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Worldline as a spin chain

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Abstract The general theoretical ground for models based on compact angle coordinates is presented. It is observed that the proper dependence on compact coordinates has to be through the group elements and is achieved most naturally in a discrete-time formulation of the theory. By the construction, the discrete worldline inlaid by compact coordinates resembles the spin chains of magnetic systems. As examples, the models based on the groups U(1), $\mathbb{Z}_N$ and SU(2) are explicitly constructed and their exact energy spectra are obtained. As the consequence of the minima in the spectra, the models exhibit a phase transition of first order. We attempt to fit the dynamics by the U(1) group to the proposed role for monopoles in the dual Meissner effect of the confinement mechanism.

1 Introduction

It is well known that treating the gauge fields as compact angle variables would reveal some non-trivial aspects of gauge field theories [1–4]. Among studies based on the compactness of gauge fields, the lattice formulation of gauge theories has provided an explanation for the confinement mechanism as well as a basis for numerical studies at the strong coupling regime [4–6].

By now there are specific instances of the affinities between coordinates and gauge fields. The oldest example is the one of the special theory of relativity, by which it is understood that both space-time coordinates and gauge fields transform as 4-vectors under the Lorentz transformations. As another example, based on the duality proposed in [7], one can formulate the coordinate/field correspondence in both Abelian and non-Abelian gauge theories [8]. The other instance of the relation between coordinates and gauge fields is provided by the T-duality of string theory. Accordingly, the transverse coordinates of Dp-branes in the dual theory are represented by the gauge fields of open string states, leading to the correspondence [9,10]

$$A_i \longleftrightarrow X_i/l_s^2, \tag{1}$$

in which l_s is the string theory length. At weak coupling the dynamics of X_i's is captured by the theory resulting from dimensional reduction of the ordinary U(1) gauge theory [9,10]. In particular the reduction on all spatial components of the gauge field yields the D0-brane dynamics, namely [9]

$$S_{\rm D0} = \int {\rm d}t\ \frac{m_0}{2}\,\dot{x}_i^2, \tag{2}$$

in which $m_0 \propto 1/g^2$ (g: gauge coupling) [9]. The transverse coordinates of N, the number of Dp-branes, are represented by N dimensional hermitian matrices [11].

In [12] the dimensional reduction of pure U(1) lattice gauge theory is considered to model the dynamics of 0-branes at strong coupling regime. The model by [12] might be considered as a result of the combination of two themes mentioned earlier, (1) treating gauge fields as compact angle variables [1–4], (2) assuming similar characters between coordinates and gauge fields [7–10]. The explicit form of the action after the dimensional reduction of U(1) lattice gauge theory is

$$S_0 = \frac{1}{g^2}\sum_n \left(\cos\frac{x_{n+1}-x_n}{R} - 1\right), \tag{3}$$

in which the coordinates appear as the compact angle variables depending on discrete imaginary time label n. By this form the worldline theory takes the form of the 1D plane-rotator model of spin lattice systems [13]. Based on the prescription for the original lattice gauge theory [4], using the transfer-matrix method the quantization of the model is formulated. The exact energy spectrum as the function of gauge coupling is obtained, with a minimum at critical coupling $g_c = 1.125$ in the lowest energy. As a direct consequence of the minimum, the model exhibits a first-order phase tran-

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sition between coexistent phases with small and large couplings [12]. Based on the discontinuous nature of the first-order phase transition, for $g < g_c$ and $T \approx 0$ the effective zero mean-square velocity $\langle v^2 \rangle$ is zero.

The purpose of the present work is to provide a general theoretical ground for the models based on compact angle coordinates. It is clarified that the dependence on the group elements rather than the algebra ones would lead to invariance of the action under the total shifts in the compact domain, like what is happening in (3). It is observed that the proper dependence is achieved most naturally in a discrete-time formulation of the theory. The worldline action by the formulation resembles the 1d spin chain Hamiltonian of magnetic systems, with coordinates appearing as the spin degrees of freedom. The transfer-matrix method is used to define the quantum theory [14]. As now the 1d spin chain is used for a "particle-like" dynamics interpretation, there should be a square root of the mass in the definition of the transfer-matrix elements [14]. As a direct consequence of the presence of the square root, opposite to the 1d chains of magnetic systems, here the energy spectra develop minima. In a path-integral representation of the formulation, it is emphasized that the square root prefactor in the definition of the transfer-matrix elements, in contrast to the case with infinite-extent coordinates, cannot be absorbed by a change of the integration variables.

It is well known that the 1d spin systems with short range interactions do not exhibit the second-order phase transition expected for these systems. However, the present model, as a consequence of the minima in the spectrum, exhibits a first-order phase transition. As mentioned earlier, although the worldline according to the model looks like a 1d spin chain, due to the square root prefactor in the matrix elements the spectrum is different. The phase transition nature of the model will be discussed based on the behavior of the Gibbs free energy. In particular, the plot of G versus the thermodynamical variable M, as the effective mean-square velocity, develops a cusp below a critical temperature T_c. At the cusp the derivative $\frac{\partial G}{\partial M}$ is discontinuous, as expected in a first-order phase transition.

As examples for the formulation, the models based on the U(1), $\mathbb{Z}_N$ and SU(2) groups are explicitly constructed. In all examples the exact energy eigenvalues are obtained, leading to the first-order phase transition.

The organization of the rest of the paper is as follows. In Sect. 2 the basic assumptions and ingredients for the formulation based on compact coordinates are presented. In Sects. 3, 4 and 5 three examples based on the groups U(1), $\mathbb{Z}_N$ and SU(2) are presented explicitly. For all the three groups the energy eigenvalues are obtained exactly, together with the discussion of the nature of the phase transitions in them. Section 6 is devoted to our conclusion and a discussion.

2 Basics and formulation

The transition amplitude between positions x_0 and x_N at times t_0 and t_N is represented by the path-integral [15]

$$\langle x_N, t_N | x_0, t_0 \rangle = \lim_{N\to\infty} \int_{-\infty}^{\infty} \prod_{n=1}^{N-1} \sqrt{\frac{m}{2\pi\, \mathrm{i}\, \hbar\, \epsilon}}\, \mathrm{d}x_n\, \mathrm{e}^{\,\mathrm{i}\, S[t_0,t_N]/\hbar}, \tag{4}$$

in which $\epsilon = (t_N - t_0)/N$, tending to zero in the limit $N \to \infty$. It is noticed that in the above all the intermediate positions x_n's have infinite extents, $-\infty < x_n < \infty$. From the considerations mentioned in Sect. 1, here the main concern is the finite-extent coordinates. The finite-extent coordinates are quite well known in physics. The most familiar ones are the angle variables of polar coordinates in 2D and 3D problems. The other example occurs when the system is defined inside a finite volume, like a box or a sphere. As a definite example, let us consider a free particle on a circle of radius R. Defining the angle variable $\phi = x/R$ with $-\pi \le \phi \le \pi$, we have

$$L = \frac{1}{2} m R^2 \dot{\phi}^2 \quad \to \quad H = \frac{p_\phi^2}{2mR^2}, \tag{5}$$

leading to the eigenfunctions and eigenvalues

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\, n\phi}, \quad E_n = \frac{n^2\hbar^2}{2mR^2}. \tag{6}$$

Then the transition amplitude between two different positions is known to be

$$\begin{aligned} \langle \phi_N, t_N | \phi_0, t_0 \rangle &= \sum_{n=-\infty}^{\infty} \psi_n(\phi_N) \psi_n^*(\phi_0)\, \mathrm{e}^{-\mathrm{i}\, E_n (t_N - t_0)/\hbar} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\, n(\phi_N - \phi_0)} \mathrm{e}^{-\mathrm{i}\, n^2 \hbar (t_N - t_0)/2mR^2}. \end{aligned} \tag{7}$$

The above can be expressed in terms of the third Jacobi theta function,

$$\vartheta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\pi\,\tau\, n^2 + 2\,\mathrm{i}\, n\, z}, \tag{8}$$

by which we have [16]

$$\langle \phi_N, t_N | \phi_0, t_0 \rangle = \frac{1}{2\pi} \vartheta_3 \left(\frac{\phi_N - \phi_0}{2}, \frac{-\hbar (t_N - t_0)}{2\pi m R^2} \right). \tag{9}$$

Using the modular property of ϑ_3,

$$\vartheta_3(z,\tau) = (-\mathrm{i}\,\tau)^{-1/2} \mathrm{e}^{-\mathrm{i}\, z^2/\pi\tau}\, \vartheta_3 \left(\frac{z}{\tau}, -\frac{1}{\tau} \right), \tag{10}$$

the transition amplitude is recast as [16]

$$\langle\phi_N, t_N|\phi_0, t_0\rangle = \sum_{n=-\infty}^{\infty}\sqrt{\frac{mR^2}{2\pi\,\mathrm{i}\,\hbar(t_N-t_0)}} \times \exp\left(\frac{\mathrm{i}}{\hbar}\frac{mR^2(\phi_N-\phi_0-2\pi n)^2}{2(t_N-t_0)}\right), \tag{11}$$

in which the summand is easily recognized as the transition amplitude of a free particle experiencing the position difference $x_N - x_0 = R(\phi_N - \phi_0 - 2\pi n)$ during time $t_N - t_0$ [15]. There is a nice interpretation for the sum as well. As the particle moves on the circle from ϕ_0 to ϕ_N, it matters how many times it rounds the circle. The sum on n, the so-called winding number, is responsible for taking into account the contributions from different rounds to the amplitude. So, although the coordinate ϕ has finite extent, practically the particle may travel long distances $\Delta x = R|\phi_N - \phi_0 - 2\pi n|$ for $n = 0, \pm1, \pm2, \ldots$. In fact (11) may attain a form similar to (4) with only an extra summation. First let us present a time-sliced form of (11) [16]:

$$\langle\phi_N, t_N|\phi_0, t_0\rangle = \lim_{N\to\infty}\int_{-\pi}^{\pi}\prod_{j=1}^{N-1}\sqrt{\frac{mR^2}{2\pi\,\mathrm{i}\,\hbar\,\epsilon}}\mathrm{d}\phi_j \times \prod_{l=0}^{N-1}\sum_{n_l=-\infty}^{\infty}\exp\left[\frac{\mathrm{i}}{\hbar}\frac{mR^2(\phi_{l+1}-\phi_l-2\pi n_l)^2}{2\,\epsilon}\right]. \tag{12}$$

It is noticed that at lth time-slice the winding number n_l is introduced [16]. Now, by the change in the integral variables [16]

$$\sum_{n_l=-\infty}^{\infty}\int_{(2n_l-1)\pi}^{(2n_l+1)\pi}\mathrm{d}\phi_l \to \int_{-\infty}^{\infty}\mathrm{d}\Phi_l \tag{13}$$

Eq. (12) is recast to [16]

$$\langle\phi_N, t_N|\phi_0, t_0\rangle = \lim_{N\to\infty}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{N-1}\sqrt{\frac{mR^2}{2\pi\,\mathrm{i}\,\hbar\,\epsilon}}\mathrm{d}\Phi_j \times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{i=0}^{N-1}\frac{mR^2(\Phi_{i+1}-\Phi_i)^2}{2\,\epsilon}\right], \tag{14}$$

in which $\Phi_N = \phi_N$ and $\Phi_0 = \phi_0 + 2\pi n$. In the above all the intermediate angles Φ_j's are integrated over the whole real-line $\mathbb{R}$. In fact by the change of variable (13), integrating the intermediate angle ϕ_l over $[-\pi, \pi]$ together with a sum over infinite possible rounds n_l is replaced by integration of Φ_l over $(-\infty, \infty)$.

The above example shows that it is insufficient to merely use finite-extent coordinates to be granted the non-trivial aspects. The key point as regards the above treatment of a motion on a circle is that the time-sliced form of the action appearing in the path-integral (12) is not invariant under the multi-round shifts,

$$\phi_l \to \phi_l + 2\pi\,k_l, \quad k_l = 0, \pm1, \pm2, \ldots \tag{15}$$

with $k_l \neq k_j$ for $l \neq j$. It is noticed that the above shift is directly related to the different number of possible rounds on the circle represented earlier by n_l in (12). Recalling that the group space of U(1) is a circle, we can express the main idea in a group theoretical language. Defining the U(1) group element by $U_\phi = \exp(\mathrm{i}\,\phi)$, the action by the Lagrangian (5) can be expressed as [16]

$$S = \frac{1}{2}mR^2\int \mathrm{d}t\,(U_\phi^\dagger\dot{U}_\phi)(U_\phi^\dagger\dot{U}_\phi)^\dagger = \frac{1}{2}mR^2\int \mathrm{d}t\,\dot{\phi}^2. \tag{16}$$

In the above, although the starting point is taken to be the group element U_ϕ, the Lagrangian depends in fact on the algebra element ϕ. This observation for the group U(1) is general and holds for other groups as well [16]. Obviously the situation changes if the action would be invariant under the shift (15), namely due to the dependence on the group elements $\exp(\mathrm{i}\,\phi_l)$'s instead of the algebra elements ϕ_l's. Interestingly, once the time parameter is assumed to be discrete the desired dependence is obtained. By taking the time as $t_n = n\,a$ for some finite value a and integer n, the worldline looks like a chain or 1D lattice with spacing parameter a. On the nth site of this chain we have the angle ϕ_n. So at time step n we have $U_n = \exp(\mathrm{i}\,\phi_n)$, by which the discrete-time version of the action (16) is

$$\begin{aligned} S &= \frac{1}{2}mR^2\,a^{-1}\sum_n (U_n^\dagger(U_{n+1}-U_n))(U_n^\dagger(U_{n+1}-U_n))^\dagger \\ &= \frac{1}{2}mR^2\,a^{-1}\sum_n (U_n^\dagger U_{n+1}-1)(U_{n+1}^\dagger U_n-1) \\ &= -\frac{1}{2}mR^2\,a^{-1}\sum_n (U_n^\dagger U_{n+1}+U_{n+1}^\dagger U_n-2), \end{aligned} \tag{17}$$

which obviously keeps invariance under the shift (15). The phenomenon observed here is partially in the reverse direction of what has happened in the lattice formulation of gauge theories. Namely, once one tries to introduce gauge symmetry to the theory on the lattice the algebra elements A_μ's are to be replaced by the group elements $\exp(\mathrm{i}\,a\,A_\mu)$'s in the action [4]. Here as we were going to keep the invariance under the shift (15) the natural solution appears to be defining the action on the discrete-time worldline. In the next sections we will use this as a basis for model building.

Before ending this section it is helpful to discuss the prominent role of the imaginary time in the quantization of models with compact domain support in the sense described above. In particular let us consider the matrix element by (4) [15],

$$\langle x|\widehat{U}|x'\rangle = \langle x|\exp(-\mathrm{i}\,\Delta t\,\widehat{H}/\hbar)\,|x'\rangle = \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar\,\Delta t}}\,\mathrm{e}^{\mathrm{i}\,S[t,t+\Delta t]/\hbar}, \tag{18}$$

in which $\widehat{U}$ is the unitary time evolution operator, $\widehat{H}$ is the Hamiltonian, and $S[t, t+\Delta t]$ is the action between times t and $t+\Delta t$ [15]. The basic observation is that the above representation in terms of the action is not possible when the dynamical variables are to take values inside a compact domain in the sense mentioned earlier. The reason can easily be understood for a system with one dynamical variable λ (field or coordinate). The generalization to systems with more variables is then straightforward. The identity $\widehat{U}\widehat{U}^\dagger = \mathbf{1}$ for the unitary evolution operator $\widehat{U}$, defining $U(\lambda,\lambda'') = \langle\lambda|\widehat{U}|\lambda''\rangle$, in the λ-basis takes the form

$$\int_\Lambda \mathrm{d}\lambda''\,U(\lambda,\lambda'')\,U^\star(\lambda',\lambda'') = \delta(\lambda-\lambda'), \tag{19}$$

in which Λ is the compact domain in which λ takes values. Now, by the representation like (19), as the integrand consists of only the ordinary regular functions and not the distribution ones, there is no way that the integral over a compact domain can develop a δ-function. Lacking the representation (18) for a unitary time evolution operator, the alternative is to assume that time is imaginary ($t \to -\mathrm{i}\,t$). In the models with discrete time ($\Delta t = a$), by this alternative option the one-step unitary operator $\widehat{U}_1 = \exp(-\mathrm{i}\,a\widehat{H}/\hbar)$ is replaced by the so-called transfer-matrix operator $\widehat{V}$ whose matrix element between two adjacent times n and $n+1$ is given by [14]

$$\langle\lambda_{n+1}|\widehat{V}|\lambda_n\rangle \propto \sqrt{m}\,\exp(S_E(n,n+1)/\hbar), \tag{20}$$

in which $S_E(n,n+1)$ is the Euclidean action. Then by common eigenstates for $\widehat{H}$ and $\widehat{V}$, the eigenvalues of $\widehat{H}$ are defined by [4,14]

$$E_s = -\hbar\,a^{-1}\,\ln v_s \tag{21}$$

where v_s is the corresponding eigenvalue of $\widehat{V}$. Provided that $\widehat{V}$ does not have negative eigenvalues, the above would give a consistent description of the quantum theory based on an action with discrete imaginary time [4,14]. This approach is exactly the one that is chosen in a lattice formulation of gauge theories [4], turning space-time to a Euclidean one, and it will be used in the present work as well.

In the formulation presented above the discrete worldline is inlaid by numbers as spin variables. An interesting extension is to consider the case in which the site n on the worldline is equipped with the spin operator $\hat{S}_n$, promoting the worldline to a quantum spin chain. As an example, let us consider the Heisenberg XYZ model, defined by the Hamiltonian operator for two arbitrary adjacent sites,

$$\widehat{H}_{\mathrm{XYZ}} = -\frac{1}{2}\left(\kappa_x\,\hat{S}^x_n\hat{S}^x_{n+1} + \kappa_y\,\hat{S}^y_n\hat{S}^y_{n+1} + \kappa_z\,\hat{S}^z_n\hat{S}^z_{n+1}\right). \tag{22}$$

The spin operators in the above are not restricted to a specific representation, and they generally belong to the $2s+1$ dimensional representation for $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The transfer matrix $\widehat{V}$ of the model, in analogy with (20), is then defined by

$$\widehat{V} \propto \sqrt{\kappa_x\kappa_y\kappa_z}\,\exp(-a\,\hat{H}_{\mathrm{XYZ}}). \tag{23}$$

The matrix $\widehat{V}$ in the above is hermitian by construction, as it should. The energy spectrum by the model can be obtained by the prescription (21). The detailed nature of the spectrum and the phase structure by the model based on the quantum spin chain is not discussed here, and it is left for future studies.

In summary, by the considerations mentioned above, the following are the basis for model building:

1. Time is assumed to be discrete and imaginary, taking values $t_n = n\,a$ for integer n.
2. The action with discrete time depends on the group elements to enhance the features by the compact nature of group.

3 U(1) group

For the ith direction with coordinate $-\pi R^i \le x^i \le \pi R^i$, the U(1) group element at nth time step is taken as $U^i_n = \exp(\mathrm{i}\,x^i_n/R)$. Here for simplicity we take all radii R^i's equal to R. Following (17) the Euclidean action takes the form

$$S_E = \frac{\kappa}{2}\sum_{n,i}(U^{i\,\dagger}_n\,U^i_{n+1} + U^{i\,\dagger}_{n+1}\,U^i_n - 2) = \kappa\sum_{n,i}\left(\cos\frac{x^i_{n+1}-x^i_n}{R} - 1\right). \tag{24}$$

The dimensionless constant κ is the defining parameter of the model (we have set $\hbar = c = 1$). The above, as discussed in previous section, is invariant under the shift

$$x^i_n \to x^i_n + 2\pi\,k^i_n R, \tag{25}$$

with the k^i_n being integer numbers. In the first place let us check the limits:

$$x/R \ll 1$$
$$x_{n+1} - x_n \to a\,\dot{x}$$
$$\sum_n \to a^{-1}\int \mathrm{d}t, \tag{26}$$

leading to

$$S_E \simeq -\frac{a\kappa}{2R^2}\int \mathrm{d}t\;\dot{x}_i^2, \tag{27}$$

which describes the dynamics of an ordinary free particle with mass $m_0 = a\kappa/R^2$ in the imaginary time formalism. As mentioned in the Introduction, the action (24) is used in

[12] to model the dynamics of 0-branes in the strong coupling limit. The action (24) is the result of dimensional reduction of U(1) lattice gauge theory along spatial directions, by setting

$$\kappa = 1/g^2$$
$$a\,A^i \to x^i/R. \tag{28}$$

Equation (24) for the action is also known as the 1D plane-rotator model of magnetic systems [13], although here it is interpreted as a discrete worldline defined by the angle variable coordinates x^i. In this section we review the construction by [12]. It is useful to define the new variables

$$y^i = x^i/R, \tag{29}$$

taking values in $[-\pi, \pi]$, by which the action (24) takes the form

$$S_0 = \kappa \sum_{n,i} (\cos(y^i_{n+1} - y^i_n) - 1). \tag{30}$$

As the action is fully separable for each direction, it is sufficient to consider only one copy, dropping the index i hereafter. As mentioned in Sect. 2, the action with discrete imaginary time can be used to define the quantum theory based on the transfer-matrix $\widehat{V}$, defined by its matrix elements

$$\langle y_{n+1}|\widehat{V}|y_n\rangle = \sqrt{\frac{\kappa}{2\pi}}\,\exp[\kappa(\cos(y_{n+1} - y_n) - 1)], \tag{31}$$

in which, recalling $m_0 \propto \kappa$, the normalization prefactor has to be inserted to match the propagator (18) (see also (20)) [14]

$$\langle x_2, t_2|x_1, t_1\rangle \propto \sqrt{\frac{m_0}{2\pi}}\,\exp\left(\frac{-m_0(x_2 - x_1)^2}{2\,(t_2 - t_1)}\right). \tag{32}$$

Using the identity for the modified Bessel function of the first kind,

$$\exp[\kappa\cos(y' - y)] = \sum_{s=-\infty}^{\infty} I_s(\kappa)\,\mathrm{e}^{\mathrm{i}\,s\,(y'-y)}, \tag{33}$$

we have for (31)

$$\langle y_{n+1}|\widehat{V}|y_n\rangle = \sum_{s=-\infty}^{\infty} \sqrt{\frac{\kappa}{2\pi}}\mathrm{e}^{-\kappa} I_s(\kappa)\,\mathrm{e}^{\mathrm{i}\,s\,(y_{n+1}-y_n)}, \tag{34}$$

by which one reads the normalized plane wave,

$$\psi_s(x) = \frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}\,s\,y), \quad -\pi \le y \le \pi, \tag{35}$$

as eigenfunction with the eigenvalue

$$v_s(\kappa) = \sqrt{2\pi\kappa}\,\mathrm{e}^{-\kappa} I_s(\kappa). \tag{36}$$

By the well-known properties of the I_s-functions we have $v_s = \sqrt{2\pi\kappa}\,\mathrm{e}^{-\kappa} I_s(\kappa) \ge 0$. This guarantees that the transfer-matrix method defined by (20) and (21) would lead to a consistent quantum theory. Also by $I_s(z) = I_{-s}(z)$ the spectrum is doubly degenerate for $s \ne 0$. The energy eigenvalues are found by (21) and (36):

$$E_s(\kappa) = -\frac{1}{a}\ln\left[\sqrt{2\pi\kappa}\,\mathrm{e}^{-\kappa} I_s(\kappa)\right]. \tag{37}$$

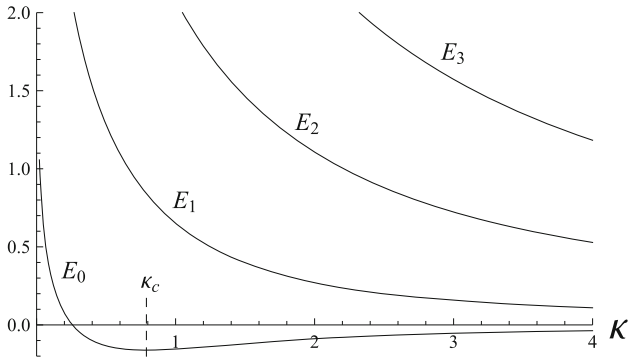

Fig. 1 The few lowest energies by (37) versus κ (E unit: a^{-1})

The behavior of the above in the limit $\kappa \to \infty$ can be checked by the saddle point approximation for Bessel functions,

$$I_s(\kappa) = \lim_{\kappa\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}y\ \exp(\kappa\cos y + \mathrm{i}\,s\,y)$$
$$\simeq \frac{\mathrm{e}^{\kappa}}{\sqrt{2\pi\kappa}}\exp\left(-\frac{s^2}{2\kappa}\right), \tag{38}$$

by which for (37) we obtain

$$E_s \simeq \frac{s^2}{2a\kappa}, \tag{39}$$

matching the energy $E = p^2/(2m_0)$ of a free particle with momentum $p = s/R$ along the compact direction, and mass $m_0 = \kappa\,a/R^2$ by (27). So in the limit $\kappa \to \infty$ the spectrum approaches that of an ordinary particle. For the intermediate κ the spectrum is discrete. In the limit $\kappa \to 0$, using

$$I_s(z) \simeq \frac{1}{s!}\left(\frac{z}{2}\right)^s, \quad z \ll 1, \tag{40}$$

we have

$$E_s = -\left(s + \frac{1}{2}\right)\frac{\ln\kappa}{a} + O(s\ln s) + O(\kappa), \tag{41}$$

in which the second term is independent of κ. Also at $\kappa \to 0$

$$E_{s+1} - E_s \simeq -\frac{\ln\kappa}{a} \gg \frac{1}{a}. \tag{42}$$

The interesting observation by the spectrum (37) is about the energy of the ground state, which has a minimum at $\kappa_c = 0.790$; see Fig. 1. As expected the existence of a minimum leads to a first-order phase transition. The one-particle partition function may be evaluated by the definition

$$Z_1(\beta, \kappa) := \sum_{s=-\infty}^{\infty} \mathrm{e}^{-\beta\,E_s(\kappa)} \tag{43}$$

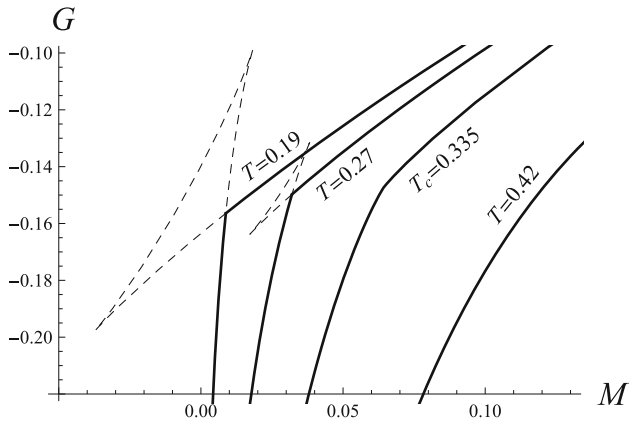

Fig. 2 The G–M plots at four temperatures. The *dashed pieces* are not followed by the system due to the minimization of G

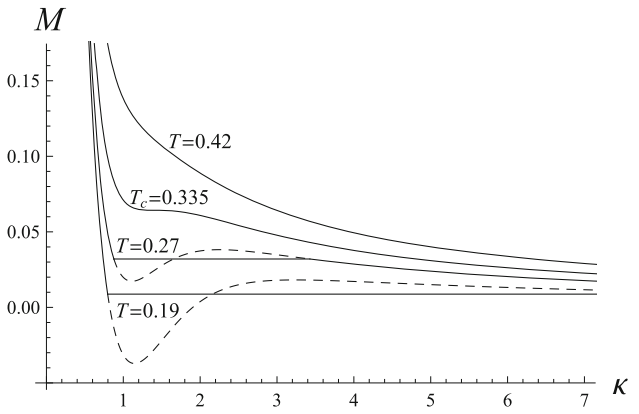

Fig. 3 The isothermal M–κ plots. The *straight lines* are due to the Maxwell construction, replacing the *dashed parts*

or by means of the transfer-matrix operator (β in a units) [14]

$$Z_1(\beta,\kappa) = \mathrm{Tr}\,\widehat{V}^{\beta} = \int_{-\pi}^{\pi} \prod_{m=0}^{\beta-1} \sqrt{\frac{\kappa}{2\pi}}\,\mathrm{d}y_m \times \exp\left[\kappa \sum_{n=0}^{\beta-1} (\cos(y_{n+1}-y_n)-1)\right] \tag{44}$$

supplemented by the periodic condition $y_0 = y_\beta$. In the present case the equivalence of (43) and (44) is checked by numerical evaluations. The basic observation by the compact angle variable in the above is, in contrast to the situation with infinite-extent coordinates, the normalization factor cannot be absorbed by a change of integration variable. As the minimum of E_0 is in the variable κ, we need the thermodynamical conjugate variable M, defined by ($T = \beta^{-1}$)

$$M(\beta,\kappa) := T\,\frac{\partial \ln Z_1(\beta,\kappa)}{\partial\,\kappa}, \tag{45}$$

which is also interpreted as the equation of state of the system. The Gibbs free energy can represent the exact nature of the phase transition,

$$G_1 = A_1 + \kappa\, M, \tag{46}$$

in which $A_1 = -T \ln Z_1$ is the free energy per particle. The isothermal G–M plots are presented in Fig. 2. Evidently below the critical temperature $T_c = 0.335\ a^{-1}$ the plots develop cusps, at which by the minimization of G at equilibrium, the system follows the path with lower G (solid lines in Fig. 2). As the consequence, for $T < T_c$ there is a jump in first derivative of $\partial G/\partial M$, indicating that the phase transition is a first-order one. It is evident by now that the above phase structure is quite similar to the gas/liquid transition, for which G–P plots show exactly the same behavior. In a similar way the equation of state (45) should be modified by the so-called Maxwell construction for P–V diagram, by which during isothermal condensation the pressure (here M) is fixed. The results of the Maxwell construction for the present model are plotted as isothermal M–κ curves in Fig. 3. The flat part at T_c corresponds to the values

$$T_c = 0.335: \quad \kappa^* = 1.403, \quad M^* = 0.064. \tag{47}$$

For isothermal curves below T_c, the straight horizontal parts describe the coexistent phases of lower and higher κ's during the phase transition. The interesting fact about the equation of state modified by Maxwell construction is that M always remains non-negative, that is, $M \geq 0$. This is specially important by expectations from the variable M in the limit $\kappa \gg 1$, at which we expect the ordinary behavior for particles. In this limit, going back to (27) and (39), we have

$$M \simeq \frac{1}{2}\langle \dot{y}^2 \rangle \propto \frac{T}{m_0} \tag{48}$$

where the proportionality is by the properties of free ordinary particles. In fact the asymptotic tails in Fig. 3 for large $m_0 \propto \kappa$ are explained by (48). There are also asymptotes at $\kappa \to 0$, although with different slopes. In fact the main difference between the case with the present model and that of ordinary particles is in the existence of a phase transition. In particular, by the present model and below the critical temperature T_c, the two asymptotes by large and small masses (large and small κ) are connected with a first-order phase transition.

One may define the order parameter for the present model as well. For ordinary magnetic systems with a second-order phase transition the order parameter is the magnetization as the derivative of G [17]. The non-zero magnetization is interpreted as the magnetic ordering phase. However, due to the different nature of the first-order phase transitions, the derivative of G is discontinuous at cusps in Fig. 2. This situation is again quite similar to the case with gas/liquid system, in which the volume difference of the coexisting phases, as the jump in the derivative $\partial G/\partial P$, is taken as the order parameter [17]. Similarly, in the present case the jump in $\partial G/\partial M$ defines the order parameter, being simply the difference of

the κ of the coexisting phases (κ difference at the ends of the fixed-M line in Fig. 3). Like the gas/liquid system, the order parameter tends to zero at the critical point (47), and larger values of the order parameter (larger κ-difference) at lower temperatures corresponds to lower fixed-M line in Fig. 3. In terms of the magnetic ordering $y_n \simeq y_{n+1}$, this is the expected behavior with $M \propto \langle \dot{y}^2 \rangle \simeq 0$ by (48).

4 $\mathbb{Z}_N$ group

The coordinates with a compact domain may form a discrete group such as $\mathbb{Z}_N$. In this case the worldline looks like a spin chain with discrete spin degrees sitting on its sites. In general the worldline resembles the spin chain of the Potts model ($\mathbb{Z}_2$ as of the Ising model). The members of the group $\mathbb{Z}_N$ are represented by

$$\{1, \varrho, \varrho^2, \ldots, \varrho^{N-1}\}, \tag{49}$$

in which

$$\varrho = \exp(\mathrm{i}\, 2\pi/N), \quad \varrho^N = 1. \tag{50}$$

At time step n the position may be represented by r_n as

$$U_n = \varrho^{r_n} = \exp(\mathrm{i}\, 2\pi\, r_n/N), \quad r_n = 0, 1, 2, \ldots, N-1, \tag{51}$$

by which the action takes the form

$$\begin{aligned} S_E &= \frac{\kappa}{2} \sum_n (U_n^\dagger\, U_{n+1} + U_{n+1}^\dagger\, U_n - 2) \\ &= \kappa \sum_n \left(\cos \frac{2\pi (r_{n+1} - r_n)}{N} - 1 \right). \end{aligned} \tag{52}$$

The action is invariant under the shifts by k_n being any integer

$$r_n \to r_n + k_n\, N. \tag{53}$$

It is convenient to define the new variable

$$\frac{2\pi\, r}{N} = w = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{(N-1)2\pi}{N}, \tag{54}$$

by which the action gets the form

$$S_E = \kappa \sum_n (\cos(w_{n+1} - w_n) - 1). \tag{55}$$

The transfer-matrix element then easily reads

$$\langle n+1 | \widehat{V} | n \rangle = \frac{1}{N} \sqrt{2\pi\kappa}\, \exp[\kappa(\cos(w_{n+1} - w_n) - 1)] \tag{56}$$

$$= \sum_{s=0}^{N-1} \mathrm{e}^{-a\, E_s}\, \frac{1}{N} \mathrm{e}^{\mathrm{i}\, s(w_{n+1} - w_n)}, \tag{57}$$

in which the plane waves

$$\psi_s(w) = \frac{1}{\sqrt{N}}\, \mathrm{e}^{\mathrm{i}\, s\, w} \tag{58}$$

satisfy the orthonormality condition

$$\sum_w \psi_s^\star(w) \psi_{s'}(w) = \delta_{ss'}. \tag{59}$$

By the identity (33) for Bessel functions, and using the condition (59), the sum on Bessel functions can be partitioned into N cyclic ones, leading to

$$\mathrm{e}^{-a\, E_s} = \sqrt{2\pi\kappa}\, \mathrm{e}^{-\kappa} \sum_{q=-\infty}^{\infty} I_{s+qN}(\kappa), \tag{60}$$

in which the sum is converging due to properties by I_s-functions. So the energy eigenvalues are simply given by

$$E_s(\kappa) = -a^{-1} \ln \left[\sqrt{2\pi\kappa}\, \mathrm{e}^{-\kappa} \sum_{q=-\infty}^{\infty} I_{s+qN}(\kappa) \right] \tag{61}$$

for $s = 0, 1, 2, \ldots, N-1$. Due to $I_s = I_{-s}$ we have the following degeneracy:

$$E_s = E_{N-s}. \tag{62}$$

In fact the lowest and highest eigenvalues are

$$\begin{aligned} E_{\min} &= E_0, \\ E_{\max} &= E_{N/2}, \quad N : \text{even}, \\ E_{\max} &= E_{(N\pm1)/2}, \quad N : \text{odd}. \end{aligned} \tag{63}$$

In the limit $N \to \infty$ we expect to recover the spectrum (37) by U(1) group. This is in fact the case using

$$\frac{I_{\pm\infty}}{I_{\text{finite}}} \to 0. \tag{64}$$

It can easily be checked that for $N = 2$ and 3 there is no extremum in E_0. For $N \geq 4$ there are both a minimum and a maximum. The energy eigenvalues are plotted in Fig. 4 for the group $\mathbb{Z}_4$, in which the extrema are at

$$\kappa_{\min} = 0.815, \quad \kappa_{\max} = 1.87. \tag{65}$$

Again the one-particle partition function (43), together with the thermodynamical functions (45) and (46), can be defined for the system. Like the case with the U(1) group, the appearance of a minimum leads to a phase transition. The isothermal G–M plots for the $\mathbb{Z}_4$ are plotted in Fig. 5. As expected, below the critical temperature $T_c = 0.44$ the plots develop cusps, exhibiting a first-order phase between two coexisting phases with low and high κ. However, at $T \approx 0$ there is a difference between the $\mathbb{Z}_N$ and U(1) groups, that is, there is a finite higher κ at which the systems follows the path with finite M. This behavior is evident by the $T \approx 0$ curve in Fig. 5, in contrast to the U(1) group with $M = 0$ toward

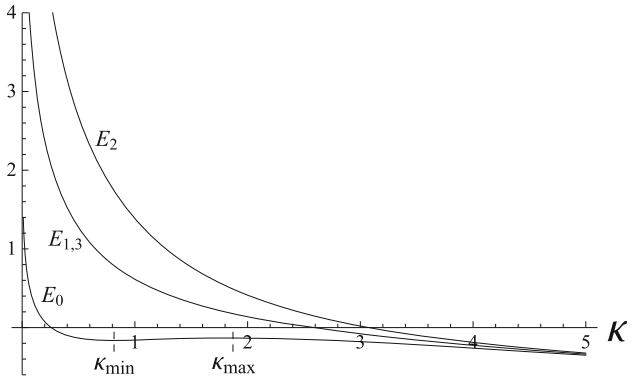

Fig. 4 The energies by (61) for $\mathbb{Z}_4$ group

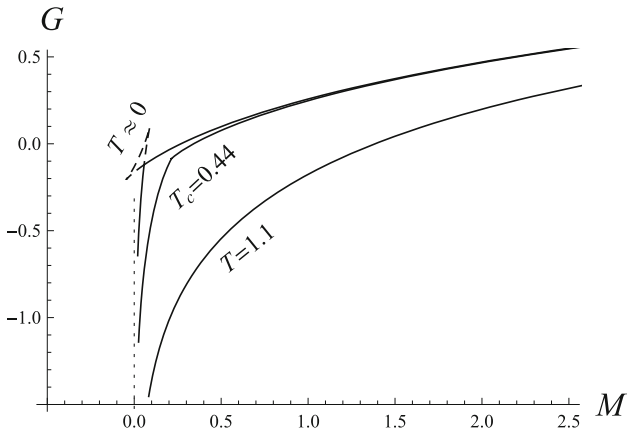

Fig. 5 The G–M plots for $\mathbb{Z}_4$ group

$\kappa \to \infty$ at $T \approx 0$. The two critical κ of $T \approx 0$ for the $\mathbb{Z}_4$ group at which the cusp starts and ends are as follows:

$$\kappa_{c1} = 0.69, \quad \kappa_{c2} = 8.8. \tag{66}$$

5 SU(2) group

As the case for a non-Abelian group here we consider the SU(2) group in one spatial direction. Then at time step n the group element is represented by

$$U_n = \exp(\mathrm{i}\, \boldsymbol{x}_n \cdot \boldsymbol{\sigma}/2R), \tag{67}$$

in which $\boldsymbol{x}_n = (x_n^1, x_n^2, x_n^3)$ represents the three components in the SU(2) sector, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. The action then simply gets the form

$$S_E = \frac{\kappa}{2} \sum_n \mathrm{Tr}(U_n^\dagger\, U_{n+1} + U_{n+1}^\dagger\, U_n \ - \ 2\mathbb{1}_2), \tag{68}$$

in which Tr is the trace over the matrix structure, with $\mathrm{Tr}(\sigma_\alpha \sigma_\beta) = 2\,\delta_{\alpha\beta}$. As the requirement mentioned earlier, the above action is invariant under the shift:

$$|\boldsymbol{x}_n| \to |\boldsymbol{x}_n| + 4\pi k_n R, \tag{69}$$

with the k_n being integer numbers. Using the identity

$$\begin{aligned} &\frac{1}{2}\,\mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\,\boldsymbol{x}_{n+1}\cdot\boldsymbol{\sigma}/2R}\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}_n\cdot\boldsymbol{\sigma}/2R}\right) \\ &\quad = \cos\frac{r_{n+1}}{2R}\cos\frac{r_n}{2R} + \hat{\boldsymbol{x}}_{n+1}\cdot\hat{\boldsymbol{x}}_n\,\sin\frac{r_{n+1}}{2R}\sin\frac{r_n}{2R} \\ &\quad =: \cos\frac{\gamma_{n+1,n}}{2}, \end{aligned} \tag{70}$$

in which $r_n = |\boldsymbol{x}_n|$ and $\hat{\boldsymbol{x}}_n = \boldsymbol{x}_n/r_n$, the action is simplified as

$$S_E = 2\kappa \sum_n \left(\cos\frac{\gamma_{n+1,n}}{2} - 1\right). \tag{71}$$

In the limits r_n and $r_{n+1} \ll R$ we have

$$\gamma_{n+1,n}^2 \simeq \frac{1}{R^2}(\boldsymbol{x}_{n+1} - \boldsymbol{x}_n)^2 + O\left(\frac{r}{R}\right)^4, \tag{72}$$

by which we have in the continuum limit

$$S_E \simeq -\frac{a\kappa}{4R^2}\int \mathrm{d}t\ \dot{\boldsymbol{x}}^2. \tag{73}$$

In the above again the minus sign is due to the use of imaginary time in the formalism. The action (73) represents the free motion of a free particle with mass $m_0 = a\kappa/(2R^2)$.

As before, the action (71) can be used to define the quantum theory based on the transfer-matrix method. The group manifold of SU(2) is known to be the 3-sphere S^3, for which the initial parametrization $\boldsymbol{x} = (x_1, x_2, x_3)$ in the spherical coordinates $\boldsymbol{x} = (r, \theta, \phi)$ is defined in the intervals

$$0 \le r \le 2\pi R, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi. \tag{74}$$

Also it is convenient to use the replacement

$$\chi = \frac{r}{2R}, \quad 0 \le \chi \le \pi. \tag{75}$$

By the above parametrization the SU(2) invariant measure takes the form [18]

$$\mathrm{d}\Omega_3 = \sin^2\chi\ \mathrm{d}\chi\ \mathrm{d}\Omega, \quad \text{with } \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{76}$$

satisfying

$$\int_{S^3} \mathrm{d}\Omega_3 = 2\pi^2. \tag{77}$$

As the case with U(1) group, by the complete orthonormal spherical harmonics on S^3 as the eigenfunctions, we can read the eigenvalues of the matrix $\widehat{V}$. The normalized spherical harmonics on S^3 are known to be [18]

$$\begin{aligned} \mathbb{Y}_{s\ell m}(\chi, \Omega) = &\sqrt{\frac{2^{2\ell+1}(s+1)\,(s-\ell)!\,\ell!^2}{\pi(s+\ell+1)!}} \\ &\times \sin^\ell\chi\ C_{s-\ell}^{(\ell+1)}(\cos\chi)\ Y_{\ell m}(\Omega), \end{aligned} \tag{78}$$

in which $C_{s-\ell}^{(\ell+1)}(x)$ are the Gegenbauer polynomials and $Y_{\ell m}(\Omega) = Y_{\ell m}(\theta, \phi)$ are the ordinary spherical harmonics

on 2-sphere S^2. In the above all indices are integers and obey the ordering [18]

$$|m| \le \ell \le s = 0, 1, 2, \ldots \tag{79}$$

The above harmonics are the normalized ones [18]

$$\int_{S^3} \mathrm{d}\Omega_3 \; \mathbb{Y}_{s\ell m}(\chi, \Omega)\; \mathbb{Y}^{\star}_{s'\ell' m'}(\chi, \Omega) = \delta_{ss'}\, \delta_{\ell\ell'}\, \delta_{mm'}. \tag{80}$$

Using (70) and in the (χ, θ, ϕ) parametrization of S^3, the action (71) between the adjacent times n and $n+1$ takes the form

$$\begin{aligned} S_E(n, n+1) = 2\kappa\, (&\cos\chi_{n+1}\, \cos\chi_n \\ &+ \hat{\boldsymbol{x}}_{n+1} \cdot \hat{\boldsymbol{x}}_n\, \sin\chi_{n+1}\, \sin\chi_n\, - 1), \end{aligned} \tag{81}$$

in which

$$\hat{\boldsymbol{x}}_{n+1} \cdot \hat{\boldsymbol{x}}_n = \cos\theta_{n+1} \cos\theta_n + \cos(\phi_{n+1} - \phi_n) \sin\theta_{n+1} \sin\theta_n. \tag{82}$$

Then the matrix element of the transfer matrix is given by

$$\langle \boldsymbol{x}_{n+1} | \widehat{V} | \boldsymbol{x}_n \rangle = \sqrt{\frac{\kappa}{4\pi}}\, \exp[S_E(n, n+1)]. \tag{83}$$

Using the identity for real w

$$\mathrm{e}^{w\, \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}'} = \sqrt{\frac{\pi}{2w}}\, 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} I_{\ell+1/2}(w)\; Y_{\ell m}(\Omega)\; Y^{\star}_{\ell m}(\Omega'), \tag{84}$$

in which the direction of the two unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}'$ are given by ordinary solid-angles Ω and Ω', respectively, and $I_{\ell+1/2}$ as before is the modified Bessel function. Taking $w = 2\kappa \sin\chi_n \sin\chi_{n+1}$, by (84) we have

$$\begin{aligned} &\sqrt{\frac{\kappa}{4\pi}}\, \exp[S_E(n, n+1)] \\ &= \sqrt{\frac{\kappa}{4\pi}}\, \mathrm{e}^{2\kappa(\cos\chi_n \cos\chi_{n+1} - 1)} \sqrt{\frac{\pi}{4\kappa \sin\chi_n \sin\chi_{n+1}}} \\ &\quad \times 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} I_{\ell+1/2} \\ &\quad \times (2\kappa \sin\chi_n \sin\chi_{n+1})\, Y_{\ell m}(\Omega_n)\, Y^{\star}_{\ell m}(\Omega_{n+1}), \end{aligned} \tag{85}$$

for which we also have an expansion based on the energy eigenvalues and the eigenfunctions as

$$\begin{aligned} &\sqrt{\frac{\kappa}{4\pi}}\, \exp[S_E(n, n+1)] \\ &= \sum_{s=0}^{\infty} \sum_{\ell=0}^{s} \sum_{m=-\ell}^{\ell} \mathrm{e}^{-a E_{s\ell m}}\, \mathbb{Y}_{s\ell m}(\chi_n, \Omega_n)\, \mathbb{Y}^{\star}_{s\ell m}(\chi_{n+1}, \Omega_{n+1}) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{s=\ell}^{\infty} \mathrm{e}^{-a E_{s\ell m}}\, \mathbb{Y}_{s\ell m}(\chi_n, \Omega_n)\, \mathbb{Y}^{\star}_{s\ell m}(\chi_{n+1}, \Omega_{n+1}). \end{aligned} \tag{86}$$

Using the orthonormality relation of $Y_{\ell m}$, by (85) and (86), after the changes $\chi_n \to \chi$ and $\chi_{n+1} \to \chi'$, the explicit expression (76) gives

$$\begin{aligned} &\sum_{s=\ell}^{\infty} \frac{2^{2\ell+1}(s+1)\,(s-\ell)!\,\ell!^2}{\pi(s+\ell+1)!} \mathrm{e}^{-a E_{s\ell}} \\ &\quad \times \sin^{\ell}\chi \sin^{\ell}\chi'\, C^{(\ell+1)}_{s-\ell}(\cos\chi) C^{(\ell+1)}_{s-\ell}(\cos\chi') \\ &= \sqrt{\frac{\kappa}{4\pi}} \sqrt{\frac{\pi}{4\kappa \sin\chi\, \sin\chi'}}\, 4\pi\, \mathrm{e}^{2\kappa(\cos\chi\, \cos\chi' - 1)} \\ &\quad \times I_{\ell+1/2}(2\kappa \sin\chi\, \sin\chi'), \end{aligned} \tag{87}$$

in which we have dropped the index m in $E_{s\ell}$, as it is now an irrelevant one. This indicates that the energy eigenvalues has at least an $(2\ell+1)$-level degeneracy. Using the orthogonality of the Gegenbauer polynomials,

$$\begin{aligned} &\int_0^{\pi} C^{(\ell+1)}_{s-\ell}(\cos\alpha)\, C^{(\ell+1)}_{s'-\ell}(\cos\alpha)\, \sin^{2\ell+2}\alpha\, \mathrm{d}\alpha \\ &= \frac{\pi\,(s+\ell+1)!}{2^{2\ell+1}(s+1)(s-\ell)!\,\ell!^2}\, \delta_{ss'}, \end{aligned} \tag{88}$$

by multiplication of the l.h.s. of (87) by $\sin^{\ell+2}\chi'\, C^{(\ell+1)}_{s-\ell}(\cos\chi')$ and integration over χ', we have

$$\begin{aligned} &\mathrm{e}^{-a E_{s\ell}}\, \sin^{\ell}\chi\, C^{(\ell+1)}_{s-\ell}(\cos\chi) \\ &= \int_0^{\pi} \mathrm{d}\chi' \sqrt{\frac{\kappa}{4\pi}} \sqrt{\frac{\pi}{4\kappa \sin\chi\, \sin\chi'}}\, 4\pi\, \mathrm{e}^{2\kappa(\cos\chi\, \cos\chi' - 1)} \\ &\quad \times \sin^{\ell+2}\chi'\, C^{(\ell+1)}_{s-\ell}(\cos\chi') I_{\ell+1/2}(2\kappa \sin\chi\, \sin\chi'). \end{aligned} \tag{89}$$

Using the identity [19,20]

$$\begin{aligned} &\int_0^{\pi} \mathrm{e}^{z \cos\chi\, \cos\chi'}\, \sin^{\ell+3/2}\chi'\, C^{(\ell+1)}_{s-\ell}(\cos\chi') \\ &\quad \times I_{\ell+1/2}(z \sin\chi\, \sin\chi')\, \mathrm{d}\chi' \\ &= \sqrt{\frac{2\pi}{z}}\; \sin^{\ell+1/2}\chi\; C^{(\ell+1)}_{s-\ell}(\cos\chi)\; I_{s+1}(z) \end{aligned} \tag{90}$$

the integration in the r.h.s. of (89) can be calculated, setting $z = 2\kappa$, leading to

$$\mathrm{e}^{-a\, E_s} = \sqrt{\frac{\kappa}{4\pi}}\, \frac{2\pi^2}{\kappa}\, \mathrm{e}^{-2\kappa}\, I_{s+1}(2\kappa), \tag{91}$$

in which we have dropped the index l from E_s as well, since there is no ℓ-dependence in spectrum. By (91) we find the energy with an $(s+1)^2$-level degeneracy:

$$E_s = -\frac{1}{a} \ln\left[\sqrt{\frac{\kappa}{4\pi}}\, \frac{2\pi^2}{\kappa}\, \mathrm{e}^{-2\kappa}\, I_{s+1}(2\kappa)\right], \quad s = 0, 1, 2, \ldots \tag{92}$$

In Fig. 6 the four lowest energies are plotted, all having a minimum. This is in contrast to the cases with U(1) and $\mathbb{Z}_N$ groups where only E_0 has a minimum. The one-particle partition function is given by

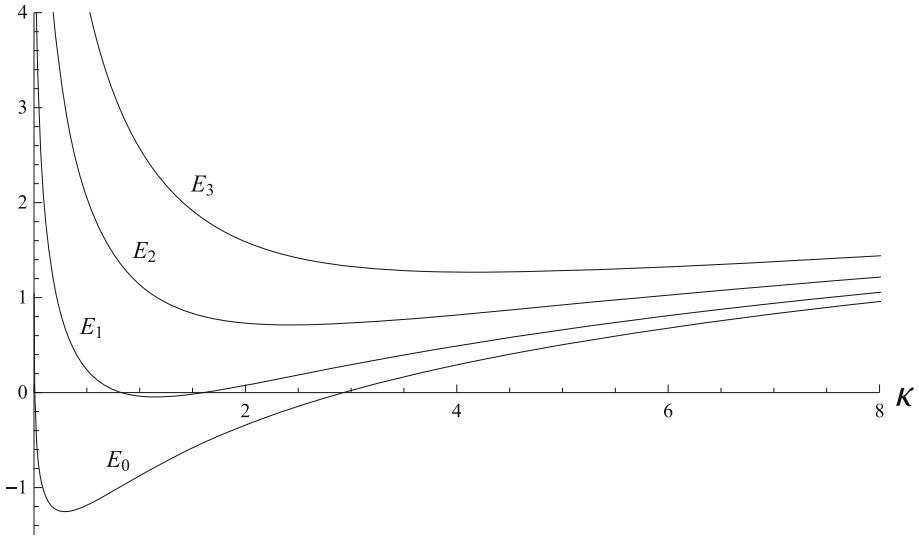

Fig. 6 The few lowest energies by (92) for SU(2) group

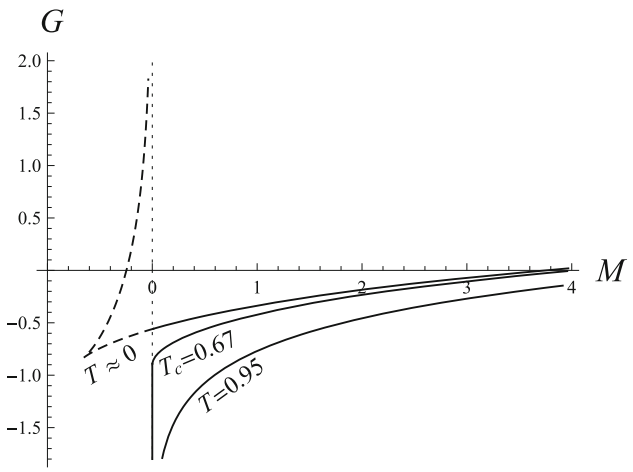

Fig. 7 The isothermal G–M plots for SU(2) group

$$Z_1(\beta,\kappa) = \sum_{s=0}^{\infty}(s+1)^2\,\mathrm{e}^{-\beta E_s}. \tag{93}$$

Again the equation of state as well as the Gibbs free energy can be obtained by (45) and (46). The G–M plots given in Fig. 7 develop a cusp, indicating that the system exhibits a first-order phase transition below the critical temperature $T_c = 0.67$. At $T \approx 0$ the critical $\kappa_c = 0.29$ is obtained above which M gets non-zero values.

6 Conclusion and discussion

The general theoretical ground as well as specific examples are presented for models based on compact angle coordinates. The present construction might be considered as a continuation of the theme by which the gauge fields are treated as compact angle variables [1–4]. In the present formulation the action depends on the group elements rather than the algebra elements, leading to the invariance under the total shifts inside the compact domain. It is observed that a discrete-time formulation of the theory is the natural way to obtain the desired dependence on the group elements. The present formulation in some sense develops in a reverse direction to what has happened in the lattice formulation of gauge theories. In particular, in the formulation of gauge theories on a lattice the invariance under the gauge transformations requires that the gauge fields are introduced into the theory via the group elements. Here the require invariance under the total shifts of compact coordinates treating the time parameter as discrete. The worldline action by the formulation resembles the spin chain Hamiltonian of magnetic systems, with coordinates appearing as the spin degrees of freedom.

The quantization of the model is formulated based on the transfer-matrix method [4,14]. The prefactor in the definition of the elements of transfer-matrix, in contrast to the case with infinite-extent coordinates, cannot be absorbed by a change of the path-integral integration variables. This particularly causes the energy eigenvalues to develop minima as functions of the defining parameter of the theory.

As examples for the formulation, the models based on the U(1), $\mathbb{Z}_N$, and SU(2) groups are explicitly constructed. In all the models based on the three groups the exact energy eigenvalues are obtained. As a consequence of the minima in the spectrum all the models exhibit first-order transition between the coexistent phases.

As mentioned earlier, by setting $\kappa = 1/g^2$ the model by the U(1) group is in fact the result of the dimensional reduction of the pure U(1) lattice gauge theory. As one possible application of the present construction here we mention the attempt in [12] to fit the model by the U(1) group to the expectations from monopole dynamics. In particular, the phase transition for the particles with mass $m_0 \propto 1/g^2$ by (27) may lead to a better understanding of the role of

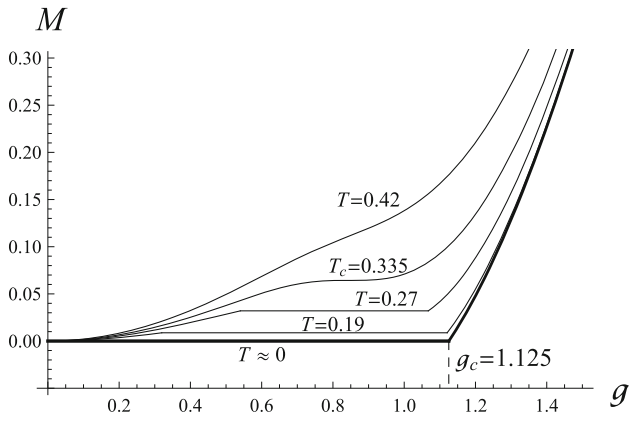

Fig. 8 The isothermal M–g plots by the model based on the U(1) group

monopoles in the confinement mechanism based on the dual Meissner effect in superconductors [21–26]. Based on the proposed mechanism, in the strong coupling limit, at which the monopoles have tiny masses, the motion of monopoles around the electric fluxes prevents the fluxes to spread, leading to the confinement of the electric charges. Instead, in the small coupling limit, where the monopoles are highly massive, the electric fluxes originating from source charges are likely to spread over space, leading to the Coulomb law. It is expected that there is a critical coupling g_c at which the transition from confined phase to the Coulomb phase occurs.

According to the model with the group U(1), the two regimes with weak and strong couplings constants are related by a first-order phase transition. The behavior of the system at low temperatures, where the main contribution to the partition function is from the ground state, is of particular interest. In the limit $T \to 0$, due to the Maxwell construction, we have $M = 0$ for $g < g_c = 1.125$; see Fig. 8. So as a consequence of the discontinuous nature of the first-order transition, at low temperatures and below g_c we have $M \propto \langle v^2 \rangle \approx 0$. This behavior is to be compared with (48) for ordinary particles, by which there is an asymptotic reduction of M by increasing the mass at constant T. According to the present model, at low temperatures and below g_c, the particles with mass $m_0 \propto g^{-2}$ are hardly moving ($\langle v^2 \rangle \approx 0$), leading to an exact Coulomb phase. On the other hand, exhibiting a high-slope increase of $\langle v^2 \rangle$ at g_c, the confined phase is instantiated once g exceeds g_c at low temperatures. This picture and specially the value of the critical coupling constant are in agreement with theoretical and numerical studies [12].

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