Eur. Phys. J. C (2015) 75:62
DOI 10.1140/epjc/s10052-015-3289-9

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Dynamical instability and expansion-free condition in $f(R,T)$ gravity

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Received: 26 November 2014 / Accepted: 21 January 2015 / Published online: 10 February 2015

Abstract A dynamical analysis of a spherically symmetric collapsing star surrounded by a locally anisotropic environment under an expansion-free condition is presented in $f(R,T)$ gravity, where R corresponds to the Ricci scalar and T stands for the trace of the energy momentum tensor. The modified field equations and evolution equations are reconstructed in the framework of $f(R,T)$ gravity. In order to acquire the collapse equation we implement the perturbation on all matter variables and dark source components comprising the viable $f(R,T)$ model. The instability range is described in the Newtonian and post-Newtonian approximation. It is observed that the unequal stresses and density profile define the instability range rather than the adiabatic index. However, the physical quantities are constrained to maintain positivity of the energy density and a stable stellar configuration.

1 Introduction

The astrophysics and astronomical theories are invigorated largely by the gravitational collapse and instability range explorations of self-gravitating objects. Celestial objects tend to collapse when they exhaust all their nuclear fuel, and gravity takes over as the inward governing force. The gravitating bodies undergoing collapse face contraction to a point, which results in high energy dissipation in the form of heat flux or radiation transport [1]. The end state of stellar collapse has been studied extensively, a continual evolution of a compact object might end up as a naked singularity or as a black hole depending upon the size of a collapsing star and also on the background that plays an important role in pressure-to-gravity imbalances [2–4].

The gravitating objects are interesting only when they are stable against fluctuations; supermassive stars tend to be more unstable in comparison to the less massive stars [5]. The instability problem in a star's evolution is of fundamental importance; Chandrasekhar [6] presented the primary explorations on the dynamical instability of spherical stars. He identified the instability range of a star having mass M and radius r by a factor Γ pertaining to the inequality $\Gamma \geq \frac{4}{3} + n\frac{M}{r}$. The adiabatic index measures the compressibility of the fluid i.e., the variation of the pressure with a given change in the density. The analysis of expanding and collapsing regions in a gravitational collapse was presented by Sharif and Abbas [7].

Herrera et al. [8–11] presented the dynamical analysis associated with isotropy, local anisotropy, shear, radiation, and dissipation with the help of Γ; it was established that minor alterations from an isotropic profile or a slight change in shearing effects bring about drastic changes in the range of instability. However, the instability range of stars with zero expansion does not depend on the stiffness of the fluid, but rather on other physical parameters [12–14], such as the mass distribution, the energy density profile, and the radial and tangential pressure. The impact of the local anisotropy on the plane expansion-free gravitational collapse is studied in [15].

General Relativity (GR) facilitates in providing the field equations that lead to the dynamics of the universe in accordance with its material ingredients. The predictions of GR are suitable for small distances, however, there are some limitations of GR in the description of the late time universe. Modified gravity theories have been widely used to incorporate dark energy components of the universe by inducing alterations in the Einstein–Hilbert (EH) action. Due to modifications in the laws of gravity at long distances, dark source terms of modified gravity leave phenomenal observational signatures, such as the cosmic microwave background, weak lensing, and galaxy clustering [16–21]. Many people inves-

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tigated the dynamics of collapse and the instability range in modified theories of gravity, Cembranos et al. [22] studied the collapse of self-gravitating dust particles. Sharif and Rani [23] established the instability range of a locally anisotropic non-dissipative evolution in $f(T)$ theory.

Among modified gravity theories, $f(R)$ exhibits the most elementary modifications to the EH action by adopting a general function $f(R)$ of the Ricci scalar. Ghosh and Maharaj [24] indicated that null dust non-static collapse in $f(R)$ for a de Sitter higher dimensional background leads to a naked singularity. The combined effects of electromagnetic field and a viable $f(R)$ model have been investigated in [25], the authors concluding that inclusion of a Maxwell source tends to enhance the stability range. Borisov et al. [26] investigated the spherically symmetric collapse of $f(R)$ models with a non-linear coupling scalar by execution of one-dimensional numerical simulations. The dynamical instability of an extremal Schwarzschild de Sitter background framed in $f(R)$ is investigated in [27].

Another modification of GR, a generalization of $f(R)$, was presented in 2011 by Harko et al. [28] termed $f(R, T)$ gravity theory constituting the matter and geometry coupling. Here the EH action is modified in such a way that the gravitational Lagrangian includes higher order curvature terms along with the trace of the energy momentum tensor T. Shabani and Farhoudi [29] explained the weak field limit by applying a dynamical system approach and they analyzed the cosmological implications of $f(R, T)$ models with a variety of cosmological parameters, such as the Hubble parameter, its inverse, snap parameters, weight function, deceleration, jerk, and equation of state parameter. Ayuso et al. [30] worked on a consistency criterion for a non-minimally coupled class of modified theories of gravity. Sharif and Zubair [31–35] discussed the laws of thermodynamics and energy conditions, and they analyzed the anisotropic universe models in the $f(R, T)$ framework.

Chakraborty [36] explored various aspects of homogeneous and isotropic cosmological models in $f(R, T)$ and formulated the energy conditions for a perfect fluid. The dynamics of scalar perturbations in $f(R, T)$ is explored in [37]. Jamil et al. [38] reconstructed some cosmological models and studied the laws of thermodynamics in $f(R, T)$. In a recent paper [39], the dynamical instability of an isotropic collapsing fluid in the context of $f(R, T)$ is considered. We have also discussed the stability analysis of a spherically symmetric collapsing star surrounded by a locally anisotropic environment in $f(R, T)$ gravity [40]. Furthermore, the conditions on physical quantities are constructed for Newtonian and post-Newtonian eras to address the instability problem.

Herein, we intend to develop the instability range of the $f(R, T)$ model for an anisotropic background constrained by zero expansion. The expansion-free condition necessarily implies the appearance of a cavity within the fluid distribution that might help in modeling of voids at cosmological scales. Also, such distributions must bear energy density inhomogeneities, which are incorporated here by inducing a non-constant energy density and a pressure anisotropy. The dynamical analysis of various fluid distributions with the expansion-free condition has been studied in $f(R)$ [41–43], however, such situations have not been covered yet in $f(R, T)$. Recently, Noureen and Zubair [44] discussed the implications of the extended Starobinsky model on the dynamical instability of an axially symmetric gravitating body.

To develop the collapse equation in $f(R, T)$, we construct the corresponding field equations constituting an expansion-free fluid. The action in $f(R, T)$ is as in [28],

$$\int dx^4\sqrt{-g}\left[\frac{f(R,T)}{16\pi G}+\mathcal{L}_{(m)}\right], \tag{1.1}$$

where $\mathcal{L}_{(m)}$ is the matter Lagrangian and g denotes the metric tensor. For the Lagrangian $\mathcal{L}_{(m)}$ one can make various choices, each choice corresponding to a set of field equations for some special form of fluid. Here, we have chosen $\mathcal{L}_{(m)}=\rho$, $8\pi G=1$, and upon variation of above action with metric g_{uv} the field equations are formed as

$$\begin{aligned}G_{uv}=\frac{1}{f_R}\Bigg[&(f_T+1)T^{(m)}_{uv}-\rho g_{uv}f_T+\frac{f-Rf_R}{2}g_{uv}\\&+(\nabla_u\nabla_v-g_{uv}\Box)f_R\Bigg],\end{aligned} \tag{1.2}$$

where $T^{(m)}_{uv}$ denotes the energy momentum tensor for the usual matter.

The matter Lagrangian is configured in such a way that it depends only on the components of metric tensor [45]. In order to present the dynamical analysis we implement the linear perturbation on the collapse equation, assuming that initially all physical quantities are in static equilibrium. The paper is arranged as follows: Einstein's field equations and the dynamical equations for $f(R, T)$ are constructed in Sect. 2, which leads to the collapse equation. In Sect. 3 a perturbation scheme is implemented for the dynamical equations. Section 4 covers the discussion of the expansion-free condition and the components affecting the stability of gravitating objects, extracted from the perturbed Bianchi identities along with corrections to the Newtonian and post-Newtonian eras and the GR solution. Section 5 comprises a summary and is followed by an appendix.

2 Dynamical equations in $f(R, T)$

We choose a three dimensional external spherical boundary surface Σ that pertains to two regions of spacetime, termed interior and exterior regions. The line element for the region inside the boundary Σ is of the form

$$ds_-^2 = W^2(t,r)dt^2 - X^2(t,r)dr^2 - Y^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

The domain beyond (lying outside) Σ is the exterior region, with the following line element [43]:

$$ds_+^2 = \left(1 - \frac{2M}{r}\right)d\nu^2 + 2drd\nu - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.2}$$

where ν is the corresponding retarded time and M is the total mass. To arrive at the onset of the field equations given in Eq. (1.2), we choose $T_{uv}^{(m)}$, describing the anisotropic fluid distribution of the usual matter, given as

$$T_{uv}^{(m)} = (\rho + p_\perp)V_uV_v - p_\perp g_{uv} + (p_r - p_\perp)\chi_u\chi_v, \tag{2.3}$$

where ρ denotes the energy density, V_u is the four-velocity of the fluid, χ_u corresponds to the radial four-vector, p_r and $p_\perp$ represent the radial and tangential pressures, respectively. The physical quantities appearing in the energy momentum tensor are in accordance with the following identities:

$$V^u = W^{-1}\delta_0^u, \quad V^uV_u = 1, \quad \chi^u = X^{-1}\delta_1^u, \quad \chi^u\chi_u = -1. \tag{2.4}$$

The expansion scalar Θ defines the rate of change of small volumes of the fluid, given by

$$\Theta = V^u_{;u} = \frac{1}{W}\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right), \tag{2.5}$$

where the dot and prime denote the time and radial derivatives, respectively. The components of the field equations for a spherically symmetric interior spacetime are of the form

$$G_{00} = \frac{1}{f_R}\left[\rho + \frac{f - Rf_R}{2} + \frac{f_R''}{X^2} - \frac{\dot{f}_R}{W^2}\left(\frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y}\right) - \frac{f_R'}{X^2}\left(\frac{X'}{X} - \frac{2Y'}{Y}\right)\right], \tag{2.6}$$

$$G_{01} = \frac{1}{f_R}\left[\dot{f_R}' - \frac{W'}{W}\dot{f}_R - \frac{\dot{X}}{X}f_R'\right], \tag{2.7}$$

$$G_{11} = \frac{1}{f_R}\left[p_r + (\rho + p_r)f_T - \frac{f - Rf_R}{2} + \frac{\ddot{f}_R}{W^2} - \frac{\dot{f}_R}{W^2}\left(\frac{\dot{W}}{W} - \frac{2\dot{Y}}{Y}\right) - \frac{f_R'}{X^2}\left(\frac{W'}{W} + \frac{2Y'}{Y}\right)\right], \tag{2.8}$$

$$G_{22} = \frac{1}{f_R}\left[p_\perp + (\rho + p_\perp)f_T - \frac{f - Rf_R}{2} + \frac{\ddot{f}_R}{W^2} - \frac{f_R''}{X^2} - \frac{\dot{f}_R}{W^2}\left(\frac{\dot{W}}{W} - \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right) - \frac{f_R'}{X^2}\left(\frac{W'}{W} - \frac{X'}{X} + \frac{Y'}{Y}\right)\right]. \tag{2.9}$$

The dynamical equations are important in the establishment of the instability range of collapsing stars. The Misner–Sharp mass function furnishes the total amount of energy in a spherical star of radius ”Y” and facilitates the formulation of the dynamical equations, given by [46]

$$m(t,r) = \frac{Y}{2}\left(1 + \frac{\dot{Y}^2}{W^2} - \frac{Y'^2}{B^2}\right). \tag{2.10}$$

The matching conditions of the adiabatic sphere on the boundary surface of an exterior spacetime result from the continuity of differential forms as [47]

$$M \overset{\Sigma}{=} m(t,r). \tag{2.11}$$

The dynamical analysis can be established by using conservation laws; we have taken conservation of the Einstein tensor because the energy momentum tensor bears a non-vanishing divergence in $f(R,T)$ gravity. The contracted Bianchi identities imply dynamical equations, which further leads to the collapse equation, given by

$$G^{uv}_{;v}V_u = 0, \quad G^{uv}_{;v}\chi_u = 0. \tag{2.12}$$

The Bianchi identities, taking into account Eq. (2.5), become

$$\dot{\rho} + \rho\left\{[1 + f_T]\Theta - \frac{\dot{f}_R}{f_R}\right\} + [1 + f_T]\left\{p_r\frac{\dot{X}}{X} + 2p_\perp\frac{\dot{Y}}{Y}\right\} + Z_1(r,t) = 0, \tag{2.13}$$

$$(\rho + p_r)f_T' + (1 + f_T) \times \left\{p_r' + \rho\frac{W'}{W} + p_r\left(\frac{W'}{W} + 2\frac{Y'}{Y} - \frac{f_R'}{f_R}\right) - 2p_\perp\frac{Y'}{Y}\right\} + f_T\left(\rho' - \frac{f_R'}{f_R}\right) + Z_2(r,t) = 0, \tag{2.14}$$

where $Z_1(r,t)$ and $Z_2(r,t)$ are the corresponding terms including dark matter components provided in the appendix as Eqs. (6.1) and (6.2), respectively. These equations are useful in the description of variation from equilibrium leading to the evolution.

3 $f(R,T)$ Model and perturbation scheme

The $f(R,T)$ model we have considered for evolution analysis is

$$f(R,T) = R + \alpha R^2 + \lambda T, \tag{3.1}$$

where α and λ correspond to positive real values. Generally, a viable model represents the choice of parameters whose variation shall be in accordance with the observational situations [47]. Astrophysical models are selected by checking their cosmological viability, which must be fulfilled to extract a consistent matter domination phase, and to assemble solar system tests and a stable high-curvature configuration recovering the standard GR. The model under consideration is consistent with the stable stellar configuration because the second-order derivative with respect to R remains positive for the assumed choice of parameters.

The field equations in $f(R,T)$ are highly complicated, their general solution is a heavy task and has not been accomplished yet. The evolution of linear perturbations can always be used to study the gravitational modifications by avoiding such discrepancies. The concerned collapse equation can be furnished by the application of a linear perturbation on the dynamical equations along with the static configuration of field equations leading to the instability range. The dynamical analysis can be anticipated either by the following fixed or co-moving coordinates i.e., via an Eulerian or a Lagrangian approach, respectively [47,48]. Since the universe is almost homogeneous at large scale structures, we have used co-moving coordinates.

Initially all physical quantities are considered to be in static equilibrium so that with the passage of time these have both a time and a radial dependence. Taking $0<\varepsilon\ll 1$ the perturbed form of the quantities along with their initial form can be written as

$$W(t,r)=W_0(r)+\varepsilon D(t)w(r), \tag{3.2}$$
$$X(t,r)=X_0(r)+\varepsilon D(t)x(r), \tag{3.3}$$
$$Y(t,r)=Y_0(r)+\varepsilon D(t)\bar{y}(r), \tag{3.4}$$
$$\rho(t,r)=\rho_0(r)+\varepsilon\bar{\rho}(t,r), \tag{3.5}$$
$$p_r(t,r)=p_{r0}(r)+\varepsilon\bar{p}_r(t,r), \tag{3.6}$$
$$p_\perp(t,r)=p_{\perp 0}(r)+\varepsilon\bar{p_\perp}(t,r), \tag{3.7}$$
$$m(t,r)=m_0(r)+\varepsilon\bar{m}(t,r), \tag{3.8}$$
$$R(t,r)=R_0(r)+\varepsilon D(t)e_1(r), \tag{3.9}$$
$$T(t,r)=T_0(r)+\varepsilon D(t)e_2(r), \tag{3.10}$$

$$\begin{aligned}f(R,T)&=[R_0(r)+\alpha R_0^2(r)+\lambda T_0]\\&\quad+\varepsilon D(t)e_1(r)[1+2\alpha R_0(r)]+\eta D(t)e_2(r),\end{aligned} \tag{3.11}$$

$$f_R=1+2\alpha R_0(r)+\varepsilon 2\alpha D(t)e_1(r), \tag{3.12}$$
$$f_T=\lambda, \tag{3.13}$$
$$\Theta(t,r)=\varepsilon\bar{\Theta}. \tag{3.14}$$

Without loss of generality, we have taken the Schwarzschild coordinate $Y_0(r)=r$ and apply the perturbation scheme on the dynamical equations i.e., Eqs. (2.13) and (2.14), and the perturbed Bianchi identities turn out to be

$$\begin{aligned}&\dot{\bar{\rho}}+\left[\frac{2e\rho_0}{1+2\alpha R_0}+\lambda_1\left\{\frac{2\bar{y}}{r}(\rho_0+2p_{\perp 0})+\frac{x}{X_0}(\rho_0+p_{r0})\right\}\right.\\&\left.\quad+(1+2\alpha R_0)Z_{1p}\right]\dot{D}=0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\lambda_1\left\{\bar{p_r}'+\bar{\rho}\frac{W_0'}{W_0}+\bar{p_r}\left(\frac{W_0'}{W_0}+\frac{2}{r}-\frac{2\alpha R_0'}{1+2\alpha R_0}\right)-\frac{2\bar{p_\perp}}{r}\right\}\\&\quad+2\alpha\left[\frac{1}{W_0^2}\left(e'+2e\frac{X_0'}{X_0}-\frac{x}{X_0}R_0'\right)+X_0^2(1+2\alpha R_0)\right.\\&\quad\left.\times\left\{\frac{e}{X_0^2(1+2\alpha R_0)}\right\}'\right]\ddot{D}+D\left[\lambda_1\left[(\rho_0+p_{r0})\left(\frac{w}{W_0}\right)'\right.\right.\\&\quad\left.-2(p_{r0}+p_{\perp 0})\left(\frac{\bar{y}}{r}\right)'\right]+\lambda\bar{\rho}'-\frac{2\alpha}{1+2\alpha R_0}\\&\quad\times\left\{\lambda_1\left(p_{r0}'+\rho_0\frac{W_0'}{W_0}+p_{r0}\left(\frac{2}{r}+\frac{W_0'}{W_0}-\frac{2\alpha R_0'}{1+2\alpha R_0}\right)\right)\right\}\\&\quad+\lambda\left(e'+e\left[\rho_0'-\frac{2\alpha R_0'}{1+2\alpha R_0}\right]\right)\\&\quad\left.+(1+2\alpha R_0)Z_{2p}\right]=0.\end{aligned} \tag{3.16}$$

For the sake of simplicity we assume that $e_1=e_2=e$ and set $\lambda_1=\lambda+1$; Z_{1p} and Z_{2p} are provided in the appendix.

The elimination of $\dot{\bar{\rho}}$ from Eq. (3.15) and integration of the resultant with respect to time yields an expression for $\bar{\rho}$ of the following form:

$$\begin{aligned}\bar{\rho}=&-\left[\frac{2e\rho_0}{1+2\alpha R_0}+\lambda_1\left\{\frac{2\bar{y}}{r}(\rho_0+2p_{\perp 0})+\frac{x}{X_0}(\rho_0+p_{r0})\right\}\right.\\&\left.+(1+2\alpha R_0)Z_{1p}\right]D.\end{aligned} \tag{3.17}$$

The perturbed field equation (2.9) leads to the expression for $\bar{p}_\perp$, which turns out to be

$$\begin{aligned}\bar{p}_\perp=&\frac{\ddot{D}}{W_0^2}\left\{\frac{(1+2\alpha R_0)\bar{y}}{r}-2\alpha e\right\}\\&-\frac{\lambda\bar{\rho}}{\lambda_1}+\left\{\left(p_{\perp 0}-\frac{\lambda}{\lambda_1}\rho_0\right)\frac{2\alpha e}{1+2\alpha R_0}+\frac{Z_3}{\lambda_1}\right\}D,\end{aligned} \tag{3.18}$$

the effective part of the field equation is denoted by Z_3, and it is given in appendix Eq. (6.5). The matching conditions at the boundary surface reveal

$$p_r\overset{\Sigma}{=}0,\quad p_\perp\overset{\Sigma}{=}0. \tag{3.19}$$

The above equation together with the perturbed form of Eq. (2.9) can be written in the following form:

$$\ddot{D}(t)-Z_4(r)D(t)=0, \tag{3.20}$$

provided that

$$Z_4 = \frac{rW_0^2}{(1+2\alpha R_0)\bar{y} - 2\alpha er}\left[\frac{2\alpha e}{1+2\alpha R_0}p_{\perp 0} + \lambda\left\{\frac{2\bar{y}}{r}(\rho_0 + 2p_{\perp 0}) + \frac{x}{X_0}(\rho_0 + p_{r0})\right\} + (1+2\alpha R_0)Z_{1p} + \frac{Z_3}{\lambda_1}\right]. \quad (3.21)$$

The solution of Eq. (3.20) takes the form

$$D(t) = -e^{\sqrt{Z_4}t}. \quad (3.22)$$

The terms appearing in Z_4 are presumed in such a way that all terms remain positive to have a valid solution for D. The expansion-free condition and the stability range are discussed in the following section.

4 Expansion-free condition with Newtonian and post-Newtonian limits

The models with an additional zero expansion condition delimit two hypersurfaces; one separates the external Schwarzschild solution from the fluid distribution and the other is the boundary between fluid distribution and internal cavity. Such models have extensive astrophysical applications where a cavity within the fluid distribution exists and these models are significant in the investigation of voids at cosmological scales [49]. The spongelike structures are termed voids; they exist in different sizes i.e., we have mini-voids up to super-voids [50,51] for almost 50 % of the universe, considered as vacuum spherical cavities within fluid distribution.

The implementation of a linear perturbation on Eqs. (2.5) and (2.10), respectively, implies

$$\bar{\Theta} = \frac{\dot{D}}{W_0}\left(\frac{x}{X_0} + 2\frac{\bar{y}}{r}\right), \quad (4.1)$$

$$m_0 = \frac{r}{2}\left(1 - \frac{1}{X_0^2}\right), \quad (4.2)$$

$$\bar{m} = -\frac{D}{X_0^2}\left[r\left(\bar{y}' - \frac{x}{X_0}\right) + (1 - X_0^2)\frac{\bar{y}}{2}\right]. \quad (4.3)$$

The expansion-free condition implies a vanishing expansion scalar i.e., $\Theta = 0$, implying

$$\frac{x}{X_0} = -2\frac{\bar{y}}{r}. \quad (4.4)$$

In order to present the dynamical analysis in the Newtonian (N) and post-Newtonian (pN) limits, we assume

$$\rho_0 \gg p_{r0}, \quad \rho_0 \gg p_{\perp 0}. \quad (4.5)$$

The metric coefficients up to the pN approximation in c.g.s. units are taken as

$$W_0 = 1 - \frac{Gm_0}{c^2 r}, \quad X_0 = 1 + \frac{Gm_0}{c^2 r}, \quad (4.6)$$

where c denotes the speed of light and G stands for the gravitational constant. The expression for $\frac{X_0'}{X_0}$ can be obtained from Eq. (4.2) as

$$\frac{X_0'}{X_0} = \frac{-m_0}{r(r-2m_0)}; \quad (4.7)$$

Eq. (4.2) together with (2.8) implies

$$\frac{W_0'}{W_0} = \frac{1}{2r(r-2m_0)(1+2\alpha R_0 + r\alpha R_0')}\left[r^3(\lambda_1 p_{r0} + \lambda\rho_0 - R_0 - 3\alpha R_0^2) + 2\alpha r(R_0 - 2rR_0' + 4R_0' m_0) + 2m_0\right]. \quad (4.8)$$

The static configuration of the first Bianchi identity is identically satisfied, while the second one provides a fruitful result in terms of the dynamical equation. Substitution of Eqs. (4.7) and (4.8) in the statically configured equation (2.14) after some manipulation of the dynamical equation in relativistic units yields

$$\begin{aligned} p_{r0}' = &-\Bigg[\frac{\lambda}{\lambda_1}\rho_0' + \frac{r(1+2\alpha R_0)}{r-2m_0}\Bigg[\frac{r-2m_0}{r(1+2\alpha R_0)} \\ &\times\Bigg\{\frac{\alpha R_0^2}{2} - \frac{2\alpha R_0'(r-2m_0)}{r}\Bigg(\frac{2}{r} \\ &+ \frac{1}{2r(r-2m_0)(1+2\alpha R_0 + r\alpha R_0')} \\ &\times\Big[r^3(\lambda_1 p_{r0} + \lambda\rho_0 - R_0 - 3\alpha R_0^2) \\ &+ 2\alpha r(R_0 - 2rR_0' + 4R_0' m_0) + 2m_0\Big]\Bigg)\Bigg\}\Bigg]_{,1} \\ &+ \frac{2m_0}{r(r-2m_0)}\left(p_{r0} - p_{\perp 0} + \frac{\alpha R_0^2}{4} - \frac{3}{r}\right) \\ &+ \frac{2}{r^2} - \frac{2\alpha R_0'}{1+2\alpha R_0}\left(p_{r0} + \frac{\lambda}{\lambda_1}\right) + \frac{2\alpha R_0''(r-2m_0)}{r^2} \\ &+ \frac{1}{2r(r-2m_0)(1+2\alpha R_0 + r\alpha R_0')} \\ &\times\Big[r^3(\lambda_1 p_{r0} + \lambda\rho_0 - R_0 - 3\alpha R_0^2) \\ &+ 2\alpha r(R_0 - 2rR_0' + 4R_0' m_0) + 2m_0\Big] \\ &\times\Bigg[\rho_0 + p_{r0} + \frac{2\alpha R_0'(r-2m_0)}{r}\Bigg\{\frac{3m_0}{r(r-2m_0)} \end{aligned}$$

$$-\frac{1}{2r(r-2m_0)(1+2\alpha R_0+r\alpha R_0')}\times\Big[r^3(\lambda_1 p_{r0}+\lambda\rho_0-3\alpha R_0^2)+2\alpha r\times(R_0-2rR_0'+4R_0'm_0)+2m_0-R_0\Big]\Big\}\Big]\Big]. \tag{4.9}$$

In c.g.s. units, we may write the above equation as

$$p_{r0}'=-\Bigg[\frac{c^{-2}r(1+2\alpha R_0)}{r-2Gc^{-2}m_0}\Bigg[\frac{r-2Gc^{-2}m_0}{rc^{-2}(1+2\alpha R_0)}\times\Bigg\{-\frac{2\alpha R_0'(r-2Gc^{-2}m_0)}{rc^{-2}}\Bigg(\frac{2}{r}+\frac{1}{2r(r-2c^{-2}m_0)(1+2\alpha R_0+r\alpha R_0')}\times\big[r^3(\lambda_1c^{-2}p_{r0}+\lambda\rho_0-R_0-3\alpha R_0^2)+2\alpha r(R_0-2rR_0'+4R_0'Gc^{-2}m_0)+2Gc^{-2}m_0\big]+\frac{\alpha R_0^2}{2}\Bigg)\Bigg\}\Bigg]_{,1}$$
$$+\frac{2Gc^{-2}m_0}{rc^{-2}(r-2Gc^{-2}m_0)}\Bigg(p_{r0}-p_{\perp0}+\frac{\alpha R_0^2}{4}-\frac{3}{r}\Bigg)-\frac{2\alpha R_0'}{1+2\alpha R_0}\Bigg(c^{-2}p_{r0}+\frac{\lambda}{\lambda_1}\Bigg)+\frac{2}{c^{-2}r^2}$$
$$+\frac{1}{2r(r-2Gc^{-2}m_0)(1+2\alpha R_0+r\alpha R_0')}\times\big[r^3(\lambda_1c^{-2}p_{r0}+\lambda\rho_0-R_0-3\alpha R_0^2)+2\alpha r(2Gc^{-2}m_0-2rc^{-2}R_0'+4R_0'Gc^{-2}m_0)+R_0\big]$$
$$\times\Bigg[\rho_0+p_{r0}+\Bigg\{\frac{-1}{2r(r-2Gc^{-2}m_0)(1+2\alpha R_0+rc^{-2}\alpha R_0')}\times\big[r^3(\lambda\rho_0+\lambda_1c^{-2}p_{r0}-3\alpha R_0^2)+2\alpha rc^{-2}\times(R_0-2rR_0'+4R_0'Gc^{-2}m_0)-R_0+2Gc^{-2}m_0\big]+\frac{3Gc^{-2}m_0}{r(r-2Gc^{-2}m_0)}\Bigg\}\frac{2\alpha R_0'(r-2Gc^{-2}m_0)}{rc^{-2}}\Bigg]+\frac{\lambda}{\lambda_1}\rho_0'+\frac{2\alpha R_0''(r-2Gc^{-2}m_0)}{r^2}\Bigg]. \tag{4.10}$$

The terms of order c^0 and c^{-2} belong to the N and pN approximations, respectively. One can expand Eq. (4.10) up to c^{-2} and separate the terms of the N and pN limits to distinguish the physical quantities lying in various regimes.

The use of an expansion-free condition in Eq. (3.17) modifies $\bar{\rho}$ to the following form:

$$\bar{\rho}=-\Bigg[\frac{2e\rho_0}{1+2\alpha R_0}+\lambda_1\frac{2\bar{y}}{r}((p_{r0}-p_{\perp0}))+(1+2\alpha R_0)Z_{1p}\Bigg]D. \tag{4.11}$$

The Harrison–Wheeler type equation of state describing the second law of thermodynamics relates $\bar{\rho}$ and $\bar{p}_r$ in terms of adiabatic index Γ as

$$\bar{p}_r=\Gamma\frac{p_{r0}}{\rho_0+p_{r0}}\bar{\rho}. \tag{4.12}$$

Γ measures the fluid's compressibility entailing its stiffness. Inserting $\bar{\rho}$ from Eq. (4.11) in Eq. (4.12), we have

$$\bar{p}_r=-\Gamma\frac{p_{r0}}{\rho_0+p_{r0}}\Bigg[\frac{2e\rho_0}{1+2\alpha R_0}+\lambda_1\frac{2\bar{y}}{r}((p_{r0}-p_{\perp0}))+(1+2\alpha R_0)Z_{1p}\Bigg]D. \tag{4.13}$$

In view of a dimensional analysis, it is found that terms of $\bar{p}_r$ and $\rho_0\frac{W_0'}{W_0}$ lie in the post-post-Newtonian (ppN) era and thus can be ignored in the terms of the N and pN approximations. Since we are going to exclude $\bar{p}_r$, it is intuitively clear that the instability range is independent of Γ, and no compression is introduced. By use of the expansion-free condition together with the expression found for D, it follows that

$$2\alpha Z_4\Bigg[\frac{1}{W_0^2}\Bigg(e'+2e\frac{X_0'}{X_0}+2\frac{\bar{y}}{r}R_0'\Bigg)+X_0^2(1+2\alpha R_0)\Bigg\{\frac{e}{X_0^2(1+2\alpha R_0)}\Bigg\}'\Bigg]+\lambda_1\Bigg[\rho_0\Bigg(\frac{w}{W_0}\Bigg)'-2(p_{r0}+p_{\perp0})\Bigg(\frac{\bar{y}}{r}\Bigg)'\Bigg]+\lambda\bar{\rho}'-\frac{2\alpha}{1+2\alpha R_0}\times\Bigg\{\lambda_1\Bigg(p_{r0}'+p_{r0}\Bigg(\frac{2}{r}-\frac{2\alpha R_0'}{1+2\alpha R_0}\Bigg)\Bigg)\Bigg\}+\lambda\Bigg(e'+e[\rho_0'-\frac{2\alpha R_0'}{1+2\alpha R_0}]\Bigg)+(1+2\alpha R_0)Z_{2p}=0. \tag{4.14}$$

For simplification of above expression, we take relativistic units and assume that $\rho_0\gg p_{r0}$, $\rho_0\gg p_{\perp0}$. Substitution of the expressions for Z_4, W_0, X_0, $\frac{X_0'}{X_0}$ and Z_2p, respectively, from Eqs. (3.21), (4.6), (4.7), and (6.4) yields a very lengthy expression defining the factors affecting the instability range at the N and pN limits. The expanded version of Eq. (4.14) is large enough; therefore we are quoting only the results obtained from the collapse equation together with the restrictions to be imposed on the physical parameters. It is clear from Eq. (4.9) that $p_{r0}'<0$, provided that all the terms maintain positivity to fulfill the stability criterion. Negative values of p'_{r0} show a decrease in pressure with the passing of time, leading to the collapse of a gravitating star. Furthermore, using the c.g.s. units, it is found that the terms of $\bar{p}_r$ and $\rho_0\frac{W'_0}{W_0}$ do not take part in the evolution for the N and pN approximations since these terms belong to the ppN limit. The analysis of terms lying in the N and pN limits imply few restrictions to be imposed on physical quantities for a discussion of the instability range. These are listed as follows:

- **Newtonian regime**: The constraints on the material parameters are

$$p_{r0} > p_{\perp 0}, \quad \alpha^2 r R'_0 < 1 + 2\alpha R_0,$$
$$\frac{2\alpha R'_0}{1+2\alpha R_0} > \rho'_0 - e'.$$

  The gravitating body remains unstable as long as the inequalities hold in the N approximation.
- **Post-Newtonian regime**: In the pN limits the following restrictions are found to effect the instability range:

$$p_{r0} > p_{\perp 0}, \quad r > 2m_0,$$
$$\frac{r}{r+m_0}\left(xR'_0 + \frac{2em_0}{r}\right) < e' + \lambda,$$
$$2\alpha e - (1+2\alpha R_0)\frac{\bar{y}}{r} > \frac{(r^2-m_0^2)^2}{r^4}\left\{\frac{er^2}{(r+m_0)}\right\}',$$
$$(r-2m_0) > \frac{R'_0}{2R_0}.$$

5 Summary and results

The mysterious content named dark energy (DE), occupying the major part of the universe is significant in the description of cosmic speed-up. The modified gravity theories are assumed to be effective in understanding the cosmic acceleration by induction of the so-called dark matter components in the form of higher order curvature invariants. Among such theories, $f(R,T)$ represents a non-minimal coupling of matter and geometry. It provides an alternative to incorporating the dark energy components and cosmic acceleration [52]. Thus consideration of $f(R,T)$ for the dynamical analysis is worthwhile, covering the impact of the higher order curvature terms and the trace of energy momentum tensor T. This manuscript is based on the role of a viable $f(R,T)$ model in the establishment of the instability range of a spherically symmetric star.

Our exploration regarding the viability of the $f(R,T)$ model reveals that the selection of the $f(R,T)$ model for the dynamical analysis is constrained to the form $f(R,T) = f(R) + \lambda T$, where λ is an arbitrary positive constant. The restriction on the $f(R,T)$ form originates from the complexities of non-linear terms of the trace in the analytical formulation of the field equations. The $f(R,T)$ form we have chosen mainly is $f(R,T) = R + \alpha R^2 + \lambda T$, in agreement with the stable stellar configuration, and it satisfies the cosmic viability. The matter configuration is assumed to have unequal stresses, i.e., being anisotropic with a central vacuum cavity evolving under an expansion-free condition. The zero expansion condition on an anisotropic background reveals the significance of the energy density profile and pressure inhomogeneity in the structure formation and evolution.

The field equations framed in $f(R,T)$ gravity are formulated and their conservation is considered to study the evolution. Conservation laws yield dynamical equations that are significant in the formation of the collapse equation. In order to examine the variation from a static equilibrium, we introduced a linear perturbation for all physical parameters. The expressions of a perturbed configuration of the field equations reveal expressions for the energy density $\bar{\rho}$ and tangential pressure $\bar{p}_\perp$. The second law of thermodynamics relating the radial pressure and the density with the help of the adiabatic index Γ is considered to be useful to extract $\bar{p}_r$.

On account of the zero expansion it is found that the fluid's evolution is independent of Γ; rather the instability range depends on higher order curvature corrections and the static pressure anisotropy. Recently, the dynamical analysis of isotropic and anisotropic spherical stars in $f(R,T)$ has been studied in [39,40]. It is found that the perturbed form of dark source terms of the collapse equation also has the contribution of the trace T, affecting the stability range. Thus a non-minimal coupling of the higher order curvature terms and the trace of the energy momentum tensor imply a wider range of stability; however, the fluid evolving with zero expansion might cause drastic and unexpected variations. As the expansion-free condition produces a shear blow-up in the gravitating system, it is very captivating to extend this work for the shearing expansion-free case. The results are in accordance with [43] for vanishing λ, for vanishing α, and λ corrections to the GR solution can be found.

In addition to the model (3.1), the nature of various $f(R,T)$ models i.e., $f(R,T) = R+\alpha R^n+\lambda T$, $f(R,T) = R+\alpha R^2+\frac{\mu^4}{R}+\lambda T$, and $f(R,T) = R+\frac{\mu^4}{R}+\lambda T$ has been briefly discussed in this section, as follows:

- $f(R,T) = R+\alpha R^n+\lambda T$: The model $f(R,T) = R+\alpha R^n+\lambda T$ is viable for any $n \geq 2$ and positive constants α and λ. The collapse equation for such a model with zero expansion is of the form

$$\lambda\bar{\rho}' + \lambda_1\left\{\bar{p}_r{}' + (\bar{\rho}+\bar{p}_r)\frac{W'_0}{W_0} + \bar{p}_r\right.$$
$$\left.\times\left(\frac{2}{r} - \frac{\alpha n(n-1)R_0^{n-2}R'_0}{1+\alpha nR_0^{n-1}}\right) - \frac{2}{r}\bar{p_\perp}\right\}$$

$$+D\left[\lambda_1(\rho_0+p_{r0})\frac{w'}{W_0}+\lambda_1\frac{2}{r}(p_{r0}-p_{\perp 0})\right.$$
$$\left.-(\lambda+\lambda_1 p_{r0})\frac{\alpha n(n-1)(R_0^{n-2}e)'}{1+\alpha nR_0^{n-1}}\right]$$
$$+Z_{3p}, \tag{5.1}$$

where pressure stresses $\bar{p_r}, \bar{p_\perp}$ can be generated from the perturbed field equations and the perturbed energy density is

$$\bar{\rho}=-\left[\frac{\alpha n(n-1)(R_0^{n-2}e)}{1+\alpha nR_0^{n-1}}\rho_0+\lambda_1(p_{r0}-p_{\perp 0})\frac{x}{X_0}+Z_{4p}\right]D, \tag{5.2}$$

where Z_{3p} and Z_{4p} depict the perturbed dark source terms. The N and pN limits of this model reveal that the terms $\bar{p_r}$ and $\rho_0\frac{W_0'}{W_0}$ belong to the ppN limit and so do not contribute to the evolution. In the Newtonian limit the physical quantities must satisfy the following conditions:

$$p_{r0}>p_{\perp 0},\quad \alpha^2 rn(n-1)R_0^{n-2}R_0'<1+n\alpha R_0^{n-1},$$
$$\frac{n\alpha n(n-1)R_0^{n-2}R_0'}{1+n\alpha R_0^{n-1}}>\rho_0'-e'.$$

The constraints on the physical quantities in pN regime are

$$r>2m_0,\quad \frac{r}{r+m_0}\left(xn(n-1)R_0^{n-1}R_0'+\frac{2em_0}{r}\right)<e'+\lambda,$$
$$2\alpha e-(1+n\alpha R_0^{n-1})\frac{\bar{y}}{r}>\frac{(r^2-m_0^2)^2}{r^4}\left\{\frac{er^2}{(r+m_0)}\right\}',$$
$$(r-2m_0)>\frac{(n-1)R_0^{n-1}R_0'}{R_0^{n-1}},\quad p_{r0}>p_{\perp 0}.$$

The dynamical analysis of various models involving higher order curvature terms, combined with the trace of the energy momentum tensor, can be presented for $n\geq 2$.

- $f(R,T)=R+\alpha R^2+\frac{\mu^4}{R}+\lambda T$ and $f(R,T)=R+\frac{\mu^4}{R}+\lambda T$: The perturbed form of the Bianchi identity for $f(R,T)=R+\alpha R^2+\frac{\mu^4}{R}+\lambda T$, where μ is an arbitrary constant leading to the expression for $\bar{\rho}$ as follows:

$$\bar{\rho}=-\left[\frac{2e\rho_0}{1+2\alpha R_0-\mu^4R_0^{-2}}+\lambda_1\left\{\frac{x}{X_0}(p_{r0}-p_{\perp 0})\right\}\right.$$
$$\left.+(1+2\alpha R_0-\mu^4R_0^{-2})Z_{1p}\right]D. \tag{5.3}$$

The evolution equation becomes

$$\lambda_1\left\{\bar{p_r}'+\bar{\rho}\frac{W_0'}{W_0}+\bar{p_r}\left(\frac{W_0'}{W_0}+\frac{2}{r}\right.\right.$$
$$\left.\left.-2\frac{\alpha R_0'+\mu^4R_0^{-3}R_0'}{1+2\alpha R_0-\mu^4R_0^{-2}}\right)-\frac{2\bar{p_\perp}}{r}\right\}$$
$$+D\left[\lambda_1\left[(\rho_0+p_{r0})\left(\frac{w}{W_0}\right)'-2(p_{r0}+p_{\perp 0})\left(\frac{\bar{y}}{r}\right)'\right]\right.$$
$$+\lambda\bar{\rho}'-\frac{2\alpha+\mu^4R_0^{-3}}{1+2\alpha R_0-\mu^4R_0^{-2}}\left\{\lambda_1\left(p_{r0}'+\rho_0\frac{W_0'}{W_0}\right.\right.$$
$$\left.\left.+p_{r0}\left(\frac{2}{r}+\frac{W_0'}{W_0}-2\frac{\alpha R_0'+\mu^4R_0^{-3}R_0'}{1+2\alpha R_0-\mu^4R_0^{-2}}\right)\right)\right\}$$
$$+\lambda\left(e'+e\left[\rho_0'-\frac{2\alpha R_0'}{1+2\alpha R_0}\right]\right)$$
$$\left.+(1+2\alpha R_0-\mu^4R_0^{-2})Z_{5p}[\right]=0. \tag{5.4}$$

Z_{5p} denotes the perturbed dark source entries. The N and pN limits are obtained by avoiding the terms lying in the ppN region. To maintain the viability of the model in the Newtonian era, the following inequalities must hold:

$$\alpha R_0'+\mu^4R_0^{-3}R_0'<1+2\alpha R_0-\mu^4R_0^{-2},$$
$$\frac{\alpha R_0'+1+2\alpha R_0-\mu^4R_0^{-2}R_0'}{1+2\alpha R_0-\mu^4R_0^{-2}}>\rho_0'-e'.$$

In the pN regime the system remains stable as long as the following ordering relations are satisfied:

$$\frac{r}{r+m_0}(\alpha R_0'+\mu^4R_0^{-3}R_0')<e'+\lambda T_0,$$
$$2\alpha e-\mu^4R_0^{-2}-(1+2\alpha R_0-\mu^4R_0^{-2})\frac{\bar{y}}{r}$$
$$>\frac{(r^2-2m_0^2)^2}{r^4}\left\{\frac{er^2}{(r+m_0)}\right\}'.$$

The collapse equation for $f(R,T)=R+\frac{\mu^4}{R}+\lambda T$ can be obtained by setting $\alpha=0$ in Eq. (5.4), and likewise the restrictions on physical quantities can be found.

Acknowledgments I.N. acknowledges Dr. Hafiza Rizwana Kausar for her support and fruitful discussions.

6 Appendix

We have

$$Z_1(r,t) = \Bigg[\Bigg\{\frac{1}{f_R W^2}\Bigg(\frac{f-Rf_R}{2}-\frac{\dot f_R}{W^2}\Theta - \frac{f'_R}{X^2}\left(\frac{X'}{X}-\frac{2Y'}{Y}\right)+\frac{f''_R}{X^2}\Bigg)\Bigg\}_{,0} + \Bigg\{\frac{1}{f_R W^2 X^2}\left(\dot f'_R - \frac{W'}{W}\dot f_R - \frac{\dot X}{X} f'_R\right)\Bigg\}_{,1}\Bigg] \times f_R W^2 - \left\{\left(\frac{\dot X}{X}\right)^2 + 2\left(\frac{\dot Y}{Y}\right)^2 + \frac{3\dot W}{W}\Theta\right\}\frac{\dot f_R}{W^2} + \frac{\ddot f_R}{W^2}\Theta - \frac{2 f'_R}{X^2}\Bigg\{\frac{\dot W}{W}\left(\frac{X'}{X}-\frac{Y'}{Y}\right) + \frac{\dot X}{X}\left(\frac{2W'}{W}+\frac{X'}{X}+\frac{Y'}{Y}\right) + \frac{\dot Y}{Y}\left(\frac{W'}{W}-\frac{3Y'}{Y}\right)\Bigg\} + \frac{\dot W}{W}(f-Rf_R) + \frac{f''_R}{X^2}\left(\frac{2\dot W}{W}+\frac{\dot X}{X}\right) + \frac{1}{X^2}\left(\dot f'_R - \frac{W'}{W}\dot f_R\right)\times\left(\frac{3W'}{W}+\frac{X'}{X}+\frac{2Y'}{Y}\right), \tag{6.1}$$

$$Z_2(r,t) = \Bigg[\Bigg\{\frac{1}{f_R W^2 X^2}\left(\dot f_R{}' - \frac{W'}{W}\dot f_R - \frac{\dot X}{X}f'_R\right)\Bigg\}_{,0} + \Bigg\{\frac{1}{f_R X^2}\Bigg(\frac{Rf_R-f}{2} - \frac{\dot f_R}{W^2}\left(\frac{\dot W}{W}-\frac{2\dot Y}{Y}\right) - \frac{f'_R}{X^2}\left(\frac{W'}{W}+\frac{2Y'}{Y}\right)+\frac{\ddot f_R}{W^2}\Bigg)\Bigg\}_{,1}\Bigg] f_R X^2 - \frac{\dot f_R}{W^2}\Bigg\{\frac{W'}{W}\left(\frac{\dot W}{W}+\frac{\dot X}{X}\right)+\frac{X'}{X}\left(\frac{\dot W}{W}-\frac{2\dot Y}{Y}\right) + \frac{2Y'}{Y}\left(\frac{\dot X}{X}-\frac{\dot Y}{Y}\right)\Bigg\} + (Rf_R - f)\frac{X'}{X} - \frac{1}{W^2}\times\left(\frac{\dot W}{W}+\frac{3\dot X}{X}+\frac{2\dot Y}{Y}\right)\left(\frac{W'}{W}\dot f_R + \frac{\dot X}{X}f'_R - \dot f_R{}'\right) - \frac{f'_R}{X^2}\Bigg\{\frac{W'}{W}\left(\frac{W'}{W}+\frac{3X'}{X}\right) + \frac{2Y'}{Y}\left(\frac{3X'}{X}+\frac{Y'}{Y}\right)\Bigg\} + \frac{\ddot f_R}{W^2}\times\left(\frac{W'}{W}+\frac{2X'}{X}\right) + \frac{f''_R}{X^2}\left(\frac{W'}{W}+\frac{2Y'}{Y}\right), \tag{6.2}$$

$$Z_{1p} = 2\alpha W_0^2\left[\frac{1}{W_0^2X_0^2(1+2\alpha R_0)}\left\{e' - e\frac{W'_0}{W_0} - \frac{x}{X_0}R'_0\right\}\right]_{,1} + \frac{1}{1+2\alpha R_0}\Bigg[e - [\lambda T_0 - \alpha R_0^2]\left(\frac{w}{W_0}+\frac{e}{1+2\alpha R_0}\right) - \frac{2\alpha}{X_0^2}\Bigg\{\left(\frac{X'_0}{X_0}-\frac{2}{r}\right)\Bigg(2R'_0\left(\frac{w}{W_0}+\frac{x}{X_0}\right) - e' - \frac{2\alpha e}{1+2\alpha R_0}R'_0\Bigg) + R''_0\left(\frac{2w}{W_0}+\frac{x}{X_0}\right) - 2R'_0\Bigg(\frac{x}{X_0}\left(\frac{2W'_0}{W_0}+\frac{X'_0}{X_0}+\frac{1}{r}\right) + \frac{\bar y}{r}\left(\frac{W'_0}{W_0}-\frac{3}{r}\right)\Bigg) + \left(e' - e\frac{W'_0}{W_0}\right)\left(\frac{3W'_0}{W_0}+\frac{X'_0}{X_0}+\frac{2}{r}\right)\Bigg\}\Bigg], \tag{6.3}$$

$$Z_{2p} = X_0^2(1+2\alpha R_0)\Bigg[\frac{1}{X_0^2(1+2\alpha R_0)}\times\Bigg\{e + \frac{2\alpha}{X_0^2}\Bigg\{\left(\frac{W'_0}{W_0}+\frac{2}{r}\right)\Bigg([\frac{2\alpha e}{1+2\alpha R_0} + \frac{4x}{X_0}]R'_0 - e\Bigg) - R'_0[\left(\frac{w}{W_0}\right)' + \left(\frac{\bar y}{r}\right)']\Bigg\} + [\lambda T_0 - \alpha R_0^2]\left(\frac{e}{1+2\alpha R_0}+\frac{x}{X_0}\right)\Bigg\}\Bigg]_{,1} + xX_0(1+2\alpha R_0)\Bigg[\frac{-1}{X_0^2(1+2\alpha R_0)}\times\left\{\frac{4\alpha R'_0}{X_0^2}\left(\frac{W'_0}{W_0}+\frac{2}{r}\right)+\alpha R_0^2 - \lambda T_0\right\}\Bigg]_{,1} + \frac{2\alpha}{X_0^2}\Bigg[R''_0\Bigg\{\left(\frac{w}{W_0}\right)' - 2\left(\frac{W'_0}{W_0}+\frac{2}{r}\right)\times\left(\frac{e}{1+2\alpha R_0}+\frac{x}{X_0}\right) + \left(\frac{\bar y}{r}\right)'\Bigg\} - R'_0\Bigg\{\frac{W'_0}{W_0}\left[\left(\frac{2w}{W_0}\right)' + \left(\frac{3x}{X_0}\right)'\right] + 3\frac{X'_0}{X_0}\left[\left(\frac{x}{X_0}\right)' + 2\left(\frac{\bar y}{r}\right)'\right] + \frac{2}{r}\left[\left(3\frac{x}{X_0}\right)' + 2\left(\frac{\bar c}{r}\right)'\right]\Bigg\} + \left(\frac{2x}{X_0}R'_0 - e\right)\Bigg\{\left(\frac{W'_0}{W_0}\right)^2 3\frac{X'_0}{X_0}\left(\frac{W'_0}{W_0}+\frac{2}{r}\right) + \frac{2}{r^2}\Bigg\}\Bigg] + e\frac{X'_0}{X_0} - [\lambda T_0 - \alpha R_0^2]\times\left(\frac{2e}{1+2\alpha R_0}\frac{X'_0}{X_0}+\frac{x}{X_0}\right), \tag{6.4}$$

$$Z_3 = \frac{1+2\alpha R_0}{X_0^2}\Bigg[\frac{w''}{W_0} + \frac{\bar y''}{r} - \frac{W''_0}{W_0}\left(\frac{w}{W_0}+\frac{2x}{X_0}\right) + \frac{W'_0}{W_0}\Bigg\{\frac{2x}{X_0}\left(\frac{X'_0}{X_0}-\frac{1}{r}\right) + \left(\frac{\bar y}{r}\right)' - \left(\frac{x}{X_0}\right)'\Bigg\} + \frac{X'_0}{X_0}\Bigg\{\frac{2xX'_0}{rX_0} - \left(\frac{w}{W_0}\right)' - \left(\frac{\bar y}{r}\right)'\Bigg\} + \Bigg\{\left(\frac{w}{W_0}\right)' - \left(\frac{x}{X_0}\right)'\Bigg\}\frac{1}{r}\Bigg] - \frac{2\alpha e}{1+2\alpha R_0}\Bigg\{\frac{\lambda T_0 - \alpha R_0^2}{2}$$

$$-\frac{2\alpha}{X_0^2}\left(R_0'\left(\frac{W_0'}{W_0}-\frac{X_0'}{X_0}+\frac{1}{r}\right)-R_0''\right)\Bigg\}$$
$$-\frac{2\alpha}{X_0^2}\left\{e''+\frac{2x}{X_0}R_0''+\left(\frac{W_0'}{W_0}-\frac{X_0'}{X_0}+\frac{1}{r}\right)\right.$$
$$\left.\times\left(\frac{2x}{X_0}R_0'-e'\right)\right\}. \tag{6.5}$$

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