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BRST symmetry for Regge–Teitelboim-based minisuperspace models

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Abstract The Einstein–Hilbert action in the context of higher derivative theories is considered for finding their BRST symmetries. Being a constraint system, the model is transformed in the minisuperspace language with the FRLW background and the gauge symmetries are explored. Exploiting the first order formalism developed by Banerjee et al. the diffeomorphism symmetry is extracted. From the general form of the gauge transformations of the field, the analogous BRST transformations are calculated. The effective Lagrangian is constructed by considering two gauge-fixing conditions. Further, the BRST (conserved) charge is computed, which plays an important role in defining the physical states from the total Hilbert space of states. The finite field-dependent BRST formulation is also studied in this context where the Jacobian for the functional measure is illustrated specifically.

1 Introduction

In field theory, higher derivative terms are often considered to describe renormalization of the corresponding theory. In [1] Stelle showed that higher derivative gravity models can be renormalized. Being a simple model, the Einstein–Hilbert (EH) action, when expressed with respect to the metric components shows the higher derivative nature. A careful examination of the action shows that one can take out a total derivative term, which is known as the Gibbons–Hawking term [2]. This makes the Einstein–Hilbert theories first order ones. Although in the usual theories the surface terms are neglected, in gravitation they serve as important information as regards the entropy of the system. Therefore, removing the surface term trivially for the gravity models may lead to losing important information as regards the thermodynamics of the system. This motivates us to keep the total derivative term intact to the action, thereby making it a higher derivative theory.

In the minisuperspace version one considers finite degrees of freedom of a model to study the full theory. With a Friedmann–Robertson–Lamaitre–Walker (FRLW) background we construct the minisuperspace version of the EH action which is analogous to the minisuperspace version of the Regge–Teitelboim (RT) model [3]. A Hamiltonian analysis of this minisuperspace model has already been performed in different ways and quantization was further explored in [4–9]. Particularly, in [4] the same model with the higher derivative term was analyzed without neglecting the surface term. There the authors have explored the reparammetrization symmetries of the model exploiting the first order formalism developed earlier in [10]. To the best of our knowledge, the BRST analysis for this model is still lacking in the literature, which can be important from the view of quantization of the gravity models. Therefore, this paper will serve the purpose of visualizing the role of BRST symmetries in the cosmological models which will be of significant service in the study of the purely higher derivative cosmological models.

The BRST symmetry and the associated concept of BRST cohomology provide the most used covariant quantization method for constrained systems such as gauge [11], string, and gravity theories [12,13]. There appear geometrical constraints, for instance in topological solitons such as the nonlinear sigma model, the CP(N) model, the skyrmion model, and the chiral bag model, which can be rigorously treated in the Hamiltonian quantization scheme in extended phase spaces [14]. The extended phase space includes ghost and antighost fields of the theories. This symmetry guarantees the quantization, renormalizability, unitarity, and other aspects of the first-class constrained theories. The derivation of the Slavnov–Taylor identities utilizes the BRST symmetry transformation. The generalization of the BRST transformation

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by allowing the parameter to be finite and field dependent, known as a FFBRST symmetry [15], is studied extensively for diverse areas of research [16–38].

In the present paper, the minisuperspace version of the EH action is considered to study its BRST symmetries. For that we first perform the Hamiltonian formulation exploiting the first order formalism developed in [10]. Dirac's method for constraint analysis is followed to find the full constraint structure of the model [12,39–41]. The theory is found to have only one primary first-class constraint and one secondary first-class constraint. The existence of the primary first-class constraint is in conformity with one independent gauge symmetry, which is in the course of the calculation identified as reparametrization invariance. To canonically quantize the model two gauges were proposed in [4]. We considered the same gauge-fixing conditions to find their implications in the BRST symmetries. We found the nilpotent symmetries which are in line with the standard characteristics of the BRST symmetries. Also the BRST charge is calculated using the Noether charge definition. Further the BRST symmetry is generalized by integrating the transformation parameter after making it field dependent. By doing so we obtain a new symmetry of the theory. The main feature of this symmetry is that this leads to a non-trivial Jacobian for the functional measure under the change of variables, however, the effective action remains unchanged. We compute the Jacobian systematically for an arbitrary and for some specific choices. We observe that such an analysis amounts to a certain change in the effective action. Since this change appears in the BRST exact part, it is not harmful at all for the theory on physical grounds.

The paper is organized as follows. In Sect. 2 we give a very brief introduction to the RT model. In Sect. 3 we study the gauge symmetries using the first order formalism. Section 4 is our main contribution which we devoted to the exploration of the BRST symmetries. In this section we calculated the BRST transformation using the same gauges as considered in [4]. Also, we calculate the BRST charge and the Jacobian of the path integral under FFBRST symmetry, in Sect. 5. Finally, we conclude in Sect. 6.

2 Minisuperspace version of the Einstein–Hilbert action

In this section we convert the Einstein–Hilbert action to the minisuperspace version keeping the FRLW background. As mentioned earlier, this has a direct representation as the Regge–Teitelboim model (see [4,8]). As the universe has degrees of freedom, limiting the degrees of freedom can make life easier. So we reduce the dimension of spacetime which is the underlying concept for minisuperspace. Let us take a d-dimensional brane Σ evolving in a N-dimensional bulk spacetime with fixed Minkowski metric $\eta_{\mu\nu}$. The world volume swept out by the brane is a $d+1$-dimensional manifold m defined by the embedding $x^\mu = X^\mu(\xi^a)$ where x^μ are the local coordinates of the background spacetime and ξ^a are local coordinates for m. The theory is given by the action functional

$$S[X] = \int_m d^{d+1}\xi \sqrt{-g}\left(\frac{\beta}{2}\mathcal{R} - \Lambda\right). \tag{1}$$

Clearly, in this action integral β should have the dimension $[L]^{1-d}$. Here g is the determinant of the metric, Λ is the cosmological constant, and $\mathcal{R}$ is the Ricci scalar.

Next, we take a homogeneous and isotropic universe in four dimensions and embed it locally in a five-dimensional Minkowski spacetime. The reason behind taking this choice of dimension is worth mentioning. Algebraically, the embedding of a four-dimensional space in a 5D Minkowski space is correct but on physical grounds it has a limitation only to the vacuum solutions with cosmological constant. This is in line with the findings of Kasner, who was first to point out that most curved solutions in 4D cannot be embedded in five dimensions [46]. On the other hand, we can embed all four dimensions with vacuum solutions to six or higher dimensions. This helps us to understand the dynamics of 4D space with the help of an extra dimension.

Now we decompose the Lagrangian in the action (1) in the ADM formalism [42]. In $(4+1)$ dimensions if we take the normal vector ($N = \sqrt{\dot{t}^2 - \dot{a}^2}$ is the lapse function and the over dot '$\cdot$' is the derivative with respect to τ)

$$n_\mu = \frac{1}{N}\left(-\dot{a}, \dot{t}, 0, 0, 0\right), \tag{2}$$

the induced metric on the world volume is

$$\mathrm{d}s^2 = -N^2\mathrm{d}\tau^2 + a^2\mathrm{d}\Omega_2^2. \tag{3}$$

We will consider the above metric for the analysis of the system. The system is parametrized by τ. For a spherically symmetric system only $t(\tau)$ and $a(\tau)$ are the dynamical variables and the other variables can be integrated out. In this situation, the Ricci scalar is found to be

$$\mathcal{R} = \frac{6\dot{t}}{a^2N^4}\left(a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + N^2\dot{t}\right). \tag{4}$$

From (1) the Lagrangian takes the form in four dimensions of [4,8][1]

$$L(a, \dot{a}, \ddot{a}, \dot{t}, \ddot{t}) = \frac{a\dot{t}}{N^3}\left(a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + N^2\dot{t}\right) - Na^3H^2. \tag{5}$$

Note that the Lagrangian (5) contains higher derivative terms of the field a. However, we can write it as [8]

[1] Here $H^2 = \frac{\Lambda}{3\beta}$ is a constant quantity.

$$L = -\frac{a\dot{a}^2}{N} + aN\left(1 - a^2H^2\right) + \frac{d}{d\tau}\left(\frac{a^2\dot{a}}{N}\right). \tag{6}$$

It is worthy to note that the boundary term in this case plays no role in the equations of motion. However, being a gravitational system, it is better to keep it in the first place as it carries important information as regards entropy. Also, this will enable us to understand a HD gravity theory. For the Hamiltonian analysis of the system we will consider the HD version, i.e. (5).

3 Hamiltonian analysis

In this section we will perform the Hamiltonian formulation to extract the inherent symmetries of the higher derivative system (5). Particularly, we will follow the method of the first order formalism developed in [10]. This is a different approach from the well-known Ostrogradski method. We give results rather briefly as details can be gathered from [4].

To begin with, we convert the HD Lagrangian (5) into a first order Lagrangian by defining the time derivative of the fields as new fields. Thus in the equivalent first order formalism, we define the new fields by

$$\begin{aligned} \dot{a} &= A, \\ \dot{t} &= T. \end{aligned} \tag{7}$$

These definitions introduce new constraints in the system given by

$$\begin{aligned} A - \dot{a} &\approx 0, \\ T - \dot{t} &\approx 0. \end{aligned} \tag{8}$$

Enforcing the constraints (8) through the Lagrange multipliers λ_a and λ_t,

$$\begin{aligned} L' = &\frac{aT}{(T^2 - A^2)^{\frac{3}{2}}}\left(aT\dot{A} - aA\dot{T} + \left(T^2 - A^2\right)T\right) \\ &- \left(T^2 - A^2\right)^{\frac{1}{2}} a^3H^2 + \lambda_a(A - \dot{a}) + \lambda_t(T - \dot{t}), \end{aligned} \tag{9}$$

we get the first order Lagrangian, known as the auxiliary Lagrangian. The Euler–Lagrange equation of motion, obtained from the first order Lagrangian (9), by varying w.r.t. a, A, t, T, λ_a, and λ_t, are, respectively, given by

$$\begin{aligned} &\frac{2a(\dot{A}T^2 - AT\dot{T})}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{T^2}{(T^2 - A^2)^{\frac{1}{2}}} \\ &-3a^2H^2(T^2 - A^2)^{\frac{1}{2}} + \dot{\lambda}_a = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &\frac{3a^2A(\dot{A}T^2 - AT\dot{T})}{(T^2 - A^2)^{\frac{5}{2}}} - \frac{d}{d\tau}\left(\frac{a^2T^2}{(T^2 - A^2)^{\frac{3}{2}}}\right) - \frac{a^2T\dot{T}}{(T^2 - A^2)^{\frac{3}{2}}} \\ &+ \frac{aAT^2}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{a^3AH^2}{(T^2 - A^2)^{\frac{1}{2}}} + \lambda_a = 0, \end{aligned} \tag{11}$$

$$\dot{\lambda}_t = 0,$$

$$\begin{aligned} &\frac{3a^2T(\dot{A}T^2 - AT\dot{T})}{(T^2 - A^2)^{\frac{5}{2}}} + \frac{2a^2\dot{A}T}{(T^2 - A^2)^{\frac{3}{2}}} - \frac{d}{d\tau}\left(\frac{a^2AT}{(T^2 - A^2)^{\frac{3}{2}}}\right) \\ &- \frac{a^2A\dot{T}}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{2aT}{(T^2 - A^2)^{\frac{1}{2}}} \\ &- \frac{aT^3}{(T^2 - A^2)^{\frac{1}{2}}} + \lambda_t = 0, \end{aligned} \tag{12}$$

$$A - \dot{a} = 0, \tag{13}$$

$$T - \dot{t} = 0. \tag{14}$$

To perform the Hamiltonian formulation we notice that the phase space is spanned by $q_\mu = a, t, A, T, \lambda_a, \lambda_t$ and their corresponding momenta as $\Pi_{q_\mu} = \Pi_a, \Pi_t, \Pi_A, \Pi_T, \Pi_{\lambda_a}, \Pi_{\lambda_t}$ with $\mu = 0, 1, 2, 3, 4, 5$. Here the momenta are defined by

$$\Pi_{q_\mu} = \frac{\partial L'}{\partial \dot{q}_\mu}. \tag{15}$$

This is the point of departure of our Hamiltonian formulation from the Ostrogradsky convention of [8].

Using the definition of the momenta we find the following set of primary constraints:

$$\begin{aligned} \Phi_1 &= \Pi_t + \lambda_t \approx 0, \\ \Phi_2 &= \Pi_a + \lambda_a \approx 0, \\ \Phi_3 &= \Pi_T + \frac{a^2TA}{\left(T^2 - A^2\right)^{\frac{3}{2}}} \approx 0, \\ \Phi_4 &= \Pi_A - \frac{a^2T^2}{\left(T^2 - A^2\right)^{\frac{3}{2}}} \approx 0, \\ \Phi_5 &= \Pi_{\lambda_t} \approx 0, \\ \Phi_6 &= \Pi_{\lambda_a} \approx 0. \end{aligned} \tag{16}$$

To obtain the full set of primary first-class constraints we consider the constraint combination

$$\Phi_3' = T\Phi_3 + A\Phi_4 \approx 0. \tag{17}$$

Computing the Poisson brackets we find that only Φ_3' gives zero Poisson brackets with all the constraints. The nonzero Poisson brackets between the newly defined primary set of constraints, $\Phi_1, \Phi_2, \Phi_3', \Phi_4, \Phi_5, \Phi_6$, become

$$\{\Phi_1, \Phi_5\} = 1,$$
$$\{\Phi_2, \Phi_4\} = \frac{2aT^2}{\left(T^2 - A^2\right)^{\frac{3}{2}}}, \tag{18}$$
$$\{\Phi_2, \Phi_6\} = 1.$$

Now, the expression for the canonical Hamiltonian is given by

$$H_{\text{can}} = -\frac{aT^2}{\left(T^2 - A^2\right)^{\frac{1}{2}}} + \left(T^2 - A^2\right)^{\frac{1}{2}} a^3 H^2 - \lambda_a A - \lambda_t T, \tag{19}$$

while the total Hamiltonian becomes

$$H_T = H_{\text{can}} + \Lambda_1\Phi_1 + \Lambda_2\Phi_2 + \Lambda_3\Phi_3' + \Lambda_4\Phi_4 + \Lambda_5\Phi_5 + \Lambda_6\Phi_6. \tag{20}$$

Here $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6$ are undetermined Lagrange multipliers. Demanding the time evolution of the constraints Φ_1, Φ_5, Φ_6 to be zero ($\{\Phi_i, H_T\} \approx 0$) the following Lagrange multipliers get fixed:

$$\Lambda_5 = 0,$$
$$\Lambda_1 = T, \tag{21}$$
$$\Lambda_2 = A,$$

whereas conservation of Φ_2 gives the following condition between Λ_4 and Λ_6:

$$\frac{T^2}{\left(T^2 - A^2\right)^{\frac{1}{2}}} - 3a^2H^2\left(T^2 - A^2\right)^{\frac{1}{2}} + \Lambda_6 + \Lambda_4\frac{2aT^2}{\left(T^2 - A^2\right)^{\frac{3}{2}}} = 0. \tag{22}$$

Time preservation of the constraint Φ_3' gives rise to the following secondary constraint:

$$\Psi_1 = \frac{aT^2}{\left(T^2 - A^2\right)^{\frac{1}{2}}} - a^3H^2\left(T^2 - A^2\right)^{\frac{1}{2}} + \lambda_t T + \lambda_a A \approx 0. \tag{23}$$

Likewise, Φ_4 yields the following secondary constraint:

$$\Psi_2 = \frac{aAT^2}{\left(T^2 - A^2\right)^{\frac{3}{2}}} - \frac{a^3H^2A}{\left(T^2 - A^2\right)^{\frac{1}{2}}} - \lambda_a \approx 0. \tag{24}$$

Time preservation of Ψ_1 trivially gives $0 = 0$. A similar analysis involving Ψ_2 yields, on exploiting (22),

$$\Lambda_4 = -\frac{\left(T^2 - 3a^2H^2\left(T^2 - A^2\right)\right)\left(T^2 - A^2\right)}{a\left(3T^2 - a^2H^2\left(T^2 - A^2\right)\right)},$$
$$\Lambda_6 = -\frac{\left(T^2 - 3a^2H^2\left(T^2 - A^2\right)\right)\left(T^2 - a^2H^2\left(T^2 - A^2\right)^{\frac{1}{2}}\right)}{\left(T^2 - A^2\right)^{\frac{1}{2}}\left(3T^2 - a^2H^2\left(T^2 - A^2\right)\right)}. \tag{25}$$

The iterative procedure is thus closed and no more secondary constraints or other relations are generated.

Finally, after manipulation of the constraints we arrive at the following set of constraints:

$$F_1 = \Phi_3' = T\Phi_3 + A\Phi_4 \approx 0,$$
$$F_2 = \Psi_1' = \Psi_1 - \Lambda_4\Phi_4 \approx 0,$$
$$S_1 = \Phi_4 \approx 0, \tag{26}$$
$$S_2 = \Psi_2 = \frac{aAT^2}{\left(T^2 - A^2\right)^{\frac{3}{2}}} - \frac{a^3AH^2}{\left(T^2 - A^2\right)^{\frac{1}{2}}} + \Pi_a \approx 0.$$

Here $\{F_1, F_2\}$ are the pair of first-class constraints and $\{S_1, S_2\}$ are the pair of second-class constraints. F_1 is the primary first-class constraint which is consistent with the fact that there is only one undetermined multiplier in the theory. For removal of the second-class constraints the Poisson brackets will be replaced by the Dirac brackets, defined by

$$\{f, g\}_D = \{f, g\} - \{f, S_i\}\Delta_{ij}^{-1}\{S_j, g\}, \tag{27}$$

where we have taken

$$\Delta_{ij} = \{S_i, S_j\} = -\frac{aT^2\left(3T^2 - a^2H^2\left(T^2 - A^2\right)\right)}{\left(T^2 - A^2\right)^{\frac{5}{2}}}\epsilon_{ij}, \tag{28}$$

with $\epsilon_{12} = 1$ and $i, j = 1, 2$.

We calculate the Dirac brackets between the basic fields which are given now (only the nonzero brackets are listed):

$$\{a, A\}_D = -\frac{\left(T^2 - A^2\right)^{\frac{5}{2}}}{aT^2\left(3T^2 - a^2H^2\left(T^2 - A^2\right)\right)},$$
$$\{a, \Pi_a\}_D = \frac{T^2 + 2A^2 - a^2H^2\left(T^2 - A^2\right)}{\left(3T^2 - a^2H^2\left(T^2 - A^2\right)\right)},$$
$$\{a, \Pi_A\}_D = -\frac{3aA}{3T^2 - a^2H^2\left(T^2 - A^2\right)},$$
$$\{a, \Pi_T\}_D = \frac{a\left(T^2 + 2A^2\right)}{T\left(3T^2 - a^2H^2(T^2 - A^2)\right)},$$
$$\{t, \Pi_t\}_D = 1,$$
$$\{A, \Pi_a\}_D = -\frac{A\left(T^2 - A^2\right)\left(T^2 - 3a^2H^2\left(T^2 - A^2\right)\right)}{aT^2\left(3T^2 - a^2H^2\left(T^2 - A^2\right)\right)},$$
$$\{A, \Pi_A\}_D = \frac{2\left(T^2 - A^2\right)}{3T^2 - a^2H^2\left(T^2 - A^2\right)},$$

$$\{A,\Pi_T\}_D = \frac{A\left(T^2+2A^2-a^2H^2\left(T^2-A^2\right)\right)}{T\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)},$$
$$\{T,\Pi_T\}_D = 1,$$
$$\{\Pi_a,\Pi_A\}_D = -\frac{a\left(2T^4+A^2T^2+a^2H^2(T^2-A^2)(9A^2-2T^2)\right)}{\left(T^2-A^2\right)^{\frac{3}{2}}\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)},$$
$$\{\Pi_a,\Pi_T\}_D = \frac{aA\left(T^4+2T^2A^2+a^2H^2(T^2-A^2)(T^2+6A^2)\right)}{T\left(T^2-A^2\right)^{\frac{3}{2}}\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)},$$
$$\{\Pi_A,\Pi_T\}_D = -\frac{a^2T\left(T^2+2A^2-a^2H^2(T^2-A^2)\right)}{\left(T^2-A^2\right)^{\frac{3}{2}}\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}. \quad (29)$$

Due to the Dirac brackets all the second-class constraints become strongly zero. So in this theory there is only two first-class constraint. Now we proceed to show the gauge symmetry of the system.

3.1 Construction of the gauge generator

To construct the BRST symmetries, it is important to find the gauge symmetries of the theory. So, we construct the gauge generator which is the linear combination of all the first-class constraints given by

$$G = \epsilon_1 F_1 + \epsilon_2 F_2. \quad (30)$$

Here $\{\Phi_a\}$ is the whole set of constraints and ϵ_a, $a = 1, 2$ are the gauge parameters. However, not all the gauge parameters ϵ_a are independent. The independent gauge symmetries can be found by applying the algorithm developed in [43–45]. In addition, one should be careful about finding the independent symmetries, since it is a HD theory. For the HD theories the commutativity of gauge variation and time translation, i.e.

$$\delta q_{n,\alpha} - \frac{d}{dt}\delta q_{n,\alpha-1} = 0,\ (\alpha > 1)\,, \quad (31)$$

where $q_{n,\alpha}$ denotes the αth order time derivative of q, can play very important role in finding the independent gauge parameters [10]. Now, to find the independent gauge parameters we use [43–45] which gives

$$\epsilon_1 = -\Lambda_3\epsilon_2 - \dot{\epsilon}_2. \quad (32)$$

So here ϵ_2 may be chosen as the independent gauge parameter. This means that there is only one independent gauge transformation which essentially is in conformation with the fact that there is only one independent primary first-class constraint.

The gauge transformations of the fields are given by

$$\delta a = \{a, G\}_D = -\epsilon_2 A, \quad (33)$$
$$\delta t = -\epsilon_2 T, \quad (34)$$
$$\delta A = \epsilon_1 A - \epsilon_2\frac{\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\left(T^2-A^2\right)}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}, \quad (35)$$
$$\delta T = \epsilon_1 T. \quad (36)$$

This completes our analysis of finding the gauge symmetries of the system. We have seen that the RT model has one independent primary first-class constraint, which is consistent with the fact that there is only one independent gauge symmetry.

4 BRST quantization

As we have seen, the RT model possesses a redundant degree of freedom and has been identified as the diffeomorphism symmetry in [4]. The principle of gauge invariance is essential to constructing a workable RT model. But it is not feasible to perform a perturbative calculation without first fixing the gauge i.e. adding terms to the Lagrangian density of the action principle which break the gauge symmetry to suppress these unphysical degrees of freedom. Since there are two first-class constraints, we take two well-motivated gauge-fixing terms,

$$\varphi_1 = \sqrt{T^2-A^2} - 1 \approx 0 \quad (37)$$

and

$$\varphi_2 = T - \alpha a \approx 0. \quad (38)$$

The first gauge is the very well-known cosmic gauge and the second gauge was proposed in [4]. Here α is some number obeying the condition $\alpha \neq H$. With these choices of gauges the first-class constraints become second class. It is also known that such a gauge-fixing condition will induce ghost terms for the model which plays an important role in the proof of unitarity of the theory. Now we perform the BRST symmetry analysis of this model.

4.1 Gauge fixing and Faddeev–Popov action

For simplicity we rename the gauge parameter ϵ_2 by ϵ. From (32) we can see that the other gauge parameter becomes $-\Lambda_3\epsilon - \dot{\epsilon}$. The above gauge conditions can be incorporated at the quantum level by adding the following term to the classical action:

$$L_{gf} = \lambda(\sqrt{T^2-A^2} - 1 + T - \alpha a), \quad (39)$$

here the two first-class constraints are incorporated by a single Lagrange multiplier λ as there is only one independent gauge symmetry. To construct the ghost term in the effective Lagrangian we first compute the gauge variation of the gauges φ_1 and φ_2. They are given by

$$\delta\varphi_1 = -\sqrt{T^2-A^2}\Lambda_3\epsilon - \sqrt{T^2-A^2}\dot{\epsilon} - \frac{A\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\sqrt{T^2-A^2}}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}\epsilon, \quad (40)$$
$$\delta\varphi_2 = T(-\Lambda_3\epsilon - \dot{\epsilon}) + \alpha\epsilon A. \quad (41)$$

So we can easily calculate the ghost term for the effective Lagrangian as

$$\begin{aligned}
L_{gh} &= \bar{c}\left(\frac{\partial}{\partial\epsilon}\delta\varphi_1 + \frac{\partial}{\partial\epsilon}\delta\varphi_2\right)c \\
&= -\bar{c}\sqrt{T^2-A^2}\Lambda_3 c - \bar{c}\sqrt{T^2-A^2}\dot{c} \\
&\quad - \bar{c}\frac{A\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\sqrt{T^2-A^2}}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}c \\
&\quad - \bar{c}T\Lambda_3 c - \bar{c}T\dot{c} + \bar{c}\alpha A c, \qquad (42)
\end{aligned}$$

where c is a ghost field and $\bar{c}$ is an antighost field. These ghost and antighost fields have a geometrical interpretation in terms of Maurer–Cartan forms on a diffeomorphism. By adding the gauge-fixing and ghost terms to the classical Lagrangian, the effective Lagrangian is given by

$$\begin{aligned}
L_{eff} &= L + \lambda(\sqrt{T^2-A^2} - 1 + T - \alpha a) \\
&\quad - \bar{c}\sqrt{T^2-A^2}\Lambda_3 c - \bar{c}\sqrt{T^2-A^2}\dot{c} \\
&\quad - \bar{c}\frac{A\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\sqrt{T^2-A^2}}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}c \\
&\quad - \bar{c}T\Lambda_3 c - \bar{c}T\dot{c} + \bar{c}\alpha A c. \qquad (43)
\end{aligned}$$

Following the structure of the gauge transformations (33–36) of the fields, we are able to construct the BRST transformations, thus

$$\delta_b a = -cA\eta, \qquad (44)$$

$$\delta_b t = -cT\eta, \qquad (45)$$

$$\begin{aligned}
\delta_b A &= (-\Lambda_3 c - \dot{c})A\eta \\
&\quad - c\frac{\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\left(T^2-A^2\right)}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}\eta, \qquad (46)
\end{aligned}$$

$$\delta_b T = (-\Lambda_3 c - \dot{c})T\eta, \qquad (47)$$

$$\delta_b c = 0, \quad \delta_b \lambda = 0, \quad \delta_b \bar{c} = \lambda\eta, \qquad (48)$$

which leaves the above effective action invariant. Here η is the Grassmann parameter of the transformation. It is easy to check that the above transformation is nilpotent, i.e. $\delta_b^2 = 0$. This transformation can be used to compute the Ward identities which will yield the relation between different Green's function. Here we are also able to define the anti-BRST transformation just by replacing the ghost by an antighost field and vice versa.

4.2 BRST charge calculation

In this subsection we derive the total BRST charge corresponding to the above BRST symmetry. Utilizing the Noether formula, we calculate the BRST charges for the different fields as

$$\begin{aligned}
Q_a &= \frac{\partial\mathcal{L}}{\partial\dot{a}}\delta_b a\frac{1}{\eta} = -\lambda_a(-cA\eta)\frac{1}{\eta} = \lambda_a cA, \\
Q_t &= -\lambda_t(-cT\eta)\frac{1}{\eta} = \lambda_t cT, \\
Q_A &= \frac{a^2T^2}{(T^2-a^2)^{\frac{3}{2}}} \\
&\quad \times \left((-\Lambda_3 c - \dot{c})A - c\frac{\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\left(T^2-A^2\right)}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}\right), \\
Q_T &= -\frac{a^2T^2A}{(T^2-a^2)^{\frac{3}{2}}}(-\Lambda_3 c - \dot{c}), \\
Q_c &= Q_{\bar{c}} = 0. \qquad (49)
\end{aligned}$$

The total BRST charge, which is a hermitian operator, is given by

$$Q = \lambda_a cA + \lambda_t cT + \frac{a^2T^2}{(T^2-a^2)^{\frac{1}{2}}}c\frac{\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}.$$

This operator Q^2, which implements the BRST symmetry in quantum Hilbert space, should be nilpotent, which is evident from the above expression. In order to get probabilistic interpretation of the model we must project out all the physical states in the positive definite Hilbert space. Now, this charge (Q) annihilates the physical states of the total Hilbert space as follows:

$$Q|\text{phys}\rangle = 0, \qquad (50)$$

which helps in defining the physical states in total Hilbert space of the theory.

Similarly, it is also possible to compute the anti-BRST symmetry of this model where the role of ghost fields will be changed by antighost field. The conserved charge for anti-BRST symmetry ($\bar{Q}$) must also satisfy the Kugo–Ojima condition:

$$\bar{Q}|\text{phys}\rangle = 0. \qquad (51)$$

The solution of this equation will give us the quantum mechanical wave function of the universe. The BRST charge obtained in (50) can be mapped easily to the Wheeler– DeWitt equation obtained in [4]. Here, in this paper, in the BRST approach we have not solved the first-class constraints and hence the quantization is actually done in the extended phase space. The corresponding form of the WDW potential will also depend upon the variables of extended phase space. Here we did only the Lagrangian formulation to construct the BRST charges. There exists another process namely the BFV [47–49] formulation, by which one can construct the Hamiltonian of the model with ghost fields in the extended phase space. In this approach the ghost field and the Lagrange multipliers are treated as dynamical variable and is a gauge-independent way. The Hamiltonian thus obtained enables one to directly write down the Wheeler– DeWitt equation and the Wheeler–DeWitt potential will be straightforward.

5 Finite field-dependent BRST transformation

The purpose of this section is to study the extended BRST symmetry, known as FFBRST transformation, by making the transformation parameter finite and field dependent. Such transformations have been studied in various contexts with various important motivations. We try to build such formulation first time in RT based minisuperspace models. We achieve this goal by deriving first the methodology of the FFBRST transformation, originally advocated in Ref. [15], in a very elegant way. Then we discuss an illustration.

We start with the consideration of the fields of RT model, written collectively as ϕ, as a function of parameter κ : $0 \leq \kappa \leq 1$ in such a manner that the original fields and finitely transformed fields are described by its extremum values. Specifically, $\phi(\tau, \kappa = 0) = \phi(\tau)$ defines the original (non-transformed) fields, however, $\phi(\tau, \kappa = 1) = \phi'(\tau)$ refers to the finite field-dependent BRST transformed fields. Now, fields transform under an infinitesimal field-dependent BRST transformation as [15]

$$\frac{\mathrm{d}\phi(\tau,\kappa)}{\mathrm{d}\kappa} = s_b\phi(\tau,\kappa)\Theta'[\phi(\tau,\kappa)]. \tag{52}$$

Here $s_b\phi$ refers to the Slavnov variation. To get the FFBRST transformation, we first integrate the above equation w.r.t. κ from 0 to κ, leading to

$$\phi(\tau,\kappa) = \phi(\tau,0) + s_b\phi(\tau,0)\Theta[\phi(\tau,\kappa)], \tag{53}$$

which at the boundary ($\kappa = 1$) yields the FFBRST transformation [15],

$$\phi'(\tau) = \phi(\tau) + s_b\phi(\tau)\Theta[\phi(\tau)]. \tag{54}$$

Here $\Theta[\phi]$ is a finite field-dependent parameter related to an infinitesimal version by $\Theta[\phi] = \int \mathrm{d}\kappa\, \Theta'[\phi]$. The remarkable feature of this FFBRST transformation is that this leaves the Faddeev–Popov action invariant; however, the functional measure is not, leading to a non-trivial Jacobian. We compute the Jacobian of path integral measure under such a FFBRST transformation by following a similar procedure (up to some extent) as discussed in [15]. The Jacobian of the functional measure under an infinitesimal change is given by

$$\mathcal{D}\phi = J(\kappa)\mathcal{D}\phi(\kappa) = J(\kappa + \mathrm{d}\kappa)\mathcal{D}\phi(\kappa + \mathrm{d}\kappa), \tag{55}$$

which further reads

$$\frac{J(\kappa)}{J(\kappa+\mathrm{d}\kappa)} = \sum_\phi \pm \frac{\delta\phi(\kappa+\mathrm{d}\kappa)}{\delta\phi(\kappa)}, \tag{56}$$

where $\pm$ signs are considered according to the nature of the fields ϕ. For instance, $+$ is used for bosonic fields and $-$ for fermionic ones. Upon taking a Taylor expansion, Eq. (56) yields

$$-\frac{1}{J}\frac{\mathrm{d}J}{\mathrm{d}\kappa}\mathrm{d}\kappa = \mathrm{d}\kappa \int \mathrm{d}\tau \sum_\phi \pm s_b\phi(\tau,\kappa)\frac{\delta\Theta'[\phi(\tau,\kappa)]}{\delta\phi(\tau,\kappa)}. \tag{57}$$

This further simplifies to

$$\frac{\mathrm{d}\ln J}{\mathrm{d}\kappa} = -\int \mathrm{d}\tau \sum_\phi \pm s_b\phi(\tau,\kappa)\frac{\delta\Theta'[\phi(\tau,\kappa)]}{\delta\phi(\tau,\kappa)}. \tag{58}$$

Upon integration w.r.t. κ with its limiting values, this leads to following expression:

$$\ln J = -\int_0^1 \mathrm{d}\kappa \int \mathrm{d}\tau \sum_\phi \pm s_b\phi(\tau,\kappa)\frac{\delta\Theta'[\phi(\tau,\kappa)]}{\delta\phi(\tau,\kappa)},$$
$$= -\left(\int \mathrm{d}^4x \sum_\phi \pm s_b\phi(\tau)\frac{\delta\Theta'[\phi(\tau)]}{\delta\phi(\tau)}\right). \tag{59}$$

By exponentiating the above expression, we get the explicit form of the Jacobian of the functional measure as follows:

$$J = \exp\left(-\int \mathrm{d}\tau \sum_\phi \pm s_b\phi(\tau)\frac{\delta\Theta'[\phi(\tau)]}{\delta\phi(\tau)}\right). \tag{60}$$

Due to this Jacobian under the FFBRST transformation, an effective action $S[\phi]$ of the generating functional of the theory undergoes the following change:

$$\int \mathcal{D}\phi'\, e^{iS[\phi']} = \int \mathcal{D}\phi\, e^{iS[\phi] - \int \mathrm{d}\tau\left(\sum_\phi \pm s_b\phi\frac{\delta\Theta'}{\delta\phi}\right)}, \tag{61}$$

where ϕ' refers to the transformed fields collectively. Here we draw the conclusion that under the whole procedure, the effective action of the theory gets a precise modification in their original expression by an extra piece. We will notice that under such an analysis the theory does not change on physical grounds. The resulting action depends on an arbitrary value of Θ'. In the next subsection, we illustrate this result by considering a specific value of Θ'.

5.1 Jacobian calculation

For illustrating the above results discussed in the last section, we compute the Jacobian of the functional measure explicitly. To compute the Jacobian under a specific FFBRST transformation, we first construct an infinitesimal field-dependent parameter of the following form:

$$\Theta'[\tau] = -i\bar{c}\left[\sqrt{T^2 - A^2} - 1 + T - \alpha a - F[\varphi]\right], \tag{62}$$

where $F[\varphi]$ is an arbitrary gauge-fixing condition with $\varphi \equiv T, A, a, \alpha$. In the construction of this parameter, we take care of the ghost number of Θ', which must be -1.

Now, Eq. (60) together with (62) yields

$$\begin{aligned}J = \exp\Bigg[i\Bigg(& -\lambda(\sqrt{T^2-A^2}-1+T-\alpha a) \\ & +\bar{c}\sqrt{T^2-A^2}\Lambda_3 c+\bar{c}\sqrt{T^2-A^2}\dot{c} \\ & +\bar{c}\frac{A\left(T^2-3a^2H^2\left(T^2-A^2\right)\right)\sqrt{T^2-A^2}}{a\left(3T^2-a^2H^2\left(T^2-A^2\right)\right)}c+\bar{c}T\Lambda_3 c \\ & +\bar{c}T\dot{c}-\bar{c}\alpha Ac+\lambda F[\varphi]-\bar{c}s_b\varphi\frac{\delta F}{\delta\varphi}\Bigg)\Bigg]. \end{aligned} \tag{63}$$

This corresponds to the Jacobian for the path integral measure under the FFBRST transformation with the parameter in (62). With this Jacobian, the functional integral (under FFBRST transformation) changes to

$$\begin{aligned}\int \mathcal{D}\phi'\, e^{i\int \mathrm{d}\tau L_{\mathrm{eff}}[\phi']} &= \int J[\phi]\mathcal{D}\phi\, e^{i\int \mathrm{d}\tau L_{\mathrm{eff}}[\phi]}, \\ &= \int \mathcal{D}\phi\, \exp\left[i\int \mathrm{d}\tau\left(L+\lambda F[\varphi]-\bar{c}s_b\varphi\frac{\delta F}{\delta\varphi}\right)\right], \end{aligned} \tag{64}$$

which is nothing but the expression for the generating functional of the minisuperspace model in an arbitrary gauge. This ensures that the FFBRST transformation maps the minisuperspace model in one specific gauge to any other arbitrary gauge. Since the gauge choice from one to another does not change the theory on physical grounds the gauge-fixed action is BRST exact. Thus, we conclude that the FFBRST transformation with any particular parameter does not change the theory numerically.

6 Conclusions

In this paper, we analyzed the BRST symmetries of the Einstein–Hilbert (EH) action in the minisuperspace representation. Einsitein's theory includes first-class constraints for which BRST symmetries play a significant role during quantization. We constrain the generalized EH action, to 4 + 1 dimensions. So the bulk we considered is five-dimensional, while gravity is induced on the 4D hypersurface. The cosmological constant has been considered here for the reason that five-dimensional theory cannot have a vacuum solution [46]. For the homogeneous and isotropic FRLW background, the Lagrangian which was obtained appears to have a pseudo higher derivative nature. The Lagrangian we derived with respect to the minisuperspace variables contains a total derivative term in addition to the first order term. Without avoiding the surface term we considered the higher derivative version as the existence of the surface term, which is directly linked to the thermodynamics of the system. A Dirac constraint analysis showed that there are one primary first-class constraint and one secondary first-class constraint. We have seen that the model consists of only one gauge symmetry i.e. the reparametrization invariance. The gauge symmetry is due to the existence of the primary first-class constraints. Looking at the gauge structure of the system, analogous BRST symmetries were constructed. We considered the cosmic gauge and the gauge proposed in [4] to construct the effective Lagrangian. As there is only one gauge symmetry, these two constraints were added to the effective Lagrangian by a single Lagrange multiplier. The BRST transformations for all the fields in the extended phase space were calculated. Further we have calculated the BRST charges utilizing the Noether theorem. The charge so constructed annihilates the physical states. The finite field-dependent BRST (FFBRST) transformations were also analyzed for the theory. We have computed the Jacobian for the path integral under a FFBRST transformation with an arbitrary parameter, which amounts to a precise change in the effective action. The results were also illustrated with a particular construction of the parameter which maps the action in a specific gauge to an arbitrary gauge.

The aim to construct the BRST symmetries for a higher derivative system has been completed in the previous sections. The utility of the first order formalism to extract the gauge symmetries for the higher derivative system was invoked in the present paper. In fact, apart from being just a mathematical tool the method described here can be helpful for other higher-dimensional models like the Randall–Sundrum background or more likely the Dvali–Gabadadze–Porrati extensions [50,51]. As the Ransdall–Sundrum (RS) model deals with the extra dimensions, one can intriguingly find connection of the model discussed in this paper. The induced metric (3), which is four-dimensional, here can give the results as obtained only after consideration of the normal vector (2). So in the embedded space the choice of the normal vector is crucial. Consequently this will lead to the metric discussed in [50,51]. One can further use these result to quantize the system and use more tools, like the renormalization from quantum field theories and can see the outcomes. The analysis is left here with options for future consideration taking a step toward quantization in the path integral method for better understanding of the more complicated higher derivative actions.

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References

1. K.S. Stelle, Phys. Rev. D **16**, 953 (1977)
2. G.W. Gibbons, S.W. Hawking, Phys. Rev. D **15**, 2752 (1977)

3. T. Regge, C. Teitelboim, *Proceedings of the Marcel Grossman Meeting*, Trieste, Italy, 1975, ed. by R. Ruffini (North-Holland, Amsterdam, 1977), p. 77
4. R. Banerjee, P. Mukherjee, B. Paul, Phys. Rev. D **89**, 043508 (2014)
5. A. Davidson, D. Karasik, Y. Lederer, Class. Quant. Grav. **16**, 1349 (1999)
6. A. Davidson, D. Karasik, Y. Lederer, Phys. Rev. D **72**, 064011 (2005)
7. D. Karasik, A. Davidson, Phys. Rev. D **67**, 064012 (2003)
8. R. Cordero, A. Molgado, E. Rojas, Phys. Rev. D **79**, 024024 (2009)
9. R. Cordero, A. Molgado, E. Rojas, Gen. Relativ. Grav. **46**, 1761 (2014). arXiv:1309.3031
10. R. Banerjee, P. Mukherjee, B. Paul, JHEP **08**, 085 (2011)
11. R. Banerjee, B. Paul, S. Upadhyay, Phys. Rev. D **88**, 065019 (2013). arXiv:1306.0744
12. M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992)
13. S. Weinberg, *The Quantum Theory of Fields, Vol. II: Modern Applications* (UK University Press, Cambridge, 1996)
14. S.-T. Hong, Y.-J. Park, Phys. Rep. **358**, 143 (2002)
15. S.D. Joglekar, B.P. Mandal, Phys. Rev. D **51**, 1919 (1995)
16. S. Upadhyay, Phys. Lett. B **740**, 341 (2015)
17. S. Upadhyay, Ann. Phys. **340**, 110 (2014)
18. S. Upadhyay, Ann. Phys. **344**, 290 (2014)
19. S. Upadhyay, EPL **105**, 21001 (2014)
20. S. Upadhyay, EPL **104**, 61001 (2013)
21. S. Upadhyay, Phys. Lett. B **727**, 293 (2013)
22. S. Upadhyay, Ann. Phys. **356**, 299 (2015)
23. S. Upadhyay, Mod. Phys. Lett. A **30**, 1550072 (2015)
24. S. Upadhyay, M.K. Dwivedi, B.P. Mandal, Int. J. Mod. Phys. A **30**, 1550178 (2015)
25. S. Upadhyay, M.K. Dwivedi, B.P. Mandal, Int. J. Mod. Phys. A **28**, 1350033 (2013)
26. S. Upadhyay, D. Das, Phys. Lett. B **733**, 63 (2014)
27. R. Banerjee, S. Upadhyay, Phys. Lett. B **734**, 369 (2014)
28. M. Faizal, S. Upadhyay, B.P. Mandal, Eur. Phys. J. C **76**, 189 (2016)
29. M. Faizal, S. Upadhyay, B.P. Mandal, Phys. Lett. B **738**, 201 (2014)
30. M. Faizal, S. Upadhyay, B.P. Mandal, Int. J. Mod. Phys. A **30**, 1550032 (2015)
31. S. Upadhyay, B.P. Mandal, Phys. Lett. B **744**, 231 (2015)
32. S. Upadhyay, B.P. Mandal, Eur. Phys. J. C **75**, 327 (2015)
33. S. Upadhyay, B.P. Mandal, Int. J. Theor. Phys. **55**, 1 (2016)
34. S. Upadhyay, B.P. Mandal, Prog. Theor. Exp. Phys. **053B04**, 1 (2014)
35. S. Upadhyay, B.P. Mandal, Eur. Phys. J. C **72**, 2065 (2012)
36. S. Upadhyay, B.P. Mandal, Ann. Phys. **327**, 2885 (2012)
37. S. Upadhyay, B.P. Mandal, EPL **93**, 31001 (2011)
38. S. Upadhyay, B.P. Mandal, Mod. Phys. Lett. A **25**, 3347 (2010)
39. P.A.M. Dirac, Can. J. Math. **2**, 129 (1950)
40. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964)
41. A. Hanson, T. Regge, C. Tietelboim, *Constrained Hamiltonian System* (Accademia Nazionale Dei Lincei, Roma, 1976)
42. R. Arnowitt, S. Deser, C.W. Misner, Gen. Relativ. Grav. **40**, 1997 (2008)
43. R. Banerjee, H.J. Rothe, K.D. Rothe, Phys. Lett. B **463**, 248 (1999). arXiv:hep-th/9906072
44. R. Banerjee, H.J. Rothe, K.D. Rothe, Phys. Lett. B **479**, 429 (2000). arXiv:hep-th/9907217
45. R. Banerjee, H.J. Rothe, K.D. Rothe, J. Phys. A **33**, 2059 (2000). arXiv:hep-th/9909039
46. E. Kasner, Am. J. Math. **43**, 126 (1921)
47. E.S. Fradkin, G. Vilkovisky, Phys. Lett. B **55**, 224 (1975)
48. I.A. Batalin, G. Vilkovisky, Phys. Lett. B **69**, 309 (1977)
49. I.A. Batalin, E.S. Fradkin, Phys. Lett. B **122**, 157 (1983)
50. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999). arXiv:hep-ph/9905221
51. G. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208 (2000). arXiv:hep-th/0005016