

The sigma meson from QCD sum rules for large- N_c Regge spectra

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Abstract The QCD sum rules in the large- N_c limit for the light non-strange vector, axial and scalar mesons are considered assuming a string-like linear spectrum for the radially excited states. We propose an improved method for combined analysis of these channels that gives a reasonable description of the observed spectrum. In the vector-axial case, fixing the pion decay constant and the gluon condensate we obtain more or less physical values for the masses of ground states and the quark condensate. Thus a typical need for this method to fix the mass of some ground state is overcome. Using in the scalar channel the values of presumably universal slope of radial trajectories and the quark condensate obtained in the vector-axial channel, we find that, in contrast to some strong claims in the literature, a prediction of light scalar state with a mass close to the mass of $f_0(500)$ seems to be natural in the considered approach and may follow in a natural way from the Regge phenomenology.

1 Introduction

It is widely known that the physics of non-perturbative strong interactions is encoded in the hadron masses. This largely unknown physics is most pronounced in the hadrons consisting of u - and d -quarks as the masses $m_{u,d}$ are much less than the non-perturbative scale Λ_{QCD} . At the same time, these hadrons shape the surrounding world. Aside from the nucleons and pions, an important role is played by the scalar σ -meson which is responsible for the main part of the nucleon attraction potential. In the particle physics, the given resonance is identified as $f_0(500)$ [1] and is indispensable for description of the chiral symmetry breaking in many phenomenological field models for the strong interactions. In spite of the great efforts invested in the study of this non-ordinary resonance in the last 60 years, its nature remains disputable [2].

The physical characteristics of hadrons are encoded in various correlation functions of corresponding hadron currents. Perhaps the most important characteristic is the hadron mass. The calculation of a hadron mass from first principles consists in finding the relevant pole of two-point correlator $\langle JJ \rangle$, where the current J is built from the quark and gluon fields and interpolates the given hadron. For instance, if the scalar isoscalar state f_0 represents an ordinary light non-strange quark–antiquark meson, its current should be interpolated by the quark bilinear $J = \bar{q}q$, where q stays for the u or d quark. In the real QCD, the straightforward calculations of correlators are possible only in the framework of lattice simulations which are still rather restricted.

A well-known phenomenological way for extraction of masses and other characteristics from the correlators is provided by various QCD sum rules. This method exploits some information from QCD via the operator product expansion (OPE) of correlation functions [3]. On the other hand, one assumes a certain spectral representation for a correlator in question. Typically the representation is given by the ansatz “one infinitely narrow resonance + perturbative continuum”. Such an approximation is very rough but works well phenomenologically in many cases [3–6]. From the theoretical viewpoint, the zero-width approximation (and simultaneously the absence of multiparticle cuts) arises in the large- N_c (called also planar) limit of QCD [7, 8]. In this limit, the only singularities of the two-point correlation function of a hadron current J are one-hadron states [8]. In the case of mesons, the two-point correlator has the following form to lowest order in $1/N_c$ (in the momentum space):

$$\langle J(q)J(-q) \rangle = \sum_n \frac{F_n^2}{q^2 - M_n^2}, \quad (1)$$

where the large- N_c scaling of quantities is $M_n = \mathcal{O}(1)$ for the masses, $F_n^2 = \langle 0|J|n \rangle^2 = \mathcal{O}(N_c)$ for the residues, $\Gamma = \mathcal{O}(1/N_c)$ for the full decay width [8]. Due to asymptotic freedom, the left-hand side of Eq. (1) behaves logarithmically

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at large q^2 . This behavior is only possible if the number of terms in the sum is infinite [8].

The logarithmic behavior of the right-hand side of (1) emerges naturally if one has the following large- n asymptotics: $F_n^2 \sim \text{const}$, $M_n^2 \sim \Lambda^2 n$. Such a Regge-like behavior for masses of radially excited states appears in the two-dimensional QCD in the planar limit [9], Veneziano dual amplitudes [10, 11], and various hadron string models [12–17]. In addition, the relation $F_n^2 = \text{const}$ can be regarded as a natural consequence of the string picture even without assumption on Regge behavior [18]. Within the aforementioned approaches, the slope Λ^2 is independent of the quantum numbers. This can be explained by the universality of gluodynamics, which determines the slope. The radial Regge behavior in the light non-strange mesons has some experimental evidence [19, 20]. The experimental slopes do demonstrate an approximately universal behavior. Within the accuracy of the large- N_c limit (10–20%), the universality of slopes is a quite adequate assumption.

Considering the linear ansatz $M_n^2 = \Lambda^2 n + M_0^2$ for the radial mass spectrum, the sum in (1) can be taken, expanded at large $Q^2 = -q^2$, and compared with the corresponding OPE in QCD. The ensuing planar sum rules were considered many times in the past (see, e.g., [21–41]). Later it became clear that the given sum rules are tightly related with a popular bottom-up holographic approach to QCD (see, e.g., discussions in [42]). On the other hand, the phenomenological understanding of spectral regularities has improved recently (an incomplete list of references is [43–58]). It seems timely to refresh the method of planar sum rules and exploit it again in the hadron phenomenology.

The main focus of our work will be concentrated on the enigmatic σ -meson. It is usually believed that the mass of the lightest scalar quark–antiquark state lies near 1 GeV or higher [2, 4]. The σ -meson, also referred to as $f_0(500)$ in Particle Data [1], is much lighter. Various phenomenological approaches insist on a highly unusual (likely tetraquark) nature of the σ -particle [2]. Our intention was to confirm the absence of a light scalar particle among usual mesons using the QCD sum rules in the large- N_c limit combined with the Regge phenomenology. Our conclusion, however, turned out to be opposite – a light scalar state can be predicted in a natural way within the considered framework. We will also comment briefly why this result was not obtained earlier in various QCD sum rules.

The paper is organized as follows. In Sect. 2, we recall the derivation of planar sum rules in the vector case. This derivation is extended to the axial channel in Sect. 3. In Sect. 4, we propose a solution of combined vector-axial sum rules. This solution is then used in the scalar channel in Sect. 5. Section 6 is devoted to some discussions. We conclude in Sect. 7.

2 Vector mesons

Due to conservation of the vector current $J_\mu^V = \bar{q}\gamma_\mu q$, the vector two-point correlator is transverse and depends on one scalar function only,

$$\langle J_\mu^V(q) J_\nu^V(-q) \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi_V(q^2). \quad (2)$$

Following the discussions in the Introduction, we will assume the simplest linear Regge ansatz for the vector spectrum,

$$M_V^2(n) = \Lambda^2 n + M_V^2, \quad n = 0, 1, 2, \dots \quad (3)$$

Since the isosinglet and isotriplet states are degenerate in the large- N_c (and chiral) limit [8], the spectra of ω and ρ mesons are indistinguishable in our framework. We will discuss the isosinglet states.

There are at least two reasons to separate the ground state out of the linear trajectory (3). First, the available experimental data show that the ground state lies noticeably below the linear trajectory in all unflavored vector quarkonia [57, 58]. An example for the ω -mesons is depicted in Fig. 1. Second, the ground ω and ρ mesons belong to the leading angular Regge trajectory. It is well known that the meson states on this trajectory do not have parity (and chiral) partners [43–47]. Hence, the vector channel should have one additional state with respect to the axial channel, which will be considered in the next section.

Using the spectral representation (1), definition (2), and ansatz (3) we get in the euclidean domain, $Q^2 = -q^2$,

$$\Pi_V(Q^2) = \frac{F_\omega^2}{Q^2 + M_\omega^2} + \sum_{n=0}^{\infty} \frac{F^2}{Q^2 + \Lambda^2 n + M_V^2}. \quad (4)$$

As we motivated in the Introduction, the residues of excited states in (4) are assumed to be constant and universal. In addition, it can easily be demonstrated that the asymptotics “logarithm + power terms” (5) holds only if $F_n^2 \sim \frac{dM_n^2}{dn}$ [38, 39] which gives a constant for the linear ansatz (3).

In the chiral and planar limits (with setting $N_c = 3$ at the end), the operator product expansion (OPE) of the vector correlator at large Q^2 reads [3]

$$\begin{aligned} \Pi_V(Q^2) = & -\frac{C_0}{8\pi^2} \log \frac{Q^2}{\mu^2} + \frac{\alpha_s}{24\pi} \frac{\langle G^2 \rangle}{Q^4} \\ & - \frac{14}{9} \pi \alpha_s \frac{\langle \bar{q}q \rangle^2}{Q^6} + \dots, \end{aligned} \quad (5)$$

where $\langle G^2 \rangle$ and $\langle \bar{q}q \rangle$ denote the gluon and quark vacuum condensate, respectively. According to the tenets of classical QCD sum rules [3], these vacuum characteristics are universal, i.e., their values do not depend on the quantum numbers of a hadron current J (the method is not applicable otherwise). The factor C_0 includes the perturbative correction to the leading logarithm, $C_0 = 1 + \frac{\alpha_s}{\pi}$. Within the accuracy of

the large- N_c limit, the correction is rather small and cannot be taken into account reliably. We set $C_0 = 1$ in the following.

The expression (4) can be rewritten via the ψ -function (a logarithmic derivative of Γ -function),

$$\sum_{n=0}^{\infty} \frac{1}{n+a} = -\psi(a) + \text{const}, \tag{6}$$

which has the following asymptotic expansion at large argument:

$$\psi(z) = \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kz^{2k}}. \tag{7}$$

Here B_{2k} are Bernoulli numbers. With the help of these formulas, the correlator (4) can be expanded at large Q^2 . In terms of the dimensionless variables

$$m_v = \frac{M_V}{\Lambda}, \quad m_\omega = \frac{M_\omega}{\Lambda}, \quad f = \frac{F}{\Lambda}, \quad f_\omega = \frac{F_\omega}{\Lambda}, \tag{8}$$

the result is

$$\begin{aligned} \Pi_V(Q^2) = & -f^2 \log \frac{Q^2}{\mu^2} + \frac{\Lambda^2}{Q^2} \left[f_\omega^2 - f^2 \left(m_v^2 - \frac{1}{2} \right) \right] \\ & + \frac{\Lambda^4}{Q^4} \left[-f_\omega^2 m_\omega^2 + \frac{1}{2} f^2 \left(m_v^4 - m_v^2 + \frac{1}{6} \right) \right] \\ & + \frac{\Lambda^6}{Q^6} \left[f_\omega^2 m_\omega^4 - \frac{1}{3} f^2 m_v^2 \left(m_v^2 - \frac{1}{2} \right) \left(m_v^2 - 1 \right) \right] \\ & + \dots \end{aligned} \tag{9}$$

The planar sum rules for the linear spectrum (3) follow from the comparison of (9) with (5). But first let us consider the axial-vector channel.

3 Axial mesons

As the axial-vector current $J_\mu^A = \bar{q} \gamma_\mu \gamma_5 q$ is not conserved, the axial two-point correlator has two independent contributions,

$$\langle J_\mu^A(q) J_\nu^A(-q) \rangle = \Pi_A(q^2) q_\mu q_\nu - \tilde{\Pi}_A(q^2) g_{\mu\nu}. \tag{10}$$

The sum rules for Π_A and $\tilde{\Pi}_A$ are different because the longitudinal part Π_A contains an extra contribution from the pion pole due to PCAC, $J_\mu^A \sim f_\pi \partial_\mu \pi$. In our normalization, the value of the pion weak decay constant is $f_\pi = 93$ MeV. Since the classical Weinberg paper [59] one traditionally extracts the transverse part in (10) (by adding and subtracting the term $g_{\mu\nu} q^2 \Pi_A$) and considers the sum rules for Π_A in conjunction with the sum rules for Π_V .

As was motivated in the Introduction, we assume the linear ansatz for the radial axial spectrum with universal slope. The axial analogue of the correlator (4) is

$$\Pi_A(Q^2) = \frac{f_\pi^2}{Q^2} + \sum_{n=0}^{\infty} \frac{F^2}{Q^2 + \Lambda^2 n + M_A^2}. \tag{11}$$

Strictly speaking, we should consider the isosinglet η -meson in place of the pion. In the two-flavor case, however, the difference is not substantial. The OPE of the correlator (11) reads [3]

$$\Pi_A(Q^2) = -\frac{C_0}{8\pi^2} \log \frac{Q^2}{\mu^2} + \frac{\alpha_s}{24\pi} \frac{\langle G^2 \rangle}{Q^4} + \frac{22}{9} \pi \alpha_s \frac{\langle \bar{q}q \rangle^2}{Q^6} + \dots \tag{12}$$

It should be noted that only the last term in (5) and (12) is different. Proceeding as in the vector case, in terms of dimensionless notations (8) ($m_a = \frac{M_A}{\Lambda}$) we get

$$\begin{aligned} \Pi_A(Q^2) = & -f^2 \log \frac{Q^2}{\mu^2} + \frac{\Lambda^2}{Q^2} \left[\frac{f_\pi^2}{\Lambda^2} - f^2 \left(m_a^2 - \frac{1}{2} \right) \right] \\ & + \frac{\Lambda^4}{Q^4} \frac{f^2}{2} \left(m_a^4 - m_a^2 + \frac{1}{6} \right) \\ & - \frac{\Lambda^6}{Q^6} \frac{f^2}{3} m_a^2 \left(m_a^2 - \frac{1}{2} \right) \left(m_a^2 - 1 \right) + \dots \end{aligned} \tag{13}$$

As in the vector case, the pure axial sum rules follow from comparison of (12) with (13).

4 Vector sum rules

As was indicated above, the combined set of vector-axial sum rules emerges from equating terms at $\log Q^2$, $1/Q^2$, $1/Q^4$, and $1/Q^6$ in (5) and (9) and in (12) and (13). Our inputs will be the pion decay constant f_π and the gluon condensate $\frac{\alpha_s}{\pi} \langle G^2 \rangle$. The quark condensate will be a prediction. More precisely, we predict the value of the dim-6 condensate $\alpha_s \langle \bar{q}q \rangle^2$, which has a rather small but non-zero anomalous dimension. The sum rules are consistent at some definite value of the dim-6 condensate. The quark condensate at certain normalization point can be deduced from this value. Thus at $1/Q^6$ we will have only one sum rule, which follows from equating the $1/Q^6$ -terms in (9) and (13) with the factor $-7/11$ (as prescribed by the OPE (5) and (12)). The resulting set of equations is

$$f^2 = \frac{1}{8\pi^2}, \tag{14}$$

$$f^2 \left(m_v^2 - \frac{1}{2} \right) = f_\omega^2, \tag{15}$$

$$\Lambda^2 f^2 \left(m_a^2 - \frac{1}{2} \right) = f_\pi^2, \tag{16}$$

$$\Lambda^4 \left[-f_\omega^2 m_\omega^2 + \frac{1}{2} f^2 \left(m_v^4 - m_v^2 + \frac{1}{6} \right) \right] = \frac{\alpha_s}{24\pi} \langle G^2 \rangle, \tag{17}$$

$$\Lambda^4 f^2 \left(m_a^4 - m_a^2 + \frac{1}{6} \right) = \frac{\alpha_s}{12\pi} \langle G^2 \rangle, \tag{18}$$

Table 1 The numerical solutions in GeV (see text)

	$f_\pi = 93$ MeV	$f_\pi = 87$ MeV
Λ	1.43 (2)	1.32 (2)
M_V	1.60 (4)	1.45 (4)
M_A	1.31 (1)	1.21 (1)
M_ω	0.79 (3)	0.69 (3)
F	0.16	0.15
F_ω	0.14	0.13
$(-\langle\bar{q}q\rangle)^{\frac{1}{3}}$	0.30 (1)	0.27 (1)

$$f_\omega^2 m_\omega^4 - \frac{1}{3} f^2 m_v^2 \left(m_v^2 - \frac{1}{2}\right) (m_v^2 - 1) = \frac{7}{33} f^2 m_a^2 \times \left(m_a^2 - \frac{1}{2}\right) (m_a^2 - 1). \tag{19}$$

Thus we arrive at the system of six polynomial equations with six variables $\Lambda^2, m_v^2, m_\omega^2, m_a^2, f^2,$ and f_ω^2 . This system can be solved numerically. The values of inputs are $f_\pi = 93$ MeV and $\frac{\alpha_s}{\pi} \langle G^2 \rangle = (360 \pm 20 \text{ MeV})^4$. To demonstrate the sensitivity of solutions to a choice of inputs we try also $f_\pi = 87$ MeV (a presumable value of f_π in the chiral limit [60]) and show the uncertainty caused by the uncertainty in the value of the gluon condensate. The physical solutions after rescaling (8) are given in Table 1. Concerning the dim-6 condensate, a self-consistent interpretation of the value of $\langle\bar{q}q\rangle$ appears at the choice $\alpha_s \simeq 1/\pi \simeq 0.3$, which corresponds to the scale $\mu \simeq 2$ GeV. So the obtained value of the quark condensate refers to that scale. The predicted masses of the first three states are displayed in Table 2.

Taking into account all rough approximations that we have made, the resulting solution is surprisingly good. First of all, the masses of ground states are close to the experimental masses of unflavored vector $\omega(782)$ and axial $f_1(1285)$ mesons [1]. The agreement looks excellent for the large- N_c limit. Second, the obtained value of $\langle\bar{q}q\rangle$ is also unexpectedly reasonable. Indeed, the numerical solution yields the product $\alpha_s \langle\bar{q}q\rangle^2$, where both α_s and the quark condensate $\langle\bar{q}q\rangle$ depend on the normalization scale μ . Excluding μ one can draw a “physical” curve on the $(\alpha_s, \langle\bar{q}q\rangle)$ plane. The fact that our solution approximately belongs to this curve is non-trivial.

As to the radially excited states, there is still a large controversy in the interpretation of the relevant experimental data and in the determination of real physical masses [1]. In view of a rather qualitative character of our model we do not want to delve into the corresponding speculations. Our predictions refer to the large- N_c limit. The real masses of excited states must be shifted by various effects which are beyond the scope of our model. We just mention that the obtained masses seem to lie in the correct mass ranges and thereby look reasonable.

Table 2 The masses of the first three predicted states in GeV (central values)

n	0	1	2
$f_\pi = 93$ MeV			
$M_V(n)$	0.79	1.60	2.15
$M_A(n)$	1.31	1.93	2.41
$f_\pi = 87$ MeV			
$M_V(n)$	0.69	1.45	1.96
$M_A(n)$	1.21	1.79	2.22

5 Scalar sum rules

Consider the two-point correlator of the scalar isoscalar current $J^S = \bar{q}q$. Its resonance representation reads (up to two contact terms)

$$\Pi_S(q^2) = \langle J^S(q) J^S(-q) \rangle = \sum_n \frac{G_n^2 M_S^2(n)}{q^2 - M_S^2(n)}, \tag{20}$$

where the residues stem from the definition $\langle 0 | J^S | n \rangle = G_n M_S(n)$. As in the vector cases, we assume the linear radial spectrum with universal slope

$$M_S^2(n) = \Lambda^2 n + M_S^2, \quad n = 0, 1, 2, \dots \tag{21}$$

And as in the vector channels, within the linear ansatz (21), the analogues of decay constant must be equal for consistency with the OPE: $G_n = G$.

As a priori we do not know reliably the radial Regge behavior of scalar masses, we will consider two simple possibilities: (I) the ground $n = 0$ state lies on the linear trajectory (21); (II) the state $n = 0$, below called σ , is not described by the linear spectrum (21). The second assumption looks more physical; see Fig. 1. The corresponding spectral representations in the Euclidean space are

$$\Pi_S^{(I)}(Q^2) = \sum_{n=0}^{\infty} \frac{G^2 (\Lambda^2 n + M_S^2)}{Q^2 + \Lambda^2 n + M_S^2}, \tag{22}$$

$$\Pi_S^{(II)}(Q^2) = \frac{G_\sigma^2 M_\sigma^2}{Q^2 + M_\sigma^2} + \sum_{n=1}^{\infty} \frac{G^2 (\Lambda^2 n + M_S^2)}{Q^2 + \Lambda^2 n + M_S^2}. \tag{23}$$

Proceeding as in the vector case, we expand (22) and (23) at large Q^2 and compare the expansions with the OPE of the scalar correlator (20). Introducing the dimensionless variables

$$m_s = \frac{M_S}{\Lambda}, \quad g = \frac{G}{\Lambda}, \tag{24}$$

the expansions have the form

$$\Pi_S^{(I)}(Q^2) = g^2 Q^2 \log \frac{Q^2}{\mu^2} - \frac{\Lambda^4 g^2}{Q^2} \frac{1}{2} \left(m_s^4 - m_s^2 + \frac{1}{6} \right)$$

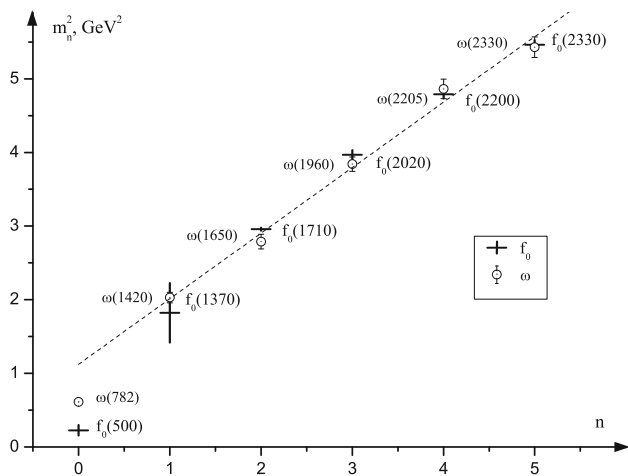


Fig. 1 A presumable spectrum of non-strange ω (circles) and f_0 (crosses) mesons [1]. A rather large fixed horizontal size of crosses is drawn to indicate better the position of scalar resonances. The $f_0(1500)$ is excluded as the available data on this state are poorly compatible with the $q\bar{q}$ -assignment (see the mini-review “Non- $q\bar{q}$ Mesons” in Particle Data [1]). The plot is taken from Ref. [61]

$$+ \frac{\Lambda^6 g^2}{Q^4} \frac{m_s^2}{3} \left(m_s^2 - \frac{1}{2}\right) (m_s^2 - 1) + \dots, \tag{25}$$

$$\begin{aligned} \Pi_S^{(II)}(Q^2) &= g^2 Q^2 \log \frac{Q^2}{\mu^2} + \frac{G_\sigma^2 M_\sigma^2}{Q^2} - \frac{\Lambda^4 g^2}{Q^2} \frac{1}{2} \\ &\times \left(m_s^4 + m_s^2 + \frac{1}{6}\right) \\ &- \frac{G_\sigma^2 M_\sigma^4}{Q^4} + \frac{\Lambda^6 g^2}{Q^4} \frac{m_s^2}{3} \left(m_s^2 + \frac{1}{2}\right) (m_s^2 + 1) + \dots \end{aligned} \tag{26}$$

The OPE of the correlator (20) in the chiral and large- N_c limits reads [4]

$$\begin{aligned} \Pi_S(Q^2) &= \frac{3C_0}{16\pi^2} Q^2 \log \frac{Q^2}{\mu^2} + \frac{\alpha_s}{16\pi} \frac{\langle G^2 \rangle}{Q^2} \\ &- \frac{11}{3} \pi \alpha_s \frac{\langle \bar{q}q \rangle^2}{Q^4} + \dots, \end{aligned} \tag{27}$$

where

$$C_0 = 1 + \frac{11\alpha_s}{3\pi}. \tag{28}$$

Now the perturbative correction can contribute more than 30% to the factor in front of the logarithm. This contribution has a much stronger impact than in the vector channels and should be taken into account. Matching the logarithmic terms we obtain

$$g^2 = \frac{3C_0}{16\pi^2}. \tag{29}$$

Consider the assumption (I). From (25) and (27) we have two sum rules,

$$\frac{3C_0}{2\pi^2} \Lambda^4 \left(m_s^4 - m_s^2 + \frac{1}{6}\right) = -\frac{\alpha_s}{\pi} \langle G^2 \rangle, \tag{30}$$

$$\frac{3C_0}{16\pi^2} \Lambda^6 m_s^2 \left(m_s^2 - \frac{1}{2}\right) (m_s^2 - 1) = -11\pi\alpha_s \langle \bar{q}q \rangle^2. \tag{31}$$

Substituting the numerical values of Λ and $\langle \bar{q}q \rangle$ from the solution of vector sum rules (Table 1), we arrive at two independent polynomial equations. If we neglect the perturbative correction in (28), $C_0 = 1$, the Eqs. (30) and (31) share an approximately common solution¹ $m_s^2 \simeq 0.74$ leading to the radial scalar spectrum $M_S(n) \simeq 1.23, 1.89, 2.37, \dots$ GeV. If we include the perturbative correction, a miracle with the common solution disappears.

Consider a more physical assumption (II). Matching (26) with the OPE (27) leads to the following sum rules:

$$G_\sigma^2 M_\sigma^2 - \frac{3C_0}{32\pi^2} \Lambda^4 \left(m_s^4 + m_s^2 + \frac{1}{6}\right) = \frac{\alpha_s}{16\pi} \langle G^2 \rangle, \tag{32}$$

$$-3G_\sigma^2 M_\sigma^4 + \frac{3C_0}{16\pi^2} \Lambda^6 m_s^2 \left(m_s^2 + \frac{1}{2}\right) (m_s^2 + 1) = -11\pi\alpha_s \langle \bar{q}q \rangle^2. \tag{33}$$

Now we have two equations with three variables m_s , M_σ , and G_σ . Excluding G_σ we get a relation for the mass of σ -meson as a function of the intercept parameter m_s^2 ,

$$M_\sigma^2 = \frac{\frac{C_0}{16\pi^2} \Lambda^6 m_s^2 \left(m_s^2 + \frac{1}{2}\right) (m_s^2 + 1) + \frac{11}{3} \pi \alpha_s \langle \bar{q}q \rangle^2}{\frac{3C_0}{32\pi^2} \Lambda^4 \left(m_s^4 + m_s^2 + \frac{1}{6}\right) + \frac{\alpha_s}{16\pi} \langle G^2 \rangle}. \tag{34}$$

The “decay constant” G_σ as a function of m_s^2 can be obtained by substituting (34) to (32) or (33). The quantities M_σ , G_σ , $G = \Lambda g$ (where g is defined in (29)), and mass of the first state on the scalar trajectory are plotted in Fig. 2 using the inputs from Table 1 for $f_\pi = 93$ MeV and $\alpha_s \simeq 1/\pi$ in (28) that was obtained in the vector case. The intercept m_s^2 can be negative as the sum in (23) begins with $n = 1$.

We checked also other variants with inputs corresponding to $f_\pi = 87$ MeV in Table 1 and with $\alpha_s = 0$ in (28). They result in a shift within 70–80 MeV for masses that lies within the accuracy of the large- N_c limit. The general picture displayed in Fig. 2 remains, however, the same for all variants. Going to negative intercept an unphysical behavior emerges already at relatively small values. The mass $M_S(1)$ is rather stable and seems to reproduce the mass of $a_0(1450)$ -meson, $M_{a_0(1450)} = 1474 \pm 19$ MeV [1]. Its isosinglet partner (the candidates is $f_0(1370)$) should be degenerate with $a_0(1450)$ in the planar limit.

¹ At $f_\pi = 93$ MeV. Using $f_\pi = 87$ MeV, the solution is $m_s^2 \simeq 0.72$ resulting in the spectrum $M_S(n) \simeq 1.12, 1.73, 2.17, \dots$ GeV.

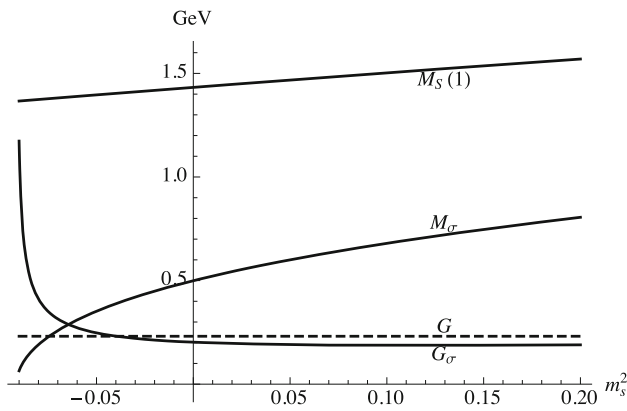


Fig. 2 The values of M_σ , G_σ , G , and the first state on the scalar trajectory $M_S(1)$ as a function of dimensionless intercept m_s^2 from the relations (34), (32) (or (33)), (29), and $M_S^2(1) = \Lambda^2(1 + m_s^2)$ from (21)

The plot in Fig. 2 demonstrates that the actual prediction for M_σ is rather sensitive to the intercept of scalar linear trajectory, though initially M_σ is not described by the linear spectrum (21). And *vice versa*, the expected value of M_σ (around 0.5 GeV [1]) imposes a strong bound on the allowed values of intercept m_s^2 . The plot in Fig. 2 shows that m_s^2 is close to zero.

Although both the ground ω -meson and σ lie out of the corresponding linear trajectories (as is suggested, e.g., by Fig. 1) there is a difference between them in our analysis. In the vector case, it was important to start the sum in (4) from $n = 0$ in order to relate the resonance representation in the vector case to the axial one in (11). If we started from $n = 0$ in the scalar channel (23), the sign of both numerator and denominator in (34) would depend on the value of m_s^2 ,

$$M_\sigma^2 = \frac{\frac{C_0}{16\pi^2} \Lambda^6 m_s^2 (m_s^2 - \frac{1}{2}) (m_s^2 - 1) + \frac{11}{3} \pi \alpha_s (\bar{q}q)^2}{\frac{3C_0}{32\pi^2} \Lambda^4 (m_s^4 - m_s^2 + \frac{1}{6}) + \frac{\alpha_s}{16\pi} (G^2)}, \quad (35)$$

making the prediction of M_σ highly unstable and uncertain. In this sense, the σ -meson is not unusual since it belongs to the radial scalar trajectory. Just its mass is not described by the linear ansatz (21). The given interpretation can also be motivated by a comparison of residues – G_σ lies only slightly below G . Physically this means that an external source (some scalar current) of scalar mesons creates the lightest state with a probability close to the probabilities of creation of other scalar resonances. Within our accuracy, the coupling of σ -meson to that source is barely suppressed.

The observation above suggests to check the hypothesis $G_\sigma = G$ explicitly. After this substitution, the sum rules (32) and (33) (or their analogues if we start from $n = 0$ in (23), the shift does not change the results) represent two equations with two unknown variables M_σ^2 and m_s^2 . This system has four numerical solutions. Two of them are unphysical (give

tachyonic masses). The third one is (in terms of dimensionless quantities $m_\sigma = M_\sigma/\Lambda$, $m_s = M_S/\Lambda$ and for $f_\pi = 93$ MeV): $(m_\sigma^2, m_s^2) \simeq (0.742, 0.739)$. This case reproduces the solution of the assumption (I) above. The given solution corresponds to the branch where the condensates in the r.h.s. of Eqs. (32) and (33) can be neglected. The fourth solution, $(m_\sigma^2, m_s^2) \simeq (0.074, -0.040)$, predicts $M_\sigma \simeq 0.39$ GeV and the radial spectrum $M_S(n) \simeq 1.40, 2.01, \dots$ GeV. These values (except masses of higher excitations) are visually seen in Fig. 2 – they correspond to the point where the lines G and G_σ intersect. The given solution is the most interesting: The obtained mass of σ -meson lies close to the expected mass range [1] and the radial spectrum looks reasonable. For instance, the first radial state can be identified with $f_0(1370)$ in Fig. 1. This state has a natural isovector partner $a_0(1450)$ [1]. Also the second radial excitation has a natural interpretation – the resonance $f_0(2020)$ in Fig. 1. Setting $f_\pi = 87$ MeV, the corresponding predictions are $M_\sigma \simeq 0.38$ GeV and $M_S(n) \simeq 1.30, 1.85, \dots$ GeV.

When one predicts some quark–antiquark state it is important to indicate its place on the angular Regge trajectory as well. In other words, what are the f_2, f_4, \dots companions of $f_0(500)$ on this trajectory? In order to answer this question we must know the slope of the trajectory under consideration. According to the analysis of Ref. [19], the slope of f_0 trajectory, most likely, lies in the interval $1.1 \div 1.2$ GeV². Several independent estimations made in some papers of Refs. [43–56] seem to confirm this value. Consider our preferable estimate on the σ -meson mass obtained above, $m_\sigma \approx 390$ MeV. Then we obtain $m_{f_2} \approx 1.53 \div 1.60$ GeV. The PDG contains a well-established resonance $f_2(1565)$ [1] with mass $m_{f_2(1565)} = 1562 \pm 13$ MeV. It is a natural companion of σ -meson on the corresponding angular Regge trajectory. The next state would have the mass $m_{f_4} \approx 2.13 \div 2.23$ GeV. The discovery of the predicted tensor meson f_4 (and perhaps the next companion f_6 with $m_{f_6} \approx 2.60 \div 2.71$ GeV) would confirm our conjecture about the form of Regge trajectory with the σ -meson on the top. A tentative candidate for our f_4 in the Particle Data is the resonance $f_J(2220)$, having a still undetermined spin – its value is either $J = 2$ or $J = 4$ [1]. Our model would favor the second possibility.

It is interesting to note that the predicted trajectory is drawn in Ref. [19] among numerous angular Regge trajectories for isosinglet P -wave states of even spin. But the resonance $f_2(1565)$ is replaced there by $f_2'(1525)$ (and it is absent on other trajectories). As a result, $m_{f_0}^2$ has a very small negative value leading to the disappearance of a scalar state from this trajectory. The predicted f_4 -companion is labeled $f_4(2150)$ [19]. The modern PDG contains the state $f_2'(1525)$ but this resonance is typically produced in reactions with K -mesons, which evidently indicates the dominant strange component. For this reason we should exclude it from our estimates.

Our prediction of the Regge trajectory containing the σ -meson on the top seems to contradict studies of the σ -state on the complex Regge trajectory which claim that because of the very large width the corresponding state cannot belong to the usual Regge trajectories [2, 62]. It is not excluded, however, that this observation may simply indicate limitations of the usual methods which are applied to the description of the $\pi\pi$ -scattering. These methods are based on analyticity and unitarity of the S -matrix and do not contain serious dynamical inputs. The generation of a huge width for $f_0(500)$ represents, most likely, some dynamical effect. For this reason, uncovering the genuine nature of σ -meson requires dynamical approaches.

Thus our analysis demonstrates that the existence of a light scalar state is well compatible with the structure of the planar sum rules in the scalar channel and may follow in a natural way from the Regge phenomenology.

6 Discussions

There exists a widespread belief that a natural mass of the lightest quark–antiquark scalar state in the QCD sum rules lies near 1 GeV. This prediction follows both from the standard borelized spectral sum rules [4] and from the planar sum rules [38–40]. It should be emphasized that the given prediction is not definitive but rather represents a consequence of some specific assumptions and tricks. As was demonstrated in a recent paper [63], if one uses the Borel transform and the typical ansatz “one narrow resonance plus continuum”, the extracted mass of the quark–antiquark scalar state cannot be less than about 0.8 GeV independently of any further assumptions. This turns out to be a specific internal restriction of the method itself.² In the planar sum rules, the reason was different. In case of Refs. [38, 39], the result seems to be related to the fact that one studied the scalar sum rules in conjunction with the pseudoscalar ones with some shared parameters. In the scheme considered, the ground scalar state cannot be significantly lighter than $\pi(1300)$ whose mass was taken as an input. The pseudoscalar channel is notoriously problematic and the applicability of the sum rules in this channel is questionable [3, 4]. Thus the assumption made in Refs. [38, 39] was rather strong. In the planar analysis of Ref. [40], the resonance $f_0(980)$ was placed as the first state on the scalar trajectory and alternative possibilities were not studied.

In our consideration, the assumptions above are not used. Making the standard sum rule analysis of the two-point correlator for the simplest quark–antiquark scalar current in the

planar limit, we have demonstrated that the existence of scalar state compatible with $f_0(500)$ can be rather natural. But a concrete prediction for its mass is uncertain, mainly because the form of experimental radial scalar trajectory is controversial. We have advocated that the most consistent value of m_σ within our scheme lies near $m_\sigma \approx 0.4$ GeV. One should keep in mind that our predictions refer to the large- N_c limit where meson mixings and decays are suppressed. In the real world with $N_c = 3$, a strong coupling to two pions should enhance the observable mass of σ -meson. A phenomenological way to exclude the mixing with other meson (typically pion) states in the propagation of resonances consists in extracting the K -matrix poles where the corresponding “bare states” emerge. Albeit the procedure is model-dependent, it could make sense to compare the large- N_c masses with the relevant K -matrix poles. For instance, the relevant scalar radial trajectory in Ref. [19] has $f_0(1300)$ (called $f_0(1370)$ in the PDG [1]) on the top. The corresponding “bare” trajectory, according to Ref. [19], has a scalar state with the mass $m_{f_0(\text{bare})} = 1240 \pm 50$ MeV on the top. The slope of the “bare” trajectory is about $\Lambda^2 \approx 1.38$ GeV². We propose to interpret the σ -meson as the lightest state on this trajectory. Extending the “bare” trajectory to lower mass, we obtain an approximate estimate $m_\sigma \approx \sqrt{m_{f_0(\text{bare})}^2 - \Lambda^2} \approx 400 \pm 100$ MeV. This estimate agrees with our result. In Ref. [19], however, the σ -meson was claimed to be alien to the classification of $\bar{q}q$ -states.

Since the used sum rule method is based on the narrow-width approximation, a direct translation of our predictions to the physical parameters of a broad resonance looks questionable. As a matter of fact, we claim only that a scalar isoscalar pole in the range 400–600 MeV can naturally exist in the large- N_c limit.

Another pertinent question is why the σ -meson lies below the linear radial Regge trajectory like the ground vector states. In the latter case, one can propose a simple qualitative explanation. The ground vector states are S -wave, so they represent relatively compact hadrons. In this case, a contribution from the Coulomb part of the Cornell confinement potential, $V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r$, is not small, effectively “decreasing” the tension σ at smaller distances and, hence, the masses of the ground S -wave states. In the case of the σ -meson, one can imagine the following situation: this state represents a tetraquark but the admixture of an additional $q\bar{q}$ -pair is small and gives a small direct contribution to the mass. For this reason, we may use the large- N_c limit as a first approximation. However, due to the extra $q\bar{q}$ -pair, the σ -meson (originally representing a scalar P -wave state) can exist as a S -wave state. Due to this phenomenon, on the one hand, the decay of this state becomes OZI-superallowed, explaining thereby its abnormally large width, on the other hand, its mass decreases similarly to the masses of ground S -wave vector mesons.

² Another objection against the σ -meson within this method was the observation that unmixed scalar quarkonia ground states are not wide [64]. We note that this result also refers to a scalar state near 1 GeV where the exploited method was reliable.

In our scheme, the value of slope of linear radial scalar trajectory is taken from the solution of vector and axial planar sum rules. This solution differs from the solution of Ref. [37]. According to the assumptions of Ref. [37], the slopes of the vector and axial trajectories are different and, as a consequence, the residues are also different, and the quark condensate represents an input parameter (together with f_π and the gluon condensate). As a result, one has a system of 8 polynomial equations for 8 variables: $\Lambda_{V,A}^2$, $M_{V,A}^2$, $F_{V,A}^2$, M_ρ^2 , and F_ρ^2 . This system, however, cannot be solved since it consists of two independent groups of equations – four equations for the vector channel and four for the axial one. The first group contains five variables and the second one contains three variables. An approximate solution was found by fixing M_ρ and playing with F_ρ in some range. We believe that our ansatz and solution are more compact and natural.

The σ -meson within the large- N_c Regge approach was also studied in Ref. [41], where it was found that the given state represents a usual meson (it survives in the large- N_c limit) and its mass lies in the interval 450–600 MeV. These conclusions agree with our results. However, the analysis made in Ref. [41] is completely different. First, the interpolating operator for the scalar isoscalar states was the energy-momentum tensor in QCD. The results and conclusions were heavily based on an analysis of the corresponding OPE of its correlation function and some gravitational form factors. Second, all such states were placed on a single radial Regge trajectory with half the standard slope. The existence of this possibility is interesting but we believe that the predominantly non-strange and predominantly strange isosinglet scalar mesons should form two separate trajectories with approximately standard slope, as advocated in Ref. [19]. Third, in the case of the energy-momentum tensor, the scalar correlator should contain additional poles corresponding to the glueball states [64]. They should cause a distortion of the pole positions corresponding to the quark–antiquark states. For our choice of the scalar interpolating current, we expect a suppression of the glueball admixture in the planar limit. From the phenomenological side, there is only one scalar candidate with a presumably rich gluonic content – the resonance $f_0(1500)$ [1]. We do not describe this state (in particular, it is excluded from Fig. 1). The agreement in estimating the σ -meson mass between our analysis and Ref. [41] may be due to the fact that a natural glueball scale where the distortion is maximal lies about 1 GeV higher than the σ -mass.

It is interesting to observe that the old dual models incorporating the chiral symmetry predict the degeneracy of the radial vector and f_0 trajectories [10, 11]. This might be not far from reality; see Fig. 1.

The lattice calculations of m_σ are still inconclusive. Simulations with the simplest scalar quark current $J = \bar{q}q$ by SCALAR Collaboration yielded a mass of the lightest ordinary scalar isoscalar meson close to $m_\rho \simeq 0.77$ GeV [65].

An old simulation by Detar and Kogut arrived at lower values [66]. The work of SCALAR Collaboration has recently been continued and the conclusion was that the σ -meson may be a molecular state [67]. This conclusion, however, cannot be regarded as serious evidence against our results. The main findings of SCALAR Collaboration consisted in the observation of a strong significance of disconnected diagrams in the scalar isoscalar channel. In addition, as correctly noticed in the Introduction of Ref. [67], “the quark masses used in the present work are admittedly not small, and hence it may not be straightforward to extract direct implications regarding the nature of the sigma”. Indeed, in the simulations the authors had $m_\rho/m_\pi = 1.5$ while in the real world $m_\rho/m_\pi = 5.5$. One of the conclusions of the given analysis stated that for the comprehensive understanding of the isosinglet scalar mesons the interpolation operators including two-quark states and others should be taken into account [67].

Our analysis was based on the standard OPE and the use of the simplest scalar quark current. It is well known that the scalar correlator has also the so-called “direct instantons” contribution (see, e.g., discussions in Ref. [68]). This contribution is not seen in the OPE because of exponential fall-off. In principle, the given contribution might lead to some non-linear corrections to our linear spectrum. Perhaps the exponentially decreasing corrections to the string-like spectrum introduced phenomenologically in Refs. [38, 39] could have an instanton origin. A clarification of this issue represents an interesting problem deserving a separate study.

It would be interesting to extend our analysis to the sector with hidden strangeness. The combined sum rules for vector and axial states will have a different numerical solution because the dim-4 condensate $m_s \langle s\bar{s} \rangle$ is not negligible, moreover, an effective dim-2 condensate m_s^2 emerges from the quark loop. Also the isovector sector with inclusion of the scalar mesons a_0 should be considered. A study of these problems is left for future.

7 Conclusions

We have considered the QCD sum rules in the large- N_c limit assuming for the radial excitations a linear Regge spectrum with universal slope for the isosinglet vector, axial and scalar mesons. The choice of spectrum is motivated by hadron string models and related approaches and also by the meson spectroscopy. The considered ansatz allows one to solve the arising sum rules with a minimal number of inputs. Since the QCD sum rules do not describe microscopically neither the generation of QCD mass scale nor spontaneous chiral symmetry breaking, the minimal number of free parameters is two and they parametrize numerically the given two phenomena. In our scheme, the corresponding inputs are the gluon condensate and the pion decay constant. The numerical solution of arising equations reproduces the physical mass of $\omega(782)$ -

meson and a consistent value for the quark condensate. The excited spectrum of vector and axial states looks reasonable as well.

The obtained values of the slope of radial trajectories and quark condensate are then used for the analysis of scalar channel. We arrived at the conclusion that, interpolating the scalar states by the simplest quark bilinear current, a prediction of light scalar resonance with mass about 500 ± 100 MeV can be quite natural. We indicated on the reasons of absence of this pole in the QCD sum rules considered in the past. The coupling of this light scalar meson to an external source does not reveal any unusual features. It looks tempting to identify the given scalar state with $f_0(500)$, which is commonly interpreted as a highly unusual particle [2]. This identification would mean that at least the value of mass of $f_0(500)$ is not unusual. We also observed that the mass of the lightest scalar meson, although not being a part of the scalar radial Regge trajectory, correlates strongly with the mass parameters of that trajectory. Concerning the usual angular Regge trajectories for the quark–antiquark states, we proposed a corresponding angular trajectory with $f_0(500)$ on the top.

In summary, there is a possibility that the σ -meson represents a “turn-skin” resonance showing features of ordinary and non-ordinary hadrons simultaneously. This makes revealing its genuine nature even more challenging.

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