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Decay of massive scalar field in a black hole background immersed in magnetic field

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Abstract We evaluate quasinormal modes of a massive scalar field of the Ernst spacetime, an exact solution of the Einstein–Maxwell equations, describing a black hole immersed in a uniform magnetic field B. It is well known that the quasinormal spectrum for a massive scalar field in the vicinity of the magnetized black holes acquires an effective mass $\mu_{\text{eff}} = \sqrt{4B^2m^2 + \mu^2}$, where m is the azimuthal number and μ is the mass of scalar field. The numerical result shows that increasing of the field effective mass and the magnetic field B gives rise to decreasing of the imaginary part of the quasinormal modes until reaching a vanishing damping rate.

1 Introduction

It has been well understood that when a classical black hole is perturbed by an exterior field, the dynamical evolution of the field will undergo three stages [1]. The first one is an initial wave burst coming directly from the source, and it is dependent on the initial form of the original wave field. The second one involves the damped oscillations called the quasinormal modes (QNMs), which do not depend on the initial values of the wave but are characteristic of the background black hole spacetimes. The QNMs are defined as the complex solutions to the perturbation wave equations under certain boundary conditions. The QNM frequencies have complex values because of radiation damping. The last stage is the power-law tail behavior. Here we would like to concentrate on the intermediate stage of the evolution of the massive scalar field where the QNMs dominate.

Astrophysical interest in QNMs originated from their relevance in gravitational wave analysis. The QNMs of black holes are expected to be detected by gravitational wave detectors such as LISA [2]. Recently the study of the QNMs has gained considerable attention coming from the AdS/CFT correspondence in string theory. The lowest quasinormal frequencies of black holes have a direct interpretation as dispersion relations of hydrodynamic excitations in the ultra-relativistic heavy ion collisions [3]. All of these aspects motivated the extensive numerical and analytical study of QNMs for different spacetimes and different fields (both massive and massless) around black holes. We refer the reader to the reviews [4–7] where a lot of references to the recent research of QNMs can be found.

It is well known that the massive QNMs decay more slowly than the massless ones, as shown both by frequency domain methods [8–10] and by time domain methods [11–13]. In Ref. [8], Simone and Will have investigated massive QNMs on a Schwarzschild black hole spacetime using the WKB method [14–16]. They studied the dependence of QNM frequencies on the mass of the scalar field, which is restricted to a narrow range due to the restriction required by the WKB method. A full revelation of the dependence of QNM frequencies on a wide range of field masses is still expected. Ohashi and Sakagami [17] investigated QNMs for the decay of the massive scalar field on the Reissner–Nordström black hole spacetime by using the continued fraction method [18] and found that there are QNMs with arbitrarily long life when the field mass has special values. They named these modes quasi-resonance modes (QRMs).

The magnetic field is one of the most important constituents of the cosmic space and one of the main sources of the dynamics of interacting matter in the universe. It was found that the equation of state of compact stars is strongly affected by a strong magnetic field [19,20]. Moreover, strong magnetic fields of up to 10^4–$10^8 G$ are supposed to exist near supermassive black holes in the active galactic nuclei and even around stellar mass black holes [21]. The interaction between a black hole and a magnetic field can happen in a lot of physical situations, such as a charged accretion disk

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or other charged matter distributions around a black hole. If mini-black holes are created in ultra-relativistic particle collisions in brane-world scenarios, there exists a possibility of interaction between strong magnetic fields and mini-black holes in a great variety of high energy processes when quantum gravity states are excited. So astrophysicists are highly interested in investigating the magnetic fields around black holes. In Ref. [22], Konoplya and Fontana have found the QNMs for the Ernst black hole [23], that is, a black hole immersed in an external magnetic field. They described the influence of the magnetic field on the characteristic quasinormal spectrum of black holes.

In this paper, we consider the massive scalar field perturbations around the Ernst black hole. We shall find the QNMs of the Ernst black holes in the frequency domain. Our numerical investigation shows that the magnetic field B increases with the decrement of the imaginary part of the QNM until reaching a vanishing damping rate. When some threshold values of B are exceeded, the particular QNMs disappear. In Sect. 2, we consider the Klein–Gordon equation in the Ernst spacetime and its reduction into a Schrödinger-like equation with a particular effective potential. Then we evaluate the quasinormal frequencies of the massive scalar field using the continued fraction method. The last section ends this paper with a summary and a conclusion.

2 The basic equations and numerical results

In 1976 Ernst found a class of exact black hole solutions of the Einstein–Maxwell equations [23]. The simplest of these solutions corresponds to a magnetized Schwarzschild black hole, described by the Ernst metric, which has the form

$$\mathrm{d}s^2 = \Lambda^2\left(-\left(1-\frac{2M}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\theta^2\right) +\frac{r^2\sin^2\theta}{\Lambda^2}\mathrm{d}\phi^2, \tag{1}$$

where the external magnetic field B is determined by the relation

$$\Lambda^2 = 1 + B^2 r^2 \sin^2\theta. \tag{2}$$

The vector potential giving rise to the homogeneous magnetic field reads

$$A_\mu \mathrm{d}x^\mu = -\frac{B r^2 \sin^2\theta}{\Lambda}\mathrm{d}\phi. \tag{3}$$

As a magnetic field is assumed to exist everywhere in space, the above metric is not asymptotically flat.

The Klein–Gordon equation describing the evolution of the massive scalar perturbation field outside the Ernst black hole is given by

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi) + \mu^2\Phi = 0. \tag{4}$$

Generally, the Klein–Gordon equation for the Ernst black hole spacetime does not allow for separation of radial and angular variables. Yet because of a small B in our problem, one can safely neglect terms higher than B^2 in Eq. (4). In this way it is known that we have only a dominant correction due to the magnetic field to the effective potential of the Schwarzschild black hole approximately [22]. The Klein–Gordon equation for the angular part has the form

$$\frac{P_{\mathrm{sch}}(\theta,\phi)\Phi}{r^2} + \frac{\Lambda^4 - 1}{r^2\sin^2\theta}\partial_{\phi\phi}\Phi = 0, \tag{5}$$

with $P_{\mathrm{sch}}(\theta,\phi)$ meaning the corresponding pure-Schwarzschild part of the angular equation, $P_{\mathrm{sch}}(\theta,\phi)\Phi = -l(l+1)\Phi$; then one finds

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial^2\phi} = \left(-l(l+1) + 4B^2m^2r^2\right)Y_{lm}. \tag{6}$$

After separation of the angular variables, one can reduce the wave equation (4) in the Ernst background to the Schrödinger wave equation,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} + \omega^2 - V(r^*)\right)\Psi(r^*) = 0, \tag{7}$$

with the effective potential $V(r)$:

$$V(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + 4B^2m^2 + \mu^2\right), \tag{8}$$

where

$$f(r) = 1 - \frac{2M}{r}, \quad \mathrm{d}r^* = \frac{\mathrm{d}r}{f(r)}, \tag{9}$$

and m is the azimuthal quantum number. One can see that the effective potential Eq. (8) coincides with the potential for the massive scalar field with the effective mass $\mu_{\mathrm{eff}} = \sqrt{4B^2m^2 + \mu^2}$ in the Schwarzschild background.

Note that the wave equation with the obtained potential Eq. (8) satisfies the quasinormal mode boundary condition at spatial infinity, which in our particular case takes the form

$$\psi(r^*) \sim C_+ e^{i\chi r^*} r^{iM\mu_{\mathrm{eff}}^2/\chi} (r, r^* \to +\infty), \tag{10}$$

$$\chi = \sqrt{\omega^2 - \mu_{\mathrm{eff}}^2}. \tag{11}$$

Within the continued fraction method we can calculate the singular factor of the solution of Eq. (7) that satisfies an ingoing wave boundary condition at the horizon and Eq. (10) at infinity, and we expand the remaining part into the Frobenius series that are convergent in the R-region ($-\infty < r^* < +\infty$). The solution of Eq. (7) is expanded as follows:

$$\psi(r) = e^{i\chi r} r^{(2iM\chi + iM\mu_{\text{eff}}^2/\chi)} \left(1 - \frac{2M}{r}\right)^{-2iM\omega} \times \sum_n a_n \left(1 - \frac{2M}{r}\right)^n . \quad (12)$$

Substituting Eq. (12) into Eq. (7) we obtain the following recursion relation for the coefficients a_n:

$$\alpha_0 a_1 + \beta_0 a_0 = 0, \quad \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad n > 0, \quad (13)$$

where

$$\alpha_n = (n+1)(n+1-4M\omega i), \quad (14)$$

$$\beta_n = -2n(n+1) - 1 - l(l+1) + \frac{M(\omega+\chi)(4M(\omega+\chi)^2 + i(2n+1)(\omega+3\chi))}{\chi}, \quad (15)$$

$$\gamma_n = (n - Mi(\omega+\chi)^2/\chi)^2. \quad (16)$$

Since the series are convergent at infinity, the ratio of successive a_n will be given by the infinite continued fraction:

$$\frac{a_{n+1}}{a_n} = \frac{\gamma_n}{\alpha_n} \frac{\alpha_{n-1}}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} - \alpha_{n-3}\gamma_{n-2}/\ldots}} - \frac{\beta_n}{\alpha_n} = -\frac{\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \alpha_{n+2}\gamma_{n+3}/\ldots}}. \quad (17)$$

Thus the QNM frequencies are given by the vanishing point of the following continued fraction equation:

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} - \alpha_{n-3}\gamma_{n-2}/\ldots}} = -\frac{\gamma_{n+1}\alpha_n}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \alpha_{n+2}\gamma_{n+3}/\ldots}}. \quad (18)$$

The quasinormal modes for massive scalar fields were studied for the first time by Simone and Will [8] and later in [9,10] with the help of the WKB method [14–16]. The massive QNMs are characterized by the growing of the damping time with the mass until the appearance of the infinitely long lived modes called quasi-resonances [17].

We obtained the QNMs for different values of B, l, and m. The results are summarized in Table 1. From the available data we can see that the $\text{Re}\,\omega$ grows and $\text{Im}\,\omega$ decreases when the magnetic field B increases. Therefore a black hole is a better oscillator in the presence of a magnetic field, i.e., in this case the quality factor $Q \sim \text{Re}\,\omega/\text{Im}\,\omega$ increases, compared to the nonmagnetic circumstance. Figure 1 shows that increasing of the magnetic field B gives rise to decreasing of the imaginary part of the QNM until reaching a vanishing damping rate. When some threshold values of B are exceeded, the particular QNMs disappear.

Then for a given magnetic field, we calculated the QNMs for Ernst black holes for different values of the scalar field mass in the two cases of $l = 1$ and 2. We list our numerical results in Table 2. The numerical investigation shows that increasing of the field mass μ also gives rise to decreasing of the imaginary part of the QNM until reaching a vanishing damping rate. When some threshold values of μ are exceeded, the quasinormal modes also disappear. For a given l, for example $l = 1$ and 2, the larger magnetic field B is, the smaller the threshold value of μ becomes. In order to illustrate our results more transparently, we show our numerical results in Fig. 2.

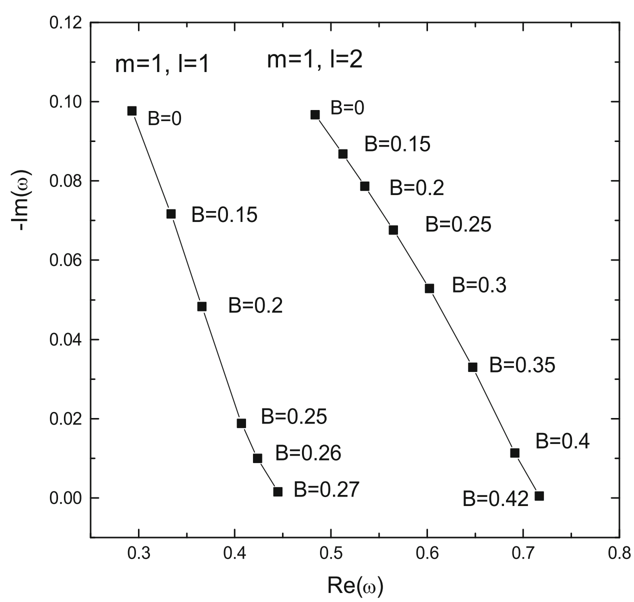

Fig. 1 Fundamental quasinormal modes of the massless scalar field for Ernst black holes under different magnetic field B, for $m = 1$, $l = 1, 2$. Given $l = 1$ for example, the larger the magnetic field B is, the smaller the imaginary part of QNM becomes. When the threshold value of $B = 0.27$ is exceeded, the particular QNMs disappear

Table 1 Fundamental quasinormal modes for Ernst black holes for different values of the magnetic field B. Here, $M = 1$, $\mu = 0$

B	$l = 1, m = 1$	$l = 2, m = 1$	$l = 2, m = 2$
0.005	0.292981–0.097633i	0.483675–0.096748i	0.483770–0.096716i
0.025	0.294054–0.096988i	0.484433–0.096488i	0.486804–0.095675i
0.050	0.297416–0.094957i	0.486804–0.095675i	0.496327–0.092389i
0.075	0.303040–0.091521i	0.490764–0.094312i	0.512346–0.086795i
0.100	0.310957–0.086593i	0.496327–0.092389i	0.535100–0.078676i
0.125	0.321199–0.080040i	0.503512–0.089891i	0.564937–0.067625i

Table 2 Fundamental quasinormal modes for Ernst black holes for different values of the scalar field mass for given B. The parameter m is set to be 1

l	B	μ	ω
1	0.1	0	0.310957–0.086593i
		0.1	0.315507–0.083712i
		0.2	0.329199–0.074756i
		0.3	0.352055–0.058663i
		0.4	0.383561–0.033446i
		0.5	0.415513–0.010131i
1	0.2	0	0.365601–0.048286i
		0.1	0.370064–0.044731i
		0.2	0.383561–0.033446i
		0.3	0.406864–0.018802i
		0.4	0.431849–0.006525i
2	0.1	0	0.496327–0.092389i
		0.1	0.499516–0.091283i
		0.2	0.509127–0.087926i
		0.3	0.525301–0.082200i
		0.4	0.548279–0.073859i
		0.5	0.578409–0.062441i
		0.6	0.616040–0.047068i
		0.7	0.660590–0.026323i
		0.8	0.712166–0.008647i
2	0.2	0	0.535100–0.078676i
		0.1	0.538382–0.077485i
		0.2	0.548279–0.073859i
		0.3	0.564937–0.067625i
		0.4	0.588593–0.058421i
		0.5	0.619495–0.045577i
		0.6	0.657356–0.027823i
		0.7	0.696345–0.010999i
2	0.3	0	0.602270–0.052867i
		0.1	0.605705–0.051442i
		0.2	0.616040–0.047068i
		0.3	0.633343–0.039449i
		0.4	0.657356–0.027823i
		0.5	0.695061–0.010504i

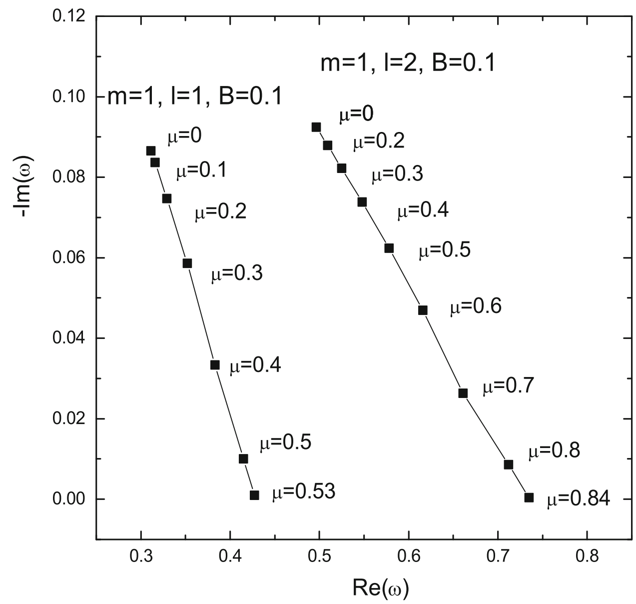

Fig. 2 Fundamental quasinormal modes of the massive scalar field for Ernst black holes, for magnetic field $B = 0.1$, $m = 1$, $l = 1, 2$. Given $l = 1$ for example, the larger the scalar field mass μ is, the smaller the imaginary part of QNM becomes. When the threshold value of μ ($\mu = 0.53$) is exceeded, the particular QNMs disappear

3 Summary

In summary, we have presented in this paper the quasinormal modes of massive scalar field of the Ernst black holes. Calculations show that the real oscillation frequency grows when the magnetic field increases. The damping rate decreases with the increment of the magnetic field, so that the magnetized black hole is characterized by longer lived modes with higher oscillation frequencies, i.e., by a larger quality factor. It is shown that the effective mass $\mu_{\text{eff}} = \sqrt{4B^2m^2 + \mu^2}$ of the scalar field has a crucial influence on the damping rate of the QNMs. In particular, the greater the effective mass of the field is, the smaller the damping rate becomes. As a result, purely real modes which correspond to non-damping oscillations appear, and when the field's effective mass is greater than a certain threshold value, the corresponding QNMs disappear. In Ref. [24], Brito et al. have made a fully consistent study of massless scalar field perturbations of Ernst black holes (including the rotating version of these solutions), without performing the small-B approximation. Next we will research their fully consistent linear analysis and consider how their method influences the behavior of the QRMs.

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