)(\chi^b K^b)(z) \\
&\quad + a_{23}\, (\psi^a K^a)(\psi^b K^b)(z) \\
&\quad + a_{24}\, (\psi^a K^a)(\chi^b J^b)(z) + a_{25}\, (\psi^a K^a)(\chi^b K^b)(z) \\
&\quad + a_{26}\, (\chi^a J^a)(\chi^b J^b)(z) \\
&\quad + a_{27}\, (\chi^a J^a)(\chi^b K^b)(z) + a_{28}\, (\chi^a K^a)(\chi^b K^b)(z).
\end{aligned} \tag{4.7}$$

The $U(1)$ charge $\frac{4}{3}$ will be determined later.

As done in previous sections, one can use two requirements in order to fix the above coefficients. One of them is the regularity with the diagonal spin 1 current as follows:

$$J'^a(z)\, W_{\frac{4}{3}}^{(3)}(w) = + \cdots. \tag{4.8}$$

Here the diagonal spin 1 current in (4.8) is the sum of (3.3) and (4.2). The other is given by the primary condition, which can be described as follows together with (4.7):

$$\hat{T}(z)\, W_{\frac{4}{3}}^{(3)}(w) \Big|_{\frac{1}{(z-w)^n},\, n=3,4,5} = 0. \tag{4.9}$$

Here the stress energy tensor is given by (2.24) substituted by (3.3) and (4.2).

It turns out, from (4.8) and (4.9), that the above higher spin 3 current with the explicit coefficients is given by

$$\begin{aligned}
W_{\frac{4}{3}}^{(3)}(z) &= \Big[-\frac{i}{4}(a_7 - a_8)\, J^a M_1^a + \frac{i}{2}(a_7 - a_8)\, K^a M_1^a \\
&\quad - a_8\, L^a M_1^a - \frac{i}{4}(a_7 - a_8)\, K^a M_2^a \\
&\quad - a_7\, L^a M_2^a + a_7\, J^a M_3^a + a_8\, K^a M_3^a \\
&\quad + i(a_7 - a_8)\, L^a M_3^a \\
&\quad + \frac{3i}{4}(a_7 - a_8)\, J^a \partial L^a + \frac{3i}{4}(a_7 - a_8)\, K^a \partial L^a \\
&\quad + \frac{3i}{4}(a_7 - a_8)\, \partial J^a L^a + \frac{3i}{4}(a_7 - a_8)\, \partial K^a L^a \\
&\quad + (-a_7 + a_8)\, (\psi^a J^a)(\psi^b K^b) \\
&\quad + \frac{i}{2}(a_7 - a_8)\, (\psi^a J^a)(\chi^b J^b) \\
&\quad - \frac{i}{2}(a_7 - a_8)\, (\psi^a J^a)(\chi^b K^b)
\end{aligned}$$

$$+\frac{3i}{2}(a_7-a_8)\,(\psi^a K^a)(\chi^b J^b)$$
$$+\frac{i}{2}(a_7-a_8)\,(\psi^a K^a)(\chi^b K^b)$$
$$+(a_7-a_8)\,(\chi^a J^a)(\chi^b K^b)\Big](z). \quad (4.10)$$

Note that there exist also $a_4, a_{11}, a_{14}, a_{20}, a_{24}$, and a_{28} dependent terms (other coefficients depend on these six coefficients and a_7 and a_8 after the above two conditions are used) but they are identically zero, respectively. From the definitions of (4.6), the first eight terms in (4.10) contain the rank 4 d symbol. One can see the common nonderivative expression in the third term and sixth term and then one can combine them with the coefficient (a_7-a_8). Similarly, the fifth term and the seventh term share the common nonderivative term with the coefficient $-(a_7-a_8)$. Furthermore, the composite fields appearing in (4.10) contain the various derivative terms (it is obvious that the ninth–twelfth terms do have the derivative terms and also they can appear from the ordering for the composite fields) but the precise coefficients will lead to the vanishing of these derivative terms.

For the extended $\mathcal{N}=2$ superconformal algebra, there is one additional condition for the higher spin current which is the $U(1)$ charge (i.e., the coefficient of the first-order pole of the OPE with the spin 1 current). That is [44],

$$J(z)\,W_q^{(3)}(w)=\frac{1}{(z-w)}\,q\,W_q^{(3)}(w)+\cdots. \quad (4.11)$$

It turns out that the $U(1)$ charge is fixed and for $q=\frac{4}{3}$, there is a relation $a_{12}=\frac{3i}{4}(a_7-a_8)$. This relation is used in (4.10). For $q=-\frac{4}{3}$, there is a relation $a_{12}=-\frac{3i}{4}(a_7-a_8)$. It is useful to express the above higher spin 3 current in a manifestly $U(1)$ charge symmetric way. Let us focus on the first term in (4.10). If one substitutes the definition of M_1^a in (4.6), one has $f^{abc}\psi^b\psi^c d^{adef}\psi^d\chi^e J^f(w)$ where J^a is replaced by the fermions. One substitutes for the J^f using the relation (3.3) and obtains $f^{abc}d^{adef}f^{fgh}\psi^b\psi^c\psi^d\chi^e\psi^g\psi^h(z)$. Now move the composite field $\psi^d\chi^e$ to the right. One obtains $f^{abc}d^{adef}f^{fgh}\psi^b\psi^c\psi^g\psi^h\psi^d\chi^e(z)$, which can be written in terms of $\frac{i}{2}f^{abc}d^{adef}f^{fgh}\psi^b\psi^c\psi^g\psi^h(\psi^d+i\chi^d)(\psi^e-i\chi^e)(z)$ from the symmetric property of d^{adef}. Then the overall factor is given by $\frac{1}{8}(a_7-a_8)$ by considering the numerical factor $-\frac{i}{4}(a_7-a_8)$.[6] Let us describe the eighth term which is the last term which contains the d symbol. So one has $f^{abc}d^{adef}f^{fgh}\psi^b\chi^c\psi^g\chi^h\psi^d\chi^e(z)$, which can be identified with $-\frac{i}{2}f^{abc}d^{adef}f^{fgh}\psi^b i\chi^c\psi^g i\chi^h(\psi^d+i\chi^d)(\psi^e-i\chi^e)(z)$. By multiplying the overall factor $i(a_7-a_8)$, one obtains $\frac{1}{2}(a_7-a_8)f^{abc}d^{adef}f^{fgh}\psi^b i\chi^c\psi^g i\chi^h(\psi^d+i\chi^d)(\psi^e-i\chi^e)(z)$. This can be further rewritten in terms of $\frac{1}{8}(a_7-a_8)f^{abc}d^{adef}f^{fgh}(\psi^b i\chi^c\psi^g i\chi^h+\psi^b i\chi^c i\chi^g\psi^h+i\chi^b\psi^c\psi^g i\chi^h+\psi^b i\chi^c i\chi^g\psi^h)(\psi^d+i\chi^d)(\psi^e-i\chi^e)(z)$. Finally, one can summarize the first eight terms in (4.10) that are given by $\frac{1}{8}(a_7-a_8)d^{abcd}f^{aef}f^{bgh}(\psi^e+i\chi^e)(\psi^f+i\chi^f)(\psi^g+i\chi^g)(\psi^h+i\chi^h)(\psi^c+i\chi^c)(\psi^d-i\chi^d)(z)$.

Now we are considering the last six terms in (4.10). The first term is given by $-(a_7-a_8)f^{acd}\psi^a\psi^c\psi^d f^{bef}\psi^b\chi^e\chi^f(z)$. This can be rewritten as $-\frac{1}{4}(a_7-a_8)(f^{acd}\psi^a\psi^c\psi^d f^{bef}\psi^b\chi^e\chi^f + 3f^{acd}\psi^a i\chi^c i\chi^d f^{bef}\psi^b\psi^e\psi^f)(z)$ where we use the fact that there exists a minus sign when the first three factors $\psi^a i\chi^c i\chi^d$ move to the right. Therefore, there should be an overall factor $\frac{1}{4}$. One can analyze the other four terms.[7] Finally, one can summarize the last six terms in (4.10) are given by $-\frac{1}{4}(a_7-a_8)f^{abc}f^{def}(\psi^a+i\chi^a)(\psi^b+i\chi^b)(\psi^c+i\chi^c)(\psi^d+i\chi^d)(\psi^e+i\chi^e)(\psi^f-i\chi^f)(z)$.

By putting $(a_7-a_8)=1$, one obtains the higher spin 3 current with $U(1)$ charge $\frac{4}{3}$ as follows:

$$\begin{aligned}W_{\frac{4}{3}}^{(3)}(z)&=\frac{1}{8}d^{abcd}f^{aef}f^{bgh}(\psi^e+i\chi^e)(\psi^f+i\chi^f)\\&\times(\psi^g+i\chi^g)(\psi^h+i\chi^h)(\psi^c+i\chi^c)(\psi^d-i\chi^d)(z)\\&-\frac{1}{4}f^{abc}f^{def}(\psi^a+i\chi^a)(\psi^b+i\chi^b)(\psi^c+i\chi^c)\\&\times(\psi^d+i\chi^d)(\psi^e+i\chi^e)(\psi^f-i\chi^f)(z).\end{aligned} \quad (4.13)$$

One can calculate the $U(1)$ charges for the adjoint fermions with (4.3) as follows:

$$\begin{aligned}J(z)\,(\psi^a+i\chi^a)(w)&=\frac{1}{(z-w)}\,\frac{1}{3}\,(\psi^a+i\chi^a)(w)+\cdots,\\J(z)\,(\psi^a-i\chi^a)(w)&=\frac{1}{(z-w)}\,(-1)\frac{1}{3}\,(\psi^a-i\chi^a)(w)+\cdots.\end{aligned} \quad (4.14)$$

[6] Similarly, the second term can be analyzed also. The relevant term can be written in terms of $f^{abc}d^{adef}f^{fgh}\chi^b\chi^c\psi^d\chi^e\psi^g\psi^h(z)$, which can also be expressed as $f^{abc}d^{adef}f^{fgh}\chi^b\chi^c\psi^g\psi^h\psi^d\chi^e(z)$. Once again this can be described as $-\frac{i}{2}f^{abc}d^{adef}f^{fgh}i\chi^b i\chi^c\psi^g\psi^h(\psi^d+i\chi^d)(\psi^e-i\chi^e)(z)$ as done before. The overall factor of the second term is given by $\frac{i}{2}(a_7-a_8)$. Then the total overall factor gives $\frac{1}{4}(a_7-a_8)$. Intentionally, we rewrite the above as $\frac{1}{8}(a_7-a_8)f^{abc}d^{adef}f^{fgh}(i\chi^b i\chi^c\psi^g\psi^h+i\psi^b i\psi^c\chi^g\chi^h)(\psi^d+i\chi^d)$

Footnote 6 continued
$(\psi^e-i\chi^e)(z)$ using the property of the d symbol. One can analyze the other terms up to the seventh term.

[7] Let us describe the last term, which is given by $-(a_7-a_8)f^{acd}i\chi^a\psi^c\psi^d f^{bef}i\chi^b\chi^e\chi^f(z)$. As above, this can be written as $-\frac{1}{4}(a_7-a_8)(3f^{acd}i\chi^a\psi^c\psi^d f^{bef}i\chi^b\chi^e\chi^f - f^{acd}i\chi^a\chi^c\chi^d f^{bef}i\chi^b\psi^e\psi^f)(z)$.
There are also identities as follows:

$$\begin{aligned}&f^{abc}\psi^a\psi^b\psi^c f^{def}\psi^d\psi^e\psi^f=0,\quad f^{abc}\psi^a\chi^b\chi^c f^{def}\psi^d\chi^e\chi^f=0,\\&f^{abc}\chi^a\psi^b\psi^c f^{def}\chi^d\psi^e\psi^f=0,\quad f^{abc}\chi^a\chi^b\chi^c f^{def}\chi^d\chi^e\chi^f=0.\end{aligned} \quad (4.12)$$

As explained before, Eq. (4.12) can be checked by moving the first three fermions to the right; there exists a minus sign.

Then it is obvious that the above higher spin 3 current (4.13) has $U(1)$ charge $\frac{4}{3}$: there exist five factors with $U(1)$ charge $\frac{1}{3}$ and one factor with $U(1)$ charge $-\frac{1}{3}$ according to (4.14).

For the unitary case [44], one sees the factor $f^{aef}\, f^{bgh}(\psi^e+i\chi^e)(\psi^f+i\chi^f)(\psi^g+i\chi^g)(\psi^h+i\chi^h)$ and the other factor is given by $d^{abc}\, f^{chi}(\psi^h-i\chi^h)(\psi^i-i\chi^i)$ of the $U(1)$ charge $-\frac{2}{3}$ in the nonderivative terms. However, the orthogonal case contains the different factor $d^{abcd}(\psi^c+i\chi^c)(\psi^d-i\chi^d)$ of $U(1)$ charge 0 in (4.13).

In order to obtain the other higher spin currents, it is useful to calculate the following OPEs:

$$
\begin{aligned}
&G^+(z)\,(\psi^a+i\chi^a)(w) = +\cdots,\\
&G^+(z)\,(\psi^a-i\chi^a)(w) = \frac{1}{(z-w)}\\
&\quad\times\frac{1}{2\sqrt{3(2N-2)}}\, f^{abc}(\psi^b+i\chi^b)(\psi^c+i\chi^c)(w)+\cdots,\\
&G^-(z)\,(\psi^a-i\chi^a)(w) = +\cdots,\\
&G^-(z)\,(\psi^a+i\chi^a)(w) = \frac{1}{(z-w)}\\
&\frac{1}{2\sqrt{3(2N-2)}}\, f^{abc}(\psi^b-i\chi^b)(\psi^c-i\chi^c)(w)+\cdots.
\end{aligned}
\tag{4.15}
$$

We will use this property to calculate the OPEs for the particular singular terms. One sees the $U(1)$ charge conservation in (4.15).

How does one determine the other higher spin currents related to the lowest one? Let us recall the following OPE [44–46]:

$$
G^+(z)\, W^{(3)}_{\frac{4}{3}}(w) = -\frac{1}{(z-w)}\, W^{(\frac{7}{2})}_{\frac{7}{3}}(w)+\cdots. \tag{4.16}
$$

Here the higher spin current appears in the first-order pole. Once we have obtained the first-order pole in the above OPE, then we obtain the higher spin current. See also the relevant work in [47]. Because the lowest higher spin 3 current is written in terms of adjoint fermions, it is better to calculate the OPE between $G^+(z)$ and fermions appearing in (4.13). According to the observations of (4.15), the spin $\frac{3}{2}$ current $G^+(z)$ has non-trivial OPE with the spin $\frac{1}{2}$ current of $U(1)$ charge $-\frac{1}{3}$, while the spin $\frac{3}{2}$ current $G^-(z)$ has non-trivial OPE with the spin $\frac{1}{2}$ current of $U(1)$ charge $\frac{1}{3}$. Then it is obvious that when one calculates the left-hand side of (4.16), the only non-trivial singular terms appear at the location of the last factors, $(\psi^d-i\chi^d)(w)$ and $(\psi^f-i\chi^f)(w)$ in (4.13). This leads to the following higher spin $\frac{7}{2}$ current of $U(1)$ charge $\frac{7}{3}$:

$$
\begin{aligned}
W^{(\frac{7}{2})}_{\frac{7}{3}}(z) = \frac{1}{2\sqrt{3(2N-2)}}\Bigg[&\frac{1}{8} d^{abcd}\, f^{aef}\, f^{bgh}\, f^{dij}\\
&\times(\psi^e+i\chi^e)(\psi^f+i\chi^f)(\psi^g+i\chi^g)(\psi^h+i\chi^h)\\
&\times(\psi^c+i\chi^c)(\psi^i+i\chi^i)(\psi^j+i\chi^j)\\
&-\frac{1}{4} f^{abc}\, f^{def}\, f^{fgh}\\
&\times(\psi^a+i\chi^a)(\psi^b+i\chi^b)(\psi^c+i\chi^c)(\psi^d+i\chi^d)\\
&\times(\psi^e+i\chi^e)(\psi^g+i\chi^g)(\psi^h+i\chi^h)\Bigg](z).
\end{aligned}
\tag{4.17}
$$

In (4.17), the N dependence appears in the overall factor rather than the relative coefficients. One easily sees that the above two expressions preserve the $U(1)$ charge by counting the $U(1)$ charge at each factor. In other words, each factor has a $U(1)$ charge of $\frac{1}{3}$.

From the OPE [44]

$$
G^-(z)\, W^{(3)}_{\frac{4}{3}}(w) = \frac{1}{(z-w)}\, W^{(\frac{7}{2})}_{\frac{1}{3}}(w)+\cdots, \tag{4.18}
$$

one can obtain the other higher spin $\frac{7}{2}$ current of $U(1)$ charge $\frac{1}{3}$. It turns out, from the first-order pole of (4.18), that

$$
\begin{aligned}
W^{(\frac{7}{2})}_{\frac{1}{3}}(z) = &\frac{1}{2\sqrt{3(2N-2)}}\frac{1}{8} d^{abcd}\, f^{aef}\, f^{bgh}\\
&\times[+f^{eij}((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^f+i\chi^f)(\psi^g+i\chi^g)\\
&\times(\psi^h+i\chi^h)(\psi^c+i\chi^c)(\psi^d-i\chi^d)\\
&-f^{fij}(\psi^e+i\chi^e)((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^g+i\chi^g)\\
&\times(\psi^h+i\chi^h)(\psi^c+i\chi^c)(\psi^d-i\chi^d)\\
&+f^{gij}(\psi^e+i\chi^e)(\psi^f+i\chi^f)((\psi^i-i\chi^i)(\psi^j-i\chi^j))\\
&\times(\psi^h+i\chi^h)(\psi^c+i\chi^c)(\psi^d-i\chi^d)\\
&-f^{hij}(\psi^e+i\chi^e)(\psi^f+i\chi^f)(\psi^g+i\chi^g)\\
&\times((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^c+i\chi^c)(\psi^d-i\chi^d)\\
&+f^{cij}(\psi^e+i\chi^e)(\psi^f+i\chi^f)(\psi^g+i\chi^g)(\psi^h+i\chi^h)\\
&\times((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^d-i\chi^d)](z)\\
&-\frac{1}{2\sqrt{3(2N-2)}}\frac{1}{4} f^{abc}\, f^{def}\\
&\times[+f^{aij}((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^b+i\chi^b)\\
&\times(\psi^c+i\chi^c)(\psi^d+i\chi^d)(\psi^e+i\chi^e)(\psi^f-i\chi^f)\\
&-f^{bij}(\psi^a+i\chi^a)((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^c+i\chi^c)\\
&\times(\psi^d+i\chi^d)(\psi^e+i\chi^e)(\psi^f-i\chi^f)\\
&+f^{cij}(\psi^a+i\chi^a)(\psi^b+i\chi^b)((\psi^i-i\chi^i)(\psi^j-i\chi^j))\\
&\times(\psi^d+i\chi^d)(\psi^e+i\chi^e)(\psi^f-i\chi^f)\\
&-f^{dij}(\psi^a+i\chi^a)(\psi^b+i\chi^b)(\psi^c+i\chi^c)\\
&\times((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^e+i\chi^e)(\psi^f-i\chi^f)\\
&+f^{eij}(\psi^a+i\chi^a)(\psi^b+i\chi^b)(\psi^c+i\chi^c)(\psi^d+i\chi^d)\\
&\times((\psi^i-i\chi^i)(\psi^j-i\chi^j))(\psi^f-i\chi^f)](z).
\end{aligned}
\tag{4.19}
$$

From (4.15), the OPE between $G^-(z)$ and $(\psi^a-i\chi^a)(w)$ does not have any singular terms and the contribution from this OPE in (4.19) vanishes. Note that the big bracket stands for the normal ordered product [19,20]. Of course, one can move those factors to the right in order to simplify further. Each term has the $U(1)$ charge $\frac{1}{3}$ because there are four factors for the $U(1)$ charge $\frac{1}{3}$ and three factors for the $U(1)$ charge $-\frac{1}{3}$. Totally one has $\frac{1}{3}$ $U(1)$ charge.

From the relation [44]

$$G^-(z)\, W^{(\frac{7}{2})}_{\frac{7}{3}}(w) = \frac{1}{(z-w)^2}\,(-1)\frac{7}{3}W^{(3)}_{\frac{4}{3}}(w) + \frac{1}{(z-w)}\left[W^{(4)}_{\frac{4}{3}} - \frac{1}{2}\partial W^{(3)}_{\frac{4}{3}}\right](w) + \cdots, \tag{4.20}$$

one obtains, by calculating the left-hand side of (4.20) with (4.3) and (4.17) and reading off the first order pole,

$$\begin{aligned}
(W^{(4)}_{\frac{4}{3}} - \frac{1}{2}\partial W^{(3)}_{\frac{4}{3}})(z) &= \frac{1}{2\sqrt{3(2N-2)}}\frac{1}{8}d^{abcd} f^{aef} f^{bgh} f^{dij}\\
&\times[+f^{ekl}((\psi^k - i\chi^k)(\psi^l - i\chi^l))(\psi^f + i\chi^f)(\psi^g + i\chi^g)\\
&\times(\psi^h + i\chi^h)(\psi^c + i\chi^c)(\psi^i + i\chi^i)(\psi^j + i\chi^j)\\
&- f^{fkl}(\psi^e + i\chi^e)((\psi^k - i\chi^k)(\psi^l - i\chi^l))(\psi^g + i\chi^g)\\
&\times(\psi^h + i\chi^h)(\psi^c + i\chi^c)(\psi^i + i\chi^i)(\psi^j + i\chi^j)\\
&+ f^{gkl}(\psi^e + i\chi^e)(\psi^f + i\chi^f)((\psi^k - i\chi^k)(\psi^l - i\chi^l))\\
&\times(\psi^h + i\chi^h)(\psi^c + i\chi^c)(\psi^i + i\chi^i)(\psi^j + i\chi^j)\\
&- f^{hkl}(\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)((\psi^k - i\chi^k)\\
&\times(\psi^l - i\chi^l))(\psi^c + i\chi^c)(\psi^i + i\chi^i)(\psi^j + i\chi^j)\\
&+ f^{ckl}(\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&\times((\psi^k - i\chi^k)(\psi^l - i\chi^l))(\psi^i + i\chi^i)(\psi^j + i\chi^j)\\
&- f^{ikl}(\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&\times(\psi^c + i\chi^c)((\psi^k - i\chi^k)(\psi^l - i\chi^l))(\psi^j + i\chi^j)\\
&+ f^{jkl}(\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&\times(\psi^c + i\chi^c)(\psi^i + i\chi^i)\left((\psi^k - i\chi^k)(\psi^l - i\chi^l)\right)](z)\\
&- \frac{1}{2\sqrt{3(2N-2)}}\frac{1}{4}f^{abc} f^{def} f^{fgh}\\
&\times[+f^{aij}((\psi^i - i\chi^i)(\psi^j - i\chi^j))(\psi^b + i\chi^b)(\psi^c + i\chi^c)\\
&\times(\psi^d + i\chi^d)(\psi^e + i\chi^e)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&- f^{bij}(\psi^a + i\chi^a)((\psi^i - i\chi^i)(\psi^j - i\chi^j))(\psi^c + i\chi^c)\\
&\times(\psi^d + i\chi^d)(\psi^e + i\chi^e)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&+ f^{cij}(\psi^a + i\chi^a)(\psi^b + i\chi^b)((\psi^i - i\chi^i)(\psi^j - i\chi^j))\\
&\times(\psi^d + i\chi^d)(\psi^e + i\chi^e)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&- f^{dij}(\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c)((\psi^i - i\chi^i)\\
&\times(\psi^j - i\chi^j))(\psi^e + i\chi^e)(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&+ f^{eij}(\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c)(\psi^d + i\chi^d)\\
&\times((\psi^i - i\chi^i)(\psi^j - i\chi^j))(\psi^g + i\chi^g)(\psi^h + i\chi^h)\\
&- f^{gij}(\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c)(\psi^d + i\chi^d)\\
&\times(\psi^e + i\chi^e)((\psi^i - i\chi^i)(\psi^j - i\chi^j))(\psi^h + i\chi^h)\\
&+ f^{hij}(\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c)(\psi^d + i\chi^d)\\
&\times(\psi^e + i\chi^e)(\psi^g + i\chi^g)((\psi^i - i\chi^i)(\psi^j - i\chi^j))](z).
\end{aligned} \tag{4.21}$$

The properties in (4.15) are used. One can check that the $U(1)$ charge of each term is equal to $\frac{4}{3}$ where there are six positive ones and two negative ones. In order to obtain the primary current, one should consider $(W^{(4)}_{\frac{4}{3}} - \frac{1}{9}\partial W^{(3)}_{\frac{4}{3}})(z)$ [44] which can be obtained from (4.21) and (4.13).

Then the higher spin 3, $\frac{7}{2}$, $\frac{7}{2}$, and 4 currents are summarized by (4.13), (4.17), (4.19) and (4.21) with addition of the derivative of (4.13).

4.3 Other higher spin 3, $\frac{7}{2}$, $\frac{7}{2}$, 4 currents

In the description of (4.11), for the opposite $U(1)$ charge, there exists also another solution for the higher spin 3 current. One obtains the higher spin 3 current of $U(1)$ charge $-\frac{4}{3}$ as follows:

$$W^{(3)}_{-\frac{4}{3}}(w) = W^{(3)}_{\frac{4}{3}}(w)\Big|_{\chi^a \to -\chi^a}. \tag{4.22}$$

More explicitly, one can read off the explicit expression which can be obtained from (4.13) by replacing the second adjoint fermions with those together with minus sign. It is obvious that the $U(1)$ charge $-\frac{4}{3}$ of this higher spin current can be seen during this process: five factors of $U(1)$ charge $-\frac{5}{3}$ and one factor with $U(1)$ charge $\frac{1}{3}$.

Let us calculate the other higher spin currents. From the well-known OPE [44]

$$G^+(z)\, W^{(3)}_{-\frac{4}{3}}(w) = -\frac{1}{(z-w)}\, W^{(\frac{7}{2})}_{-\frac{1}{3}}(w) + \cdots, \tag{4.23}$$

one obtains the higher spin $\frac{7}{2}$ current together with (4.22) and (4.23) as follows:

$$W^{(\frac{7}{2})}_{-\frac{1}{3}}(w) = -W^{(\frac{7}{2})}_{\frac{1}{3}}(w)\Big|_{\chi^a \to -\chi^a}. \tag{4.24}$$

Note that under the change of $\chi^a \to -\chi^a$, the original $U(1)$ charge is changed into the negative one. More explicitly, one can take this operation in (4.18). Then the left-hand side of (4.18) leads to the left-hand side of (4.23) with the help of (4.22) and the right-hand side of (4.18) becomes $W^{(\frac{7}{2})}_{\frac{1}{3}}(w)\Big|_{\chi^a \to -\chi^a}$. By using the first-order pole from (4.23), then we are left with (4.24).

Similarly, the OPE [44] with (4.22),

$$G^-(z)\, W^{(3)}_{-\frac{4}{3}}(w) = \frac{1}{(z-w)}\, W^{(\frac{7}{2})}_{-\frac{7}{3}}(w) + \cdots, \tag{4.25}$$

provides the following result for the higher spin current, by considering Eq. (4.16), where the operation $\chi^a \to -\chi^a$ is taken, and Eq. (4.22),

$$W^{(\frac{7}{2})}_{-\frac{7}{3}}(w) = -W^{(\frac{7}{2})}_{\frac{7}{3}}(w)\Big|_{\chi^a \to -\chi^a}. \tag{4.26}$$

In other words, the left-hand side of (4.25) is equal to the left-hand side of (4.16) with the additional operation $\chi^a \rightarrow -\chi^a$. We also use Eq. (4.22). Then the right-hand side of (4.25) can be read off from this relation and we arrive at (4.26).

From the relation [44],

$$G^+(z)\, W^{(\frac{7}{2})}_{-\frac{7}{3}}(w) = \frac{1}{(z-w)^2}\,\frac{7}{3}\,W^{(3)}_{-\frac{4}{3}}(w) + \frac{1}{(z-w)}\left[W^{(4)}_{-\frac{4}{3}} + \frac{1}{2}\partial W^{(3)}_{-\frac{4}{3}}\right](w) + \cdots, \tag{4.27}$$

one sees that the first-order pole of (4.27) leads to

$$\left(W^{(4)}_{-\frac{4}{3}} + \frac{1}{2}\partial W^{(3)}_{-\frac{4}{3}}\right)(w) = -\left(W^{(4)}_{\frac{4}{3}} - \frac{1}{2}\partial W^{(3)}_{\frac{4}{3}}\right)(w)\Big|_{\chi^a \rightarrow -\chi^a}, \tag{4.28}$$

where Eq. (4.20) together with the operation $\chi^a \rightarrow -\chi^a$ is used. Moreover, Eq. (4.26) is used also. As described before, the field (4.28) is not a primary under the stress energy tensor. The primary current is given by $(W^{(4)}_{-\frac{4}{3}} + \frac{1}{9}\partial W^{(3)}_{-\frac{4}{3}})(w)$, which can be obtained from $(-W^{(4)}_{\frac{4}{3}} + \frac{1}{9}\partial W^{(3)}_{\frac{4}{3}})(w)$ by changing of $\chi^a(w) \rightarrow -\chi^a(w)$.

Therefore, the higher spin 3, $\frac{7}{2}$, $\frac{7}{2}$, 4 currents are summarized by (4.22), (4.24), (4.26), and (4.28). They are obtained from the higher spin currents appearing in previous subsection by simple change of the adjoint fermions $\chi^a(z)$ up to signs.

4.4 The OPE between the two lowest higher spin currents in $\mathcal{N}=2$ superspace

Because the coset with the critical levels has the $\mathcal{N}=2$ supersymmetry, one can describe the OPE between the two lowest higher spin multiplets in the $\mathcal{N}=2$ superspace. Let us consider the OPE between the two $\mathcal{N}=2$ lowest higher spin 3 multiplets where they have two opposite $U(1)$ charges. That is,

$$\mathbf{W}^{(3)}_{\frac{4}{3}}(Z_1)\, \mathbf{W}^{(3)}_{-\frac{4}{3}}(Z_2), \tag{4.29}$$

where each four component current, obtained in the previous subsection, is given by

$$\mathbf{W}^{(3)}_{\frac{4}{3}} \equiv \left(W^{(3)}_{\frac{4}{3}},\ W^{(\frac{7}{2})}_{\frac{7}{3}},\ W^{(\frac{7}{2})}_{\frac{1}{3}},\ W^{(4)}_{\frac{4}{3}}\right),$$
$$\mathbf{W}^{(3)}_{-\frac{4}{3}} \equiv \left(W^{(3)}_{-\frac{4}{3}},\ W^{(\frac{7}{2})}_{-\frac{1}{3}},\ W^{(\frac{7}{2})}_{-\frac{7}{3}},\ W^{(4)}_{-\frac{4}{3}}\right). \tag{4.30}$$

In principle, in order to obtain the explicit OPE in (4.29), one should calculate only the four OPEs between the four component currents living in $\mathbf{W}^{(3)}_{\frac{4}{3}}(Z_1)$ in (4.30) and the lowest component current in $\mathbf{W}^{(3)}_{-\frac{4}{3}}(Z_2)$ in (4.30), due to the $\mathcal{N}=2$ supersymmetry. See also the relevant work in [48–50] where the various $\mathcal{N}=2$ multiplets in different coset models are studied. From the four OPEs, one can realize that the right-hand sides of these OPEs should have $U(1)$ charges 0, 1, or -1 by adding the $U(1)$ charges. Recall that the four currents characterized by the $\mathcal{N}=2$ stress energy tensor $\mathbf{T} \equiv (J, G^+, G^-, T)$ of the $\mathcal{N}=2$ superconformal algebra have 0, 1, -1, and 0, respectively. It is natural to consider the right-hand side of (4.29) in terms of the $\mathcal{N}=2$ stress energy tensor $\mathbf{T}(Z_2)$ with its various descendant fields in a minimal way.

Inside of the package of [51], one can introduce the OPEs, $\mathbf{T}(Z_1)\,\mathbf{T}(Z_2)$, which is the standard OPE corresponding to the $\mathcal{N}=2$ superconformal algebra, $\mathbf{T}(Z_1)\,\mathbf{W}^{(3)}_{\frac{4}{3}}(Z_2)$, which is the $\mathcal{N}=2$ primary condition with $U(1)$ charge $\frac{4}{3}$ and $\mathbf{T}(Z_1)\,\mathbf{W}^{(3)}_{-\frac{4}{3}}(Z_2)$, which is the $\mathcal{N}=2$ primary condition with $U(1)$ charge $-\frac{4}{3}$. All the coefficients appearing in these OPEs are constants except the central charge c, which is a function of N in (4.5). Then one can write down the right-hand side of OPE (4.29) with arbitrary coefficients which depend on c or N. After using the Jacobi identities, we summarize the structure constants in Appendix L explicitly. See also the relevant work in [52].

One expects that there should be present other higher spin multiplets in the various OPEs. For example, $\mathbf{W}^{(3)}_{\frac{4}{3}}(Z_1)\,\mathbf{W}^{(3)}_{\frac{4}{3}}(Z_2)$ or $\mathbf{W}^{(3)}_{-\frac{4}{3}}(Z_1)\,\mathbf{W}^{(3)}_{-\frac{4}{3}}(Z_2)$ as in the unitary case [44]. It would be interesting to obtain these higher spin multiplets explicitly further.

5 Conclusions and outlook

In the coset model (1.1), we have constructed the higher spin 4 current for general levels. For $k_1 = 1$ with arbitrary N and k_2, the eigenvalue equations of the zero mode of the higher spin 4 current acting on the states are obtained. The corresponding three-point functions are also determined. The $\mathcal{N}=1$ higher spin multiplet characterized by $(\frac{7}{2}, 4)$ for $k_1 = 2N-2$ in terms of adjoint fermions and spin 1 current is obtained. The two $\mathcal{N}=2$ higher spin multiplets denoted by $(3, \frac{7}{2}, \frac{7}{2}, 4)$ for $k_1 = k_2 = 2N-2$ in terms of two adjoint fermions are determined. Some of the OPEs in the $\mathcal{N}=1$ or $\mathcal{N}=2$ coset models are given explicitly.

We consider the possible related open problems as follows:

- One can also try to obtain the higher spin currents in the following coset model:

$$\frac{\hat{SO}(2N+1)_{k_1} \oplus \hat{SO}(2N+1)_{k_2}}{\hat{SO}(2N+1)_{k_1+k_2}}. \tag{5.1}$$

It seems that the minimum value of N for the non-trivial existence of the d symbol (and corresponding higher spin 4 current) is given by $N = 2$. In the present paper, the minimum value of N is given by $N = 4$ and the number of independent fields in the higher spin currents is rather big, implying that it is rather non-trivial to extract the corresponding OPEs. In the coset model (5.1), for the $N = 2$ or $N = 3$ case, one expects that one can analyze the OPEs further and observe more structures in the right-hand sides of the OPEs.

- Further algebraic structures.

In order to observe the algebraic structures living in the bosonic, $\mathcal{N} = 1$ or $\mathcal{N} = 2$ higher spin multiplets for generic N (and generic k_2), one should calculate the various OPEs between them manually. In practice, this is rather involved because, for example, the higher spin 4 current in the bosonic coset model consists of 20 terms and the number of OPEs is greater than 200. In [28], one can try to obtain the various OPEs for the fixed low N values (for example, $N = 4, 5, 6, 7, \ldots$) and expect the N dependence of structure constants appearing in the right-hand side of the OPEs.

- $\mathcal{N} = 2$ enhancement of [25].

One considers the critical level condition in [25,53]. It would be interesting to observe any $\mathcal{N} = 2$ enhancement or not. One can easily see the breaking of the adjoint representation in $SO(2N + 1)$ into the adjoint representation of $SO(2N)$ plus the vector representation of $SO(2N)$. The first step is to construct the $\mathcal{N} = 2$ superconformal algebra realization.

- The additional numerator factors.

For example, one considers the following coset model where the extra numerator factor exists in the coset:

$$\frac{\hat{SO}(2N)_{2N-2} \oplus \hat{SO}(2N)_{2N-2} \oplus \hat{SO}(2N)_{2N-2}}{\hat{SO}(2N)_{6N-6}}. \tag{5.2}$$

It is an open problem to see whether one constructs the $\mathcal{N} = 3$ superconformal algebra [54] from the three kinds of adjoint fermions or not. It is non-trivial to obtain the three spin $\frac{3}{2}$ currents satisfying the standard OPEs between them. Then one can try to obtain the higher spin currents living in the above coset model (5.2). Furthermore, one can describe another coset model where the additional numerical factor occurs. It is an open problem to construct the linear (or nonlinear) $\mathcal{N} = 4$ superconformal algebra from the four kinds of adjoint fermions.

- Further identities between f and d tensors of $SO(2N)$.

One can analyze the various identities involving f and d tensors by following the description of [17,18]. They will be useful in order to calculate the OPEs between the higher spin currents in the context of Sects. 3 and 4.

- Zero mode eigenvalue equations in other representations

There exists an adjoint representation of $SO(2N)$. It is an open problem to describe the eigenvalue equations for the zero mode of the higher spin 4 current acting on the states associated with the adjoint representation. For the $SO(8)$ generators in the adjoint representation, one has 28×28 matrices whose elements are given by the structure constant.

- Marginal operator.

One of the motivations in Sect. 4 is based on the presence of the perturbing marginal operator [55], which breaks the higher spin symmetry but preserving the $\mathcal{N} = 2$ supersymmetry. It would be interesting to obtain this operator and calculate the mass terms with the explicit eigenvalues along the lines of [56–59]. In the large c limit, the right-hand side of the OPE has the simple linear terms.

- $\mathcal{N} = 2$ superspace description for the adjoint fermions.

We obtained the two $\mathcal{N} = 2$ higher spin multiplets. It is an open problem to see whether one can write down the two adjoint fermions in $\mathcal{N} = 2$ superspace. This will allow us to write down the $\mathcal{N} = 2$ higher spin multiplets in $\mathcal{N} = 2$ superspace.

- Asymptotic quantum symmetry algebra.

We have obtained the eigenvalue equations and three-point functions at finite N and k_2 in Sect. 2. Along the lines of [9], it is an open problem to study the asymptotic quantum symmetry algebra of the higher spin theory on the AdS_3 space. See also [24] where a brief sketch for the large N 't Hooft limit is given.

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