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Equations of motion with respect to the $(1+1+3)$ threading of a $5D$ universe

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Abstract We continue our research work started in Bejancu (Eur Phys J C 75:346, 2015), and obtain in a covariant form the equations of motion with respect to the $(1+1+3)$ threading of a $5D$ universe $(\bar{M}, \bar{g})$. The natural splitting of the tangent bundle of $\bar{M}$ leads to the study of three categories of geodesics: spatial geodesics, temporal geodesics, and vertical geodesics. As an application of the general theory, we introduce and study what we call the $5D$ Robertson–Walker universe.

1 Introduction

This paper is a continuation of our previous paper [1] on kinematic quantities and Raychaudhuri equations in a $5D$ universe. According to the new approach presented in [1], the $5D$ universe $\bar{M} = M \times K$ is studied by means of the submersion of $\bar{M}$ on the $4D$ spacetime M. Note that in all the other theories of a $5D$ universe the study was performed via an immersion of M in $\bar{M}$ (cf. [2–5]).

The kinematic quantities together with the spatial tensor fields and the Riemannian spatial connection enable us to obtain, in a covariant form, the equations of motion in $(\bar{M}, \bar{g})$. By using the natural splitting of the tangent bundle of $\bar{M}$ we introduce into the study three categories of geodesics: spatial geodesics, temporal geodesics and vertical geodesics. We apply the general theory to what we call the $5D$ Robertson–Walker universe, which can be thought of as a disjoint union of $4D$ Robertson–Walker spacetimes. In this case, the above three categories of geodesics are completely determined.

Now, we outline the content of the paper. In Sect. 2 we recall from [1] the kinematic quantities with respect to the $(1+1+3)$ threading of the $5D$ universe $(\bar{M}, \bar{g})$, and the Riemannian spatial connection ∇ on the spatial distribution $\mathcal{S}\bar{M}$. The complete characterization of the Levi-Civita connection on $(\bar{M}, \bar{g})$ (cf. (2.18)) enables us to write down in Sect. 3, for the first time in the literature, the splitting of the equations of motions into three groups (cf. (3.6)). As an example, we present the $5D$ Robertson–Walker universe (see (3.7)) together with its equations of motion (cf. (3.14)). In Sect. 4 we introduce spatial, temporal, and vertical geodesics and state their characterizations via the geometric objects defined on $(\bar{M}, \bar{g})$ (cf. Theorems 4.1, 4.5). In case $\mathcal{T}\bar{M} \oplus \mathcal{V}\bar{M}$ is a Killing vector bundle, we show that spatial geodesics coincide with autoparallel curves of ∇ (cf. Theorem 4.3). Finally, we describe explicitly the above three categories of geodesics in a $5D$ Robertson–Walker universe (cf. Theorem 4.4, Corollary 4.2). The conclusions on the research developed in the paper are presented in Sect. 5.

2 Kinematic quantities and the Riemannian spatial connection in a $5D$ universe

In this section we describe the geometric configuration of a $5D$ universe that has been presented in [1]. Let $\bar{M} = M \times K$ be a product bundle over M, where M and K are manifolds of dimensions four and one, respectively. The existence of two vector fields η and U on $\bar{M}$ and M, respectively, induces a coordinate system (x^a) on $\bar{M}$ such that $\eta = \frac{\partial}{\partial x^4}$ and $U = \frac{\partial}{\partial x^0}$. The coordinate transformations on $\bar{M}$ are given by

$$\begin{aligned} x^\alpha &= \widetilde{x}^\alpha(x^1, x^2, x^3); \quad \widetilde{x}^0 = x^0 + f(x^1, x^2, x^3) \\ \widetilde{x}^4 &= x^4 + \bar{f}(x^0, x^1, x^2, x^3). \end{aligned} \tag{2.1}$$

Throughout the paper we use the ranges of indices: $a, b, c, \ldots \in \{0, 1, 2, 3, 4\}$, $i, j, k, \ldots \in \{0, 1, 2, 3\}$, $\alpha, \beta, \gamma, \ldots \in \{1, 2, 3\}$. Also, for any vector bundle E over $\bar{M}$ denote by $\Gamma(E)$ the $\mathcal{F}(\bar{M})$-module of the smooth sections of E, where $\mathcal{F}(\bar{M})$ is the algebra of smooth functions on $\bar{M}$.

Next, suppose that $\bar{M}$ is endowed with a Lorentz metric $\bar{g}$ such that

$$\bar{g}(\eta, \eta) = \Psi^2. \tag{2.2}$$

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Denote by $\mathcal{V}\bar{M}$ the line bundle over $\bar{M}$ spanned by η, and by $\mathcal{H}\bar{M}$ its complementary orthogonal vector bundle in $T\bar{M}$. Then, suppose that the lift of $\partial/\partial x^0$ to $\bar{M}$ is timelike with respect to $\bar{g}$, and denote by $\delta/\delta x^0$ its projection on $\mathcal{H}(\bar{M})$, that is, we have

$$\frac{\delta}{\delta x^0} = \frac{\partial}{\partial x^0} - A_0 \frac{\partial}{\partial x^4}. \tag{2.3}$$

It is proved that there exists a globally defined vector field ξ on $\bar{M}$ which is locally given by $\delta/\delta x^0$, and we have

$$\bar{g}(\xi, \xi) = -\Phi^2. \tag{2.4}$$

Thus the tangent bundle of $\bar{M}$ admits the orthogonal decomposition

$$T\bar{M} = \mathcal{T}\bar{M} \oplus \mathcal{S}\bar{M} \oplus \mathcal{V}\bar{M}, \tag{2.5}$$

where $\mathcal{T}\bar{M}$ is the line bundle spanned by ξ, and $\mathcal{S}\bar{M}$ is the complementary orthogonal distribution to $\mathcal{T}\bar{M}$ in $\mathcal{H}\bar{M}$. We call $\mathcal{T}\bar{M}$, $\mathcal{S}\bar{M}$, and $\mathcal{V}\bar{M}$ the *temporal distribution*, the *spatial distribution*, and the *vertical distribution*, respectively. According to (2.5) there exists an *adapted frame field* $\{\delta/\delta x^0, \delta/\delta x^\alpha, \partial/\partial x^4\}$ on $\bar{M}$, where we put

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - B_\alpha \frac{\delta}{\delta x^0} - A_\alpha \frac{\partial}{\partial x^4}. \tag{2.6}$$

Its dual frame field $\{\delta x^0, \mathrm{d}x^\alpha, \delta x^4\}$, where we put

$$\delta x^0 = \mathrm{d}x^0 + B_\alpha \mathrm{d}x^\alpha, \quad \delta x^4 = \mathrm{d}x^4 + A_i \mathrm{d}x^i, \tag{2.7}$$

is called an *adapted coframe field* on $\bar{M}$. The pair $(\bar{M}, \bar{g})$ with the geometric configuration presented above is called a $5D$ *universe*, and it is the main object of study in the present paper.

Now, denote by h the Riemannian metric induced by $\bar{g}$ on $\mathcal{S}\bar{M}$, and put

$$h_{\alpha\beta} = h\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right), \ \alpha, \beta \in \{1, 2, 3\}. \tag{2.8}$$

Then the line element with respect to the adapted coframe field is given by

$$\mathrm{d}\bar{s}^2 = -\Phi^2(\delta x^0)^2 + h_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \Psi^2(\delta x^4)^2. \tag{2.9}$$

The $4D$ *vorticity* $\omega_{\alpha\beta}$ and $5D$ *vorticity* $\eta_{\alpha\beta}$ in the $5D$ universe are given by

$$\begin{aligned} \omega_{\alpha\beta} &= \tfrac{1}{2}\left\{\tfrac{\delta B_\beta}{\delta x^\alpha} - \tfrac{\delta B_\alpha}{\delta x^\beta}\right\}, \\ \eta_{\alpha\beta} &= \tfrac{1}{2}\left\{\tfrac{\delta A_\beta}{\delta x^\alpha} - \tfrac{\delta A_\alpha}{\delta x^\beta} + B_\alpha \tfrac{\delta A_0}{\delta x^\beta} - B_\beta \tfrac{\delta A_0}{\delta x^\alpha}\right\}. \end{aligned} \tag{2.10}$$

Also, the $4D$ *expansion tensor field* $\Theta_{\alpha\beta}$ and the $5D$ *expansion tensor field* $K_{\alpha\beta}$ are given by

$$\text{(a)} \ \ \Theta_{\alpha\beta} = \frac{1}{2}\frac{\delta h_{\alpha\beta}}{\delta x^0}, \quad \text{(b)} \ \ K_{\alpha\beta} = \frac{1}{2}\frac{\partial h_{\alpha\beta}}{\partial x^4}. \tag{2.11}$$

The $4D$ *expansion function* Θ and the $5D$ *expansion function* K are the traces of the spatial tensor fields from (2.11), expressed as follows:

$$\text{(a)} \ \ \Theta = \Theta_{\alpha\beta}h^{\alpha\beta}, \quad \text{(b)} \ \ K = K_{\alpha\beta}h^{\alpha\beta}. \tag{2.12}$$

Remark 2.1 It is worth mentioning that $\omega_{\alpha\beta}$, $\eta_{\alpha\beta}$, $\Theta_{\alpha\beta}$, and $K_{\alpha\beta}$ define spatial tensor fields of type (0, 2) on the $5D$ universe $(\bar{M}, \bar{g})$. According to the study presented in [1], this means that with respect to the transformations (2.1) they change like tensor fields of type (0,2) on a 3-dimensional manifold. □

An important geometric object introduced in [1] is the *Riemannian spatial connection* on $\bar{M}$, which is a linear connection ∇ on the spatial distribution $\mathcal{S}\bar{M}$, given by

$$\nabla_X \mathcal{S}Y = \mathcal{S}\bar{\nabla}_X \mathcal{S}Y, \quad \forall \ \ X, Y \in \Gamma(T\bar{M}), \tag{2.13}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and $\mathcal{S}$ is the projection morphism of $T\bar{M}$ on $\mathcal{S}\bar{M}$ with respect to (2.5). Locally, ∇ is given by

$$\begin{aligned} &\text{(a)} \ \ \nabla_{\frac{\delta}{\delta x^\beta}} \tfrac{\delta}{\delta x^\alpha} = \Gamma_{\alpha\ \beta}^{\ \gamma} \tfrac{\delta}{\delta x^\gamma}, \quad \text{(b)} \ \ \nabla_{\frac{\delta}{\delta x^0}} \tfrac{\delta}{\delta x^\alpha} = \Gamma_{\alpha\ 0}^{\ \gamma} \tfrac{\delta}{\delta x^\gamma} \\ &\text{(c)} \ \ \nabla_{\frac{\partial}{\partial x^4}} \tfrac{\delta}{\delta x^\alpha} = \Gamma_{\alpha\ 4}^{\ \gamma} \tfrac{\delta}{\delta x^\gamma}, \end{aligned} \tag{2.14}$$

where we put

$$\begin{aligned} &\text{(a)} \ \ \Gamma_{\alpha\ \beta}^{\ \gamma} = \tfrac{1}{2}h^{\gamma\mu}\left\{\tfrac{\delta h_{\mu\alpha}}{\delta x^\beta} + \tfrac{\delta h_{\mu\beta}}{\delta x^\alpha} - \tfrac{\delta h_{\alpha\beta}}{\delta x^\mu}\right\}, \\ &\text{(b)} \ \ \Gamma_{\alpha\ 0}^{\ \gamma} = \Theta_\alpha^\gamma + \Phi^2\omega_\alpha^\gamma, \quad \text{(c)} \ \ \Gamma_{\alpha\ 4}^{\ \gamma} = K_\alpha^\gamma - \Psi^2\eta_\alpha^\gamma. \end{aligned} \tag{2.15}$$

Next, we express the Lie brackets of vector fields from the adapted frame field, as follows:

$$\begin{aligned} &\text{(a)} \ \ \left[\tfrac{\delta}{\delta x^\alpha}, \tfrac{\delta}{\delta x^0}\right] = b_\alpha \tfrac{\delta}{\delta x^0} + a_\alpha \tfrac{\partial}{\partial x^4}, \\ &\text{(b)} \ \ \left[\tfrac{\delta}{\delta x^0}, \tfrac{\partial}{\partial x^4}\right] = a_0 \tfrac{\partial}{\partial x^4}, \\ &\text{(c)} \ \ \left[\tfrac{\delta}{\delta x^\alpha}, \tfrac{\partial}{\partial x^4}\right] = d_\alpha \tfrac{\delta}{\delta x^0} + c_\alpha \tfrac{\partial}{\partial x^4}, \\ &\text{(d)} \ \ \left[\tfrac{\delta}{\delta x^\beta}, \tfrac{\delta}{\delta x^\alpha}\right] = 2\omega_{\alpha\beta} \tfrac{\delta}{\delta x^0} + 2\eta_{\alpha\beta} \tfrac{\partial}{\partial x^4}, \end{aligned} \tag{2.16}$$

where we put

$$\begin{aligned} a_\alpha &= \tfrac{\delta A_\alpha}{\delta x^0} - \tfrac{\delta A_0}{\delta x^\alpha} - B_\alpha \tfrac{\delta A_0}{\delta x^0}, \\ b_\alpha &= \tfrac{\delta B_\alpha}{\delta x^0}, \quad c_\alpha = \tfrac{\partial A_\alpha}{\partial x^4} - B_\alpha \tfrac{\partial A_0}{\partial x^4}, \quad d_\alpha = \tfrac{\partial B_\alpha}{\partial x^4}. \end{aligned} \tag{2.17}$$

Note that $a_\alpha, b_\alpha, c_\alpha$, and d_α define spatial tensor fields of type (0, 1) on $\bar{M}$.

Finally, the Levi-Civita connection $\bar{\nabla}$ on $(\bar{M}, \bar{g})$, is expressed as follows:

$$
\begin{aligned}
\bar{\nabla}_{\frac{\delta}{\delta x^{\beta}}} \frac{\delta}{\delta x^{\alpha}} &= \Gamma_{\alpha\ \beta}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}} + \left(\omega_{\alpha\beta} + \Phi^{-2}\Theta_{\alpha\beta}\right) \frac{\delta}{\delta x^{0}} \\
&\quad + \left(\eta_{\alpha\beta} - \Psi^{-2}K_{\alpha\beta}\right) \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\delta}{\delta x^{0}}} \frac{\delta}{\delta x^{\alpha}} &= \Gamma_{\alpha\ 0}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}} + (\Phi_{\alpha} - b_{\alpha}) \frac{\delta}{\delta x^{0}} \\
&\quad + \tfrac{1}{2}\left(\Phi^{2} d_{\alpha} \Psi^{-2} - a_{\alpha}\right) \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\partial}{\partial x^{4}}} \frac{\delta}{\delta x^{\alpha}} &= \Gamma_{\alpha\ 4}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}} + \tfrac{1}{2}\left(\Psi^{2} a_{\alpha} \Phi^{-2} - d_{\alpha}\right) \frac{\delta}{\delta x^{0}} \\
&\quad + (\Psi_{\alpha} - c_{\alpha}) \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\delta}{\delta x^{\alpha}}} \frac{\delta}{\delta x^{0}} &= \Gamma_{\alpha\ 0}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}} + \Phi_{\alpha} \frac{\delta}{\delta x^{0}} \\
&\quad + \tfrac{1}{2}\left(\Phi^{2} d_{\alpha} \Psi^{-2} + a_{\alpha}\right) \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\delta}{\delta x^{\alpha}}} \frac{\partial}{\partial x^{4}} &= \Gamma_{\alpha\ 4}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}} + \tfrac{1}{2}\left(\Psi^{2} a_{\alpha} \Phi^{-2} + d_{\alpha}\right) \frac{\delta}{\delta x^{0}} + \Psi_{\alpha} \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\partial}{\partial x^{4}}} \frac{\delta}{\delta x^{0}} &= \tfrac{1}{2}\left(\Psi^{2} a^{\gamma} - \Phi^{2} d^{\gamma}\right) \frac{\delta}{\delta x^{\gamma}} + \Phi_{4} \frac{\delta}{\delta x^{0}} \\
&\quad + (\Psi_{0} - a_{0}) \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\delta}{\delta x^{0}}} \frac{\partial}{\partial x^{4}} &= \tfrac{1}{2}\left(\Psi^{2} a^{\gamma} - \Phi^{2} d^{\gamma}\right) \frac{\delta}{\delta x^{\gamma}} + \Phi_{4} \frac{\delta}{\delta x^{0}} + \Psi_{0} \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\delta}{\delta x^{0}}} \frac{\delta}{\delta x^{0}} &= \Phi^{2}\left(\Phi^{\gamma} - b^{\gamma}\right) \frac{\delta}{\delta x^{\gamma}} + \Phi_{0} \frac{\delta}{\delta x^{0}} + \Phi^{2}\Phi_{4}\Psi^{-2} \frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\partial}{\partial x^{4}}} \frac{\partial}{\partial x^{4}} &= \Psi^{2}\left(c^{\gamma} - \Psi^{\gamma}\right) \frac{\delta}{\delta x^{\gamma}} \\
&\quad + \Psi^{2}(\Psi_{0} - a_{0})\Phi^{-2} \frac{\delta}{\delta x^{0}} + \Psi_{4} \frac{\partial}{\partial x^{4}},
\end{aligned}
\tag{2.18}
$$

where we put

$$
\begin{aligned}
&\Phi_{i} = \Phi^{-1}\frac{\delta \Phi}{\delta x^{i}}, \quad \Psi_{i} = \Psi^{-1}\frac{\delta \Psi}{\delta x^{i}}, \\
&\Phi_{4} = \Phi^{-1}\frac{\partial \Phi}{\partial x^{4}}, \quad \Psi_{4} = \Psi^{-1}\frac{\partial \Psi}{\partial x^{4}}.
\end{aligned}
\tag{2.19}
$$

3 Equations of motion in a 5D universe

In this section we write down, in a covariant form, the equations of motion in the 5D universe $(\bar{M}, \bar{g})$. It is first time in the literature that these equations are expressed by three groups of equations (cf. (3.6)), and in terms of kinematic quantities and of the local coefficients of the Riemannian spatial connection. As an example of such 5D universe we present the 5D Robertson–Walker universe (cf. (3.7)), and state its equations of motion (cf. (3.14)).

Let $\bar{C}$ be a smooth curve in $\bar{M}$ given by the equations

$$
x^{a} = x^{a}(t), \quad t \in [c, d], \quad a \in \{0, 1, 2, 3, 4\}. \tag{3.1}
$$

Then by direct calculations using (2.3) and (2.6), we deduce that the tangent vector field $\frac{\mathrm{d}}{\mathrm{d}t}$ to $\bar{C}$ is expressed with respect to the adapted frame field $\{\frac{\delta}{\delta x^{0}}, \frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial x^{4}}\}$, as follows:

$$
\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\delta x^{0}}{\delta t}\frac{\delta}{\delta x^{0}} + \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta}{\delta x^{\alpha}} + \frac{\delta x^{4}}{\delta t}\frac{\partial}{\partial x^{4}}, \tag{3.2}
$$

where we put

$$
\frac{\delta x^{0}}{\delta t} = \frac{\mathrm{d}x^{0}}{\mathrm{d}t} + B_{\alpha}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}, \quad \frac{\delta x^{4}}{\delta t} = \frac{\mathrm{d}x^{4}}{\mathrm{d}t} + A_{i}\frac{\mathrm{d}x^{i}}{\mathrm{d}t}. \tag{3.3}
$$

Next, after some long calculations by using (3.2) and (2.18) we obtain

$$
\begin{aligned}
\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{\delta}{\delta x^{0}} &= \Big\{\Phi^{2}\left(\Phi^{\gamma} - b^{\gamma}\right)\frac{\delta x^{0}}{\delta t} \\
&\quad + \Gamma_{\alpha\ 0}^{\ \gamma}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} + \tfrac{1}{2}\left(\Psi^{2}a^{\gamma} - \Phi^{2}d^{\gamma}\right)\frac{\delta x^{4}}{\delta t}\Big\}\frac{\delta}{\delta x^{\gamma}} \\
&\quad + \Big\{\Phi_{0}\frac{\delta x^{0}}{\delta t} + \Phi_{\alpha}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} + \Phi_{4}\frac{\delta x^{4}}{\mathrm{d}t}\Big\}\frac{\delta}{\delta x^{0}} \\
&\quad + \Big\{\Phi^{2}\Phi_{4}\Psi^{-2}\frac{\delta x^{0}}{\delta t} + \tfrac{1}{2}\left(\Phi^{2}d_{\alpha}\Psi^{-2} + a_{\alpha}\right)\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} \\
&\quad + (\Psi_{0} - a_{0})\frac{\delta x^{4}}{\delta t}\Big\}\frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{\delta}{\delta x^{\alpha}} &= \Big\{\Gamma_{\alpha\ 0}^{\ \gamma}\frac{\delta x^{0}}{\delta t} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} + \Gamma_{\alpha\ 4}^{\ \gamma}\frac{\delta x^{4}}{\delta t}\Big\}\frac{\delta}{\delta x^{\gamma}} \\
&\quad + \Big\{(\Phi_{\alpha} - b_{\alpha})\frac{\delta x^{0}}{\delta t} + \left(\omega_{\alpha\beta} + \Phi^{-2}\Theta_{\alpha\beta}\right)\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} \\
&\quad + \tfrac{1}{2}\left(\Psi^{2}a_{\alpha}\Phi^{-2} - d_{\alpha}\right)\frac{\delta x^{4}}{\delta t}\Big\}\frac{\delta}{\delta x^{0}} \\
&\quad + \Big\{\tfrac{1}{2}\left(\Phi^{2}d_{\alpha}\Psi^{-2} - a_{\alpha}\right)\frac{\delta x^{0}}{\delta t} + \left(\eta_{\alpha\beta} - \Psi^{-2}K_{\alpha\beta}\right)\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} \\
&\quad + (\Psi_{\alpha} - c_{\alpha})\frac{\delta x^{4}}{\delta t}\Big\}\frac{\partial}{\partial x^{4}}, \\
\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{\partial}{\partial x^{4}} &= \Big\{\tfrac{1}{2}\left(\Psi^{2}a^{\gamma} - \Phi^{2}d^{\gamma}\right)\frac{\delta x^{0}}{\delta t} + \Gamma_{\alpha\ 4}^{\ \gamma}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} \\
&\quad + \Psi^{2}\left(c^{\gamma} - \Psi^{\gamma}\right)\frac{\delta x^{4}}{\delta t}\Big\}\frac{\delta}{\delta x^{\gamma}} + \Big\{\Phi_{4}\frac{\delta x^{0}}{\delta t} \\
&\quad + \tfrac{1}{2}\left(\Psi^{2}a_{\alpha}\Phi^{-2} + d_{\alpha}\right)\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} + \Psi^{2}(\Psi_{0} - a_{0})\frac{\delta x^{4}}{\delta t}\Big\}\frac{\delta}{\delta x^{0}} \\
&\quad + \Big\{\Psi_{0}\frac{\delta x^{0}}{\delta t} + \Psi_{\alpha}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} + \Psi_{4}\frac{\delta x^{4}}{\delta t}\Big\}\frac{\partial}{\partial x^{4}}.
\end{aligned}
\tag{3.4}
$$

Now, by using (3.2), (3.4), (2.15b), and (2.15c), and taking into account that $\omega_{\alpha\beta}$ and $\eta_{\alpha\beta}$ are skew-symmetric spatial tensor fields on $\bar{M}$, we deduce that

$$
\begin{aligned}
\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{\mathrm{d}}{\mathrm{d}t} &= \frac{d^{2}x^{\gamma}}{\mathrm{d}t^{2}}\frac{\delta}{\delta x^{\gamma}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta x^{0}}{\delta t}\right)\frac{\delta}{\delta x^{0}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta x^{4}}{\delta t}\right)\frac{\partial}{\partial x^{4}} \\
&\quad + \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\frac{\delta}{\delta x^{\alpha}} + \frac{\delta x^{0}}{\delta t}\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\frac{\delta}{\delta x^{0}} + \frac{\delta x^{4}}{\delta t}\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\frac{\partial}{\partial x^{4}} \\
&= \Big\{\frac{\mathrm{d}^{2}x^{\gamma}}{\mathrm{d}t^{2}} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} + 2\left(\Theta_{\alpha}^{\gamma} + \Phi^{2}\omega_{\alpha}^{\gamma}\right) \\
&\quad \times \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta x^{0}}{\delta t} + 2\left(K_{\alpha}^{\ \gamma} - \Psi^{2}\eta_{\alpha}^{\ \gamma}\right)\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta x^{4}}{\delta t} \\
&\quad + \left(\Psi^{2}a^{\gamma} - \Phi^{2}d^{\gamma}\right)\frac{\delta x^{0}}{\delta t}\frac{\delta x^{4}}{\delta t} + \Phi^{2}\left(\Phi^{\gamma} - b^{\gamma}\right)\left(\frac{\delta x^{0}}{\delta t}\right)^{2} \\
&\quad + \Psi^{2}\left(c^{\gamma} - \Psi^{\gamma}\right)\left(\frac{\delta x^{4}}{\delta t}\right)^{2}\Big\}\frac{\delta}{\delta x^{\gamma}} + \Big\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta x^{0}}{\delta t}\right) \\
&\quad + \Phi^{-2}\Theta_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} \\
&\quad + (2\Phi_{\alpha} - b_{\alpha})\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta x^{0}}{\mathrm{d}t} + \Psi^{2}a_{\alpha}\Phi^{-2}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta x^{4}}{\mathrm{d}t} + 2\Phi_{4}\frac{\delta x^{0}}{\delta t}\frac{\delta x^{4}}{\mathrm{d}t} \\
&\quad + \Phi_{0}\left(\frac{\delta x^{0}}{\delta t}\right)^{2} + \Psi^{2}(\Psi_{0} - a_{0})\left(\frac{\delta x^{4}}{\delta t}\right)^{2}\Big\}\frac{\delta}{\delta x^{0}} + \Big\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta x^{4}}{\delta t}\right) \\
&\quad - \Psi^{-2}K_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} + \Phi^{2}d_{\alpha}\Psi^{-2}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta x^{0}}{\delta t} \\
&\quad + (2\Psi_{\alpha} - c_{\alpha})\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\delta x^{4}}{\delta t} \\
&\quad + (2\Psi_{0} - a_{0})\frac{\delta x^{0}}{\delta t}\frac{\delta x^{4}}{\delta t} + \Phi^{2}\Phi_{4}\Psi^{-2}\left(\frac{\delta x^{0}}{\delta t}\right)^{2} \\
&\quad + \Psi_{4}\left(\frac{\delta x^{4}}{\delta t}\right)^{2}\Big\}\frac{\partial}{\partial x^{4}}.
\end{aligned}
\tag{3.5}
$$

Finally, since $\bar{C}$ is a geodesic of $(\bar{M}, \bar{g})$ if and only if the left hand side in (3.5) vanishes identically on $\bar{M}$, we can state the following theorem.

Theorem 3.1 *Let $(\bar{M}, \bar{g})$ be a 5D universe with kinematic quantities $\{\omega_{\alpha\beta}, \eta_{\alpha\beta}, \Theta_{\alpha\beta}, K_{\alpha\beta}\}$ and with the Riemannian* *spatial connection* ∇ *given by* (2.14) *and* (2.15b). *Then the equations of motion in* $(\bar{M}, \bar{g})$ *are expressed as follows:*

$$\begin{aligned}
&\text{(a)}\quad \frac{\mathrm{d}^2x^\gamma}{\mathrm{d}t^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} + 2\left(\Theta_\alpha^\gamma + \Phi^2\omega_\alpha^\gamma\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta x^0}{\delta t}\\
&\quad +2\left(K_\alpha{}^\gamma - \Psi^2\eta_\alpha^\gamma\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta x^4}{\delta t} + \left(\Psi^2 a^\gamma - \Phi^2 d^\gamma\right)\frac{\delta x^0}{\delta t}\frac{\delta x^4}{\delta t}\\
&\quad +\Phi^2\left(\Phi^\gamma - b^\gamma\right)\left(\frac{\delta x^0}{\delta t}\right)^2 + \Psi^2\left(c^\gamma - \Psi^\gamma\right)\left(\frac{\delta x^4}{\delta t}\right)^2 = 0,\\
&\quad \textit{where}\ \ \gamma \in \{1, 2, 3\},\\
&\text{(b)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta x^0}{\delta t}\right) + \Phi^{-2}\Theta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} + \left(2\Phi_\alpha - b_\alpha\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta x^0}{\mathrm{d}t}\\
&\quad +\Psi^2 a_\alpha\Phi^{-2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta x^4}{\mathrm{d}t} + 2\Phi_4\frac{\delta x^0}{\delta t}\frac{\delta x^4}{\mathrm{d}t} + \Phi_0\left(\frac{\delta x^0}{\delta t}\right)^2\\
&\quad +\Psi^2\left(\Psi_0 - a_0\right)\left(\frac{\delta x^4}{\delta t}\right)^2 = 0,\\
&\text{(c)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta x^4}{\delta t}\right) - \Psi^{-2}K_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} + \Phi^2 d_\alpha\Psi^{-2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta x^0}{\delta t}\\
&\quad +\left(2\Psi_\alpha - c_\alpha\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta x^4}{\delta t} + \left(2\Psi_0 - a_0\right)\frac{\delta x^0}{\delta t}\frac{\delta x^4}{\delta t}\\
&\quad +\Phi^2\Phi_4\Psi^{-2}\left(\frac{\delta x^0}{\delta t}\right)^2 + \Psi_4\left(\frac{\delta x^4}{\delta t}\right)^2 = 0.
\end{aligned} \tag{3.6}$$

It is the first time in the literature when the equations of motion in a $5D$ universe are expressed in terms of kinematic quantities and of some spatial tensor fields. The first type of equations of motion was presented in Eq. (5.28) of [2], wherein the natural frame field $\{\partial/\partial x^a\}$, $a \in \{0, 1, 2, 3, 4\}$, has been used. In this way, no differences were noticed between the temporal variable x^0, the spatial variables (x^α), and the vertical variable x^4. Also, in [6] the author stated another form of equations of motion, wherein the temporal distribution was not taken into consideration. The main difference between (5.6) of [6] and (3.6) is that the latter can relate physics and geometry with observations, via the kinematic quantities.

Next, we construct an example of $5D$ universe and write down its equations of motion. Suppose that the line element of the Lorentz metric $\bar{g}$ has the particular form

$$\begin{aligned}
\mathrm{d}\bar{s}^2 = &-(\mathrm{d}x^0)^2 + f^2(x^0, x^4)g_{\alpha\beta}(x^1, x^2, x^3)\mathrm{d}x^\alpha\mathrm{d}x^\beta\\
&+(\mathrm{d}x^4)^2,
\end{aligned} \tag{3.7}$$

where f is a positive smooth function on an open region $\mathcal{R}$ of R_1^2, and $g_{\alpha\beta}$ define a positive definite symmetric spatial tensor field g on $\bar{M}$. Taking into account the $\bar{g}$ given by (3.7) satisfies

$$\bar{g}\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^4}\right) = 0,\quad \bar{g}\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^4}\right) = 0,$$
$$\bar{g}\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^0}\right) = 0,$$

and using (2.3) and (2.6), we obtain

$$\begin{aligned}
&\text{(a)}\ \ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i},\quad \text{(b)}\ \ A_i = 0,\\
&\text{(c)}\ \ B_\alpha = 0,\ \forall i \in \{0, 1, 2, 3\},\quad \alpha \in \{1, 2, 3\}.
\end{aligned} \tag{3.8}$$

By using (3.8a) we see that the distributions $\mathcal{S}\bar{M}$, $\mathcal{T}\bar{M} \oplus \mathcal{S}\bar{M}$ and $\mathcal{S}\bar{M} \oplus \mathcal{V}\bar{M}$ are integrable, and as a consequence of (2.16) we deduce that

$$\begin{aligned}
&a_i = 0,\quad b_\alpha = c_\alpha = d_\alpha = 0,\\
&\omega_{\alpha\beta} = \eta_{\alpha\beta} = 0,\ \forall i \in \{0, 1, 2, 3\},\quad \alpha, \beta \in \{1, 2, 3\}.
\end{aligned} \tag{3.9}$$

In order to obtain the other kinematic quantities, we note that

$$h_{\alpha\beta} = f^2 g_{\alpha\beta} \quad\text{and}\quad h^{\alpha\beta} = f^{-2}g^{\alpha\beta}. \tag{3.10}$$

Then, by using (3.10) into (2.11) and (2.12), we infer that

$$\begin{aligned}
&\Theta_{\alpha\beta} = f\frac{\partial f}{\partial x^0}g_{\alpha\beta},\quad K_{\alpha\beta} = f\frac{\partial f}{\partial x^4}g_{\alpha\beta},\\
&\Theta_\alpha^\gamma = f^{-1}\frac{\partial f}{\partial x^0}\delta_\alpha^\gamma,\quad K_\alpha^\gamma = f^{-1}\frac{\partial f}{\partial x^4}\delta_\alpha^\gamma,\\
&\Theta = 3f^{-1}\frac{\partial f}{\partial x^0},\quad K = 3f^{-1}\frac{\partial f}{\partial x^4}.
\end{aligned} \tag{3.11}$$

Moreover, from (2.19) and (3.3), we obtain

$$\Phi_a = \Psi_a = 0,\quad \forall a \in \{0, 1, 2, 3, 4\}, \tag{3.12}$$

and

$$\frac{\delta x^0}{\delta t} = \frac{\mathrm{d}x^0}{\mathrm{d}t},\quad \frac{\delta x^4}{\delta t} = \frac{\mathrm{d}x^4}{\mathrm{d}t}, \tag{3.13}$$

respectively. Also, note that the local coefficients $\Gamma_{\alpha\ \beta}^{\ \gamma}$ of the Riemannian spatial connection ∇ given by (2.15a) become the usual Christoffel symbols with respect to the Riemannian metric $g = (g_{\alpha\beta})$.

Finally, by using (3.7)–(3.13) into (3.6), we obtain the following equations of motion in a $5D$ universe $(\bar{M}, \bar{g})$ whose Lorentz metric is given by (3.7):

$$\begin{aligned}
&\text{(a)}\ \ \frac{\mathrm{d}^2x^\gamma}{\mathrm{d}t^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} + 2f^{-1}\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}x^\gamma}{\mathrm{d}t} = 0,\\
&\text{(b)}\ \ \frac{\mathrm{d}^2x^0}{\mathrm{d}t^2} + f\frac{\partial f}{\partial x^0}g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} = 0,\\
&\text{(c)}\ \ \frac{\mathrm{d}^2x^4}{\mathrm{d}t^2} - f\frac{\partial f}{\partial x^4}g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} = 0.
\end{aligned} \tag{3.14}$$

As leaves of $\mathcal{T}\bar{M} \oplus \mathcal{S}\bar{M}$ are locally given by $x^4 = \text{const.}$, from (3.7) we see that the metric induced on them is of Robertson–Walker metric type (cf. [7], p. 343), provided the leaves of $\mathcal{S}\bar{M}$ are 3-dimensional manifolds of the same constant curvature. This happens in the case we take $\bar{M} = I \times S \times K$, where I is an open interval in R and S is a 3-dimensional Riemannian manifold of constant curvature $k = 1, 0$ or -1. Thus, we may think such a $5D$ universe as a disjoint union of Robertson–Walker spacetimes. For this reason we call the $5D$ universe $(\bar{M}, \bar{g})$ whose metric is given by (3.7), a $5D$ *Robertson–Walker universe*, with the *warping function* f.

4 Special geodesics in a $5D$-universe

This section is devoted to the study of some particular classes of geodesics in a $5D$ universe $(\bar{M}, \bar{g})$. The existence of these geodesics is due to the splitting (2.5) of $T\bar{M}$, which has been considered first in [1].

Let $\bar{C}$ be a curve in $\bar{M}$ given by (3.1). Then we say that $\bar{C}$ is a *spatial curve*, if it is tangent to the spatial distribution at any of its points. Thus, by (3.2) and (3.3), we deduce that $\bar{C}$ is a spatial curve if and only if we have

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\delta}{\delta x^\alpha}, \tag{4.1}$$

or equivalently

$$\frac{\delta x^0}{\delta t} = \frac{\mathrm{d}x^0}{\mathrm{d}t} + B_\alpha\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = 0, \quad \text{and}$$
$$\frac{\delta x^4}{\delta t} = \frac{\mathrm{d}x^4}{\mathrm{d}t} + A_i\frac{\mathrm{d}x^i}{\mathrm{d}t} = 0. \tag{4.2}$$

If, moreover, a spatial curve $\bar{C}$ is a geodesic of $(\bar{M}, \bar{g})$, we say that it is a *spatial geodesic*. Taking into account of (4.2) into (3.6), we state the following theorem.

Theorem 4.1 *A spatial curve $\bar{C}$ is a spatial geodesic if and only if the following equations are satisfied:*

$$\begin{aligned}
&\text{(a)}\quad \tfrac{\mathrm{d}^2x^\gamma}{\mathrm{d}t^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\tfrac{\mathrm{d}x^\alpha}{\mathrm{d}t}\tfrac{\mathrm{d}x^\beta}{\mathrm{d}t} = 0,\\
&\text{(b)}\quad \Theta_{\alpha\beta}\tfrac{\mathrm{d}x^\alpha}{\mathrm{d}t}\tfrac{\mathrm{d}x^\beta}{\mathrm{d}t} = 0,\\
&\text{(c)}\quad K_{\alpha\beta}\tfrac{\mathrm{d}x^\alpha}{\mathrm{d}t}\tfrac{\mathrm{d}x^\beta}{\mathrm{d}t} = 0.
\end{aligned} \tag{4.3}$$

Next, we say that $\bar{C}$ is an *autoparallel curve* in $\bar{M}$ with respect to the Riemannian spatial connection ∇, if it is a spatial curve satisfying

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}\frac{\mathrm{d}}{\mathrm{d}t} = 0. \tag{4.4}$$

Then, by using (4.1) and (2.14a) into (4.4) we obtain the following.

Theorem 4.2 *A spatial curve $\bar{C}$ is an autoparallel curve with respect to ∇ if and only if Eq. (4.3a) is satisfied.*

Thus, the relationship between spatial geodesics and autoparallel curves with respect to ∇ can be stated in the next corollary.

Corollary 4.1 *A spatial geodesic of $(\bar{M}, \bar{g})$ must be an autoparallel curve with respect to ∇. Conversely, an autoparallel curve with respect to ∇ is a spatial geodesic if and only if* (4.3b) *and* (4.3c) *are satisfied.*

Next, we define the Lie derivative of the Riemannian metric h on $\mathcal{S}\bar{M}$ with respect to a vector field $Z \in \Gamma(\mathcal{T}\bar{M} \oplus \mathcal{V}\bar{M})$ as follows:

$$\begin{aligned}(\mathcal{L}_Z h)(\mathcal{S}X, \mathcal{S}Y) &= Z(h(\mathcal{S}X, \mathcal{S}Y)) - h(\mathcal{S}[Z, \mathcal{S}X], \mathcal{S}Y)\\ &\quad - h(\mathcal{S}[Z, \mathcal{S}Y], \mathcal{S}X), \forall X, Y \in \Gamma(T\bar{M}).\end{aligned} \tag{4.5}$$

Then take in turn $Z = \delta/\delta x^0$ and $Z = \partial/\partial x^4$ in (4.5), and by using (2.8), (2.11), (2.16a), and (2.16c), we obtain

$$\begin{aligned}
&\text{(a)}\quad (\mathcal{L}_{\frac{\delta}{\delta x^0}} h)(\tfrac{\delta}{\delta x^\alpha}, \tfrac{\delta}{\delta x^\beta}) = 2\Theta_{\alpha\beta},\\
&\text{(b)}\quad (\mathcal{L}_{\frac{\delta}{\delta x^4}} h)(\tfrac{\delta}{\delta x^\alpha}, \tfrac{\delta}{\delta x^\beta}) = 2K_{\alpha\beta}.
\end{aligned} \tag{4.6}$$

Now, we say that $\mathcal{T}\bar{M} \oplus \mathcal{V}\bar{M}$ is a *Killing vector bundle* with respect to $(\mathcal{S}\bar{M}, h)$, if the Lie derivative given by (4.5) vanishes identically on $\bar{M}$, for any Z. Then from (4.6) we see that *$\mathcal{T}\bar{M} \oplus \mathcal{V}\bar{M}$ is a Killing vector bundle if and only if both the 4D and the 5D expansion tensor fields vanish identically on $\bar{M}$*. Thus, combining Theorems 4.1 and 4.2, we obtain the following theorem.

Theorem 4.3 *Let $(\bar{M}, \bar{g})$ be a 5D universe such that $\mathcal{T}\bar{M} \oplus \mathcal{V}\bar{M}$ is a Killing vector bundle. Then a spatial curve $\bar{C}$ in $\bar{M}$ is a spatial geodesic if and only if it is an autoparallel with respect to the Riemannian spatial connection ∇.*

In particular, considering a 5D Robertson–Walker universe, and by using (3.8), (3.14), and (4.2), we obtain the following.

Theorem 4.4 *Let $(\bar{M}, \bar{g})$ be a 5D Robertson–Walker universe whose metric is given by (3.7). Then a curve $\bar{C}$ in $\bar{M}$ is a spatial geodesic if and only if the following conditions are satisfied:*

(i) *The parametric equations of $\bar{C}$ have the form*

$$x^0 = c, \quad x^\gamma = x^\gamma(t), \quad x^4 = k, \tag{4.7}$$

where c and k are constants, and $x^\gamma = x^\gamma(t)$, $\gamma \in \{1, 2, 3\}$, define a geodesic of a leaf of $\mathcal{S}\bar{M}$ with respect to the Riemannian metric $g = (g_{\alpha\beta})$.

(ii) *The warping function f admits (c, k) as a critical point, that is,*

$$\frac{\partial f}{\partial x^0}(c, k) = \frac{\partial f}{\partial x^4}(c, k) = 0. \tag{4.8}$$

The above theorem says that spatial geodesics in $(\bar{M}, \bar{g})$ exist if and only if the warping function has at least one critical point (c, k). In that case, if S is the leaf of $\mathcal{S}\bar{M}$ given by the equations $x^0 = c, x^4 = k$, then the lifts of geodesics of (S, g) are spatial geodesics of $(\bar{M}, \bar{g})$.

Finally, we say that a geodesic $\bar{C}$ of $(\bar{M}, \bar{g})$ is a *temporal geodesic* (resp. *vertical geodesic*) if it is tangent to $\mathcal{T}\bar{M}$ (resp. $\mathcal{V}\bar{M}$) at any of its points. Then, by using (3.2), (3.3), and (3.6) we state the following theorem.

Theorem 4.5 (i) *A curve $\bar{C}$ is a temporal geodesic in the 5D universe $(\bar{M}, \bar{g})$, if and only if the following conditions are satisfied:*

$$\begin{aligned}
&\text{(a)}\ \Phi_4 = 0,\ \text{(b)}\ \Phi_\alpha = b_\alpha,\ \text{(c)}\ x^\alpha = k^\alpha,\ \ \alpha \in \{1, 2, 3\},\\
&\text{(d)}\ \tfrac{\mathrm{d}^2x^0}{\mathrm{d}t^2} + \Phi_0\left(\tfrac{\mathrm{d}x^0}{\mathrm{d}t}\right)^2 = 0,\ \text{(e)}\ \tfrac{\mathrm{d}x^4}{\mathrm{d}t} + A_0\tfrac{\mathrm{d}x^0}{\mathrm{d}t} = 0,
\end{aligned} \tag{4.9}$$

where k^α are constants.

(ii) *A curve $\bar{C}$ is a vertical geodesic in $(\bar{M}, \bar{g})$ if and only if we have*

$$\begin{aligned} &(a)\quad \Psi_0 = a_0,\\ &(b)\quad \Psi_\alpha = c_\alpha,\\ &(c)\quad x^i = \lambda^i,\ \ \alpha \in \{1,2,3\},\ \ i \in \{0,1,2,3\},\\ &(d)\quad \frac{d^2x^4}{dt^2} + \Psi_4\left(\frac{dx^4}{dt}\right)^2 = 0, \end{aligned} \tag{4.10}$$

where λ^i are constants.

In general, (4.9a) and (4.9b) (resp. (4.10a) and (4.10b)) are strong constraints which should be satisfied by the solutions from (4.9c), (4.9d), and (4.9e) (resp. (4.10c) and (4.10d)). However, we note that all these constraints are satisfied in the case of a $5D$ Robertson–Walker universe. More precisely, from Theorem 4.5 we deduce the following corollary.

Corollary 4.2 *Let $(\bar{M}, \bar{g})$ be a $5D$ Robertson–Walker universe. Then we have the following assertions:*

(i) *The temporal geodesics of $(\bar{M}, \bar{g})$ exist, and they are portions of lines given by $x^u = k^u$, $u \in \{1,2,3,4\}$, where k^u are constants.*
(ii) *The vertical geodesics of $(\bar{M}, \bar{g})$ exist, and they are portions of lines given by $x^i = \lambda^i$, $i \in \{0,1,2,3\}$, where λ^i are constants.*

5 Conclusions

The present paper has its roots in [1,8], wherein we developed new approaches on the $(1+1+3)$ threading of a $5D$ universe and on the $(1+3)$ threading of a spacetime, respectively. The main geometric objects used in the paper are: the adapted frame and coframe fields, the kinematic tensor fields, and the Riemannian spatial connection. By using these geometric objects, we state in a $5D$ covariant form, the equations of motion in $(\bar{M}, \bar{g})$. The splitting of such equations in three groups (see (3.6)) enables us to consider the spatial, temporal, and vertical geodesics. We note the interrelations between spatial geodesics and autoparallel curves with respect to the Riemannian spatial connection (cf. Corollary 4.1). In particular, if $\mathcal{T}\bar{M} \oplus \mathcal{V}\bar{M}$ is a Killing vector bundle, we show that the spatial geodesics coincide with the autoparallel curves of ∇ (cf. Theorem 4.3). This shows that ∇ has an important role in the study of geometry and physics of a $5D$ universe.

As a new example of $5D$ universe in the sense considered in [1], we present what we call the $5D$ Robertson–Walker universe, whose metric is given by (3.7). We show that such a universe can be thought of as a disjoint union of $4D$ Robertson–Walker spacetimes. The equations of motion have the simple form (cf. (3.14)), wherein the first two groups remind us of the equations of motion in a $4D$ Robertson–Walker spacetime (cf. [7], p. 353). Also, we show that the projections of spatial geodesics of $(\bar{M}, \bar{g})$ on the leaves of $\mathcal{S}\bar{M}$ are just geodesics of the leaves with the Riemannian metric g (cf. Theorem 4.4).

Finally, we note that throughout the paper, the spatial tensor fields enable us to apply the principle of covariance, which is one of the most powerful ideas in modern physics. This will be seen more evidently in a forthcoming paper on the splitting of the Einstein equations in a $5D$ universe.

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