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Analysis of the $\frac{1}{2}^{\pm}$ pentaquark states in the diquark–diquark–antiquark model with QCD sum rules

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Abstract In this article, we construct both the axial vector-diquark–axialvector-diquark–antiquark type and the axialvector-diquark–scalar-diquark–antiquark type interpolating currents, then calculate the contributions of the vacuum condensates up to dimension 10 in the operator product expansion, and we study the masses and pole residues of the $J^P = \frac{1}{2}^{\pm}$ hidden-charm pentaquark states with the QCD sum rules in a systematic way. In calculations, we use the formula $\mu = \sqrt{M_P^2 - (2\mathbb{M}_c)^2}$ to determine the energy scales of the QCD spectral densities. We take into account the SU(3) breaking effects of the light quarks, and we obtain the masses of the hidden-charm pentaquark states with the strangeness $S = 0, -1, -2, -3$, which can be confronted with the experimental data in the future.

1 Introduction

Recently, the LHCb collaboration studied the $\Lambda_b^0 \to J/\psi K^- p$ decays, and one observed two pentaquark candidates $P_c(4380)$ and $P_c(4450)$ in the $J/\psi p$ mass spectrum with the significances of more than 9σ [1]. The measured masses and widths are $M_{P_c(4380)} = 4380 \pm 8 \pm 29$ MeV, $M_{P_c(4450)} = 4449.8 \pm 1.7 \pm 2.5$ MeV, $\Gamma_{P_c(4380)} = 205 \pm 18 \pm 86$ MeV, and $\Gamma_{P_c(4450)} = 39 \pm 5 \pm 19$ MeV, respectively. The decays $P_c(4380) \to J/\psi p$ take place through the relative S-wave channel, while the decays $P_c(4450) \to J/\psi p$ take place through the relative P-wave channel, the decays $P_c(4380) \to J/\psi p$ are kinematically favored and $P_c(4380)$ has larger width. The preferred spin-parities of $P_c(4380)$ and $P_c(4450)$ are $J^P = \frac{3}{2}^-$ and $\frac{5}{2}^+$, respectively. There have been several attempted assignments, such as $\Sigma_c \bar{D}^*$, $\Sigma_c^* \bar{D}^*$, $\chi_{c1} p$, $J/\psi N(1440)$, and the $J/\psi N(1520)$ molecule-like pentaquark states [2–11] (or not the molecular pentaquark states [12]), the diquark–diquark–antiquark type pentaquark states [13–22], the diquark–triquark type pentaquark states [23], re-scattering effects [24–27], etc.

The quarks have color SU(3) symmetry; we can construct the pentaquark configurations according to the routine quark $\to$ diquark $\to$ pentaquark,

$$(3 \otimes 3) \otimes (3 \otimes 3) \otimes \overline{3} = (\overline{3} \oplus 6) \otimes (\overline{3} \oplus 6) \otimes \overline{3} = \overline{3} \otimes \overline{3} \otimes \overline{3} \oplus \cdots = 1 \oplus \cdots, \tag{1}$$

where 1, 3 ($\overline{3}$), and 6 denote the color singlet, triplet (antitriplet), and sextet, respectively. The diquarks $q_j^T C\Gamma q_k'$ have five structures in Dirac spinor space, where $C\Gamma = C\gamma_5$, C, $C\gamma_\mu\gamma_5$, $C\gamma_\mu$, and $C\sigma_{\mu\nu}$ for the scalar, pseudoscalar, vector, axialvector, and tensor diquarks, respectively, and the j and k are color indices. The attractive interactions of one-gluon exchange favor formation of the diquarks in color antitriplet $\overline{3}_c$, flavor antitriplet $\overline{3}_f$, and spin singlet 1_s [28,29], while the favored configurations are the scalar ($C\gamma_5$) and axialvector ($C\gamma_\mu$) diquark states [30–32]. The calculations based on the QCD sum rules indicate that the heavy-light scalar and axialvector diquark states have almost degenerate masses [30,31], while the masses of the light axialvector diquark states lie (150–200) MeV above that of the light scalar-diquark states [32], if they have the same quark constituents.

In Refs. [19,20], we choose the light scalar diquark and heavy axialvector diquark (or heavy scalar diquark) as the basic constituents, construct both the scalar-diquark–axialvector-diquark–antiquark type and scalar-diquark–scalar-diquark–antiquark type interpolating currents, which are supposed to couple potentially to the lowest pentaquark states according to the light scalar-diquark constituent [32], then calculate the contributions of the vacuum condensates up to dimension 10 in the operator product expansion and study the masses and pole residues of $J^P = \frac{3}{2}^-$, $\frac{5}{2}^+$, and $\frac{1}{2}^{\pm}$ hidden-charm pentaquark states with the QCD sum rules. The numerical results favor assigning $P_c(4380)$ and $P_c(4450)$

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to be the $\frac{3}{2}^-$ and $\frac{5}{2}^+$ diquark–diquark–antiquark type pentaquark states, respectively [19]. In Ref. [19], we take the energy-scale formula $\mu=\sqrt{M_P^2-(2\mathbb{M}_c)^2}$ to determine the energy scales of the QCD spectral densities, the resulting pole contributions are about (40–60) %, and the contributions of the vacuum condensates of dimension 10 are less than 5 %, the two criteria (pole dominance at the phenomenological side and convergence of the operator product expansion) of the conventional QCD sum rules can be satisfied. Now we extend our previous work to study the $\frac{1}{2}^\pm$ hidden-charm pentaquark states in a systematic way, where the energy-scale formula $\mu=\sqrt{M_P^2-(2\mathbb{M}_c)^2}$ serves as an additional constraint on the predicted masses. All the predictions can be confronted with the experimental data in the future, and the assignments of $P_c(4380)$ and $P_c(4450)$ in the scenario of the diquark–diquark–antiquark type pentaquark states can be testified.

In this article, we take the light axialvector diquarks and heavy axialvector diquarks (and heavy scalar diquarks) as the basic constituents, and study the axialvector-diquark–axialvector-diquark–antiquark type and axialvector-diquark–scalar-diquark–antiquark type pentaquark configurations. Now we illustrate how to construct the pentaquark configurations in the diquark–diquark–antiquark model according to the spin-parity J^P,

$$1^+_{q_1q_2}\otimes 0^+_{q_3c}\otimes \frac{1}{2}^-_{\bar c}=\underline{\frac{1}{2}^-_{q_1q_2q_3c\bar c}}\oplus\frac{3}{2}^-_{q_1q_2q_3c\bar c}, \tag{2}$$

$$\begin{aligned}
&1^+_{q_1q_2}\otimes 1^+_{q_3c}\otimes \frac{1}{2}^-_{\bar c}\\
&=\left[0^+_{q_1q_2q_3c}\oplus 1^+_{q_1q_2q_3c}\oplus 2^+_{q_1q_2q_3c}\right]\otimes\frac{1}{2}^-_{\bar c}\\
&=\underline{\frac{1}{2}^-_{q_1q_2q_3c\bar c}}\oplus\left[\frac{1}{2}^-_{q_1q_2q_3c\bar c}\oplus\frac{3}{2}^-_{q_1q_2q_3c\bar c}\right]\\
&\quad\oplus\left[\frac{3}{2}^-_{q_1q_2q_3c\bar c}\oplus\frac{5}{2}^-_{q_1q_2q_3c\bar c}\right],
\end{aligned}\tag{3}$$

$$\begin{aligned}
&1^+_{q_1q_2}\otimes 0^+_{q_3c}\otimes\left[1^-\otimes\frac{1}{2}^-_{\bar c}\right]\\
&=1^+_{q_1q_2}\otimes 0^+_{q_3c}\otimes\left[\frac{1}{2}^+_{\bar c}\oplus\frac{3}{2}^+_{\bar c}\right]\\
&=\left[\underline{\frac{1}{2}^+_{q_1q_2q_3c\bar c}}\oplus\frac{3}{2}^+_{q_1q_2q_3c\bar c}\right]\\
&\quad\oplus\left[\frac{1}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{3}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{5}{2}^+_{q_1q_2q_3c\bar c}\right],
\end{aligned}\tag{4}$$

$$\begin{aligned}
&1^+_{q_1q_2}\otimes 1^+_{q_3c}\otimes\left[1^-\otimes\frac{1}{2}^-_{\bar c}\right]\\
&=\left[0^+_{q_1q_2q_3c}\oplus 1^+_{q_1q_2q_3c}\oplus 2^+_{q_1q_2q_3c}\right]\otimes\left[\frac{1}{2}^+_{\bar c}\oplus\frac{3}{2}^+_{\bar c}\right]\\
&=\underline{\frac{1}{2}^+_{q_1q_2q_3c\bar c}}\oplus\left[\frac{1}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{3}{2}^+_{q_1q_2q_3c\bar c}\right]\\
&\quad\oplus\left[\frac{3}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{5}{2}^+_{q_1q_2q_3c\bar c}\right]\oplus\frac{3}{2}^+_{q_1q_2q_3c\bar c}\\
&\quad\oplus\left[\frac{1}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{3}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{5}{2}^+_{q_1q_2q_3c\bar c}\right]\\
&\quad\oplus\left[\frac{1}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{3}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{5}{2}^+_{q_1q_2q_3c\bar c}\oplus\frac{7}{2}^+_{q_1q_2q_3c\bar c}\right],
\end{aligned}\tag{5}$$

where 1^- denotes the contribution of the additional P-wave to the spin-parity, the subscripts q_1q_2, q_3c, ... denote the quark constituents. The quark and antiquark have opposite parity, we usually take it for granted that the quarks have positive parity, while the antiquarks have negative parity, so the $\bar c$-quark has $J^P=\frac{1}{2}^-$.

In this article, we study the pentaquark states with the underlined spin-parity $\frac{1}{2}^\pm_{q_1q_2q_3c\bar c}$, which are supposed to be the lowest pentaquark states with the light axialvector diquarks. We construct both the axialvector-diquark–axialvector-diquark–antiquark type and the axialvector-diquark–scalar-diquark–antiquark type currents $J(x)=J^{j_Lj_H}_{q_1q_2q_3}(x)$ according to Eqs. (2)–(3), where the superscripts j_L and j_H denote the spins of the light diquark and heavy diquark, respectively, the subscripts q_1,q_2,q_3 denote the light quark constituents. We calculate the vacuum condensate up to dimension 10 in the operator product expansion, and study the masses and pole residues of the lowest pentaquark states in a systematic way.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the $\frac{1}{2}^\pm$ pentaquark states in Sect. 2; in Sect. 3, we present the numerical results and discussions; and Sect. 4 is reserved for our conclusion.

2 QCD sum rules for the $\frac{1}{2}^\pm$ pentaquark states

We write down the two-point correlation functions $\Pi^{j_Lj_H}_{q_1q_2q_3}(p)$ in the QCD sum rules,

$$\Pi^{j_Lj_H}_{q_1q_2q_3}(p)=i\int d^4x e^{ip\cdot x}\langle 0|T\{J^{j_Lj_H}_{q_1q_2q_3}(x)\bar J^{j_Lj_H}_{q_1q_2q_3}(0)\}|0\rangle, \tag{6}$$

where

$$\begin{aligned}
J^{11}_{uuu}(x)&=\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}u_j^T(x)C\gamma_\mu u_k(x)u_m^T(x)\\
&\quad\times C\gamma^\mu c_n(x)C\bar c_a^T(x),\\
J^{11}_{uud}(x)&=\frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt3}[u_j^T(x)C\gamma_\mu u_k(x)d_m^T(x)C\gamma^\mu c_n(x)\\
&\quad+2u_j^T(x)C\gamma_\mu d_k(x)u_m^T(x)C\gamma^\mu c_n(x)]C\bar c_a^T(x),
\end{aligned}$$

$$J^{11}_{udd}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[d^T_j(x)C\gamma_\mu d_k(x)u^T_m(x)C\gamma^\mu c_n(x) + 2d^T_j(x)C\gamma_\mu u_k(x)d^T_m(x)C\gamma^\mu c_n(x)]C\bar{c}^T_a(x),$$
$$J^{11}_{ddd}(x) = \varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}d^T_j(x)C\gamma_\mu d_k(x)d^T_m(x) \times C\gamma^\mu c_n(x)C\bar{c}^T_a(x), \tag{7}$$

$$J^{11}_{uus}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[u^T_j(x)C\gamma_\mu u_k(x)s^T_m(x)C\gamma^\mu c_n(x) + 2u^T_j(x)C\gamma_\mu s_k(x)u^T_m(x)C\gamma^\mu c_n(x)]C\bar{c}^T_a(x),$$
$$J^{11}_{uds}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[u^T_j(x)C\gamma_\mu d_k(x)s^T_m(x)C\gamma^\mu c_n(x) + u^T_j(x)C\gamma_\mu s_k(x)d^T_m(x)C\gamma^\mu c_n(x) + d^T_j(x)C\gamma_\mu s_k(x)u^T_m(x)C\gamma^\mu c_n(x)]C\bar{c}^T_a(x),$$
$$J^{11}_{dds}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[d^T_j(x)C\gamma_\mu d_k(x)s^T_m(x)C\gamma^\mu c_n(x) + 2d^T_j(x)C\gamma_\mu s_k(x)d^T_m(x)C\gamma^\mu c_n(x)]C\bar{c}^T_a(x), \tag{8}$$

$$J^{11}_{uss}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[s^T_j(x)C\gamma_\mu s_k(x)u^T_m(x)C\gamma^\mu c_n(x) + 2s^T_j(x)C\gamma_\mu u_k(x)s^T_m(x)C\gamma^\mu c_n(x)]C\bar{c}^T_a(x),$$
$$J^{11}_{dss}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[s^T_j(x)C\gamma_\mu s_k(x)d^T_m(x)C\gamma^\mu c_n(x) + 2s^T_j(x)C\gamma_\mu d_k(x)s^T_m(x)C\gamma^\mu c_n(x)]C\bar{c}^T_a(x), \tag{9}$$

$$J^{11}_{sss}(x) = \varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}s^T_j(x)C\gamma_\mu s_k(x)s^T_m(x) \times C\gamma^\mu c_n(x)C\bar{c}^T_a(x), \tag{10}$$

$$J^{10}_{uuu}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}u^T_j(x)C\gamma_\mu u_k(x) \times u^T_m(x)C\gamma_5 c_n(x)\gamma_5\gamma^\mu C\bar{c}^T_a(x),$$
$$J^{10}_{uud}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[u^T_j(x)C\gamma_\mu u_k(x)d^T_m(x)C\gamma_5 c_n(x) + 2u^T_j(x)C\gamma_\mu d_k(x)u^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x),$$
$$J^{10}_{udd}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[d^T_j(x)C\gamma_\mu d_k(x)u^T_m(x)C\gamma_5 c_n(x) + 2d^T_j(x)C\gamma_\mu u_k(x)d^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x),$$
$$J^{10}_{ddd}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}d^T_j(x)C\gamma_\mu d_k(x)d^T_m(x) \times C\gamma_5 c_n(x)\gamma_5\gamma^\mu C\bar{c}^T_a(x), \tag{11}$$

$$J^{10}_{uus}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[u^T_j(x)C\gamma_\mu u_k(x)s^T_m(x)C\gamma_5 c_n(x) + 2u^T_j(x)C\gamma_\mu s_k(x)u^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x),$$
$$J^{10}_{uds}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[u^T_j(x)C\gamma_\mu d_k(x)s^T_m(x)C\gamma_5 c_n(x) + u^T_j(x)C\gamma_\mu s_k(x)d^T_m(x)C\gamma_5 c_n(x) + d^T_j(x)C\gamma_\mu s_k(x)u^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x),$$
$$J^{10}_{dds}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[d^T_j(x)C\gamma_\mu d_k(x)s^T_m(x)C\gamma_5 c_n(x) + 2d^T_j(x)C\gamma_\mu s_k(x)d^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x), \tag{12}$$

$$J^{10}_{uss}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[s^T_j(x)C\gamma_\mu s_k(x)u^T_m(x)C\gamma_5 c_n(x) + 2s^T_j(x)C\gamma_\mu u_k(x)s^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x),$$
$$J^{10}_{dss}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}[s^T_j(x)C\gamma_\mu s_k(x)d^T_m(x)C\gamma_5 c_n(x) + 2s^T_j(x)C\gamma_\mu d_k(x)s^T_m(x)C\gamma_5 c_n(x)]\gamma_5\gamma^\mu C\bar{c}^T_a(x), \tag{13}$$

$$J^{10}_{sss}(x) = \frac{\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}}{\sqrt{3}}s^T_j(x)C\gamma_\mu s_k(x)s^T_m(x) \times C\gamma_5 c_n(x)\gamma_5\gamma^\mu C\bar{c}^T_a(x), \tag{14}$$

i, j, k, l, m, n, and a are color indices, C is the charge conjugation matrix. In the currents $J^{10}_{q_1q_2q_3}(x)$, the light axialvector diquark combines with the heavy scalar diquark to form a tetraquark with $J^P = 1^+$ in color triplet according to Eq. (2), while in the currents $J^{11}_{q_1q_2q_3}(x)$, the light axialvector diquark combines with the heavy axialvector diquark to form a tetraquark with $J^P = 0^+$ in color triplet according to Eq. (3). Then they couple with the antiquark to form pentaquark states with $J^P = \frac{1}{2}^-$ in color singlet. In this article, we take the isospin limit, and classify the currents couple to the pentaquark states with degenerate masses into the following eight types:

$$\begin{aligned}
&J^{11}_{uuu,\mu}(x),\quad J^{11}_{uud,\mu}(x),\quad J^{11}_{udd,\mu}(x),\quad J^{11}_{ddd,\mu}(x)\,;\\
&J^{11}_{uus,\mu}(x),\quad J^{11}_{uds,\mu}(x),\quad J^{11}_{dds,\mu}(x)\,;\\
&J^{11}_{uss,\mu}(x),\quad J^{11}_{dss,\mu}(x)\,;\\
&J^{11}_{sss,\mu}(x)\,;\\
&J^{10}_{uuu,\mu}(x),\quad J^{10}_{uud,\mu}(x),\quad J^{10}_{udd,\mu}(x),\quad J^{10}_{ddd,\mu}(x)\,;\\
&J^{10}_{uus,\mu}(x),\quad J^{10}_{uds,\mu}(x),\quad J^{10}_{dds,\mu}(x)\,;\\
&J^{10}_{uss,\mu}(x),\quad J^{10}_{dss,\mu}(x)\,;\\
&J^{10}_{sss,\mu}(x).
\end{aligned} \tag{15}$$

In calculations, we choose the first current in each type.

The currents $J^{j_Lj_H}_{q_1q_2q_3}(0)$ have negative parity, and couple potentially to the $\frac{1}{2}^-$ hidden-charm pentaquark states

$P^{j_L j_H \frac{1}{2}}_{q_1q_2q_3}(\frac{1}{2}^-)$, where the superscript $\frac{1}{2}$ denotes the total angular momentum of the anti-c-quark $\bar{c}$,

$$\left\langle 0|J^{j_L j_H}_{q_1q_2q_3}(0)|P^{j_L j_H \frac{1}{2}}_{q_1q_2q_3}\left(\frac{1}{2}^-\right)(p)\right\rangle = \lambda^-_P U^-(p,s), \quad (16)$$

the λ^-_P are the pole residues, the spinors $U^-(p,s)$ satisfy the Dirac equations $(\not{p} - M_{P,-})U^-(p,s) = 0$, the s are the polarization vectors of the pentaquark states. On the other hand, the currents $J^{j_L j_H}_{q_1q_2q_3}(0)$ also couple potentially to the $\frac{1}{2}^+$ hidden-charm pentaquark states $P^{j_L j_H \frac{1}{2}}_{q_1q_2q_3}(\frac{1}{2}^+)$ as multiplying $i\gamma_5$ to the currents $J^{j_L j_H}_{q_1q_2q_3}(x)$ change their parity,

$$\left\langle 0|J^{j_L j_H}_{q_1q_2q_3}(0)|P^{j_L j_H \frac{1}{2}}_{q_1q_2q_3}\left(\frac{1}{2}^+\right)(p)\right\rangle = \lambda^+_P i\gamma_5 U^+(p,s), \quad (17)$$

the spinors $U^\pm(p,s)$ (pole residues $\lambda^\pm_i$) have analogous properties [33–40]. We can study the $J^P = \frac{1}{2}^+$ hidden-charm pentaquark states without introducing the additional P-wave explicitly; see Eqs. (4)–(5).

We insert a complete set of intermediate pentaquark states with the same quantum numbers as the current operators $J(x)$ and $i\gamma_5 J(x)$ into the correlation functions $\Pi^{j_L j_H}_{q_1q_2q_3}(p)$ to obtain the hadronic representation [41,42]. After isolating the pole terms of the lowest states of the hidden-charm pentaquark states, we obtain the following results:

$$\Pi^{j_L j_H}_{q_1q_2q_3}(p) = \lambda^{-2}_P \frac{\not{p} + M_{P,-}}{M^2_{P,-} - p^2} + \lambda^{+2}_P \frac{\not{p} - M_{P,+}}{M^2_{P,+} - p^2} + \cdots, \quad (18)$$

where the $M_\pm$ are the masses of the lowest pentaquark states with the parity $\pm$, respectively.

Now we obtain the hadronic spectral densities through the dispersion relation,

$$\begin{aligned}\frac{\mathrm{Im}\Pi^{j_L j_H}_{q_1q_2q_3}(s)}{\pi} =& \not{p}[\lambda^{-2}_P \delta(s - M^2_{P,-}) + \lambda^{+2}_P \delta(s - M^2_{P,+})] \\ &+ [M_{P,-}\lambda^{-2}_P \delta(s - M^2_{P,-}) - M_{P,+}\lambda^{+2}_P \delta(s - M^2_{P,+})], \\ =& \not{p}\, \rho^{j_L j_H,1}_{q_1q_2q_3,H}(s) + \rho^{j_L j_H,0}_{q_1q_2q_3,H}(s), \end{aligned} \quad (19)$$

where the subscript index H denotes the hadron side, then we introduce the weight function $\exp\left(-\frac{s}{T^2}\right)$ to obtain the QCD sum rules at the hadron side,

$$\begin{aligned}&\int_{4m_c^2}^{s_0} ds \left[\sqrt{s}\rho^{j_L j_H,1}_{q_1q_2q_3,H}(s) + \rho^{j_L j_H,0}_{q_1q_2q_3,H}(s)\right] \exp\left(-\frac{s}{T^2}\right) \\ &= 2M_{P,-}\lambda^{-2}_P \exp\left(-\frac{M^2_{P,-}}{T^2}\right), \end{aligned} \quad (20)$$

$$\begin{aligned}&\int_{4m_c^2}^{s_0} ds \left[\sqrt{s}\rho^{j_L j_H,1}_{q_1q_2q_3,H}(s) - \rho^{j_L j_H,0}_{q_1q_2q_3,H}(s)\right] \exp\left(-\frac{s}{T^2}\right) \\ &= 2M_{P,+}\lambda^{+2}_P \exp\left(-\frac{M^2_{P,+}}{T^2}\right), \end{aligned} \quad (21)$$

where the s_0 are the continuum threshold parameters and the T^2 are the Borel parameters. We separate the contributions of the negative-parity (positive-parity) pentaquark states from the positive-parity (negative-parity) pentaquark states explicitly.

In the following, we briefly outline the operator product expansion for the correlation functions $\Pi^{j_L j_H}_{q_1q_2q_3}(p)$ in perturbative QCD. First of all, we contract the u, s, and c quark fields in the correlation functions $\Pi^{j_L j_H}_{q_1q_2q_3}(p)$ with Wick theorem, and obtain the results:

$$\begin{aligned}\Pi^{11}_{uuu}(p) =& -i\, \varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'} \\ &\times \int d^4x e^{ip\cdot x} CC^T_{a'a}(-x)C\{2Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{jj'}(x)C] \\ &\times Tr[\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{mm'}(x)C] - 4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu \\ &\times CU^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{jm'}(x)C]\}, \end{aligned} \quad (22)$$

$$\begin{aligned}\Pi^{11}_{uus}(p) =& -\frac{i}{3}\, \varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'} \\ &\times \int d^4x e^{ip\cdot x} CC^T_{a'a}(-x)C \\ &\times \{2Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{jj'}(x)C]\, Tr[\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{mm'}(x)C] \\ &+ 4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CU^T_{jj'}(x)C]\, Tr[\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{mm'}(x)C] \\ &- 4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{jm'}(x)C] \\ &- 4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{jm'}(x)C] \\ &- 4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{jm'}(x)C]\}, \end{aligned} \quad (23)$$

$$\begin{aligned}\Pi^{11}_{uss}(p) =& -\frac{i}{3}\, \varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'} \\ &\times \int d^4x e^{ip\cdot x} CC^T_{a'a}(-x)C \\ &\times\{2Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{jj'}(x)C]\, Tr[\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{mm'}(x)C] \\ &+4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CS^T_{jj'}(x)C]\, Tr[\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{mm'}(x)C] \\ &-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CU^T_{jm'}(x)C] \\ &-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{jm'}(x)C] \\ &-4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{jm'}(x)C]\}, \end{aligned} \quad (24)$$

$$\begin{aligned}\Pi^{11}_{sss}(p) =& -i\, \varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'} \\ &\times \int d^4x e^{ip\cdot x} CC^T_{a'a}(-x)C \\ &\times\{2Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{jj'}(x)C]\, Tr[\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{mm'}(x)C] \\ &-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma^\mu C_{nn'}(x)\gamma^\nu CS^T_{jm'}(x)C]\}, \end{aligned} \quad (25)$$

$$\begin{aligned}\Pi^{10}_{uuu}(p) =& -i\, \varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'} \\ &\times \int d^4x e^{ip\cdot x} \gamma_5\gamma^\mu CC^T_{a'a}(-x)C\gamma^\nu\gamma_5 \\ &\times\{2Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{jj'}(x)C]\, Tr[\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{mm'}(x)C] \\ &-4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{jm'}(x)C]\}, \end{aligned} \quad (26)$$

$$\Pi^{10}_{uus}(p) = -\frac{i}{3}\,\varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'}$$
$$\times\int d^4x e^{ip\cdot x}\gamma_5\gamma^\mu CC^T_{a'a}(-x)C\gamma^\nu\gamma_5$$
$$\times\{2Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{jj'}(x)C]\,Tr[\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{mm'}(x)C]$$
$$+4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CU^T_{jj'}(x)C]\,Tr[\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{mm'}(x)C]$$
$$-4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{jm'}(x)C]$$
$$-4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{jm'}(x)C]$$
$$-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{jm'}(x)C]\},\quad(27)$$

$$\Pi^{10}_{uss}(p) = -\frac{i}{3}\,\varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'}$$
$$\times\int d^4x e^{ip\cdot x}\gamma_5\gamma^\mu CC^T_{a'a}(-x)C\gamma^\nu\gamma_5$$
$$\times\{2Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{jj'}(x)C]\,Tr[\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{mm'}(x)C]$$
$$+4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CS^T_{jj'}(x)C]\,Tr[\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{mm'}(x)C]$$
$$-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CU^T_{jm'}(x)C]$$
$$-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CU^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{jm'}(x)C]$$
$$-4Tr[\gamma_\mu U_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{jm'}(x)C]\},\quad(28)$$

$$\Pi^{10}_{sss}(p) = -i\,\varepsilon_{ila}\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{i'l'a'}\varepsilon_{i'j'k'}\varepsilon_{l'm'n'}$$
$$\times\int d^4x e^{ip\cdot x}\gamma_5\gamma^\mu CC^T_{a'a}(-x)C\gamma^\nu\gamma_5$$
$$\times\{2Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{jj'}(x)C]\,Tr[\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{mm'}(x)C]$$
$$-4Tr[\gamma_\mu S_{kk'}(x)\gamma_\nu CS^T_{mj'}(x)C\gamma_5 C_{nn'}(x)\gamma_5 CS^T_{jm'}(x)C]\},\quad(29)$$

where $U_{ij}(x)$, $S_{ij}(x)$, and $C_{ij}(x)$ are the full u, s, and c quark propagators, respectively,

$$U_{ij}(x) = \frac{i\delta_{ij}\not x}{2\pi^2x^4} - \frac{\delta_{ij}\langle\bar qq\rangle}{12} - \frac{\delta_{ij}x^2\langle\bar qg_s\sigma Gq\rangle}{192} - \frac{ig_sG^a_{\alpha\beta}t^a_{ij}(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)}{32\pi^2x^2} - \frac{1}{8}\langle\bar q_j\sigma^{\mu\nu}q_i\rangle\sigma_{\mu\nu}+\cdots,$$
$$S_{ij}(x) = \frac{i\delta_{ij}\not x}{2\pi^2x^4} - \frac{\delta_{ij}m_s}{4\pi^2x^2} - \frac{\delta_{ij}\langle\bar ss\rangle}{12} + \frac{i\delta_{ij}\not x m_s\langle\bar ss\rangle}{48} - \frac{\delta_{ij}x^2\langle\bar sg_s\sigma Gs\rangle}{192} + \frac{i\delta_{ij}x^2\not x m_s\langle\bar sg_s\sigma Gs\rangle}{1152} - \frac{ig_sG^a_{\alpha\beta}t^a_{ij}(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)}{32\pi^2x^2} - \frac{1}{8}\langle\bar s_j\sigma^{\mu\nu}s_i\rangle\sigma_{\mu\nu}+\cdots,\quad(30)$$

$$C_{ij}(x) = \frac{i}{(2\pi)^4}\int d^4k e^{-ik\cdot x}\left\{\frac{\delta_{ij}}{\not k-m_c} - \frac{g_sG^n_{\alpha\beta}t^n_{ij}}{4}\frac{\sigma^{\alpha\beta}(\not k+m_c)+(\not k+m_c)\sigma^{\alpha\beta}}{(k^2-m_c^2)^2} - \frac{g_s^2(t^at^b)_{ij}G^a_{\alpha\beta}G^b_{\mu\nu}(f^{\alpha\beta\mu\nu}+f^{\alpha\mu\beta\nu}+f^{\alpha\mu\nu\beta})}{4(k^2-m_c^2)^5}+\cdots\right\},$$
$$f^{\alpha\beta\mu\nu} = (\not k+m_c)\gamma^\alpha(\not k+m_c)\gamma^\beta(\not k+m_c)\gamma^\mu(\not k+m_c)\gamma^\nu(\not k+m_c),\quad(31)$$

and $t^n=\frac{\lambda^n}{2}$, λ^n is the Gell-Mann matrix [42], then compute the integrals both in the coordinate and momentum spaces to obtain the correlation functions $\Pi^{j_Lj_H}_{q_1q_2q_3}(p)$, therefore the QCD spectral densities $\rho^{j_Lj_H,1}_{q_1q_2q_3}(s)$ and $\widetilde\rho^{j_Lj_H,0}_{q_1q_2q_3}(s)$ at the quark level through the dispersion relation,

$$\frac{\mathrm{Im}\Pi^{j_Lj_H}_{q_1q_2q_3}(s)}{\pi} = \not p\,\rho^{j_Lj_H,1}_{q_1q_2q_3}(s) + m_c\widetilde\rho^{j_Lj_H,0}_{q_1q_2q_3}(s).\quad(32)$$

The explicit expressions of $\rho^{j_Lj_H,1}_{q_1q_2q_3}(s)$ and $\widetilde\rho^{j_Lj_H,0}_{q_1q_2q_3}(s)$ are given in the appendix. In Eq. (30), we retain the term $\langle\bar q_j\sigma_{\mu\nu}q_i\rangle$ ($\langle\bar s_j\sigma_{\mu\nu}s_i\rangle$) comes from the Fierz re-arrangement of $\langle q_i\bar q_j\rangle$ ($\langle s_i\bar s_j\rangle$) to absorb the gluons emitted from other quark lines to form $\langle\bar q_jg_sG^a_{\alpha\beta}t^a_{mn}\sigma_{\mu\nu}q_i\rangle$ ($\langle\bar s_jg_sG^a_{\alpha\beta}t^a_{mn}\sigma_{\mu\nu}s_i\rangle$) to extract the mixed condensate $\langle\bar qg_s\sigma Gq\rangle$ ($\langle\bar sg_s\sigma Gs\rangle$). A number of terms involving the mixed condensates $\langle\bar qg_s\sigma Gq\rangle$ and $\langle\bar sg_s\sigma Gs\rangle$ appear and play an important role in the QCD sum rules.

Once the analytical QCD spectral densities $\rho^{j_Lj_H,1}_{q_1q_2q_3}(s)$ and $\widetilde\rho^{j_Lj_H,0}_{q_1q_2q_3}(s)$ are obtained, we can take the quark–hadron duality below the continuum thresholds s_0 and introduce the weight function $\exp\left(-\frac{s}{T^2}\right)$ to obtain the following QCD sum rules:

$$2M_{P,-}\lambda_P^{-2}\exp\left(-\frac{M^2_{P,-}}{T^2}\right) = \int_{4m_c^2}^{s_0}ds\left[\sqrt{s}\rho^{j_Lj_H,1}_{q_1q_2q_3}(s)+m_c\widetilde\rho^{j_Lj_H,0}_{q_1q_2q_3}(s)\right]\exp\left(-\frac{s}{T^2}\right),\quad(33)$$

$$2M_{P,+}\lambda_P^{+2}\exp\left(-\frac{M^2_{P,+}}{T^2}\right) = \int_{4m_c^2}^{s_0}ds\left[\sqrt{s}\rho^{j_Lj_H,1}_{q_1q_2q_3}(s)-m_c\widetilde\rho^{j_Lj_H,0}_{q_1q_2q_3}(s)\right]\exp\left(-\frac{s}{T^2}\right),\quad(34)$$

where we take into account the contributions of the terms D_0, D_3, D_5, D_6, D_8, D_9, and D_{10},

$$\begin{aligned}
D_0 &= \text{perturbative terms},\\
D_3 &\propto \langle\bar qq\rangle,\ \langle\bar ss\rangle,\\
D_5 &\propto \langle\bar qg_s\sigma Gq\rangle,\ \langle\bar sg_s\sigma Gs\rangle,\\
D_6 &\propto \langle\bar qq\rangle^2,\ \langle\bar qq\rangle\langle\bar ss\rangle,\ \langle\bar ss\rangle^2,\\
D_8 &\propto \langle\bar qq\rangle\langle\bar qg_s\sigma Gq\rangle,\ \langle\bar ss\rangle\langle\bar qg_s\sigma Gq\rangle,\ \langle\bar qq\rangle\langle\bar sg_s\sigma Gs\rangle,\\
&\quad\ \langle\bar ss\rangle\langle\bar sg_s\sigma Gs\rangle,\\
D_9 &\propto \langle\bar qq\rangle^3,\ \langle\bar qq\rangle\langle\bar ss\rangle^2,\ \langle\bar qq\rangle^2\langle\bar ss\rangle,\ \langle\bar ss\rangle^3,\\
D_{10} &\propto \langle\bar qg_s\sigma Gq\rangle^2,\ \langle\bar qg_s\sigma Gq\rangle\langle\bar sg_s\sigma Gs\rangle,\ \langle\bar sg_s\sigma Gs\rangle^2.
\end{aligned}\quad(35)$$

In this article, we carry out the operator product expansion to the vacuum condensates up to dimension 10, and assume vacuum saturation for the higher dimension vacuum condensates.

We differentiate Eqs. (33)–(34) with respect to $\frac{1}{T^2}$, then eliminate the pole residues $\lambda_P^{\pm}$ and obtain the QCD sum rules for the masses of the pentaquark states,

$$M_{P,-}^2 = \frac{\int_{4m_c^2}^{s_0} dss\left[\sqrt{s}\rho_{q_1q_2q_3}^{j_Lj_H,1}(s)+m_c\widetilde{\rho}_{q_1q_2q_3}^{j_Lj_H,0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{s_0} ds\left[\sqrt{s}\rho_{q_1q_2q_3}^{j_Lj_H,1}(s)+m_c\widetilde{\rho}_{q_1q_2q_3}^{j_Lj_H,0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}, \quad (36)$$

$$M_{P,+}^2 = \frac{\int_{4m_c^2}^{s_0} dss\left[\sqrt{s}\rho_{q_1q_2q_3}^{j_Lj_H,1}(s)-m_c\widetilde{\rho}_{q_1q_2q_3}^{j_Lj_H,0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{s_0} ds\left[\sqrt{s}\rho_{q_1q_2q_3}^{j_Lj_H,1}(s)-m_c\widetilde{\rho}_{q_1q_2q_3}^{j_Lj_H,0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}. \quad (37)$$

Once the masses $M_{P,\pm}$ are obtained, we can take them as input parameters and obtain the pole residues from the QCD sum rules in Eqs. (33)–(34). The gluon condensates are associated with large numerical denominators, their contributions to total QCD spectral densities are less (or much less) than the contributions of the dimension 10 vacuum condensates D_{10} for the pentaquark currents with the axialvector-diquark constituents [19,20]. We obtain the masses through fractions, see Eqs. (36)–(37), the effects of the gluon condensates can be safely absorbed into the pole residues $\lambda_P^{\pm}$ and tiny effects on the masses can be safely neglected.

3 Numerical results and discussions

The vacuum condensates are taken to be the standard values $\langle\bar{q}q\rangle = -(0.24\pm0.01\,\text{GeV})^3$, $\langle\bar{s}s\rangle = (0.8\pm0.1)\langle\bar{q}q\rangle$, $\langle\bar{q}g_s\sigma Gq\rangle = m_0^2\langle\bar{q}q\rangle$, $\langle\bar{s}g_s\sigma Gs\rangle = m_0^2\langle\bar{s}s\rangle$, $m_0^2 = (0.8\pm0.1)\,\text{GeV}^2$ at the energy scale $\mu = 1\,\text{GeV}$ [41,42]. The quark condensates and mixed quark condensates evolve with the renormalization group equation, $\langle\bar{q}q\rangle(\mu) = \langle\bar{q}q\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{4}{9}}$, $\langle\bar{s}s\rangle(\mu) = \langle\bar{s}s\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{4}{9}}$, $\langle\bar{q}g_s\sigma Gq\rangle(\mu) = \langle\bar{q}g_s\sigma Gq\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{2}{27}}$ and $\langle\bar{s}g_s\sigma Gs\rangle(\mu) = \langle\bar{s}g_s\sigma Gs\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{2}{27}}$.

In the article, we take the $\overline{MS}$ masses $m_c(m_c) = (1.275\pm0.025)\,\text{GeV}$ and $m_s(\mu = 2\,\text{GeV}) = (0.095\pm0.005)\,\text{GeV}$ from the Particle Data Group [43], and take into account the energy-scale dependence of the $\overline{MS}$ masses from the renormalization group equation,

$$m_c(\mu) = m_c(m_c)\left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)}\right]^{\frac{12}{25}},$$

$$m_s(\mu) = m_s(2\text{GeV})\left[\frac{\alpha_s(\mu)}{\alpha_s(2\text{GeV})}\right]^{\frac{4}{9}}, \quad (38)$$

$$\alpha_s(\mu) = \frac{1}{b_0t}\left[1-\frac{b_1}{b_0^2}\frac{\log t}{t}+\frac{b_1^2(\log^2 t-\log t-1)+b_0b_2}{b_0^4t^2}\right],$$

where $t = \log\frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_f}{12\pi}$, $b_1 = \frac{153-19n_f}{24\pi^2}$, $b_2 = \frac{2857-\frac{5033}{9}n_f+\frac{325}{27}n_f^2}{128\pi^3}$, $\Lambda = 213$, 296 and 339 MeV for the flavors $n_f = 5$, 4, and 3, respectively [43]. Furthermore, we set the small masses of the u and d quarks equal zero, $m_u = m_d = 0$.

In Refs. [36–40], we study the $J^P = \frac{1}{2}^{\pm}$ and $\frac{3}{2}^{\pm}$ heavy, doubly heavy, and triply heavy baryon states with the QCD sum rules in a systematic way by subtracting the contributions from the corresponding $J^P = \frac{1}{2}^{\mp}$ and $\frac{3}{2}^{\mp}$ heavy, doubly heavy, and triply heavy baryon states, the continuum threshold parameters $\sqrt{s_0} = M_{\rm gr} + (0.6\text{–}0.8)\,\text{GeV}$ work well, where subscript gr denotes the ground state baryons. The pentaquark states are another type of baryon states according to the fractional spins, in Ref. [19], we take the continuum threshold parameters as $\sqrt{s_0} = M_{P_c(4380/4450)} + (0.6\text{–}0.8)\,\text{GeV}$, which also works well. In this article, we take the continuum threshold parameters $\sqrt{s_0} = M_P + (0.6\text{–}0.8)\,\text{GeV}$ as an additional constraint.

The hidden-charm (or bottom) five-quark systems $q_1q_2q_3Q\bar{Q}$ could be described by a double-well potential, just like the double heavy four-quark systems $q_1\bar{q}_2Q\bar{Q}$ [44–48]. We introduce the color indices i, j and k first. In the five-quark system $q_1q_2q_3Q\bar{Q}$, the light quarks q_1 and q_2 combine together to form a light diquark $\mathcal{D}^i_{q_1q_2}$ in color antitriplet,

$$q_1+q_2 \to \mathcal{D}^i_{q_1q_2}, \quad (39)$$

the Q-quark serves as a static well potential and combines with the light quark q_3 to form a heavy diquark $\mathcal{D}^j_{q_3Q}$ in color antitriplet,

$$q_3+Q \to \mathcal{D}^j_{q_3Q}, \quad (40)$$

the $\bar{Q}$-quark serves as another static well potential and combines with the light diquark $\mathcal{D}^i_{q_1q_2}$ to form a heavy triquark in the color triplet,

$$\bar{Q}^k+\mathcal{D}^i_{q_1q_2} \to \varepsilon^{jki}\bar{Q}^k\mathcal{D}^i_{q_1q_2}. \quad (41)$$

Then the heavy diquark $\mathcal{D}^j_{q_3Q}$ combines with the heavy triquark $\varepsilon^{jki}\bar{Q}^k\mathcal{D}^i_{q_1q_2}$ to form a pentaquark state in color singlet,

$$\varepsilon^{jki}\bar{Q}^k\mathcal{D}^i_{q_1q_2}+\mathcal{D}^j_{q_3Q} \to \varepsilon^{ijk}\mathcal{D}^i_{q_1q_2}\mathcal{D}^j_{q_3Q}\bar{Q}^k. \quad (42)$$

The interpolating currents in Eqs. (7)–(14) can also be understood in this way.

Now we can see that the double heavy five-quark system is characterized by the effective heavy quark masses $\mathbb{M}_Q$ and the virtuality $V = \sqrt{M_P^2-(2\mathbb{M}_Q)^2}$, just like the double heavy four-quark systems $q_1\bar{q}_2Q\bar{Q}$ [44–48]. The QCD sum rules have three typical energy scales μ^2, T^2, V^2, we set the energy scales to be $\mu^2 = V^2 = \mathcal{O}(T^2)$ and obtain the energy-scale formula,

$$\mu = \sqrt{M_P^2 - (2\mathbb{M}_Q)^2}, \tag{43}$$

to determine the energy scales of the QCD spectral densities [19,20]. In previous work [19], we take the value $\mathbb{M}_c = 1.8$ GeV determined in the double heavy four-quark systems [44–48] and obtain the values $\mu = 2.5$ GeV and $\mu = 2.6$ GeV for the hidden-charm pentaquark states $P_c(4380)$ and $P_c(4450)$, respectively. The energy-scale formula works well.

We can rewrite Eq. (43) into the following form:

$$M_P^2 = 4\mathbb{M}_c^2 + \mu^2, \tag{44}$$

which indicates that the masses increase (or decrease) with increase (or decrease) of the energy scales,

$$\begin{aligned} &\mu \uparrow \quad M_P \uparrow, \\ &\mu \downarrow \quad M_P \downarrow. \end{aligned} \tag{45}$$

On the other hand, the calculations based on the QCD sum rules in Eqs. (36)–(37) indicate that the masses decrease (or increase) with increase (or decrease) of the energy scales,

$$\begin{aligned} &\mu \uparrow \quad M_P \downarrow, \\ &\mu \downarrow \quad M_P \uparrow. \end{aligned} \tag{46}$$

We can search for a compromise and obtain the optimal energy scales μ and masses M_P.

In the present QCD sum rules, we choose the Borel parameters T^2 and continuum threshold parameters s_0 to satisfy the following four criteria:

- pole dominance at the phenomenological side;
- convergence of the operator product expansion;
- appearance of the Borel platforms;
- satisfying the energy-scale formula.

Now we search for the optimal Borel parameters T^2 and continuum threshold parameters s_0 according to the four criteria. The resulting Borel parameters T^2, continuum threshold parameters s_0, pole contributions, and contributions of the vacuum condensates of dimension 9 and 10 are shown explicitly in Table 1. From the table, we can see that the first two criteria of the QCD sum rules are satisfied, and we expect to make reasonable predictions. In Table 2, we present the contributions of different terms in the operator product expansion with the central values of the input parameters. In calculations, we observe that the main contributions come from the terms D_0, D_3, and D_6, see Table 2, the operator product expansion is well convergent. We should admit that the convergent behavior of the operator product expansion is not so good as that in the conventional case, where D_0 dominates the QCD sum rules, but we can find a comparatively reasonable work window to extract the hadronic information [49,50].

Table 1 The Borel parameters, continuum threshold parameters, pole contributions, contributions of the vacuum condensates of dimension 9 and dimension 10

	T^2 (GeV^2)	$\sqrt{s_0}$ (GeV)	Pole (%)	D_9 (%)	D_{10} (%)
$P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.2–3.6	5.1 ± 0.1	(38–60)	(16–22)	$\sim$1
$P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.3–3.7	5.2 ± 0.1	(39–60)	(10–15)	$\leq$1
$P_{uss}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.4–3.8	5.3 ± 0.1	(40–61)	(7–10)	$<$1
$P_{sss}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.5–3.9	5.4 ± 0.1	(42–62)	(5–7)	$<$1
$P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.3–3.7	5.1 ± 0.1	(39–60)	(10–13)	(3–4)
$P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.4–3.8	5.2 ± 0.1	(41–61)	(6–9)	(2–3)
$P_{uss}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.5–3.9	5.3 ± 0.1	(42–62)	(4–6)	(1–2)
$P_{sss}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.6–4.0	5.4 ± 0.1	(43–62)	(3–4)	$\sim$1
$P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.3–3.7	5.3 ± 0.1	(37–59)	(15–21)	$\leq$1
$P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.4–3.8	5.4 ± 0.1	(38–59)	(10–14)	$<$1
$P_{uss}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.5–3.9	5.5 ± 0.1	(39–60)	(7–10)	$<$1
$P_{sss}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.6–4.0	5.6 ± 0.1	(40–60)	(5–7)	$<$1
$P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.3–3.7	5.8 ± 0.1	(60–78)	$-$(3–5)	$<$1
$P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.4–3.8	5.9 ± 0.1	(61–79)	$-$(2–4)	$<$1
$P_{uss}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.5–3.9	6.0 ± 0.1	(62–79)	$-$(1–2)	$<$1
$P_{sss}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.6–4.0	6.1 ± 0.1	(63–80)	~ -1	$<$1

We take into account all uncertainties of the input parameters, and we obtain the values of the masses and pole residues of the $\frac{1}{2}^{\pm}$ hidden-charm pentaquark states, which are shown in Figs. 1, 2, and Table 3. From Figs. 1, 2, and Table 3, we can see that the last two criteria are also satisfied. In Table 3, we also present the corresponding thresholds of $J/\psi B_{10}$ and $J/\psi B_8$, where the B_8 and B_{10} denote the octet and decuplet baryons with the constituents $q_1q_2q_3$, respectively. From the table, we can see that the decays to the $J/\psi B_{10}$,

$$P_{q_1q_2q_3}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right), \quad P_{q_1q_2q_3}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right) \to J/\psi B_{10}, \tag{47}$$

for example,

$$P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right), \quad P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right) \to J/\psi \Delta^{++},$$

$$P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right), \quad P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right) \to J/\psi \Sigma^{*+},$$

Table 2 The contributions of different terms in the operator product expansion with the central values of the input parameters

Pentaquark states	Contributions
$P^{11\frac{1}{2}}_{uuu}\left(\frac{1}{2}^-\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert \approx \lvert D_8\rvert \approx D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{uus}\left(\frac{1}{2}^-\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert \approx \lvert D_8\rvert > D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{uss}\left(\frac{1}{2}^-\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert \approx \lvert D_8\rvert > D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{sss}\left(\frac{1}{2}^-\right)$	$D_0 \gg D_6 > D_3 > \lvert D_5\rvert > \lvert D_8\rvert > D_9 \gg D_{10}$
$P^{10\frac{1}{2}}_{uuu}\left(\frac{1}{2}^-\right)$	$D_6 > D_3 > D_0 > \lvert D_8\rvert \gg \lvert D_5\rvert > D_9 \gg D_{10}$
$P^{10\frac{1}{2}}_{uus}\left(\frac{1}{2}^-\right)$	$D_0 \approx D_3 > D_6 \gg \lvert D_8\rvert > \lvert D_5\rvert \gg D_9 \gg D_{10}$
$P^{10\frac{1}{2}}_{uss}\left(\frac{1}{2}^-\right)$	$D_0 \gg D_3 > D_6 \gg \lvert D_8\rvert > \lvert D_5\rvert \gg D_9 \gg D_{10}$
$P^{10\frac{1}{2}}_{sss}\left(\frac{1}{2}^-\right)$	$D_0 \gg D_3 \gg D_6 \gg \lvert D_8\rvert > \lvert D_5\rvert \gg D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{uuu}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert \approx \lvert D_8\rvert \approx D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{uus}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert \approx \lvert D_8\rvert > D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{uss}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert \approx \lvert D_8\rvert > D_9 \gg D_{10}$
$P^{11\frac{1}{2}}_{sss}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_6 > D_3 \gg \lvert D_5\rvert > \lvert D_8\rvert > D_9 \gg D_{10}$
$P^{10\frac{1}{2}}_{uuu}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_3 \gg \lvert D_6\rvert > \lvert D_5\rvert \gg D_8 > \lvert D_9\rvert \gg \lvert D_{10}\rvert$
$P^{10\frac{1}{2}}_{uus}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_3 \gg \lvert D_6\rvert > \lvert D_5\rvert \gg D_8 > \lvert D_9\rvert \gg \lvert D_{10}\rvert$
$P^{10\frac{1}{2}}_{uss}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_3 \gg \lvert D_6\rvert > \lvert D_5\rvert \gg D_8 \gg \lvert D_9\rvert \gg \lvert D_{10}\rvert$
$P^{10\frac{1}{2}}_{sss}\left(\frac{1}{2}^+\right)$	$D_0 \gg D_3 \gg \lvert D_6\rvert > \lvert D_5\rvert \gg \lvert D_9\rvert > D_8 \gg \lvert D_{10}\rvert$

$$P^{11\frac{1}{2}}_{uss}\left(\frac{1}{2}^-\right), \quad P^{10\frac{1}{2}}_{uss}\left(\frac{1}{2}^-\right) \to J/\psi\,\Xi^{*0},$$
$$P^{11\frac{1}{2}}_{sss}\left(\frac{1}{2}^-\right), \quad P^{10\frac{1}{2}}_{sss}\left(\frac{1}{2}^-\right) \to J/\psi\,\Omega^-, \tag{48}$$

may take place, but the decay widths are rather small due to the small available phase-spaces; on the other hand, the decays

$$P^{11\frac{1}{2}}_{q_1q_2q_3}\left(\frac{1}{2}^-\right), \quad P^{10\frac{1}{2}}_{q_1q_2q_3}\left(\frac{1}{2}^-\right) \to J/\psi\,B_8, \tag{49}$$
$$P^{11\frac{1}{2}}_{q_1q_2q_3}\left(\frac{1}{2}^+\right) \to J/\psi\,B_{10},\, J/\psi\,B_8, \tag{50}$$

for example,

$$P^{11\frac{1}{2}}_{uud}\left(\frac{1}{2}^\pm\right), \quad P^{10\frac{1}{2}}_{uud}\left(\frac{1}{2}^-\right) \to J/\psi\,p,$$
$$P^{11\frac{1}{2}}_{uus}\left(\frac{1}{2}^\pm\right), \quad P^{10\frac{1}{2}}_{uus}\left(\frac{1}{2}^-\right) \to J/\psi\,\Sigma^+,$$
$$P^{11\frac{1}{2}}_{uss}\left(\frac{1}{2}^\pm\right), \quad P^{10\frac{1}{2}}_{uss}\left(\frac{1}{2}^-\right) \to J/\psi\,\Xi^0,$$
$$P^{11\frac{1}{2}}_{uuu}\left(\frac{1}{2}^+\right) \to J/\psi\,\Delta^{++},$$
$$P^{11\frac{1}{2}}_{uus}\left(\frac{1}{2}^+\right) \to J/\psi\,\Sigma^{*+},$$
$$P^{11\frac{1}{2}}_{uss}\left(\frac{1}{2}^+\right) \to J/\psi\,\Xi^{*0},$$
$$P^{11\frac{1}{2}}_{sss}\left(\frac{1}{2}^+\right) \to J/\psi\,\Omega^-, \tag{51}$$

can take place more easily, the decay widths are larger due to the larger available phase-spaces; furthermore, the decays

$$P^{10\frac{1}{2}}_{q_1q_2q_3}\left(\frac{1}{2}^+\right) \to J/\psi\,B_8,\, J/\psi\,B_{10}, \tag{52}$$

for example,

$$P^{10\frac{1}{2}}_{uud}\left(\frac{1}{2}^+\right) \to J/\psi\,p,$$
$$P^{10\frac{1}{2}}_{uus}\left(\frac{1}{2}^+\right) \to J/\psi\,\Sigma^+,$$
$$P^{10\frac{1}{2}}_{uss}\left(\frac{1}{2}^+\right) \to J/\psi\,\Xi^0,$$
$$P^{10\frac{1}{2}}_{uuu}\left(\frac{1}{2}^+\right) \to J/\psi\,\Delta^{++},$$
$$P^{10\frac{1}{2}}_{uus}\left(\frac{1}{2}^+\right) \to J/\psi\,\Sigma^{*+},$$
$$P^{10\frac{1}{2}}_{uss}\left(\frac{1}{2}^+\right) \to J/\psi\,\Xi^{*0},$$
$$P^{10\frac{1}{2}}_{sss}\left(\frac{1}{2}^+\right) \to J/\psi\,\Omega^-, \tag{53}$$

can take place fluently, the decay widths are rather large due to the large available phase-spaces. We can search for the pentaquark states in the $J/\psi\,B_8$ and $J/\psi\,B_{10}$ mass spectrum in the decays of the bottom baryons to the final states $J/\psi\,B_8$ and $J/\psi\,B_{10}$ associated with the light vector mesons or pseudoscalar mesons [13,18,21], for example,

$$\Omega_b^- \to P^{11\frac{1}{2}}_{uss}\left(\frac{1}{2}^\pm\right) K^- \to J/\psi\,\Xi^{*0} K^-,$$
$$\Omega_b^- \to P^{11\frac{1}{2}}_{sss}\left(\frac{1}{2}^\pm\right) \phi \to J/\psi\,\Omega^- \phi. \tag{54}$$

In Ref. [22], the authors study the non-strange and strange pentaquarks with hidden charm in the diquark–diquark–antiquark model by considering the simple spin-spin interactions, and evaluate the masses, where the scalar and axialvector diquarks are chosen. The predicted masses are different from ours, see Refs. [19,20] and the present work, the differences originate partly from the fact that in Ref. [22] $P_c(4450)$

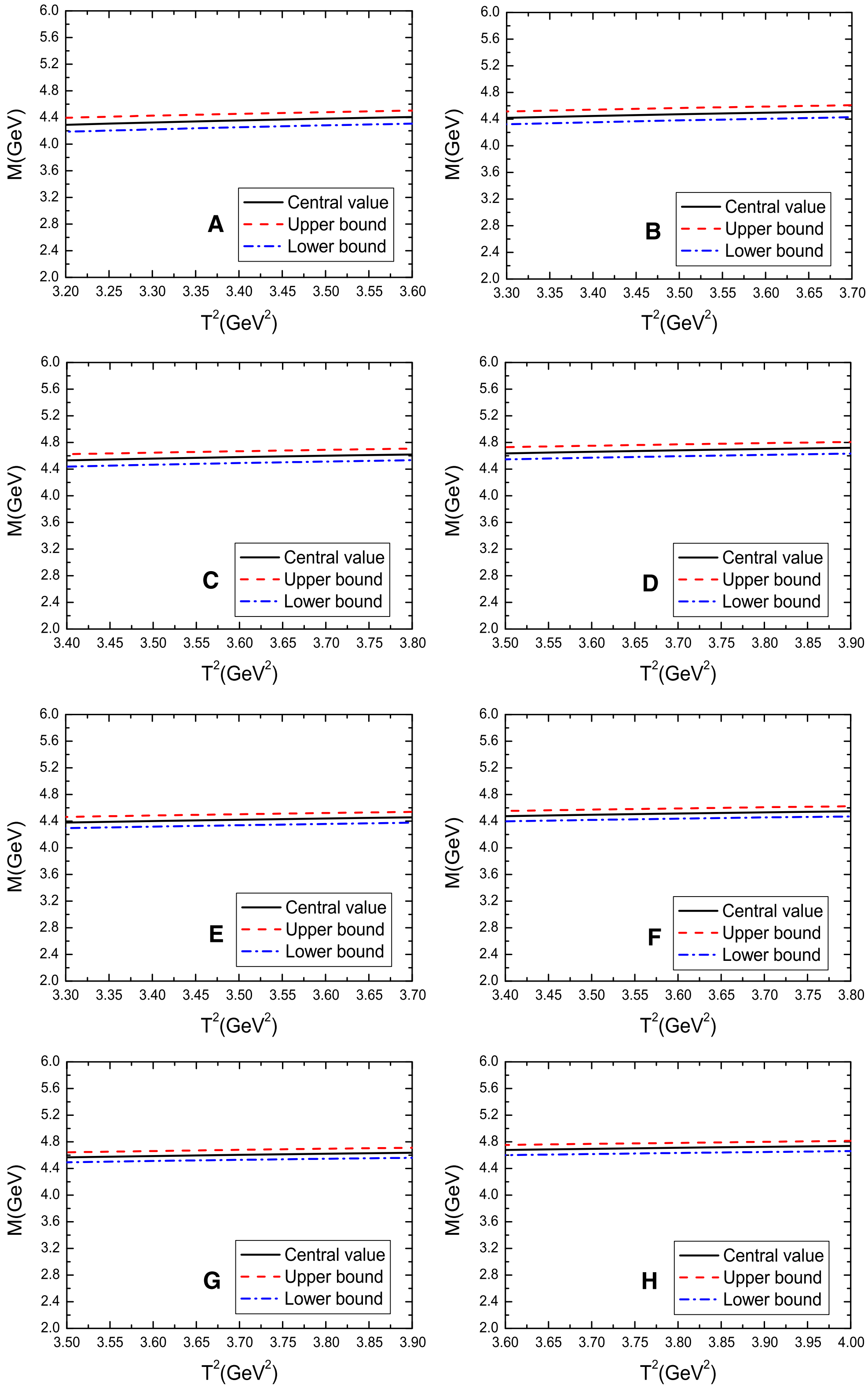

Fig. 1 The masses of the pentaquark states with variations of the Borel parameters T^2, where A, B, C, D, E, F, G, and H denote the pentaquark states $P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{uss}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{sss}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{uss}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$, and $P_{sss}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$ with negative parity, respectively

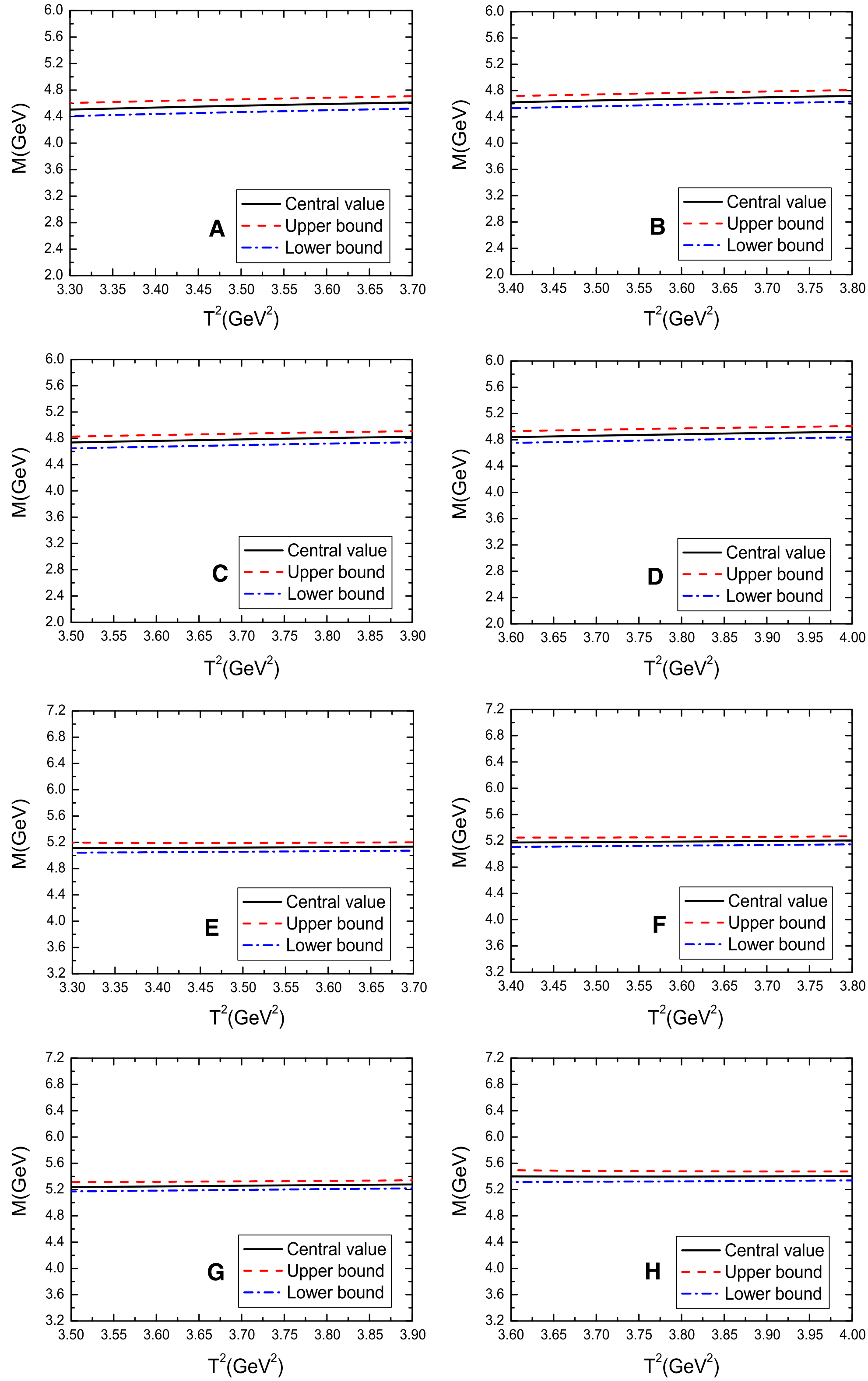

Fig. 2 The masses of the pentaquark states with variations of the Borel parameters T^2, where A, B, C, D, E, F, G, and H denote the pentaquark states $P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$, $P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$, $P_{uss}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$, $P_{sss}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$, $P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$, $P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$, $P_{uss}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$, and $P_{sss}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$ with positive parity, respectively

Table 3 The energy scales, masses, and pole residues of the pentaquark states, where B_{10} and B_8 denote the decuplet and octet baryons with the quark constituents $q_1q_2q_3$, respectively. We take the isospin limit, the pentaquark states $P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{uud}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$, $P_{udd}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$ an $P_{ddd}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$ have degenerate masses. Other states are implied

	μ (GeV)	M_P (GeV)	λ_P (GeV^6)	$M_{B_{10}J/\psi(B_8J/\psi)}$ (GeV)
$P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	2.5	4.35 ± 0.15	$(3.72 \pm 0.76) \times 10^{-3}$	4.33 (4.04)
$P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	2.6	4.47 ± 0.15	$(4.50 \pm 0.85) \times 10^{-3}$	4.48 (4.29)
$P_{uss}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	2.8	4.58 ± 0.14	$(5.43 \pm 0.96) \times 10^{-3}$	4.63 (4.41)
$P_{sss}^{11\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.0	4.68 ± 0.13	$(6.47 \pm 1.10) \times 10^{-3}$	4.77
$P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	2.5	4.42 ± 0.12	$(4.14 \pm 0.70) \times 10^{-3}$	4.33 (4.04)
$P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	2.7	4.51 ± 0.11	$(4.97 \pm 0.79) \times 10^{-3}$	4.48 (4.29)
$P_{uss}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	2.9	4.60 ± 0.11	$(5.87 \pm 0.89) \times 10^{-3}$	4.63 (4.41)
$P_{sss}^{10\frac{1}{2}}\left(\frac{1}{2}^-\right)$	3.0	4.71 ± 0.11	$(6.84 \pm 1.00) \times 10^{-3}$	4.77
$P_{uuu}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	2.8	4.56 ± 0.15	$(1.97 \pm 0.40) \times 10^{-3}$	4.33 (4.04)
$P_{uus}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.0	4.67 ± 0.14	$(2.42 \pm 0.47) \times 10^{-3}$	4.48 (4.29)
$P_{uss}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.1	4.78 ± 0.13	$(2.88 \pm 0.53) \times 10^{-3}$	4.63 (4.41)
$P_{sss}^{11\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.3	4.89 ± 0.13	$(3.44 \pm 0.61) \times 10^{-3}$	4.77
$P_{uuu}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.6	5.12 ± 0.08	$(8.01 \pm 0.92) \times 10^{-3}$	4.33 (4.04)
$P_{uus}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.7	5.19 ± 0.08	$(8.92 \pm 1.05) \times 10^{-3}$	4.48 (4.29)
$P_{uss}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	3.8	5.26 ± 0.08	$(9.93 \pm 1.21) \times 10^{-3}$	4.63 (4.41)
$P_{sss}^{10\frac{1}{2}}\left(\frac{1}{2}^+\right)$	4.0	5.40 ± 0.08	$(12.17 \pm 1.28) \times 10^{-3}$	4.77

is taken as the $P_{uud}^{11\frac{1}{2}}(\frac{5}{2}^-)$ pentaquark state, while in Refs. [19,20] $P_c(4450)$ is taken as the $P_{uud}^{11\frac{3}{2}}(\frac{5}{2}^+)$ pentaquark state.

4 Conclusion

In this article, we choose the axialvector-diquark–axialvector-diquark–antiquark type and axialvector-diquark–scalar-diquark–antiquark type pentaquark configurations, construct both the axialvector-diquark–axialvector-diquark–antiquark type and the axialvector-diquark–scalar-diquark–antiquark type interpolating currents, then calculate the contributions of the vacuum condensates up to dimension 10 in the operator product expansion, and we study the masses and pole residues of the $J^P = \frac{1}{2}^\pm$ hidden-charm pentaquark states with the QCD sum rules in a systematic way. In calculations, we use the formula $\mu = \sqrt{M_P^2 - (2\mathbb{M}_c)^2}$ to determine the energy scales of the QCD spectral densities, which works well in our previous work. We take into account the SU(3) breaking effects of the light quarks, and obtain the masses of the hidden-charm pentaquark states with the strangeness $S = 0, -1, -2, -3$, which can be confronted with the experimental data in the future. We can search for those pentaquark states in the decays of the bottom baryons to the J/ψ plus octet (decuplet) baryons plus pseudoscalar (vector) mesons, or take the pole residues as basic input parameters to study relevant processes of the pentaquark states with the three-point QCD sum rules.

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Appendix

The QCD spectral densities $\rho^{11,1}_{sss}(s)$, $\widetilde{\rho}^{11,0}_{sss}(s)$, $\rho^{11,1}_{uss}(s)$, $\widetilde{\rho}^{11,0}_{uss}(s)$, $\rho^{11,1}_{uus}(s)$, $\widetilde{\rho}^{11,0}_{uus}(s)$, $\rho^{11,1}_{uuu}(s)$, $\widetilde{\rho}^{11,0}_{uuu}(s)$, $\rho^{10,1}_{sss}(s)$, $\widetilde{\rho}^{10,0}_{sss}(s)$, $\rho^{10,1}_{uss}(s)$, $\widetilde{\rho}^{10,0}_{uss}(s)$, $\rho^{10,1}_{uus}(s)$, $\widetilde{\rho}^{10,0}_{uus}(s)$, $\rho^{10,1}_{uuu}(s)$, and $\widetilde{\rho}^{10,0}_{uuu}(s)$ of the pentaquark states,

$$
\begin{aligned}
\rho^{11,1}_{sss}(s) &= \frac{1}{61440\pi^8}\int dydz\, yz(1-y-z)^4(s-\overline{m}_c^2)^4(8s-3\overline{m}_c^2)\\
&+\frac{m_s m_c}{12288\pi^8}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^4\\
&-\frac{m_c\langle\bar{s}s\rangle}{768\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^3\\
&+\frac{m_s\langle\bar{s}s\rangle}{128\pi^6}\int dydz\, yz(1-y-z)^2(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)\\
&+\frac{3m_c\langle\bar{s}g_s\sigma Gs\rangle}{1024\pi^6}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)^2\\
&+\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{512\pi^6}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)\\
&-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{1024\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)\\
&+\frac{\langle\bar{s}s\rangle^2}{48\pi^4}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)\\
&+\frac{5m_s m_c\langle\bar{s}s\rangle^2}{48\pi^4}\int dydz\,(y+z)(s-\overline{m}_c^2)\\
&-\frac{5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{192\pi^4}\int dydz\, yz(4s-3\overline{m}_c^2)\\
&+\frac{\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{192\pi^4}\int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)\\
&-\frac{121 m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{2304\pi^4}\int dy\\
&+\frac{m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{48\pi^4}\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)-\frac{m_c\langle\bar{s}s\rangle^3}{18\pi^2}\int dy\\
&+\frac{m_s\langle\bar{s}s\rangle^3}{12\pi^2}\int dy\, y(1-y)\left[1+\frac{s}{3}\delta\left(s-\widetilde{m}_c^2\right)\right]\\
&-\frac{3\langle\bar{s}g_s\sigma Gs\rangle^2}{512\pi^4}\int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]\\
&+\frac{3\langle\bar{s}g_s\sigma Gs\rangle^2}{256\pi^4}\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]\\
&-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{192\pi^4}\int dy\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)\\
&+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{72\pi^4}\int dy\, y(1-y)\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2)\\
&-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{2304\pi^4}\int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2), \qquad (55)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\rho}^{11,0}_{sss}(s) &= \frac{1}{122880\pi^8}\int dydz\,(y+z)(1-y-z)^4(s-\overline{m}_c^2)^4\\
&\times(7s-2\overline{m}_c^2)+\frac{m_s m_c}{6144\pi^8}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^4\\
&-\frac{m_c\langle\bar{s}s\rangle}{384\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)^3\\
&+\frac{m_s\langle\bar{s}s\rangle}{768\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)\\
&+\frac{3m_c\langle\bar{s}g_s\sigma Gs\rangle}{512\pi^6}\int dydz\,(1-y-z)(s-\overline{m}_c^2)^2\\
&-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{256\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)(2s-\overline{m}_c^2)\\
&+\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{512\pi^6}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)\\
&+\frac{\langle\bar{s}s\rangle^2}{48\pi^4}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)\\
&+\frac{5m_s m_c\langle\bar{s}s\rangle^2}{24\pi^4}\int dydz\,(s-\overline{m}_c^2)\\
&-\frac{5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{384\pi^4}\int dydz\,(y+z)(3s-2\overline{m}_c^2)\\
&+\frac{\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{96\pi^4}\int dydz\,(1-y-z)(3s-2\overline{m}_c^2)\\
&-\frac{121 m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{1152\pi^4}\int dy\\
&+\frac{m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{48\pi^4}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)\\
&-\frac{m_c\langle\bar{s}s\rangle^3}{9\pi^2}\int dy+\frac{m_s\langle\bar{s}s\rangle^3}{36\pi^2}\int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]\\
&-\frac{\langle\bar{s}g_s\sigma Gs\rangle^2}{256\pi^4}\int dydz\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]\\
&-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{192\pi^4}\int dy\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)\\
&+\frac{31 m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{2304\pi^4}\int dy\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2), \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
\rho^{11,1}_{uss}(s) &= \frac{1}{61440\pi^8}\int dydz\, yz(1-y-z)^4(s-\overline{m}_c^2)^4(8s-3\overline{m}_c^2)\\
&+\frac{m_s m_c}{18432\pi^8}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^4\\
&-\frac{m_c\left[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle\right]}{2304\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^3\\
&+\frac{m_s\left[2\langle\bar{s}s\rangle-\langle\bar{q}q\rangle\right]}{192\pi^6}\int dydz\, yz(1-y-z)^2\\
&\times(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)+\frac{m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{1024\pi^6}\\
&\times\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)^2\\
&+\frac{m_s\left[\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle\right]}{384\pi^6}\\
&\times\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)\\
&+\frac{m_s\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{1536\pi^6}
\end{aligned}
$$

$$\times \int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{3072\pi^6}$$
$$\times \int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{\langle\bar{s}s\rangle\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{144\pi^4}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[11\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]}{144\pi^4}\int dydz\,(y+z)(s-\overline{m}_c^2)$$
$$-\frac{5\,[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{576\pi^4}$$
$$\times \int dydz\, yz(4s-3\overline{m}_c^2)$$
$$+\frac{\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{576\pi^4}$$
$$\times \int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)$$
$$+\frac{m_s m_c\,[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{144\pi^4}\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)$$
$$-\frac{m_s m_c\,[34\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+33\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle-5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}\int dy$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{2304\pi^4}\int dy-\frac{m_c\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{18\pi^2}\int dy$$
$$+\frac{m_s\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{18\pi^2}\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]$$
$$-\frac{\langle\bar{s}g_s\sigma Gs\rangle\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{512\pi^4}$$
$$\times \int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$+\frac{\langle\bar{s}g_s\sigma Gs\rangle\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{256\pi^4}$$
$$\times \int dy\, y(1-y)\left[1+\frac{s}{3}\delta\left(s-\widetilde{m}_c^2\right)\right]$$
$$-\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{288\pi^4}$$
$$\times \int dy\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[17\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}$$
$$\times \int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{6912\pi^4}$$
$$\times \int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2), \quad (57)$$

$$\widetilde{\rho}^{11,0}_{uss}(s)=\frac{1}{122880\pi^8}\int dydz\,(y+z)(1-y-z)^4(s-\overline{m}_c^2)^4$$
$$\times(7s-2\overline{m}_c^2)+\frac{m_s m_c}{9216\pi^8}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^4$$
$$-\frac{m_c\,[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle]}{1152\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)^3$$
$$+\frac{m_s\,[2\langle\bar{s}s\rangle-\langle\bar{q}q\rangle]}{1152\pi^6}\int dydz\,(y+z)(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)$$
$$+\frac{m_c\,[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle]}{512\pi^6}\int dydz\,(1-y-z)(s-\overline{m}_c^2)^2$$
$$-\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{768\pi^6}\int dydz\,(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)(2s-\overline{m}_c^2)+\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{384\pi^6}$$
$$\times \int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{1536\pi^6}\int dydz\,(y+z)(1-y-z)$$
$$\times(s-\overline{m}_c^2)(2s-\overline{m}_c^2)+\frac{\langle\bar{s}s\rangle\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{144\pi^4}$$
$$\times \int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[11\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]}{72\pi^4}\int dydz\,(s-\overline{m}_c^2)$$
$$-\frac{5\,[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{1152\pi^4}$$
$$\times \int dydz\,(y+z)(3s-2\overline{m}_c^2)$$
$$+\frac{[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{288\pi^4}$$
$$\times \int dydz\,(1-y-z)(3s-2\overline{m}_c^2)$$
$$+\frac{m_s m_c\,[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{144\pi^4}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)$$
$$-\frac{m_s m_c\,[34\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+33\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle-5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{864\pi^4}\int dy$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{1152\pi^4}\int dy$$
$$-\frac{m_c\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{9\pi^2}\int dy+\frac{m_s\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{54\pi^2}\int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]$$
$$-\frac{\langle\bar{s}g_s\sigma Gs\rangle\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{768\pi^4}\int dydz\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{288\pi^4}\int dy\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[17\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}$$
$$\times \int dy\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2)-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{6912\pi^4}$$
$$\times \int dy\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2), \quad (58)$$

$$
\begin{aligned}
\rho^{11,1}_{uus}(s) &= \frac{1}{61440\pi^8}\int dydz\, yz(1-y-z)^4(s-\overline{m}_c^2)^4(8s-3\overline{m}_c^2) \\
&+\frac{m_s m_c}{36864\pi^8}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^4 \\
&-\frac{m_c\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{2304\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^3 \\
&+\frac{m_s\,[3\langle\bar{s}s\rangle-2\langle\bar{q}q\rangle]}{384\pi^6}\int dydz\, yz(1-y-z)^2 \\
&\times(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)+\frac{m_c\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{1024\pi^6} \\
&\times\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)^2 \\
&-\frac{m_s\,[3\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{12288\pi^6} \\
&\times\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)(5s-3\overline{m}_c^2) \\
&+\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{384\pi^6} \\
&\times\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2) \\
&+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{1536\pi^6}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2) \\
&+\frac{\langle\bar{q}q\rangle\,[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle]}{144\pi^4}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2) \\
&+\frac{m_s m_c\langle\bar{q}q\rangle\,[6\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]}{144\pi^4}\int dydz\,(y+z)(s-\overline{m}_c^2) \\
&-\frac{5\,[\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{576\pi^4} \\
&\times\int dydz\, yz(4s-3\overline{m}_c^2) \\
&+\frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{576\pi^4} \\
&\times\int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2) \\
&+\frac{m_s m_c\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{144\pi^4}\int dydz\left(\frac{z}{y}+\frac{y}{z}\right) \\
&-\frac{m_s m_c\,[36\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle-2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle-3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{1728\pi^4}\int dy \\
&+\frac{m_s m_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{2304\pi^4}\int dy-\frac{m_c\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{18\pi^2}\int dy \\
&+\frac{m_s\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{36\pi^2}\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right] \\
&-\frac{\langle\bar{q}g_s\sigma Gq\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle]}{512\pi^4} \\
&\times\int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right] \\
&+\frac{\langle\bar{q}g_s\sigma Gq\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle]}{256\pi^4} \\
&\times\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right] \\
&-\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle^2}{576\pi^4}\int dy\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2) \\
&+\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle\,[9\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4} \\
&\times\int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2) \\
&-\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{6912\pi^4}\int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2), \qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\rho}^{11,0}_{uus}(s) &= \frac{1}{122880\pi^8}\int dydz\,(y+z)(1-y-z)^4(s-\overline{m}_c^2)^4(7s-2\overline{m}_c^2) \\
&+\frac{m_s m_c}{18432\pi^8}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^4 \\
&-\frac{m_c\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{1152\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)^3 \\
&+\frac{m_s\,[3\langle\bar{s}s\rangle-2\langle\bar{q}q\rangle]}{2304\pi^6}\int dydz\,(y+z)(1-y-z)^2 \\
&\times(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)+\frac{m_c\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{512\pi^6} \\
&\times\int dydz\,(1-y-z)(s-\overline{m}_c^2)^2-\frac{m_s\,[3\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{3072\pi^6} \\
&\times\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)(2s-\overline{m}_c^2) \\
&+\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{384\pi^6} \\
&\times\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2) \\
&+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{1536\pi^6}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2) \\
&+\frac{\langle\bar{q}q\rangle\,[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle]}{144\pi^4}\int dydz\,(y+z)(1-y-z) \\
&\times(s-\overline{m}_c^2)(2s-\overline{m}_c^2) \\
&+\frac{m_s m_c\langle\bar{q}q\rangle\,[6\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]}{72\pi^4}\int dydz\,(s-\overline{m}_c^2) \\
&-\frac{5\,[\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{1152\pi^4} \\
&\times\int dydz\,(y+z)(3s-2\overline{m}_c^2) \\
&+\frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{288\pi^4} \\
&\times\int dydz\,(1-y-z)(3s-2\overline{m}_c^2) \\
&+\frac{m_s m_c\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{144\pi^4}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right) \\
&+\frac{m_s m_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{1152\pi^4}\int dy \\
&-\frac{m_s m_c\,[36\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle-2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle-3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{864\pi^4}\int dy \\
&-\frac{m_c\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{9\pi^2}\int dy+\frac{m_s\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{108\pi^2} \\
&\times\int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right] \\
&-\frac{\langle\bar{q}g_s\sigma Gq\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle]}{768\pi^4}\int dydz\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right] \\
&-\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle^2}{576\pi^4}\int dy\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2) \\
&+\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle\,[9\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}
\end{aligned}
$$

$$\times \int dy \left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{6912\pi^4}\int dy \left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2), \qquad (60)$$

$$\rho_{uuu}^{11,1}(s) = \rho_{sss}^{11,1}(s)\mid_{m_s\to 0,\ \langle\bar{s}s\rangle\to\langle\bar{q}q\rangle,\ \langle\bar{s}g_s\sigma Gs\rangle\to\langle\bar{q}g_s\sigma Gq\rangle}, \qquad (61)$$

$$\widetilde{\rho}_{uuu}^{11,0}(s) = \widetilde{\rho}_{sss}^{11,0}(s)\mid_{m_s\to 0,\ \langle\bar{s}s\rangle\to\langle\bar{q}q\rangle,\ \langle\bar{s}g_s\sigma Gs\rangle\to\langle\bar{q}g_s\sigma Gq\rangle}, \qquad (62)$$

$$\rho_{sss}^{10,1}(s)=\frac{1}{122880\pi^8}\int dydz\, yz(1-y-z)^4(s-\overline{m}_c^2)^4(8s-3\overline{m}_c^2)$$
$$+\frac{1}{153600\pi^8}\int dydz\, yz(1-y-z)^5$$
$$\times(s-\overline{m}_c^2)^3(18s^2-16s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{m_s m_c}{12288\pi^8}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^4$$
$$+\frac{m_s m_c}{49152\pi^8}\int dydz\,(y+z)(1-y-z)^4(s-\overline{m}_c^2)^3(7s-3\overline{m}_c^2)$$
$$-\frac{m_c\langle\bar{s}s\rangle}{768\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^3$$
$$-\frac{m_c\langle\bar{s}s\rangle}{768\pi^6}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)$$
$$-\frac{m_s\langle\bar{s}s\rangle}{256\pi^6}\int dydz\, yz(1-y-z)^2(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)$$
$$+\frac{m_s\langle\bar{s}s\rangle}{128\pi^6}\int dydz\, yz(1-y-z)^3(s-\overline{m}_c^2)$$
$$\times(7s^2-8s\overline{m}_c^2+2\overline{m}_c^4)$$
$$+\frac{11m_c\langle\bar{s}g_s\sigma Gs\rangle}{4096\pi^6}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)^2$$
$$+\frac{11m_c\langle\bar{s}g_s\sigma Gs\rangle}{8192\pi^6}\int dydz\,(y+z)(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{7m_c\langle\bar{s}g_s\sigma Gs\rangle}{8192\pi^6}\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)^2(s-\overline{m}_c^2)^2$$
$$-\frac{m_c\langle\bar{s}g_s\sigma Gs\rangle}{4096\pi^6}\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)^3$$
$$\times(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{11m_s\langle\bar{s}g_s\sigma Gs\rangle}{2048\pi^6}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{256\pi^6}\int dydz\, yz(1-y-z)^2(15s^2-20s\overline{m}_c^2+6\overline{m}_c^4)$$
$$+\frac{3m_s\langle\bar{s}g_s\sigma Gs\rangle}{8192\pi^6}\int dydz\,(y+z)(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{\langle\bar{s}s\rangle^2}{48\pi^4}\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle^2}{24\pi^4}\int dydz\,(y+z)(s-\overline{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}s\rangle^2}{48\pi^4}\int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)$$
$$-\frac{19\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{768\pi^4}\int dydz\, yz(4s-3\overline{m}_c^2)$$
$$-\frac{\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{512\pi^4}\int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)$$
$$+\frac{89m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{3072\pi^4}\int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{181m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{9216\pi^4}\int dy+\frac{3m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{512\pi^4}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)-\frac{3m_s m_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{256\pi^4}$$
$$\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{m_c\langle\bar{s}s\rangle^3}{36\pi^2}\int dy+\frac{m_s\langle\bar{s}s\rangle^3}{12\pi^2}\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]$$
$$+\frac{43\langle\bar{s}g_s\sigma Gs\rangle^2}{18432\pi^4}\int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$+\frac{11\langle\bar{s}g_s\sigma Gs\rangle^2}{1024\pi^4}\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]$$
$$+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{768\pi^4}\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)\left(1+\frac{s}{2M^2}\right)\delta(s-\overline{m}_c^2)$$
$$-\frac{17m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{9216\pi^4}\int dy\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$+\frac{17m_s m_c\langle\bar{s}g_s\sigma Gs\rangle^2}{4608\pi^4}\int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2), \qquad (63)$$

$$\widetilde{\rho}_{sss}^{10,0}(s)=\frac{1}{122880\pi^8}\int dydz\,(y+z)(1-y-z)^4$$
$$\times(s-\overline{m}_c^2)^4(7s-2\overline{m}_c^2)$$
$$-\frac{1}{614400\pi^8}\int dydz\,(y+z)(1-y-z)^5(s-\overline{m}_c^2)^3$$
$$\times(28s^2-21s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{m_s m_c}{3072\pi^8}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^4$$
$$-\frac{m_s m_c}{12288\pi^8}\int dydz\,(1-y-z)^4(s-\overline{m}_c^2)^3(3s-\overline{m}_c^2)$$
$$-\frac{m_c\langle\bar{s}s\rangle}{192\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)^3$$
$$+\frac{m_c\langle\bar{s}s\rangle}{1152\pi^6}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)$$
$$-\frac{m_s\langle\bar{s}s\rangle}{768\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)$$

$$-\frac{m_s\langle\bar{s}s\rangle}{256\pi^6}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)$$
$$\times(5s^2-5s\overline{m}_c^2+\overline{m}_c^4)$$
$$+\frac{11m_c\langle\bar{s}g_s\sigma Gs\rangle}{1024\pi^6}\int dydz\,(1-y-z)(s-\overline{m}_c^2)^2$$
$$-\frac{11m_c\langle\bar{s}g_s\sigma Gs\rangle}{2048\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$-\frac{7m_c\langle\bar{s}g_s\sigma Gs\rangle}{4096\pi^6}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)^2(s-\overline{m}_c^2)^2$$
$$+\frac{m_c\langle\bar{s}g_s\sigma Gs\rangle}{2048\pi^6}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)^3$$
$$\times(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{512\pi^6}\int dydz\,(y+z)(1-y-z)^2$$
$$\times(10s^2-12s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{11m_s\langle\bar{s}g_s\sigma Gs\rangle}{1024\pi^6}\int dydz\,(y+z)(1-y-z)$$
$$\times(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{\langle\bar{s}s\rangle^2}{24\pi^4}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{m_sm_c\langle\bar{s}s\rangle^2}{6\pi^4}\int dydz\,(s-\overline{m}_c^2)$$
$$+\frac{m_sm_c\langle\bar{s}s\rangle^2}{24\pi^4}\int dydz\,(1-y-z)(3s-2\overline{m}_c^2)$$
$$-\frac{19\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{768\pi^4}\int dydz\,(y+z)(3s-2\overline{m}_c^2)$$
$$-\frac{89m_sm_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{2304\pi^4}\int dydz\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{199m_sm_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{2304\pi^4}\int dy$$
$$+\frac{3m_sm_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{256\pi^4}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)$$
$$+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{128\pi^4}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)$$
$$\times\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{m_c\langle\bar{s}s\rangle^3}{9\pi^2}\int dy+\frac{m_s\langle\bar{s}s\rangle^3}{18\pi^2}\int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]$$
$$+\frac{11\langle\bar{s}g_s\sigma Gs\rangle^2}{1536\pi^4}\int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]$$
$$-\frac{m_sm_c\langle\bar{s}g_s\sigma Gs\rangle^2}{1536\pi^4}\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)\left(1+\frac{s}{M^2}\right)\delta(s-\overline{m}_c^2)$$
$$-\frac{17m_sm_c\langle\bar{s}g_s\sigma Gs\rangle^2}{4608\pi^4}\int dy\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$+\frac{125m_sm_c\langle\bar{s}g_s\sigma Gs\rangle^2}{9216\pi^4}\int dy\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2),\qquad(64)$$

$$\rho^{10,1}_{uss}(s)=\frac{1}{122880\pi^8}\int dydz\,yz(1-y-z)^4(s-\overline{m}_c^2)^4(8s-3\overline{m}_c^2)$$
$$+\frac{1}{153600\pi^8}\int dydz\,yz(1-y-z)^5$$
$$\times(s-\overline{m}_c^2)^3(18s^2-16s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{m_sm_c}{18432\pi^8}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^4$$
$$+\frac{m_sm_c}{73728\pi^8}\int dydz\,(y+z)(1-y-z)^4$$
$$\times(s-\overline{m}_c^2)^3(7s-3\overline{m}_c^2)$$
$$-\frac{m_c\left[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle\right]}{2304\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^3$$
$$-\frac{m_c\left[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle\right]}{2304\pi^6}\int dydz\,(y+z)(1-y-z)^3$$
$$\times(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)$$
$$-\frac{m_s\left[2\langle\bar{q}q\rangle-\langle\bar{s}s\rangle\right]}{384\pi^6}\int dydz\,yz(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)$$
$$+\frac{m_s\langle\bar{s}s\rangle}{192\pi^6}\int dydz\,yz(1-y-z)^3(s-\overline{m}_c^2)(7s^2-8s\overline{m}_c^2+2\overline{m}_c^4)$$
$$+\frac{11m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{12288\pi^6}$$
$$\times\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)^2$$
$$+\frac{11m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{24576\pi^6}$$
$$\times\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{7m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{24576\pi^6}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)^2(s-\overline{m}_c^2)^2$$
$$-\frac{m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{12288\pi^6}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)^3(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{384\pi^6}\int dydz\,yz(1-y-z)^2(15s^2-20s\overline{m}_c^2+6\overline{m}_c^4)$$
$$+\frac{m_s\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{8192\pi^6}$$
$$\times\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{384\pi^6}\int dydz\,yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$+\frac{m_s\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{2048\pi^6}$$
$$\times\int dydz\,yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)+\frac{\langle\bar{s}s\rangle\left[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle\right]}{144\pi^4}$$

$$\times\int dydz\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}s\rangle\left[5\langle\bar{q}q\rangle-\langle\bar{s}s\rangle\right]}{144\pi^4}\int dydz\,(y+z)(s-\overline{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}s\rangle\left[\langle\bar{q}q\rangle+\langle\bar{s}s\rangle\right]}{144\pi^4}$$
$$\times\int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)$$
$$-\frac{19\left[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle\right]}{2304\pi^4}$$
$$\times\int dydz\, yz(4s-3\overline{m}_c^2)$$
$$-\frac{\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{1536\pi^4}$$
$$\times\int dydz\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)$$
$$+\frac{m_s m_c\left[2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle\right]}{576\pi^4}$$
$$\times\int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{1024\pi^4}$$
$$\times\int dydz(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{3072\pi^4}\int dy$$
$$-\frac{m_s m_c\left[16\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+15\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle-5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle\right]}{1728\pi^4}\int dy$$
$$+\frac{m_s m_c\left[2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle\right]}{576\pi^4}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)+\frac{m_s m_c\langle\bar{s}s\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{4608\pi^4}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)-\frac{m_s m_c\langle\bar{s}s\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{256\pi^4}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{m_c\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{36\pi^2}\int dy+\frac{m_s\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{18\pi^2}$$
$$\times\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]$$
$$+\frac{13\langle\bar{s}g_s\sigma Gs\rangle\left[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{18432\pi^4}$$
$$\times\int dydz\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]$$
$$+\frac{11\langle\bar{s}g_s\sigma Gs\rangle\left[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{3072\pi^4}$$
$$\times\int dy\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]$$
$$+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{2304\pi^4}$$
$$\times\int dydz\left(\frac{z}{y}+\frac{y}{z}\right)\left(1+\frac{s}{2M^2}\right)\delta(s-\overline{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\left[5\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle\right]}{3456\pi^4}$$
$$\times\int dy\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{27648\pi^4}$$
$$\times\int dy\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\left[7\langle\bar{q}g_s\sigma Gq\rangle-2\langle\bar{s}g_s\sigma Gs\rangle\right]}{1728\pi^4}$$
$$\times\int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{4608\pi^4}$$
$$\times\int dy\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2), \tag{65}$$

$$\widetilde{\rho}^{10,0}_{uss}(s)=\frac{1}{122880\pi^8}\int dydz\,(y+z)(1-y-z)^4(s-\overline{m}_c^2)^4(7s-2\overline{m}_c^2)$$
$$-\frac{1}{614400\pi^8}\int dydz\,(y+z)(1-y-z)^5(s-\overline{m}_c^2)^3$$
$$\times(28s^2-21s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{m_s m_c}{4608\pi^8}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^4$$
$$-\frac{m_s m_c}{18432\pi^8}\int dydz\,(1-y-z)^4(s-\overline{m}_c^2)^3(3s-\overline{m}_c^2)$$
$$-\frac{m_c\left[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle\right]}{576\pi^6}\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)^3$$
$$+\frac{m_c\left[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle\right]}{3456\pi^6}\int dydz\,(1-y-z)^3(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)$$
$$-\frac{m_s\left[2\langle\bar{q}q\rangle-\langle\bar{s}s\rangle\right]}{1152\pi^6}\int dydz\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)$$
$$-\frac{m_s\langle\bar{s}s\rangle}{384\pi^6}\int dydz\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)(5s^2-5s\overline{m}_c^2+\overline{m}_c^4)$$
$$+\frac{11m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{3072\pi^6}\int dydz\,(1-y-z)(s-\overline{m}_c^2)^2$$
$$-\frac{11m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{6144\pi^6}$$
$$\times\int dydz\,(1-y-z)^2(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$-\frac{7m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{12288\pi^6}$$
$$\times\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)^2(s-\overline{m}_c^2)^2$$
$$+\frac{m_c\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{6144\pi^6}$$
$$\times\int dydz\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)^3(s-\overline{m}_c^2)(2s-\overline{m}_c^2)+\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{768\pi^6}$$
$$\times\int dydz\,(y+z)(1-y-z)^2(10s^2-12s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{192\pi^6}\int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$

$$+\frac{m_s\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{1024\pi^6}$$
$$\times\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{\langle\bar{s}s\rangle\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{72\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[5\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]}{36\pi^4}\int \mathrm{d}y\mathrm{d}z\,(s-\overline{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{72\pi^4}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)(3s-2\overline{m}_c^2)$$
$$-\frac{19\,[\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{2304\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\,(y+z)(3s-2\overline{m}_c^2)$$
$$-\frac{m_s m_c\,[2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle+5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{432\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]-\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{768\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{768\pi^4}\int \mathrm{d}y$$
$$-\frac{m_s m_c\,[16\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+15\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle-5\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{432\pi^4}\int \mathrm{d}y$$
$$+\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{2304\pi^4}\int \mathrm{d}y\mathrm{d}z\left(\frac{1}{y}+\frac{1}{z}\right)$$
$$+\frac{m_s m_c\,[2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle]}{288\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{m_s m_c\langle\bar{s}s\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{384\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]$$
$$-\frac{m_c\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{9\pi^2}\int \mathrm{d}y+\frac{m_s\langle\bar{q}q\rangle\langle\bar{s}s\rangle^2}{27\pi^2}$$
$$\times\int \mathrm{d}y\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]$$
$$+\frac{11\langle\bar{s}g_s\sigma Gs\rangle\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{4608\pi^4}$$
$$\times\int \mathrm{d}y\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{4608\pi^4}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left(\frac{1}{y}+\frac{1}{z}\right)\left(1+\frac{s}{M^2}\right)\delta(s-\overline{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[5\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}$$
$$\times\int \mathrm{d}y\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{13824\pi^4}$$
$$\times\int \mathrm{d}y\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$
$$+\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[15\langle\bar{q}g_s\sigma Gq\rangle-3\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}$$
$$\times\int \mathrm{d}y\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2)$$
$$-\frac{m_s m_c\langle\bar{s}g_s\sigma Gs\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{9216\pi^4}$$
$$\times\int \mathrm{d}y\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2), \tag{66}$$
$$\rho_{uus}^{10,1}(s)=\frac{1}{122880\pi^8}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^4(s-\overline{m}_c^2)^4(8s-3\overline{m}_c^2)$$
$$+\frac{1}{153600\pi^8}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^5(s-\overline{m}_c^2)^3$$
$$\times(18s^2-16s\overline{m}_c^2+3\overline{m}_c^4)$$
$$+\frac{m_s m_c}{36864\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)^4$$
$$+\frac{m_s m_c}{147456\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^4$$
$$\times(s-\overline{m}_c^2)^3(7s-3\overline{m}_c^2)$$
$$-\frac{m_c\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{2304\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)^3$$
$$-\frac{m_c\,[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle]}{2304\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^3$$
$$\times(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)$$
$$-\frac{m_s\,[4\langle\bar{q}q\rangle-3\langle\bar{s}s\rangle]}{768\pi^6}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)^2(2s-\overline{m}_c^2)$$
$$+\frac{m_s\langle\bar{s}s\rangle}{384\pi^6}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^3(s-\overline{m}_c^2)$$
$$\times(7s^2-8s\overline{m}_c^2+2\overline{m}_c^4)$$
$$+\frac{11m_c\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{12288\pi^6}$$
$$\times\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)(s-\overline{m}_c^2)^2$$
$$+\frac{11m_c\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{24576\pi^6}$$
$$\times\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{7m_c\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{24576\pi^6}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)^2(s-\overline{m}_c^2)^2$$
$$-\frac{m_c\,[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle]}{12288\pi^6}$$
$$\times\int \mathrm{d}y\mathrm{d}z\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)^3(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$
$$-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{768\pi^6}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^2$$
$$\times(15s^2-20s\overline{m}_c^2+6\overline{m}_c^4)$$
$$+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{8192\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2$$
$$\times(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)$$

$$
+\frac{m_s\left[2\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle\right]}{768\pi^6}
\times\int \mathrm{d}y\mathrm{d}z\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)
+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{2048\pi^6}\int \mathrm{d}y\mathrm{d}z\, yz(1-y-z)
\times(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)
+\frac{\langle\bar{q}q\rangle\left[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle\right]}{144\pi^4}\int \mathrm{d}y\mathrm{d}z\, yz(1-y-z)(s-\overline{m}_c^2)(5s-3\overline{m}_c^2)
+\frac{m_sm_c\langle\bar{q}q\rangle\left[3\langle\bar{q}q\rangle-\langle\bar{s}s\rangle\right]}{144\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)(s-\overline{m}_c^2)
-\frac{m_sm_c\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{144\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)
-\frac{19\left[\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle\right]}{2304\pi^4}
\times\int \mathrm{d}y\mathrm{d}z\, yz(4s-3\overline{m}_c^2)
-\frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{1536\pi^4}
\times\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)(4s-3\overline{m}_c^2)
+\frac{m_sm_c\left[2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle\right]}{576\pi^4}
\times\int \mathrm{d}y\mathrm{d}z\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]
+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{1024\pi^4}\int \mathrm{d}y\mathrm{d}z(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]
+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{3072\pi^4}\int \mathrm{d}y
-\frac{m_sm_c\left[18\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle-2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle-3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle\right]}{1728\pi^4}\int \mathrm{d}y
+\frac{m_sm_c\left[2\langle\bar{q}q\rangle-\langle\bar{s}s\rangle\right]\langle\bar{q}g_s\sigma Gq\rangle}{576\pi^4}\int \mathrm{d}y\mathrm{d}z\left(\frac{z}{y}+\frac{y}{z}\right)
+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{4608\pi^4}\int \mathrm{d}y\mathrm{d}z\left(\frac{z}{y}+\frac{y}{z}\right)
-\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{256\pi^4}\int \mathrm{d}y\mathrm{d}z\left(\frac{z}{y}+\frac{y}{z}\right)(1-y-z)
\times\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]-\frac{m_c\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{36\pi^2}\int \mathrm{d}y+\frac{m_s\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{36\pi^2}
\times\int \mathrm{d}y\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]
+\frac{13\langle\bar{q}g_s\sigma Gq\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{18432\pi^4}
\times\int \mathrm{d}y\mathrm{d}z\,(y+z)\left[1+\frac{s}{3}\delta(s-\overline{m}_c^2)\right]
+\frac{11\langle\bar{q}g_s\sigma Gq\rangle\left[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle\right]}{3072\pi^4}
\times\int \mathrm{d}y\, y(1-y)\left[1+\frac{s}{3}\delta(s-\widetilde{m}_c^2)\right]
+\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{2304\pi^4}
\times\int \mathrm{d}y\mathrm{d}z\left(\frac{z}{y}+\frac{y}{z}\right)\left(1+\frac{s}{2M^2}\right)\delta(s-\overline{m}_c^2)
-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\left[3\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle\right]}{3456\pi^4}
\times\int \mathrm{d}y\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)
-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{27648\pi^4}
\times\int \mathrm{d}y\left(\frac{1-y}{y}+\frac{y}{1-y}\right)\delta(s-\widetilde{m}_c^2)
+\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\left[9\langle\bar{q}g_s\sigma Gq\rangle-4\langle\bar{s}g_s\sigma Gs\rangle\right]}{3456\pi^4}
\times\int \mathrm{d}y\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2)
-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{4608\pi^4}
\times\int \mathrm{d}y\left(1+\frac{s}{2M^2}\right)\delta(s-\widetilde{m}_c^2), \tag{67}
$$

$$
\widetilde{\rho}_{uus}^{10,0}(s)=\frac{1}{122880\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^4(s-\overline{m}_c^2)^4(7s-2\overline{m}_c^2)
-\frac{1}{614400\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^5(s-\overline{m}_c^2)^3
\times(28s^2-21s\overline{m}_c^2+3\overline{m}_c^4)+\frac{m_sm_c}{9216\pi^8}
\times\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^3(s-\overline{m}_c^2)^4-\frac{m_sm_c}{36864\pi^8}
\times\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^4(s-\overline{m}_c^2)^3(3s-\overline{m}_c^2)
-\frac{m_c\left[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle\right]}{576\pi^6}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^2(s-\overline{m}_c^2)^3
+\frac{m_c\left[2\langle\bar{q}q\rangle+\langle\bar{s}s\rangle\right]}{3456\pi^6}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^3(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)
-\frac{m_s\left[4\langle\bar{q}q\rangle-3\langle\bar{s}s\rangle\right]}{2304\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2
\times(s-\overline{m}_c^2)^2(5s-2\overline{m}_c^2)
-\frac{m_s\langle\bar{s}s\rangle}{768\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^3(s-\overline{m}_c^2)(5s^2-5s\overline{m}_c^2+\overline{m}_c^4)
+\frac{11m_c\left[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{3072\pi^6}
\times\int \mathrm{d}y\mathrm{d}z\,(1-y-z)(s-\overline{m}_c^2)^2
-\frac{11m_c\left[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{6144\pi^6}
\times\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^2(s-\overline{m}_c^2)(2s-\overline{m}_c^2)
-\frac{7m_c\left[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{12288\pi^6}
\times\int \mathrm{d}y\mathrm{d}z\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)^2(s-\overline{m}_c^2)^2
+\frac{m_c\left[2\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{s}g_s\sigma Gs\rangle\right]}{6144\pi^6}
\times\int \mathrm{d}y\mathrm{d}z\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)^3\left(s-\overline{m}_c^2\right)(2s-\overline{m}_c^2)
+\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{1536\pi^6}
$$

$$\times \int dydz\,(y+z)(1-y-z)^2(10s^2-12s\overline{m}_c^2+3\overline{m}_c^4)$$

$$+\frac{m_s\,[2\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{384\pi^6}$$

$$\times \int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$

$$+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{1024\pi^6}$$

$$\times \int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$

$$+\frac{\langle\bar{q}q\rangle\,[\langle\bar{q}q\rangle+2\langle\bar{s}s\rangle]}{72\pi^4}$$

$$\times \int dydz\,(y+z)(1-y-z)(s-\overline{m}_c^2)(2s-\overline{m}_c^2)$$

$$+\frac{m_sm_c\langle\bar{q}q\rangle\,[3\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]}{36\pi^4}$$

$$\times \int dydz\,(s-\overline{m}_c^2)+\frac{m_sm_c\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{72\pi^4}$$

$$\times \int dydz\,(1-y-z)(3s-2\overline{m}_c^2)$$

$$-\frac{19\,[\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle+\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{2304\pi^4}$$

$$\times \int dydz\,(y+z)(3s-2\overline{m}_c^2)$$

$$-\frac{m_sm_c\,[2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle+3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{432\pi^4}$$

$$\times \int dydz\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]-\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{768\pi^4}$$

$$\times \int dydz\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{768\pi^4}\int dy$$

$$-\frac{m_sm_c\,[18\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle-2\langle\bar{q}q\rangle\langle\bar{s}g_s\sigma Gs\rangle-3\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle]}{432\pi^4}\int dy$$

$$+\frac{m_sm_c\,[2\langle\bar{q}q\rangle-\langle\bar{s}s\rangle]\,\langle\bar{q}g_s\sigma Gq\rangle}{288\pi^4}$$

$$\times \int dydz\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{2304\pi^4}$$

$$\times \int dydz\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{m_sm_c\langle\bar{s}s\rangle\langle\bar{q}g_s\sigma Gq\rangle}{384\pi^4}$$

$$\times \int dydz\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)\left[1+\frac{s}{2}\delta(s-\overline{m}_c^2)\right]$$

$$-\frac{m_c\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{9\pi^2}\int dy+\frac{m_s\langle\bar{q}q\rangle^2\langle\bar{s}s\rangle}{54\pi^2}\int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]$$

$$+\frac{11\langle\bar{q}g_s\sigma Gq\rangle\,[\langle\bar{q}g_s\sigma Gq\rangle+2\langle\bar{s}g_s\sigma Gs\rangle]}{4608\pi^4}$$

$$\times \int dy\left[1+\frac{s}{2}\delta(s-\widetilde{m}_c^2)\right]-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{4608\pi^4}$$

$$\times \int dydz\left(\frac{1}{y}+\frac{1}{z}\right)\left(1+\frac{s}{M^2}\right)\delta(s-\overline{m}_c^2)$$

$$-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\,[3\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}$$

$$\times \int dy\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{13824\pi^4}$$

$$\times \int dy\left(\frac{1}{y}+\frac{1}{1-y}\right)\delta(s-\widetilde{m}_c^2)$$

$$+\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\,[9\langle\bar{q}g_s\sigma Gq\rangle-\langle\bar{s}g_s\sigma Gs\rangle]}{1728\pi^4}$$

$$\times \int dy\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2)-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}g_s\sigma Gs\rangle}{9216\pi^4}$$

$$\times \int dy\left(1+\frac{s}{M^2}\right)\delta(s-\widetilde{m}_c^2), \tag{68}$$

$$\rho_{uuu}^{10,1}(s)=\rho_{sss}^{10,1}(s)\mid_{m_s\to 0,\ \langle\bar{s}s\rangle\to\langle\bar{q}q\rangle,\ \langle\bar{s}g_s\sigma Gs\rangle\to\langle\bar{q}g_s\sigma Gq\rangle}, \tag{69}$$

$$\widetilde{\rho}_{uuu}^{10,0}(s)=\widetilde{\rho}_{sss}^{10,0}(s)\mid_{m_s\to 0,\ \langle\bar{s}s\rangle\to\langle\bar{q}q\rangle,\ \langle\bar{s}g_s\sigma Gs\rangle\to\langle\bar{q}g_s\sigma Gq\rangle}, \tag{70}$$

where $\int dydz=\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz$, $\int dy=\int_{y_i}^{y_f}dy$, $y_f=\frac{1+\sqrt{1-4m_c^2/s}}{2}$, $y_i=\frac{1-\sqrt{1-4m_c^2/s}}{2}$, $z_i=\frac{ym_c^2}{ys-m_c^2}$, $\overline{m}_c^2=\frac{(y+z)m_c^2}{yz}$, $\widetilde{m}_c^2=\frac{m_c^2}{y(1-y)}$, $\int_{y_i}^{y_f}dy\to\int_0^1 dy$, $\int_{z_i}^{1-y}dz\to\int_0^{1-y}dz$ when the δ functions $\delta(s-\overline{m}_c^2)$ and $\delta\left(s-\widetilde{m}_c^2\right)$ appear.

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