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Off-diagonal deformations of kerr black holes in Einstein and modified massive gravity and higher dimensions

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Abstract We find general parameterizations for generic off-diagonal spacetime metrics and matter sources in general relativity (GR) and modified gravity theories when the field equations decouple with respect to certain types of nonholonomic frames of reference. This allows us to construct various classes of exact solutions when the coefficients of the fundamental geometric/physical objects depend on all spacetime coordinates via corresponding classes of generating and integration functions and/or constants. Such (modified) spacetimes display Killing and non-Killing symmetries, describe nonlinear vacuum configurations and effective polarizations of cosmological and interaction constants. Our method can be extended to higher dimensions which simplifies some proofs for embedded and nonholonomically constrained four-dimensional configurations. We reproduce the Kerr solution and show how to deform it nonholonomically into new classes of generic off-diagonal solutions depending on 3–8 spacetime coordinates. Certain examples of exact solutions are analyzed and they are determined by contributions of a new type of interactions with sources in massive gravity and/or modified f(R,T) gravity. We conclude that by considering generic off-diagonal nonlinear parametric interactions in GR it is possible to mimic various effects in massive and/or modified gravity, or to distinguish certain classes of "generic" modified gravity solutions which cannot be encoded in GR.

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1 Introduction

The gravitational field equations in general relativity, GR, and modified gravity theories, MGT, are very sophisticated systems of nonlinear partial differential equations (PDEs). Advanced analytic and numerical methods are necessary for constructing exact and approximate solutions of such equations. A number of examples of exact solutions are summarized in the monographs [1,2] where the coefficients of the fundamental geometric/physical objects depend on one and/or two coordinates in four-dimensional (4-d) spacetimes and when the diagonalization of the metrics is possible via coordinate transformations. There are well-known physically important exact solutions for the Schwarzschild, Kerr, Friedman–Lemaître–Robertson–Walker (FLRW), wormhole spacetimes etc. These classes of solutions are generated by certain ansatzes when the Einstein equations are transformed into certain systems of nonlinear second order ordinary equations (ODE), 2-d solitonic equations etc. Such systems of PDEs display Killing vector symmetries which result in additional parametric symmetries [3–5].

The problem of constructing generic off-diagonal exact solutions (which cannot be diagonalized via coordinate transformations) with metric coefficients depending on three and/or four coordinates is much more difficult. There are, in general, six independent components of a metric tensor from the ten components in a 4-d (pseudo-) Riemannian spacetime.[1] Any such ansatz transforms the Einstein equations into systems of nonlinear coupled PDEs which cannot be integrated in a general analytic form if the constructions are performed in local coordinate frames.

In a series of works [5–9], we have shown that it is possible to decouple the gravitational field equations and perform formal analytic integrations in various theories of gravity with metric and nonlinear, N, and linear connection structures. To prove the decoupling property in the simplest way we have to consider spacetime fibrations with splitting of dimensions, $2(\text{or}3) + 2 + 2 + \cdots$, introduce certain adapted frames of reference, consider formal extensions/embeddings of 4-d spacetimes into higher-dimensional ones and work with necessary types of linear connections. Such an (auxiliary, in Einstein gravity) adapted connection is also completely defined in a compatible form by the metric structure and contains a nonholonomically induced torsion field. This allows us to decouple the gravitational field equations and generate various classes of exact solutions in generalized/modified gravity theories. After a class of generalized solutions has been constructed in explicit form, we can constrain to zero the induced torsion fields and "extract" solutions in Einstein gravity. We emphasize that it is important to impose the zero-torsion conditions after we found a class of generalized solutions (to the contrary, we cannot decouple the corresponding systems of PDEs).

It should be noted here that the off-diagonal solutions constructed following the above described anholonomic frame deformation method, AFDM, depend on various classes of generating and integration functions and parameters. The Cauchy problem can be formulated with respect to necessary types of N adapted frames; it is possible to generate various stable or unstable solutions with singularities, nontrivial deformed horizons, stochastic behavior, etc. which depends on the type of nonlinear couplings, prescribed symmetries, asymptotic and boundary conditions; see a number of examples in [10–13] and references therein. In general, it is not clear what physical importance (if any) these classes of such solutions may have. For some well-defined conditions, we can speculate about black hole/ellipsoid/wormhole configurations embedded, for instance, into solitonic gravitational backgrounds or to consider small ellipsoidal deformations of certain "primary" spherical/cylindrical configurations.

Our geometric techniques of constructing exact solutions can be applied to four-dimensional, 4-d, (pseudo-) Riemannian spacetimes with one and two Killing symmetries. For such configurations, the well-known Kerr solution can be generated as a particular case. Then these "primary" metrics can be subjected to nonholonomic deformations to "target" off-diagonal exact solutions depending on three, or four, spacetime coordinates.

The first goal of this paper is to show how certain primary physically important solutions depending on two coordinates can be generalized to new classes of exact solutions in Ein-

[1] Four components of the ten can be fixed to zero using coordinate transformations, and this is related to the Bianchi identities

stein gravity and (higher-dimensional) modifications, with zero or nonzero torsion, depending on all possible spacetime coordinates. We consider diagonal and off-diagonal parametrizations of primary and target solutions which are different from those in [5–8] and other works. In this way we generate new classes of Einstein spacetimes and modifications and show that the AFDM encodes various possibilities for generalization.

The second goal is to construct explicit examples of exact solutions as nonholonomic deformations of the Kerr metric determined by nontrivial sources and interactions in massive gravity and/or modified $f(R,T)$ gravity; see reviews and original results in Refs. [14–24]. For non-Hilbert Lagrangians in gravity theories, the functionals f depend on scalar curvature R (computed, in general, for a linear connection with nontrivial torsion, or for the Levi–Civita one), on various matter and effective matter sources for modified gravity theories etc. We provide a series of exact and/or small parameter-dependent solutions which for small deformations mimic rotoid Kerr–de Sitter-like black holes/ellipsoids self-consistently embedded into generic off-diagonal backgrounds of 4/6/8-dimensional spacetimes. With respect to nonholonomic frames and via the re-definition of generating and integrations functions and coefficients of the sources, modifications of Einstein gravity are modeled by effective polarized cosmological constants and off-diagonal terms in the new classes of solutions. For certain geometrically well-defined conditions, various effects in massive and f-modified gravity can be encoded into vacuum and non-vacuum, configurations (exact solutions) with nontrivial effective cosmological constants in GR. In some sense, we can mimic physically important effects in modified gravity effects (for instance, acceleration of universe, certain dark energy and dark matter locally anisotropic interactions, effective renormalization of quantum gravity models; see Refs. [13,25,26]) via nonlinear generic off-diagonal interactions on effective Einstein spaces. The main question arising from such models and solutions is whether or not we need to modify Einstein's gravitational theory, or to try and solve physically important issues in modern cosmology and quantum gravity by considering only nonlinear and generic off-diagonal interactions based on the general relativity paradigm. Necessarily additional theoretical and experimental/observational research is required in order to analyze and solve these problems. Such directions of research cannot be developed if we consider only diagonalizable metrics (and rotating ones, like the Kerr metric) generated by an ansatz with two Killing symmetries.

The plan of the paper is as follows: In Sect. 2 we provide the necessary geometric preliminaries on nonholonomic $2+2+2+\ldots$ splittings of the spacetime dimensions in GR and MGT. We summarize the key results on the AFDM for constructing generic off-diagonal solutions in gravity theories depending on all spacetime coordinates in dimensions $4, 5, \ldots, 8$.

In Sect. 3 we prove the general decoupling property of the (modified) Einstein equations which allows us to perform formal integrations of corresponding systems of nonlinear PDE. The geometric constructions are performed for the "simplest" case of one Killing symmetry in 4-d and generalized to non-Killing configurations and for higher dimensions.

Section 4 is devoted to the theory of nonholonomic deformations of exact solutions in modified gravity theories containing the Kerr solution as a "primary" configuration but with target metrics being constructed as exact solutions in massive gravity and/or f-modified gravity. We show how using the AFDM we can generate as a particular case the Kerr solution. Then we construct solutions with general off-diagonal deformations of the Kerr metrics in 4-d massive gravity, provide examples of (non-Einstein) metrics with nonholonomically induced torsions, and study small f-modifications of the Kerr metrics deformed by massive gravity. A separate subsection is devoted to ellipsoidal 4-d deformations of the Kerr metric resulting in a target vacuum rotoid or Kerr–de Sitter configuration. Another subsection is devoted to extra-dimensional massive off-diagonal modifications of the Kerr solutions, for the case of 6-d spacetime with nontrivial cosmological constant and for 8-d deformations which may model Finsler-like configurations.

Finally (in Sect. 5), we provide our conclusions and speculate on the physical meaning of the exact solutions constructed using the AFDM for massive modified gravity theories and how such effects can be modeled by nonlinear off-diagonal interactions in Einstein gravity. Some relevant formulas for the coefficients and sketches of the proofs are presented in the Appendix.

2 Nonholonomic frames with $2+2+\cdots$ splitting

In this section, we state the geometric conventions and outline the formalism which are necessary for decoupling and integrating the gravitational field equations in GR and MGTs; see relevant details in [5–8].

2.1 Geometric preliminaries

2.1.1 Conventions

For (higher-dimensional) spacetime geometric models and related exact solutions on a finite-dimensional (pseudo-) Riemannian spacetime ${}^{s}V$, we consider conventional splitting of dimensions, $\dim V = 4 + 2s = 2 + 2 + \cdots + 2 \geq$

$4; s \geq 0$.[2] The anholonomic frame deformation method, AFDM, allows us to construct exact solutions with arbitrary signatures $(\pm 1, \pm 1, \pm 1, \dots, \pm 1)$ of metrics ${}^s\mathbf{g}$. Let us establish conventions on (abstract) indices and coordinates $u^{\alpha_s} = (x^{i_s}, y^{a_s})$, for $s = 0, 1, 2, 3, \dots$ labeling the oriented number of two-dimensional, 2-d, "shells" added to a 4-d spacetime. For $s = 0$, we write $u^\alpha = (x^i, y^a)$ and consider such local systems of coordinates:

$$s = 0 : u^{\alpha_0} = (x^{i_0}, y^{a_0}) = u^\alpha = (x^i, y^a),$$
$$s = 1 : u^{\alpha_1} = (x^\alpha = u^\alpha, y^{a_1}) = (x^i, y^a, y^{a_1}),$$
$$s = 2 : u^{\alpha_2} = (x^{\alpha_1} = u^{\alpha_1}, y^{a_2}) = (x^i, y^a, y^{a_1}, y^{a_2}),$$
$$s = 3 : u^{\alpha_3} = (x^{\alpha_2} = u^{\alpha_2}, y^{a_3}) = (x^i, y^a, y^{a_1}, y^{a_2}, y^{a_3}), \dots$$

when indices run over the corresponding values $i, j, \dots = 1, 2; a, b, \dots = 3, 4; a_1, b_1 \dots = 5, 6; a_2, b_2 \dots = 7, 8; a_3, b_3 \dots = 9, 10, \dots$ and, for instance, $i_1, j_1, \dots = 1, 2, 3, 4; i_2, j_2, \dots = 1, 2, 3, 4, 5, 6; i_3, j_3, \dots = 1, 2, 3, 4, 5, 6, 7, 8; \dots$ In brief, we shall write $u = (x, y)$; ${}^1u = (u, {}^1y) = (x, y, {}^1y)$, ${}^2u = ({}^1u, {}^2y) = (x, y, {}^1y, {}^2y), \dots$.

Local frames (bases, e_{α_s}) on ${}^sV$ are denoted in the form

$$e_{\alpha_s} = e^{\underline{\alpha}_s}_{\alpha_s}({}^su)\partial/\partial u^{\underline{\alpha}_s}, \tag{1}$$

where the partial derivatives are $\partial_{\beta_s} := \partial/\partial u^{\beta_s}$, and indices are underlined if it is necessary to emphasize that such values are defined with respect to a coordinate frame. In general, the frames (1) are nonholonomic (equivalently, anholonomic, or non-integrable), $e_{\alpha_s}e_{\beta_s} - e_{\beta_s}e_{\alpha_s} = W^{\gamma_s}_{\alpha_s\beta_s}e_{\gamma_s}$, where the anholonomy coefficients $W^{\gamma_s}_{\alpha_s\beta_s} = W^{\gamma_s}_{\beta_s\alpha_s}(u)$ vanish for holonomic, i.e. integrable, configurations. The dual frames are $e^{\alpha_s} = e^{\alpha_s}_{\underline{\alpha}_s}({}^su)du^{\underline{\alpha}_s}$, which can be defined from the condition $e^{\alpha_s}\rfloor e_{\beta_s} = \delta^{\alpha_s}_{\beta_s}$ (the 'hook' operator $\rfloor$ corresponds to the inner derivative and $\delta^{\alpha_s}_{\beta_s}$ is the Kronecker symbol).

The conventional 2+2+... splitting for a metric is written in the form

$${}^s\mathbf{g} = g_{\alpha_s\beta_s}e^{\alpha_s} \otimes e^{\beta_s} = g_{\underline{\alpha}_s\underline{\beta}_s}du^{\underline{\alpha}_s} \otimes du^{\underline{\beta}_s}, \ s = 0, 1, 2, \dots, \tag{2}$$

where the coefficients of the metric transform following the rule

$$g_{\alpha_s\beta_s} = e^{\underline{\alpha}_s}_{\alpha_s}e^{\underline{\beta}_s}_{\beta_s}g_{\underline{\alpha}_s\underline{\beta}_s}. \tag{3}$$

Similar frame transformations can be considered for all tensor objects. We cannot preserve a splitting of dimensions under general frame/coordinate transformations.

2.1.2 Nonholonomic splitting with associated N connections

To prove the general decoupling property of the Einstein equations and generalizations/ modifications we have to construct a necessary type of nonholonomic $2+2+\dots$ splitting with associated nonlinear connection (N connection) structure. Such a splitting is introduced using nonholonomic distributions:[3]

1. A N connection is defined by a Whitney sum

$${}^s\mathbf{N} : T\,{}^s\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \oplus {}^1v\mathbf{V} \oplus {}^2v\mathbf{V} \oplus \dots \oplus {}^sv\mathbf{V}, \tag{4}$$

for a conventional horizontal (h) and vertical (v) "shell by shell" splitting. We shall write boldface letters for spaces and geometric objects enabled/adapted to the N connection structure. This defines a local fibered structure on ${}^s\mathbf{V}$ when the coefficients of the N connection, $N^{a_s}_{i_s}$, for ${}^s\mathbf{N} = N^{a_s}_{i_s}({}^su)dx^{i_s} \otimes \partial/\partial y^{a_s}$, induce a system of N adapted local bases, with N-elongated partial derivatives, $\mathbf{e}_{\nu_s} = (\mathbf{e}_{i_s}, e_{a_s})$, and cobases with N adapted differentials, $\mathbf{e}^{\mu_s} = (e^{i_s}, \mathbf{e}^{a_s})$. On a 4-d $\mathbf{V}$,

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N^a_i\frac{\partial}{\partial y^a}, \ e_a = \frac{\partial}{\partial y^a}, \tag{5}$$
$$e^i = dx^i, \mathbf{e}^a = dy^a + N^a_i dx^i, \tag{6}$$

and on $s \geq 1$ shells,

$$\mathbf{e}_{i_s} = \frac{\partial}{\partial x^{i_s}} - N^{a_s}_{i_s}\frac{\partial}{\partial y^{a_s}}, \ e_{a_s} = \frac{\partial}{\partial y^{a_s}}, \tag{7}$$
$$e^{i_s} = dx^{i_s}, \mathbf{e}^{a_s} = dy^{a_s} + N^{a_s}_{i_s}dx^{i_s}. \tag{8}$$

The N adapted operators (5) and (7) define a subclass of general frame transformations of type (1). The corresponding anholonomy relations,

$$[\mathbf{e}_{\alpha_s}, \mathbf{e}_{\beta_s}] = \mathbf{e}_{\alpha_s}\mathbf{e}_{\beta_s} - \mathbf{e}_{\beta_s}\mathbf{e}_{\alpha_s} = W^{\gamma_s}_{\alpha_s\beta_s}\mathbf{e}_{\gamma_s}, \tag{9}$$

[2] In a similar form, we can split odd dimensions, for instance, dim $V = 3+2+\dots+2$. Here it should be noted that it is not possible to elaborate any simplified system of notations if we want to integrate in general explicit form certain systems of PDEs related to higher-dimensional gravitational theories. It is important to distinguish indices and coordinates corresponding to higher dimensions and nonholonomically constrained variables.

[3] In modern gravity, the so-called Arnowit–Deser–Misner (ADM) formalism with a 3+1 splitting is largely used; see details in [27]. It is not possible to elaborate a technique for a general decoupling of the gravitational field equations and generating off-diagonal solutions if we use only nonholonomic frame bases determined by the "shift" and "lapse" functions. To construct exact solutions it is more convenient to work with a correspondingly defined non-integrable $2 + 2 + \cdots$ splitting [7,8].

are completely defined by the N connection coefficients and their partial derivatives, $W^{b_s}_{i_s a_s} = \partial_{a_s} N^{b_s}_{i_s}$ and $W^{a_s}_{j_s i_s} = \Omega^{a_s}_{i_s j_s}$, where the curvature of the N connection is $\Omega^{a_s}_{i_s j_s} = \mathbf{e}_{j_s}\left(N^{a_s}_{i_s}\right) - \mathbf{e}_{i_s}\left(N^{a_s}_{j_s}\right)$.

2. Any metric structure ${}^s\mathbf{g} = \{\mathbf{g}_{\alpha_s\beta_s}\}$ on ${}^s\mathbf{V}$ can be written as a distinguished metric (d-metric)[4],

$$\begin{aligned} {}^s\mathbf{g} = {} & g_{i_s j_s}({}^s u)\, e^{i_s} \otimes e^{j_s} + g_{a_s b_s}({}^s u)\mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s} \\ = {} & g_{ij}(x)\, e^i \otimes e^j + g_{ab}(u)\, \mathbf{e}^a \otimes \mathbf{e}^b \\ & + g_{a_1 b_1}({}^1 u)\, \mathbf{e}^{a_1} \otimes \mathbf{e}^{b_1} \\ & + \cdots . + g_{a_s b_s}({}^s u)\mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s}. \end{aligned} \tag{10}$$

In coordinate frames, a metric (2) is parameterized by generic off-diagonal matrices,

$$\begin{aligned} \underline{g}_{\alpha\beta}(u) &= \begin{bmatrix} g_{ij} + h_{ab} N^a_i N^b_j & h_{ae} N^e_j \\ h_{be} N^e_i & h_{ab} \end{bmatrix} \\ \underline{g}_{\alpha_1\beta_1}\left({}^1 u\right) &= \begin{bmatrix} \underline{g}_{\alpha\beta} & h_{a_1 e_1} N^{e_1}_{\beta_1} \\ h_{b_1 e_1} N^{e_1}_{\alpha_1} & h_{a_1 b_1} \end{bmatrix}, \\ \underline{g}_{\alpha_2\beta_2}\left({}^2 u\right) &= \begin{bmatrix} \underline{g}_{\alpha_1\beta_1} & h_{a_2 e_2} N^{e_2}_{\beta_1} \\ h_{b_2 e_2} N^{e_2}_{\alpha_1} & h_{a_2 b_2} \end{bmatrix}, \ldots \\ \underline{g}_{\alpha_s\beta_s}\left({}^s u\right) &= \begin{bmatrix} g_{i_s j_s} + h_{a_s b_s} N^{a_s}_{i_s} N^{b_s}_{j_s} & h_{a_s e_s} N^{e_s}_{j_s} \\ h_{b_s e_s} N^{e_s}_{i_s} & h_{a_s b_s} \end{bmatrix}. \end{aligned}$$

For extra dimensions, such parameterizations are similar to those introduced in the Kaluza–Klein theory when y^{a_s}, $s \geq 1$, are considered as extra-dimension coordinates with cylindrical compactification and $N^{e_s}_{\alpha}({}^s u) \sim A^{e_s}_{a_s\alpha}(u) y^{\alpha}$ are for certain (non-) Abelian gauge fields $A^{e_s}_{a_s\alpha}(u)$. In general, various parameterizations can be used for warped/trapped coordinates in brane gravity and modifications of GR; see examples in [10–13].

2.1.3 The Levi–Civita and auxiliary N adapted connections

There is a subclass of linear connections on ${}^s\mathbf{V}$ which are adapted to the N connection splitting (4). By definition, a distinguished connection, d-connection, $\mathbf{D} = (hD; vD)$, ${}^1\mathbf{D} = ({}^1hD;\ {}^1vD), \ldots,\ {}^s\mathbf{D} = ({}^{s-1}hD;\ {}^s vD)$, preserves under parallelism the N connection structure.[5] The coefficients

$$\begin{aligned} \Gamma^{\alpha}_{\ \beta\gamma} &= (L^i_{\ jk}, L^a_{\ bk}; C^i_{\ jc}, C^a_{\ bc}), \\ \Gamma^{\alpha_1}_{\ \beta_1\gamma_1} &= (L^{\alpha}_{\ \beta\gamma}, L^{a_1}_{\ b_1\gamma}; C^{\alpha}_{\ \beta c_1}, C^{a_1}_{\ b_1 c_1}), \\ \Gamma^{\alpha_2}_{\ \beta_2\gamma_2} &= (L^{\alpha_1}_{\ \beta_1\gamma_1}, L^{a_2}_{\ b_2\gamma_1}; C^{\alpha_1}_{\ \beta_1 c_2}, C^{a_2}_{\ b_2 c_2}), \ldots, \\ \Gamma^{\alpha_s}_{\ \beta_s\gamma_s} &= (L^{\alpha_{s-1}}_{\ \beta_{s-1}\gamma_{s-1}}, L^{a_s}_{\ b_s\gamma_{s-1}}; C^{\alpha_{s-1}}_{\ \beta_{s-1} c_s}, C^{a_s}_{\ b_s c_s}) \end{aligned} \tag{11}$$

of a d-connection ${}^s\mathbf{D} = \{\mathbf{D}_{\alpha_s}\}$ can be computed in N adapted form with respect to frames (5)–(8) following the equations $\mathbf{D}_{\alpha_s}\mathbf{e}_{\beta_s} = \Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\mathbf{e}_{\gamma_s}$ and covariant derivatives parameterized in the form

$$\begin{aligned} &\mathbf{D}_{\alpha} = (D_i; D_a), \mathbf{D}_{\alpha_1} = ({}^1 D_{\alpha}; D_{a_1}), \\ &\mathbf{D}_{\alpha_2} = ({}^2 D_{\alpha_1}; D_{a_2}), \ldots, \mathbf{D}_{\alpha_s} = ({}^s D_{\alpha_{s-1}}; D_{a_s}), \\ &\text{for } hD = (L^i_{\ jk}, L^a_{\ bk}), vD = (C^i_{\ jc}, C^a_{\ bc}), \\ &{}^1hD = (L^{\alpha}_{\ \beta\gamma}, L^{a_1}_{\ b_1\gamma}),\ {}^1vD = (C^{\alpha}_{\ \beta c_1}, C^{a_1}_{\ b_1 c_1}), \\ &{}^2hD = (L^{\alpha_1}_{\ \beta_1\gamma_1}, L^{a_2}_{\ b_2\gamma_1}),\ {}^2vD = (C^{\alpha_1}_{\ \beta_1 c_2}, C^{a_2}_{\ b_2 c_2}), \ldots, \\ &{}^s hD = \left(L^{\alpha_{s-1}}_{\ \beta_{s-1}\gamma_{s-1}}, L^{a_s}_{\ b_s\gamma_{s-1}}\right),\ {}^s vD = \left(C^{\alpha_{s-1}}_{\ \beta_{s-1} c_s}, C^{a_s}_{\ b_s c_s}\right). \end{aligned}$$

Such coefficients can be computed with respect to mixed subsets of coordinates and/or N adapted frames on different shells. It is possible always to consider such frame transformations when all shell frames are N adapted and

$${}^1 D_{\alpha} = \mathbf{D}_{\alpha},\ {}^2 D_{\alpha_1} = \mathbf{D}_{\alpha_1}, \ldots\ {}^s D_{\alpha_{s-1}} = \mathbf{D}_{\alpha_{s-1}}.$$

To perform computations in N adapted-shell form we can consider a differential connection 1-form $\Gamma^{\alpha_s}_{\ \beta_s} = \Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\mathbf{e}^{\gamma_s}$ and elaborate a differential form calculus with respect to skew symmetric tensor products of N adapted frames (5)–(8). For instance, the torsion $\mathcal{T}^{\alpha_s} = \{\mathbf{T}^{\alpha_s}_{\ \beta_s\gamma_s}\}$ and curvature $\mathcal{R}^{\alpha_s}_{\ \beta_s} = \{\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$ d-tensors of ${}^s\mathbf{D}$ can be computed, respectively,

$$\mathcal{T}^{\alpha_s} := {}^s\mathbf{D}\mathbf{e}^{\alpha_s} = d\mathbf{e}^{\alpha_s} + \Gamma^{\alpha_s}_{\ \beta_s} \wedge \mathbf{e}^{\beta_s} \tag{12}$$

$$\begin{aligned} \mathcal{R}^{\alpha_s}_{\ \beta_s} &:= {}^s\mathbf{D}\Gamma^{\alpha_s}_{\ \beta_s} = d\Gamma^{\alpha_s}_{\ \beta_s} - \Gamma^{\gamma_s}_{\ \beta_s} \wedge \Gamma^{\alpha_s}_{\ \gamma_s} \\ &= \mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\mathbf{e}^{\gamma_s} \wedge \mathbf{e}^{\delta_s}; \end{aligned} \tag{13}$$

see Refs. [8] for explicit calculation of the coefficients $\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}$ in higher dimensions.

For any (pseudo-) Riemannian metric ${}^s\mathbf{g}$, we can construct in standard form the Levi–Civita connection (LC-connection), ${}^s\nabla = \{{}_{\shortmid}\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\}$, which is completely defined by the metric coefficients following two conditions: This linear connection is metric compatible, ${}^s\nabla({}^s\mathbf{g}) = 0$, and with zero torsion, ${}_{\shortmid}\mathcal{T}^{\alpha_s} = 0$ (see (12) for ${}^s\mathbf{D} \to {}^s\nabla$). Such a linear connection is not a d-connection because it does not preserve under general coordinate transformations a N connection splitting.

To elaborate a covariant differential calculus adapted to decomposition (4) we have to introduce a different type of linear connection. This is the canonical d-connection ${}^s\widehat{\mathbf{D}}$ which is completely and uniquely defined by a (pseudo-) Riemannian metric ${}^s\mathbf{g}$ (11) for a chosen ${}^s\mathbf{N} = \{N^{a_s}_{i_s}\}$ if and only if ${}^s\widehat{\mathbf{D}}({}^s\mathbf{g}) = 0$ and the horizontal and vertical torsions

[4] Geometric objects with coefficients defined with respect to N adapted frames are called, respectively, distinguished metrics, distinguished tensors etc. (in brief, d-metrics, d-tensors etc.)

[5] In our works, certain left "up" or "low" labels are used in order to emphasize that certain geometric objects are defined on a corresponding shell and in terms of a fundamental geometric object. We shall omit such labels if that does not result in ambiguities.

are zero, i.e. $h\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^i{}_{jk}\} = 0,\ v\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^a{}_{bc}\} = 0,\ {}^1v\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{a_1}{}_{b_1c_1}\} = 0, \ldots,\ {}^sv\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{a_s}{}_{b_sc_s}\} = 0$. We can check by straightforward computations that such conditions are satisfied by ${}^s\widehat{\mathbf{D}} = \{\widehat{\Gamma}^{\gamma_s}_{\ \alpha_s\beta_s}\}$ with coefficients (11) computed recurrently,

$$
\begin{aligned}
\widehat{L}^i_{jk} &= \frac{1}{2}g^{ir}\left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}\right),\\
\widehat{L}^a_{bk} &= e_b(N^a_k) + \frac{1}{2}h^{ac}\left(\mathbf{e}_k h_{bc} - h_{dc}\, e_b N^d_k - h_{db}\, e_c N^d_k\right),\\
\widehat{C}^i_{jc} &= \frac{1}{2}g^{ik} e_c g_{jk},\ \widehat{C}^a_{bc} = \frac{1}{2}h^{ad}\left(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}\right),\\
\widehat{L}^\alpha_{\beta\gamma} &= \frac{1}{2}g^{\alpha\tau}\left(\mathbf{e}_\gamma g_{\beta\tau} + \mathbf{e}_\beta g_{\gamma\tau} - \mathbf{e}_\tau g_{\beta\gamma}\right),\\
\widehat{L}^{a_1}_{b_1\gamma} &= e_{b_1}(N^{a_1}_\gamma) + \frac{1}{2}h^{a_1c_1}\\
&\quad\times\left(\mathbf{e}_\gamma h_{b_1c_1} - h_{d_1c_1}\, e_{b_1} N^{d_1}_\gamma - h_{d_1b_1}\, e_{c_1} N^{d_1}_\gamma\right),\\
\widehat{C}^\alpha_{\beta c_1} &= \frac{1}{2}g^{\alpha\tau} e_{c_1} g_{\beta\tau},\ \widehat{C}^{a_1}_{b_1c_1} = \frac{1}{2}h^{a_1d_1}\\
&\quad\times\left(e_{c_1} h_{b_1d_1} + e_{c_1} h_{c_1d_1} - e_{d_1} h_{b_1c_1}\right),\\
&\ldots\\
\widehat{L}^{\alpha_{s-1}}_{\beta_{s-1}\gamma_{s-1}} &= \frac{1}{2}g^{\alpha_{s-1}\tau_{s-1}}\left(\mathbf{e}_{\gamma_{s-1}} g_{\beta_{s-1}\tau_{s-1}}\right.\\
&\quad\left.+\mathbf{e}_{\beta_{s-1}} g_{\gamma_{s-1}\tau_{s-1}} - \mathbf{e}_{\tau_{s-1}} g_{\beta_{s-1}\gamma_{s-1}}\right),\\
\widehat{L}^{a_s}_{b_s\gamma_{s-1}} &= e_{b_s}(N^{a_s}_{\gamma_{s-1}}) + \frac{1}{2}h^{a_sc_s}\left(\mathbf{e}_{\gamma_{s-1}} h_{b_sc_s} - h_{d_sc_s}\, e_{b_s} N^{d_s}_{\gamma_{s-1}}\right.\\
&\quad\left.-h_{d_sb_s}\, e_{c_s} N^{d_s}_{\gamma_{s-1}}\right),\\
\widehat{C}^{\alpha_{s-1}}_{\beta_{s-1}c_s} &= \frac{1}{2}g^{\alpha_{s-1}\tau_{s-1}} e_{c_s} g_{\beta_{s-1}\tau_{s-1}},\ \widehat{C}^{a_s}_{b_sc_s} = \frac{1}{2}h^{a_sd_s}\\
&\quad\times\left(e_{c_s} h_{b_sd_s} + e_{c_s} h_{c_sd_s} - e_{d_s} h_{b_sc_s}\right).
\end{aligned}\tag{14}
$$

The torsion d-tensor (12) of ${}^s\widehat{\mathbf{D}}$ is completely defined by ${}^s\mathbf{g}$ (11) for any chosen ${}^s\mathbf{N} = \{N^{a_s}_{i_s}\}$ if the above coefficients (14) are introduced "shell by shell" into the formulas

$$
\begin{aligned}
\widehat{T}^i{}_{jk} &= \widehat{L}^i_{jk} - \widehat{L}^i_{kj},\ \widehat{T}^i{}_{ja} = \widehat{C}^i_{jb},\ \widehat{T}^a{}_{ji} = -\Omega^a{}_{ji},\ \widehat{T}^c_{aj}\\
&= \widehat{L}^c_{aj} - e_a(N^c_j),\ \widehat{T}^a{}_{bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb},\\
&\ldots.\\
\widehat{T}^{\alpha_s}{}_{\beta_s\gamma_s} &= \widehat{L}^{\alpha_s}{}_{\beta_s\gamma_s} - \widehat{L}^{\alpha_s}{}_{\gamma_s\beta_s},\ \widehat{T}^{\alpha_s}{}_{\beta_s b_s} = \widehat{C}^{\alpha_s}{}_{\beta_s b_s},\ \widehat{T}^{a_s}{}_{\beta_s\gamma_s} = \Omega^{a_s}{}_{\gamma_s\beta_s}.
\end{aligned}\tag{15}
$$

The N adapted formulas (14) and (15) show that any coefficient for such objects computed in 4-d can be similarly extended shell by shell by any value $s = 1, 2, \ldots.$ redefining correspondingly the h- and v-indices. Hereafter, we shall present coordinate formulas only for $s = 0$, omitting the label s, i.e. with $\alpha = (i, a)$, or for some arbitrary coefficients $\alpha_s = (i_s, a_s)$ if that will not result in ambiguities.

Because both linear connections ${}^s\nabla$ and ${}^s\widehat{\mathbf{D}}$ are defined by the same metric structure, we can compute a canonical distortion relation,

$$
{}^s\nabla = {}^s\widehat{\mathbf{D}} + {}^s\widehat{\mathbf{Z}},\tag{16}
$$

where the distorting tensor ${}^s\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^{\alpha_s}_{\ \beta_s\gamma_s}\}$ is uniquely defined by the same metric ${}^s\mathbf{g}$ (11). The values $\widehat{\mathbf{Z}}^{\alpha_s}_{\ \beta_s\gamma_s}$ are algebraic combinations of $\widehat{\mathcal{T}}^{\alpha_s}_{\ \beta_s\gamma_s}$ and vanish for zero torsion. For instance, the GR theory in 4-d can be formulated equivalently using the connection ∇ and/or $\widehat{\mathbf{D}}$ if the distorting relation (16) is used [5,7]. The nonholonomic variables (${}^s\mathbf{g}$ (10), ${}^s\mathbf{N}$, ${}^s\widehat{\mathbf{D}}$) are equivalent to standard ones (${}^s\mathbf{g}$ (2), ${}^s\nabla$). Here we note that ${}^s\nabla$ and ${}^s\widehat{\mathbf{D}}$ are not tensor objects and such connections are subjected to different rules of coordinate transformations. It is possible to consider frame transformations with certain ${}^s\mathbf{N} = \{N^{a_s}_{i_s}\}$ when the conditions ${}_{\shortmid}\Gamma^{\gamma_s}_{\ \alpha_s\beta_s} = \widehat{\Gamma}^{\gamma_s}_{\ \alpha_s\beta_s}$ are satisfied with respect to some N adapted frames (5)–(8) even, in general, where we have ${}^s\nabla \neq {}^s\widehat{\mathbf{D}}$ and the corresponding curvature tensors, ${}_{\shortmid}R^{\alpha_s}_{\ \beta_s\gamma_s\delta_s} \neq \widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}$.

2.2 The Einstein equations in N adapted variables

An important motivation to use the linear connection ${}^s\widehat{\mathbf{D}}$ is that the Einstein equations written in variables (${}^s\mathbf{g}$, ${}^s\mathbf{N}$, ${}^s\widehat{\mathbf{D}}$) decouple with respect to N adapted frames of reference, which gives us the possibility to construct very general classes of solutions; see proofs and examples in [5–8,10–13]. We cannot "see" a general decoupling property for such nonlinear systems of PDE if we work from the very beginning with ${}^s\nabla$, for instance, in coordinate frames or with respect to arbitrary nonholonomic ones: The condition of zero torsion, ${}_{\shortmid}\mathcal{T}^{\alpha_s} = 0$ states "strong coupling" conditions between various tensor coefficients in the Einstein equations and does not allow one to decouple the equations.[6]

The main idea of the "anholonomic frame deformation method", AFDM, is to use the data (${}^s\mathbf{g}$, ${}^s\mathbf{N}$, ${}^s\widehat{\mathbf{D}}$) in order to decouple certain gravitational and matter field equations, then to solve them in very general off-diagonal form, with a possible dependence on all coordinates, and generate exact solutions with nontrivial nonholonomically induced torsion. Such integral varieties of solutions depend on a number of arbitrary generating and integration functions and possible symmetry parameters. This geometric approach can be applied for constructing exact solutions in various modified gravity theories with nonlinear effective Lagrangians and nontrivial torsion. Nevertheless, we can extract "integral subvarieties" of solutions in GR if at the end (after a class of "generalized" solutions was constructed) we impose, additionally, the condition of zero torsion (15). This constrains the set of admissible generating/integration functions but also results in generic off-diagonal solutions

[6] The condition of decoupling a system of equations to contain, for instance, only partial derivatives on a coordinate is different from that of a separation of variables for a function.

depending on all coordinates. We can impose certain symmetry/asymptotic/boundary/Cauchy conditions in order to determine certain geometrically/physically important off-diagonal configurations. Following additional assumptions, this can be related to small parametric off-diagonal, solitonic or other types of deformations of well-known solutions in GR. The goal of this work is to study possible nonholonomic transformations of the Kerr and several wormhole metrics into off-diagonal (4-d or higher-dimensional) exact solutions.

The Ricci d-tensor $Ric = \{\mathbf{R}_{\alpha_s\beta_s} := \mathbf{R}^{\tau_s}_{\ \alpha_s\beta_s\tau_s}\}$ of a d-connection ${}^s\mathbf{D}$ is introduced via a respective contracting of coefficients of the curvature tensor (13). The explicit formulas for the h-/v-components,

$$\mathbf{R}_{\alpha_s\beta_s} = \big\{R_{i_sj_s} := R^{k_s}_{\ i_sj_sk_s},\quad R_{i_1a_1} := -R^{k_1}_{\ i_1k_1a_1}, \ldots,\ R_{a_si_s} := R^{b_s}_{\ a_si_sb_s}\big\}, \tag{17}$$

are direct recurrent s-modifications of those derived in Refs. [5–8] (we do not repeat such details in this article). Contracting such values with the inverse d-metric, with coefficients computed for the inverse matrix of ${}^s\mathbf{g}$ (10), we define and compute the scalar curvature of ${}^s\mathbf{D}$,

$$\begin{aligned} {}^sR &:= \mathbf{g}^{\alpha_s\beta_s}\mathbf{R}_{\alpha_s\beta_s} = g^{i_sj_s}R_{i_sj_s} + h^{a_sb_s}R_{a_sb_s} \\ &= R + S + {}^1S + \cdots + {}^sS, \end{aligned} \tag{18}$$

with respective h- and v-components of the scalar curvature, $R = g^{ij}R_{ij},\ S = h^{ab}R_{ab},\ \ {}^1S = h^{a_1b_1}R_{a_1b_1}, \ldots, {}^sS = h^{a_sb_s}R_{a_sb_s}$.

The Einstein d-tensor ${}^s\mathcal{E} = \{\mathbf{E}_{\alpha_s\beta_s}\}$ for any data $({}^s\mathbf{g}, {}^s\mathbf{N}, {}^s\mathbf{D})$ can be defined in standard form,

$$\mathbf{E}_{\alpha_s\beta_s} := \mathbf{R}_{\alpha_s\beta_s} - \frac{1}{2}\mathbf{g}_{\alpha_s\beta_s}\ {}^sR. \tag{19}$$

It should be noted that ${}^s\mathbf{D}({}^s\mathcal{E}) \neq 0$ and the d-tensor $\mathbf{R}_{\alpha_s\beta_s}$ is not symmetric for a general ${}^s\mathbf{D}$. Nevertheless, we can always compute, for instance, ${}^s\widehat{\mathbf{D}}({}^s\widehat{\mathcal{E}})$ as a unique distortion relation determined by (16). This is a consequence of the nonholonomic splitting structure (4). It is similar to nonholonomic mechanics when the conservation laws became more sophisticated when we impose certain non-integrable constraints on the dynamical equations.

The Einstein equations for a metric $\mathbf{g}_{\beta_s\gamma_s}$ can be postulated in standard form using the LC-connection ${}^s\nabla$ (with corresponding Ricci tensor, ${}_{\shortmid}R_{\alpha_s\beta_s}$, curvature scalar, ${}^s_{\shortmid}R$, and Einstein tensor, ${}_{\shortmid}E_{\alpha_s\beta_s}$),

$${}_{\shortmid}E_{\alpha_s\beta_s} := {}_{\shortmid}R_{\alpha_s\beta_s} - \frac{1}{2}g_{\alpha_s\beta_s}\ {}^s_{\shortmid}R = \varkappa\ {}_{\shortmid}T_{\alpha_s\beta_s}, \tag{20}$$

where $\varkappa$ is the gravitational constant and ${}_{\shortmid}T_{\alpha_s\beta_s}$ is the stress–energy tensor for matter fields. In 4-d, there are well-defined geometric/variational and physically motivated procedures of constructing ${}_{\shortmid}T_{\alpha_s\beta_s}$. Such values can be similarly (at least geometrically) re-defined with respect to N adapted frames using the distorting relations (16) and introducing extra dimensions.[7]

The gravitational field equations (20) can be rewritten equivalently in N adapted form for the canonical d-connection ${}^s\widehat{\mathbf{D}}$,

$${}^s\widehat{\mathbf{R}}_{\ \beta_s\delta_s} - \frac{1}{2}\mathbf{g}_{\beta_s\delta_s}\ {}^sR = \mathbf{\Upsilon}_{\beta_s\delta_s}, \tag{21}$$

$$\widehat{L}^{c_s}_{a_sj_s} = e_{a_s}(N^{c_s}_{j_s}),\ \ \widehat{C}^{i_s}_{j_sb_s} = 0,\ \ \Omega^{a_s}_{\ j_si_s} = 0, \tag{22}$$

where the sources $\mathbf{\Upsilon}_{\beta_s\delta_s}$ are formally defined in GR but for extra dimensions when $\mathbf{\Upsilon}_{\beta_s\delta_s} \to \varkappa T_{\beta_s\delta_s}$ for ${}^s\widehat{\mathbf{D}} \to {}^s\nabla$. The solutions of (21) are found with nonholonomically induced torsion (12). If the conditions (22) are satisfied, the d-torsion coefficients (15) are zero and we get the LC-connection, i.e. it is possible to "extract" solutions of the standard Einstein equations. The decoupling property can be proved in explicit form working with ${}^s\widehat{\mathbf{D}}$ and nonholonomic torsion configurations. Having constructed certain classes of solutions in explicit form, with nonholonomically induced torsions and depending on various sets of integration and generating functions and parameters, we can "extract" solutions for ${}^s\nabla$ imposing at the end additional constraints resulting in zero torsion.

2.3 Nonholonomic massive $f(R, T)$ gravity and extra dimensions

We shall consider modified gravity theories constructed on dimension shells derived for the action

$$S = \frac{1}{16\pi}\int \delta^{4+2s}u\sqrt{|\mathbf{g}_{\alpha_s\beta_s}|}[f({}^sR,\ {}^sT) - \frac{\mu_g^2}{4}\mathcal{U} \times(\mathbf{g}_{\mu_s\nu_s}, \mathbf{K}_{\alpha_s\beta_s}) + {}^mL]. \tag{23}$$

This generalizes to nonholonomic variables the modified $f(R, T)$ gravity; see reviews in [14–17], and the ghost-free massive gravity (by de Rham, Gabadadze, and Tolley, dRGT) [18–20]. Nontrivial mass terms allow us to solve certain problems of the bimetric theory by Hassan and Rosen, [21,22], with connections to a variety of recent research in black hole physics and modern cosmology [23,24], and this allows us to model solutions of (23) in various theories with generalized Finsler branes, stochastic processes, Clifford and phase variables, fractional derivatives etc.; see details in Refs. [25,28–32,34]. For instance, y^{a_s}-coordinates can be treated as "velocity/momentum" variables, to model stochastic and fractional processes, or to be considered as "standard" extra-dimensional ones. In this paper, we shall use the units $\hbar = c = 1$ and the Planck mass M_{Pl} is defined $M_{Pl}^2 = 1/8\pi G$ via 4-d Newton constant G and similar units will be considered for higher dimensions. We write $\delta^{4+2s}u$

[7] We do not need additional field equations for torsion fields like in Einstein–Cartan, gauge or string gravity theories.

instead of $d^{4+2s}u$ because N-elongated differentials are used (5) and consider the constant μ_g as the mass parameter for gravity (for simplicity, massive gravity theories will be studied for 4-d spacetimes). The geometric and physical meaning of the values contained in this formula will be explained below.

The Lagrangian density ${}^{m}L$ in action (23) is used for computing the stress–energy tensor of matter. On nonholonomic manifolds/bundles such variations can be considered in N adapted form, using the operators (5) and (6), on inverse the metric d-tensor (10). For all shells, we can compute $\mathbf{T}_{\alpha_s\beta_s} = -\frac{2}{\sqrt{|\mathbf{g}_{\mu_s\nu_s}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu_s\nu_s}|}\ {}^{m}L)}{\delta\mathbf{g}^{\alpha_s\beta_s}}$, when the trace is (by definition) ${}^{s}T := \mathbf{g}^{\alpha_s\beta_s}\mathbf{T}_{\alpha_s\beta_s}$. The functional $f({}^{s}R,\ {}^{s}T)$ modifies the standard Einstein–Hilbert Lagrangian (with a scalar curvature R usually taken for the Levi–Civita connection ∇) to that for the modified f-gravity in various dimensions but with dependence on ${}^{s}R$ and T. For various applications in modern cosmology, we can assume that

$$\mathbf{T}_{\alpha_s\beta_s} = (\rho + p)\mathbf{v}_{\alpha_s}\mathbf{v}_{\beta_s} - p\mathbf{g}_{\alpha_s\beta_s}, \tag{24}$$

for the approximation of perfect fluid matter with the energy density ρ and the pressure p. The four-velocity $\mathbf{v}_{\alpha_s}$ is subjected to the conditions $\mathbf{v}_{\alpha_s}\mathbf{v}^{\alpha_s} = 1$ and $\mathbf{v}^{\alpha_s}\widehat{\mathbf{D}}_{\beta_s}\mathbf{v}_{\alpha_s} = 0$, for ${}^{m}L = -p$ in a corresponding local N adapted frame. For simplicity, we can parameterize

$$f({}^{s}R,\ {}^{s}T) = {}^{1}f({}^{s}R) + {}^{2}f({}^{s}T) \tag{25}$$

and denote ${}^{1}F({}^{s}R) := \partial\ {}^{1}f({}^{s}R)/\partial\ {}^{s}R$ and ${}^{2}F({}^{s}T) := \partial\ {}^{2}f({}^{s}T)/\partial\ {}^{s}T$.

A mass term with "gravitational mass" μ_g and potential

$$\begin{aligned}\mathcal{U}/4 = &-12 + 6[\sqrt{\mathcal{S}}] + [\mathcal{S}] - [\sqrt{\mathcal{S}}]^2 + \alpha_3\{18[\sqrt{\mathcal{S}}]\\ &- 6[\sqrt{\mathcal{S}}]^2 + [\sqrt{\mathcal{S}}]^3 + 2[\mathcal{S}^{3/2}] - 3[\mathcal{S}]([\sqrt{\mathcal{S}}] - 2)\\ &- 24\} + \alpha_4\{[\sqrt{\mathcal{S}}](24 - 12[\sqrt{\mathcal{S}}] - [\sqrt{\mathcal{S}}]^3)\\ &- 12[\sqrt{\mathcal{S}}][\mathcal{S}] + 2[\sqrt{\mathcal{S}}]^2(3[\mathcal{S}] + 2[\sqrt{\mathcal{S}}])\\ &+ 3[\mathcal{S}](4 - [\mathcal{S}]) - 8[\mathcal{S}^{3/2}](\sqrt{\mathcal{S}} - 1) + 6[\mathcal{S}^2] - 24\}\end{aligned} \tag{26}$$

is considered in (23) in addition to the usual f-gravity term (in particular, to the Einstein–Hilbert one). The trace of a shell extended matrix $\mathcal{S} = (S_{\mu_s\nu_s})$ is denoted by $[\mathcal{S}] := S^{\nu_s}_{\ \nu_s}$. We understand the square root of such a matrix, $\sqrt{\mathcal{S}} = (\sqrt{\mathcal{S}}^{\nu_s}_{\ \mu_s})$, to be a matrix for which $\sqrt{\mathcal{S}}^{\nu_s}_{\ \alpha_s}\sqrt{\mathcal{S}}^{\alpha_s}_{\ \mu_s} = S^{\nu_s}_{\ \mu_s}$ and α_3 and α_4 are free parameters. We use such constants which transform $\mathcal{U}$ into the standard 4-d one for $s = 0$. In [19,20], see additional arguments in [35], such a nonlinearly extended Fierz–Pauli type potential was shown to result in a theory of massive gravity which is seem to be free from ghost-like degrees of freedom (it takes a special form of total derivative in the absence of dynamics). We emphasize that the potential generating matrix $\mathcal{S}$ is constructed in a special form, which results in a d-tensor with shell decomposition, $\mathbf{K}^{\nu_s}_{\ \mu_s} = \delta^{\nu_s}_{\ \mu_s} - \sqrt{\mathcal{S}}^{\nu_s}_{\ \mu_s}$, characterizing metric fluctuations away from a fiducial (flat) 4-d spacetime and possible extra dimensions, or velocity/momentum type variables.

In 4-d, the coefficients

$$\mathbf{S}^{\nu}_{\ \mu} = \mathbf{g}^{\nu\alpha}\eta_{\overline{\nu}\overline{\mu}}\mathbf{e}_{\alpha}s^{\overline{\nu}}\mathbf{e}_{\mu}s^{\overline{\mu}}, \tag{27}$$

with the Minkowski metric $\eta_{\overline{\nu}\overline{\mu}} = diag(1,1,1,-1)$, are generated by introducing four scalar Stükelberg fields $s^{\overline{\nu}}$, which is necessary for restoring the diffeomorphism invariance. Using N adapted shell extended values $\mathbf{g}^{\nu_s\alpha_s}$ and $\mathbf{e}_{\alpha_s}$ we can always transform a tensor $S_{\mu\nu}$ into a shell distinguished d-tensor $\mathbf{S}_{\mu_s\nu_s}$ characterizing nonholonomically constrained fluctuations. This is possible for the values $\mathbf{K}^{\nu_s}_{\ \mu_s}$, $\mathbf{S}^{\nu_s}_{\ \mu_s}$, $\sqrt{\mathcal{S}}^{\nu_s}_{\ \mu_s}$ etc. even shell extended $s^{\overline{\nu_s}}$ transforms as scalar fields under coordinate and frame transformations.

For simplicity, we can consider 4-d variations of the action (23) in N adapted form for the coefficients of d-metric $\mathbf{g}_{\nu\alpha}$ (10). The corresponding generalized/effective Einstein equations for the f-modified massive gravity are

$$\widehat{\mathbf{E}}_{\alpha\beta} = \mathbf{\Upsilon}_{\beta\delta}, \tag{28}$$

where the source encodes three terms of a different nature,

$$\mathbf{\Upsilon}_{\beta\delta} = {}^{ef}\eta\ G\ \mathbf{T}_{\beta\delta} + {}^{ef}\mathbf{T}_{\beta\delta} + \mu_g^2\ {}^{K}\mathbf{T}_{\beta\delta}. \tag{29}$$

The first component is determined by the usual matter fields with energy momentum $\mathbf{T}_{\beta\delta}$ tensor but with effective polarization of the gravitational constant ${}^{ef}\eta = [1 + {}^{2}F/8\pi]/\ {}^{1}F$. The second term is for the f-modifications of the energy–momentum tensor,

$$\begin{aligned}{}^{ef}\mathbf{T}_{\beta\delta} = \Big[&\frac{1}{2}\left({}^{1}f - {}^{1}F\ \widehat{R} + 2p\ {}^{2}F + {}^{2}f\right)\mathbf{g}_{\beta\delta}\\ &- \left(\mathbf{g}_{\beta\delta}\ \widehat{\mathbf{D}}_{\alpha}\widehat{\mathbf{D}}^{\alpha} - \widehat{\mathbf{D}}_{\beta}\widehat{\mathbf{D}}_{\delta}\right)\ {}^{1}F\Big]/\ {}^{1}F.\end{aligned} \tag{30}$$

The mass gravity contribution, i.e. the third term in the source is computed as a dimensionless effective stress–energy tensor

$$\begin{aligned}{}^{K}\mathbf{T}_{\alpha\beta} :=& \frac{1}{4\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\,\mathcal{U})}{\delta\mathbf{g}^{\alpha\beta}}\\ =& -\frac{1}{12}\{\mathcal{U}\mathbf{g}_{\alpha\beta}/4 - 2\mathbf{S}_{\alpha\beta} + 2([\sqrt{\mathcal{S}}] - 3)\sqrt{\mathcal{S}}_{\alpha\beta}\\ &+ \alpha_3[3(-6 + 4[\sqrt{\mathcal{S}}] + [\sqrt{\mathcal{S}}]^2 - [\mathcal{S}])\sqrt{\mathcal{S}}_{\alpha\beta}\\ &+ 6([\sqrt{\mathcal{S}}] - 2)\mathbf{S}_{\alpha\beta} - \mathcal{S}^{3/2}_{\alpha\beta}]\\ &- \alpha_4[24\left(\mathcal{S}^2_{\alpha\beta} - ([\sqrt{\mathcal{S}}] - 1)\mathcal{S}^{3/2}_{\alpha\beta}\right)]\\ &+12(2 - 2[\sqrt{\mathcal{S}}] - [\mathcal{S}] + [\sqrt{\mathcal{S}}]^2)\mathbf{S}_{\alpha\beta}\end{aligned}$$

$$+(24 - 24[\sqrt{\mathcal{S}}] + 12[\sqrt{\mathcal{S}}]^2 - [\sqrt{\mathcal{S}}]^3 - 12[\mathcal{S}] +12[\mathcal{S}][\sqrt{\mathcal{S}}] - 8[\mathcal{S}^{3/2}])\sqrt{\mathcal{S}}_{\alpha\beta}\}.$$

The value ${}^{K}\mathbf{T}_{\alpha\beta}$ encodes bi-metric configurations when the second (fiducial) d-metric $\mathbf{f}_{\alpha\mu} = \eta_{\overline{\nu}\overline{\mu}}\mathbf{e}_{\alpha}s^{\overline{\nu}}\mathbf{e}_{\mu}s^{\overline{\mu}}$ is determined by the St ükelberg fields $s^{\overline{\nu}}$. The potential $\mathcal{U}$ (26) defines interactions between $\mathbf{g}_{\mu\nu}$ and $\mathbf{f}_{\mu\nu}$ via $\sqrt{\mathcal{S}}^{\nu}_{\ \mu} = \sqrt{\mathbf{g}^{\nu\mu}\mathbf{f}_{\alpha\nu}}$ and $\mathcal{S}^{\nu}_{\ \mu} := \mathbf{g}^{\nu\mu}\mathbf{f}_{\alpha\nu}$. We can construct exact solutions in explicit form and study bi-metric gravity models with ${}^{K}\mathbf{T}_{\alpha\beta} = \lambda(x^k)\,\mathbf{g}_{\alpha\beta}$, which can be generated by such configurations of $s^{\overline{\nu}}$ when $\mathbf{g}_{\mu\nu} = \iota^2(x^k)\mathbf{f}_{\mu\nu}$ with a possible nontrivial conformal factor ι^2. Such nonholonomic configurations allow us to compute, using (27), the diagonal matrices $\mathcal{S}^{\nu}_{\ \mu} := \iota^{-2}\delta^{\nu}_{\ \mu}$. We can express the effective polarized anisotropic constant encoding the contributions of $s^{\overline{\nu}}$ as a functional $\lambda[\iota^2(x^k)]$.

The theories with gravitational field equations (28) are similar to the Einstein one but for a different metric compatible linear connection, $\widehat{\mathbf{D}}$, and with a nonlinear "gravitationally polarized" coupling in the effective source $\mathbf{\Upsilon}_{\beta\delta}$ (29). In the next sections, we shall prove that such nonlinear systems of PDE can be integrated in general forms for any N adapted parameterizations

$$\mathbf{\Upsilon}^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_{\alpha} : \mathbf{\Upsilon}^1_{\ 1} = \mathbf{\Upsilon}^2_{\ 2} = \Upsilon(x^k, y^3); \mathbf{\Upsilon}^3_{\ 3} = \mathbf{\Upsilon}^4_{\ 4} = {}^{v}\Upsilon(x^k)]. \quad (31)$$

In particular, we can consider

$$\Upsilon = {}^{v}\Upsilon = \Lambda = \text{const}, \quad (32)$$

for an effective cosmological constant Λ; see details in [5–9]. It should be noted that $\widehat{\mathbf{D}}_{\delta}\ {}^{1}F_{|\Upsilon=\Lambda} = 0$ in (30) if we prescribe a functional dependence on $\widehat{R} = const$ (we have to choose particular types of N coefficients and respective canonical d-connection structure). For certain general distributions of the matter fields and effective matter, we can prescribe such values for (32) with $\mathbf{T}_{\beta\delta} = \check{T}(x^k)\mathbf{g}_{\beta\delta}$ and ${}^{s}R = \widehat{\Lambda}$ in (31); then we can write

$$\Upsilon = \widetilde{\Lambda} + \widetilde{\lambda},\ \text{for}\ \widetilde{\lambda} = \mu_g^2\,\lambda(x^k),$$
$$\widetilde{\Lambda} = {}^{ef}\eta\, G\, \check{T}(x^k) + \frac{1}{2}({}^{1}f(\widehat{\Lambda}) - \widehat{\Lambda}\ {}^{1}F(\widehat{\Lambda}) + 2p\ {}^{2}F(\check{T}) + {}^{2}f(\check{T})),$$
$${}^{ef}\eta = [1 + {}^{2}F(\check{T})/8\pi]/\ {}^{1}F(\widehat{\Lambda}). \quad (33)$$

In general, any term may depend on coordinates x^i but via a re-definition of the generating functions they can be transformed into certain effective constants. Prescribing the values $\widehat{\Lambda}, \check{T}, \lambda, p$ and the functionals ${}^{1}f$ and ${}^{2}f$, we describe a nonholonomically constrained matter and effective matter fields dynamics with respect to N adapted frames.

All the above constructions can be extended to extra shells $s = 1, 2, \ldots$ via a formal re-definition of indices for higher dimension. Under very general assumptions, the effective source can be parameterized in the form

$$\mathbf{\Upsilon}^{\beta_s}_{\ \delta_s} = ({}^{s}\widetilde{\Lambda} + {}^{s}\widetilde{\lambda})\delta^{\beta_s}_{\ \delta_s}. \quad (34)$$

This formal diagonal form is fixed with respect to N adapted frames and (see next section) for corresponding redefinition of certain generation functions. Such $({}^{s}\widetilde{\Lambda} + {}^{s}\widetilde{\lambda})$-terms encode via nonholonomic constraints and the canonical d-connection ${}^{s}\widehat{\mathbf{D}}$ a variety of physically important information on modifications of the GR theory by modifications in f-functional and/or massive gravity theories of various dimensions. LC-configurations can be extracted in all such types of theories by imposing additional constraints when $\widehat{\mathbf{D}}_{\mathcal{T}=0} \to \nabla$.

3 Decoupling and integration of (modified) Einstein equations

In this section, we show how the gravitational field equations (21) with possible constraints (22), or (20), can be formally integrated in very general forms for generic off-diagonal metrics with coefficients depending on all spacetime coordinates.

3.1 Off-diagonal configurations with Killing symmetries

In the simplest form, the decoupling property can be proven for certain ansatz with at least one Killing symmetry.

3.1.1 Ansatz for metrics, N connections, and gravitational polarizations

Let us consider metrics of type (10) which via frame transformations (3) (for N adapted transformations, $\mathbf{g}_{\alpha_s\beta_s} = e^{\alpha'_s}_{\ \alpha_s} e^{\beta'_s}_{\ \beta_s}\mathbf{g}_{\alpha'_s\beta'_s}$) can be parameterized in the form[8]

$$\begin{aligned} {}^{s}_{K}\mathbf{g} = &\ g_i(x^k)dx^i \otimes dx^i + h_a(x^k, y^4)\mathbf{e}^a \otimes \mathbf{e}^b \\ &+ h_{a_1}(u^{\alpha}, y^6)\,\mathbf{e}^{a_1} \otimes \mathbf{e}^{a_1} + h_{a_2}(u^{\alpha_1}, y^8) \\ &\times \mathbf{e}^{a_2} \otimes \mathbf{e}^{b_2} + \ldots + h_{a_s}(u^{\alpha_{s-1}}, y^{a_s})\mathbf{e}^{a_s} \otimes \mathbf{e}^{a_s}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \mathbf{e}^a &= dy^a + N^a_i dx^i,\ \text{for}\ N^3_i = n_i(x^k, y^4), \\ &\quad N^4_i = w_i(x^k, y^4); \\ \mathbf{e}^{a_1} &= dy^{a_1} + N^{a_1}_{\alpha} du^{\alpha},\ \text{for}\ N^5_{\alpha} = {}^{1}n_{\alpha}(u^{\beta}, y^6), \\ &\quad N^6_{\alpha} = {}^{1}w_{\alpha}(u^{\beta}, y^6); \\ \mathbf{e}^{a_2} &= dy^{a_2} + N^{a_2}_{\alpha_1} du^{\alpha_1},\ \text{for}\ N^7_{\alpha_1} = {}^{2}n_{\alpha_1}(u^{\beta_1}, y^8), \\ &\quad N^8_{\alpha_1} = {}^{2}w_{\alpha}(u^{\beta_1}, y^8); \\ &\ldots. \end{aligned}$$

[8] In our former works, we used a quite different system of notation.

$$g_{\alpha_1\beta_1} =$$

$$\begin{bmatrix}
g_1 + (n_1)^2 h_3 + (w_1)^2 h_4 + ({}^1n_1)^2 h_5 + ({}^1w_1)^2 h_6 & n_1 n_2 h_3 + w_1 w_2 h_4 + {}^1n_1\, {}^1n_2 h_5 + {}^1w_1\, {}^1w_2 h_6 & n_1 h_3 + {}^1n_1\, {}^1n_3 h_5 + {}^1w_1\, {}^1w_3 h_6 & w_1 h_4 + {}^1n_1\, {}^1n_4 h_5 + {}^1w_1\, {}^1w_4 h_6 & {}^1n_1 h_5 & {}^1w_1 h_6 \\
n_1 n_2 h_3 + w_1 w_2 h_4 + {}^1n_1\, {}^1n_2 h_5 + {}^1w_1\, {}^1w_2 h_6 & g_2 + (n_2)^2 h_3 + (w_2)^2 h_4 + ({}^1n_2)^2 h_5 + ({}^1w_2)^2 h_6 & n_2 h_3 + {}^1n_2\, {}^1n_3 h_5 + {}^1w_2\, {}^1w_3 h_6 & w_2 h_4 + {}^1n_2\, {}^1n_4 h_5 + {}^1w_2\, {}^1w_4 h_6 & {}^1n_2 h_5 & {}^1w_2 h_6 \\
n_1 h_3 + {}^1n_1\, {}^1n_3 h_5 + {}^1w_1\, {}^1w_3 h_6 & n_2 h_3 + {}^1n_2\, {}^1n_3 h_5 + {}^1w_2\, {}^1w_3 h_6 & h_3 + ({}^1n_3)^2 h_5 + ({}^1w_3)^2 h_6 & {}^1n_3\, {}^1n_4 h_5 + {}^1w_3\, {}^1w_4 h_6 & {}^1n_3 h_5 & {}^1w_3 h_6 \\
w_1 h_4 + {}^1n_1\, {}^1n_4 h_5 + {}^1w_1\, {}^1w_4 h_6 & w_2 h_4 + {}^1n_2\, {}^1n_4 h_5 + {}^1w_2\, {}^1w_4 h_6 & {}^1n_3\, {}^1n_4 h_5 + {}^1w_3\, {}^1w_4 h_6 & h_4 + ({}^1n_4)^2 h_5 + ({}^1w_4)^2 h_6 & {}^1n_4 h_5 & {}^1w_4 h_6 \\
{}^1n_1 h_5 & {}^1n_2 h_5 & {}^1n_3 h_5 & {}^1n_4 h_5 & h_5 & 0 \\
{}^1w_1 h_6 & {}^1w_2 h_6 & {}^1w_3 h_6 & {}^1w_4 h_6 & 0 & h_6
\end{bmatrix}$$

Fig. 1 Generic off-diagonal metrics with respect to coordinate frames in 6-d spaces

$$\mathbf{e}^{a_s} = dy^{a_s} + N^{a_s}_{\alpha_{s-1}} du^{\alpha_{s-1}}, \quad \text{for } N^{4+2s-1}_{\alpha_{s-1}} = {}^s n_{\alpha_1}(u^{\beta_{s-1}}, y^{4+2s}),$$
$$N^{4+2s}_{\alpha_1} = {}^s w_\alpha(u^{\beta_{s-1}}, y^{4+2s}).$$

Such ansatz contains a Killing vector $\partial/\partial y^{s-1}$ because the coordinate y^{s-1} is not contained in the coefficients of such metrics. With respect to coordinate frames, for instance, in dim ${}^s\mathbf{V} = 6;\ s = 1, u^{\alpha_1} = (x^1, x^2, y^3, y^4, y^5, y^6)$, the metrics (35) are written in a form similar to that in Fig. 1.

We note that nonholonomic $2+2+\cdots$ parameterizations of type (11) prescribe certain algebraic symmetries of metrics both with respect to N adapted and/or coordinate frames. For instance, a splitting $3+3+3+\cdots$ may contain more complex topological configurations but to integrate the Einstein gravitational equations in such cases is not possible for a general "non-Killing" ansatz.

In a more general context, a d-metric (35) can be a result of nonholonomic deformations of some "primary" geometric/physical data into certain "target" data,

$$[\text{ primary }]({}^s_\circ\mathbf{g}, {}^s_\circ\mathbf{N}, {}^s_\circ\widehat{\mathbf{D}}) \to [\text{ target }]({}^s_\eta\mathbf{g} = {}^s\mathbf{g}, {}^s_\eta\mathbf{N} = {}^s\mathbf{N}, {}^s_\eta\widehat{\mathbf{D}} = {}^s\widehat{\mathbf{D}}).$$

In this work we shall consider that the values labeled by "$\circ$" may or may not define exact solutions in a gravity theory. The metrics with "η" will be constrained always to define a solution of gravitational field equations (21), or (20). For simplicity, we shall use prime ansatz of type

$$\begin{aligned}
{}^s_\circ\mathbf{g} = {} & \mathring{g}_i(x^k) dx^i \otimes dx^i + \mathring{h}_a(x^k, y^4)\mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^b \\
& + \epsilon_{a_1} dy^{a_1} \otimes dy^{a_1} + \cdots + \epsilon_{a_s} dy^{a_s} \otimes dy^{a_s}, \\
\mathring{\mathbf{e}}^a = {} & dy^a + \mathring{N}^a_i(x^k, y^4) dx^i, \text{ with } \mathring{N}^3_i = \mathring{n}_i, \mathring{N}^4_i = \mathring{w}_i,
\end{aligned} \tag{36}$$

where the constants ϵ_{a_s} take values $+1$ and/or -1 which depends on the signature of the higher-dimensional spacetime and on $(\mathring{g}_i, \mathring{h}_a; \mathring{N}^a_i)$. Such an ansatz may define, for instance, a Kerr black hole (or a wormhole) solution trivially embedded into a $4+2s$ spacetime if the corresponding values of the coefficients are constructed respectively for different type solutions of the gravitational field equations. We choose the target metric ansatz (35) as

$$\begin{aligned}
& g_{\alpha_s} = \eta_{\alpha_s}(u^{\beta_s})\mathring{g}_{\alpha_s};\ N^{a_s}_{i_s} = {}_\eta N^{a_s}_{i_s}(u^{\beta_{s-1}}, y^{4+2s}) \\
& n_i = \eta^3_i \mathring{n}_i,\ w_i = \eta^4_i \mathring{w}_i, \text{ not summation on i;}
\end{aligned} \tag{37}$$

with so-called gravitational "polarization" functions and extra-dimensional N coefficients, $\eta_{\alpha_s}, \eta^a_i$ and ${}_\eta N^{a_s}_{i_s}$. In order to consider the limits

$$({}^s_\eta\mathbf{g}, {}^s_\eta\mathbf{N}, {}^s_\eta\widehat{\mathbf{D}}) \to ({}^s_\circ\mathbf{g}, {}^s_\circ\mathbf{N}, {}^s_\circ\widehat{\mathbf{D}}), \text{ for } \varepsilon \to 0,$$

depending on a small parameter $\varepsilon, 0 \leq \varepsilon \ll 1$, we shall introduce "small" polarizations of type $\eta = 1 + \varepsilon\chi(u...)$ and ${}_\eta N^{a_s}_{i_s} = \varepsilon n^{a_s}_{i_s}(u...)$.

It should be noted that if a target d-metric (35) is generated by a nonholonomic deformation with nontrivial η-, or χ-functions, it contains both "old" geometric/physical information on a prime metric (36) and additional data for a new class of exact solutions.

3.1.2 Ricci d-tensors and N adapted sources

Let us consider an ansatz (35) with $\partial_4 h_a \neq 0, \partial_6 h_{a_1} \neq 0, \ldots, \partial_{2s} h_{a_s} \neq 0$,[9] when the partial derivatives are denoted, for instance, $\partial_1 h = \partial h/\partial x^1$, $\partial_4 h = \partial h/\partial y^4$, and $\partial_{44} h = \partial^2 h/\partial y^4 \partial y^4$. A tedious computation of the coefficients of the canonical d-connection $\widehat{\Gamma}^{\gamma_s}_{\alpha_s\beta_s}$ (14) and then of corresponding nontrivial coefficients of the Ricci d-tensor $\boldsymbol{R}_{\alpha_s\beta_s}$ (17); see similar details in [5–8], results in such nontrivial values:

$$\widehat{R}^1_1 = \widehat{R}^2_2 = -\frac{1}{2g_1 g_2}\left[\partial_{11} g_2 - \frac{(\partial_1 g_1)(\partial_1 g_2)}{2g_1} - \frac{(\partial_1 g_2)^2}{2g_2} + \partial_{22} g_1 - \frac{(\partial_2 g_1)(\partial_2 g_2)}{2g_2} - \frac{(\partial_2 g_1)^2}{2g_1}\right], \quad (38)$$

$$\widehat{R}^3_3 = \widehat{R}^4_4 = -\frac{1}{2h_3 h_4}\left[\partial_{44} h_3 - \frac{(\partial_4 h_3)^2}{2h_3} - \frac{(\partial_4 h_3)(\partial_4 h_4)}{2h_4}\right], \quad (39)$$

$$\widehat{R}_{3k} = \frac{h_3}{2h_4}\partial_{44} n_k + \left(\frac{h_3}{h_4}\partial_4 h_4 - \frac{3}{2}\partial_4 h_3\right)\frac{\partial_4 n_k}{2h_4}, \quad (40)$$

$$\widehat{R}_{4k} = \frac{w_k}{2h_3}\left[\partial_{44} h_3 - \frac{(\partial_4 h_3)^2}{2h_3} - \frac{(\partial_4 h_3)(\partial_4 h_4)}{2h_4}\right] + \frac{\partial_4 h_3}{4h_3}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k(\partial_4 h_3)}{2h_3}, \quad (41)$$

and, on shells $s = 1, 2, \ldots,$

$$\widehat{R}^5_5 = \widehat{R}^6_6 = -\frac{1}{2h_5 h_6}\left[\partial_{66} h_5 - \frac{(\partial_6 h_5)^2}{2h_5} - \frac{(\partial_6 h_5)(\partial_6 h_6)}{2h_6}\right], \quad (42)$$

$$\widehat{R}_{5\tau} = \frac{h_5}{2h_6}\partial_{66}\,{}^1n_\tau + \left(\frac{h_5}{h_6}\partial_6 h_6 - \frac{3}{2}\partial_6 h_5\right)\frac{\partial_6\,{}^1n_\tau}{2h_6}, \quad (43)$$

$$\widehat{R}_{6\tau} = \frac{{}^1w_\tau}{2h_5}\left[\partial_{66} h_5 - \frac{(\partial_6 h_5)^2}{2h_5} - \frac{(\partial_6 h_5)(\partial_6 h_6)}{2h_6}\right] + \frac{\partial_6 h_5}{4h_5}\left(\frac{\partial_\tau h_5}{h_5} + \frac{\partial_\tau h_6}{h_6}\right) - \frac{\partial_\tau(\partial_6 h_5)}{2h_5}, \quad (44)$$

when $\tau = 1, 2, 3, 4$;

$$\widehat{R}^7_7 = \widehat{R}^8_8 = -\frac{1}{2h_7 h_8}\left[\partial_{88} h_7 - \frac{(\partial_8 h_7)^2}{2h_7} - \frac{(\partial_8 h_7)(\partial_8 h_8)}{2h_6}\right],$$

$$\widehat{R}_{7\tau} = \frac{h_7}{2h_8}\partial_{88}\,{}^2n_{\tau_1} + \left(\frac{h_7}{h_8}\partial_8 h_8 - \frac{3}{2}\partial_8 h_7\right)\frac{\partial_8\,{}^2n_{\tau_1}}{2h_7},$$

$$\widehat{R}_{8\tau_1} = \frac{{}^2w_{\tau_1}}{2h_7}\left[\partial_{88} h_7 - \frac{(\partial_8 h_7)^2}{2h_7} - \frac{(\partial_8 h_7)(\partial_8 h_8)}{2h_8}\right] + \frac{\partial_8 h_7}{4h_7}\left(\frac{\partial_{\tau_1} h_7}{h_7} + \frac{\partial_{\tau_1} h_8}{h_8}\right) - \frac{\partial_{\tau_1}(\partial_8 h_7)}{2h_7}, \quad (45)$$

when $\tau_1 = 1, 2, 3, 4, 5, 6$. Similar formulas can be written recurrently for arbitrary finite extra dimensions.

Using the above formulas, we can compute the Ricci scalar (18) for ${}^s\widehat{\mathbf{D}}$ (for simplicity, we consider $s = 1$), ${}^s\widehat{R} = 2(\widehat{R}^1_1 + \widehat{R}^3_3 + \widehat{R}^5_5)$. There are certain N adapted symmetries of the Einstein d-tensor (19) for the ansatz (35), $\widehat{E}^1_1 = \widehat{E}^2_2 = -(\widehat{R}^3_3 + \widehat{R}^5_5)$, $\widehat{E}^3_3 = \widehat{E}^4_4 = -(\widehat{R}^1_1 + \widehat{R}^5_5)$, $\widehat{E}^5_5 = \widehat{E}^6_6 = -(\widehat{R}^1_1 + \widehat{R}^3_3)$. In a similar form, we find symmetries for $s = 2$:

$$\widehat{E}^1_1 = \widehat{E}^2_2 = -(\widehat{R}^3_3 + \widehat{R}^5_5 + \widehat{R}^7_7),\ \widehat{E}^3_3 = \widehat{E}^4_4 = -(\widehat{R}^1_1 + \widehat{R}^5_5 + \widehat{R}^7_7),$$
$$\widehat{E}^5_5 = \widehat{E}^6_6 = -(\widehat{R}^1_1 + \widehat{R}^3_3 + \widehat{R}^7_7),\ \widehat{E}^7_7 = \widehat{E}^8_8 = -(\widehat{R}^1_1 + \widehat{R}^3_3 + \widehat{R}^5_5).$$

We search for solutions of the nonholonomic Einstein equations (38)–(45) with nontrivial Λ-sources written in the form

$$\widehat{R}^1_1 = \widehat{R}^2_2 = -\Lambda(x^k),\ \widehat{R}^3_3 = \widehat{R}^4_4 = -\,{}^v\Lambda(x^k, y^4),$$
$$\widehat{R}^5_5 = \widehat{R}^6_6 = -\,{}^v_1\Lambda(u^\beta, y^6),\ \widehat{R}^7_7 = \widehat{R}^8_8 = -\,{}^v_2\Lambda(u^{\beta_1}, y^8). \quad (46)$$

Similar equations can be written recurrently for arbitrary finite extra dimensions. This constrains us to define such N adapted frame transformations when the sources $\boldsymbol{\Upsilon}_{\beta_s\delta_s}$ in (21) are parameterized

$$\boldsymbol{\Upsilon}^1_1 = \boldsymbol{\Upsilon}^2_2 = {}^v\Lambda + {}^v_1\Lambda + {}^v_2\Lambda,\ \boldsymbol{\Upsilon}^3_3 = \Upsilon^4_4 = \Lambda + {}^v_1\Lambda + {}^v_2\Lambda,$$
$$\boldsymbol{\Upsilon}^5_5 = \boldsymbol{\Upsilon}^6_6 = \Lambda + {}^v\Lambda + {}^v_2\Lambda,\ \boldsymbol{\Upsilon}^7_7 = \boldsymbol{\Upsilon}^8_8 = \Lambda + {}^v\Lambda + {}^v_1\Lambda.$$

For certain models of extra-dimensional gravity, we can write ${}^v_1\Lambda = {}^v_2\Lambda = {}^\circ\Lambda = const$. Re-defining the generating functions (see below) for non-vacuum configurations, we can always introduce such effective sources.

3.1.3 Decoupling of gravitational field equations

Introducing the ansatz (35) for $g_i(x^k) = \epsilon_i e^{\psi(x^k)}$ with nonzero $\partial_4\phi, \partial_4 h_a, \partial_6\,{}^1\phi, \partial_6 h_{a_1}, \partial_8\,{}^2\phi, \partial_8 h_{a_2}, \ldots$ in (38)–(45) with respective sources, we obtain this system of PDEs:

$$\epsilon_1 \partial_{11}\psi + \epsilon_2 \partial_{22}\psi = 2\Lambda(x^k), \quad (47)$$
$$(\partial_4\phi)(\partial_4 h_3) = 2h_3 h_4\,{}^v\Lambda(x^k, y^4), \quad (48)$$
$$\partial_{44} n_i + \gamma\,\partial_4 n_i = 0, \quad (49)$$
$$\beta w_i - \alpha_i = 0, \quad (50)$$
$$(\partial_6\,{}^1\phi)(\partial_6 h_5) = 2h_5 h_6\,{}^v_1\Lambda(u^\beta, y^6), \quad (51)$$
$$\partial_{66}\,{}^1n_\tau + {}^1\gamma\,\partial_6\,{}^1n_\tau = 0, \quad (52)$$
$${}^1\beta\,{}^1w_\tau - {}^1\alpha_\tau = 0, \quad (53)$$

[9] we can construct more special classes of solutions if such conditions are not satisfied; for simplicity, we suppose that via frame transformations it is always possible to introduce necessary type parameterizations for d-metrics.

$$(\partial_6\, {}^2\phi)(\partial_6 h_7) = 2h_7 h_8\, {}^v_2\Lambda(u^{\beta_1}, y^8),$$
$$\partial_{88}\, {}^2n_{\tau_1} + {}^2\gamma\,\partial_8\, {}^2n_{\tau_1} = 0,$$
$${}^2\beta\, {}^2w_{\tau_1} - {}^2\alpha_{\tau_1} = 0, \qquad (54)$$

(similar equations can be written recurrently for arbitrary finite extra dimensions),

where the coefficients are defined, respectively, by

$$\phi = \ln\left|\frac{\partial_4 h_3}{\sqrt{|h_3 h_4|}}\right|, \qquad (55)$$

$$\gamma := \partial_4\left(\ln\frac{|h_3|^{3/2}}{|h_4|}\right),\ \ \alpha_i = \frac{\partial_4 h_3}{2h_3}\partial_i\phi,\ \ \beta = \frac{\partial_4 h_3}{2h_3}\partial_4\phi, \qquad (56)$$

$${}^1\phi = \ln\left|\frac{\partial_6 h_5}{\sqrt{|h_5 h_6|}}\right|, \qquad (57)$$

$${}^1\gamma := \partial_6\left(\ln\frac{|h_5|^{3/2}}{|h_6|}\right),\ \ {}^1\alpha_\tau = \frac{\partial_6 h_5}{2h_5}\partial_\tau\, {}^1\phi,$$
$${}^1\beta = \frac{\partial_6 h_5}{2h_5}\partial_\tau\, {}^1\phi, \qquad (58)$$

$${}^2\phi = \ln\left|\frac{\partial_8 h_7}{\sqrt{|h_7 h_8|}}\right|,$$
$${}^2\gamma := \partial_8\left(\ln\frac{|h_7|^{3/2}}{|h_8|}\right),\ \ {}^2\alpha_{\tau_1} = \frac{\partial_8 h_7}{2h_7}\partial_{\tau_1}\, {}^2\phi,$$
$${}^2\beta = \frac{\partial_8 h_7}{2h_7}\partial_{\tau_1}\, {}^2\phi,$$

and similarly for extra shells.

Equations (47)–(54) reflect a very important decoupling property of the (generalized) Einstein equations with respect to the corresponding N adapted frames. In explicit form, such formulas can be obtained for metrics with at least one Killing symmetry (the constructions can be generalized for non-Killing configurations). Let us explain in brief the decoupling property for 4-d configurations following such steps:

1. Equation (47) is just a 2-d Laplace, or d'Alembert one (depending on prescribed signature), which can be solved for any value $\Lambda(x^k)$.
2. Equation (48) contains only the partial derivative ∂_4 and is related to the formula for the coefficient (55) for the values $h_3(x^i, y^4)$, $h_4(x^i, y^4)$ and $\phi(x^i, y^4)$ and source ${}^v\Lambda(x^k, y^4)$. Prescribing any two such functions, we can define (by integrating with respect to y^4) the other two such functions.
3. Using h_3 and ϕ in the previous point, we can compute the coefficients α_i and β, see (56), which allows us to define n_i from the algebraic equations (49).
4. Having computed the coefficient γ (56), the N connection coefficients w_i can be defined after two integrations with respect to y^4 in (50).

The procedure 2–4 can be repeated step by step on the other shells for higher dimensions. We have to add the corresponding dependencies on the extra-dimensional coordinates and additional partial derivatives. For instance, (51) and (57) with partial derivative ∂_6 involve the functions $h_5(x^i, y^a, y^6)$, $h_6(x^i, y^a, y^6)$ and ${}^1\phi(x^i, y^a, y^6)$ and the source ${}^v_1\Lambda(u^\beta, y^6)$. We can compute any two such functions integrating with respect to y^6 if the two other ones are prescribed. In a similar form, we follow the steps in points 3 and 4 with ${}^1\alpha_\tau$, ${}^1\beta$, ${}^1\gamma$, see (58), and compute the higher order N connection coefficients ${}^1n_\tau$ and ${}^1w_\tau$.

3.1.4 Integration of (modified) Einstein equations by generating functions and effective sources

The system of nonlinear PDEs (47)–(54) can be integrated in general form for any finite dimension dim ${}^s\mathbf{V} \geq 4$.

4-d non-vacuum configurations:

The coefficients $g_i = \epsilon_i e^{\psi(x^k)}$ are defined by solutions of the corresponding Laplace/d'Alembert equation (47).

We can solve (48) and (55) for any $\partial_4\phi \neq 0$, $h_a \neq 0$ and ${}^v\Lambda \neq 0$ if we re-write the equations as

$$h_3 h_4 = (\partial_4\phi)(\partial_4 h_3)/2\, {}^v\Lambda \text{ and } |h_3 h_4| = (\partial_4 h_3)^2 e^{-2\phi}, \qquad (59)$$

for any nontrivial source ${}^v\Lambda$. Inserting the first equation into the second one, we find

$$|\partial_4 h_3| = \frac{\partial_4(e^{-2\phi})}{4|\, {}^v\Lambda|} = \frac{\partial_4[\Phi^2]}{2|\, {}^v\Lambda|}, \qquad (60)$$

for $\Phi := e^\phi$. This formula can be integrated with respect to y^4, which results in

$$h_3[\Phi,\, {}^v\Lambda] = {}^0h_3(x^k) + \frac{\epsilon_3\epsilon_4}{4}\int dy^4 \frac{\partial_4(\Phi^2)}{{}^v\Lambda},$$

where ${}^0h_3 = {}^0h_3(x^k)$ is an integration function and $\epsilon_3, \epsilon_4 = \pm 1$. To find h_4 we can use the first equation (59) and write

$$h_4[\Phi,\, {}^v\Lambda] = \frac{(\partial_4\phi)}{{}^v\Lambda}\partial_4(\ln\sqrt{|h_3|}) = \frac{1}{2\, {}^v\Lambda}\frac{\partial_4\Phi}{\Phi}\frac{\partial_4 h_3}{h_3}. \qquad (61)$$

These formulas for h_a can be simplified if we introduce an "effective" cosmological constant $\widetilde{\Lambda} = const \neq 0$ and re-define the generating function $\Phi \to \tilde{\Phi}$, for which $\frac{\partial_4[\Phi^2]}{{}^v\Lambda} = \frac{\partial_4[\tilde{\Phi}^2]}{\tilde{\Lambda}}$, i.e.

$$\Phi^2 = \widetilde{\Lambda}^{-1}\int dy^4 ({}^v\Lambda)\partial_4(\tilde{\Phi}^2) \text{ and}$$
$$\tilde{\Phi}^2 = \widetilde{\Lambda}\int dy^4 ({}^v\Lambda)^{-1}\partial_4(\Phi^2). \qquad (62)$$

Introducing the integration function ${}^0h_3(x^k)$ and ϵ_3 and ϵ_4 in Φ and, respectively, in ${}^{\upsilon}\Lambda$, we can express

$$h_3[\tilde{\Phi},\widetilde{\Lambda}] = \frac{\tilde{\Phi}^2}{4\widetilde{\Lambda}} \text{ and } h_4[\tilde{\Phi},\widetilde{\Lambda}] = \frac{(\partial_4\tilde{\Phi})^2}{\Xi}, \qquad (63)$$

where $\Xi = \int dy^4({}^{\upsilon}\Lambda)\partial_4(\tilde{\Phi}^2)$. We can work for convenience with two couples of generating data, $(\Phi,\ {}^{\upsilon}\Lambda)$ and $(\tilde{\Phi},\ \tilde{\Lambda})$, related by (62).

Using the values h_a (63), we compute the coefficients α_i, β and γ from (56). The resulting solutions for N coefficients can be expressed recurrently,

$$n_k = {}_1n_k + {}_2n_k\int dy^4 h_4/(\sqrt{|h_3|})^3 = {}_1n_k + {}_2\widetilde{n}_k\int dy^4(\partial_4\tilde{\Phi})^2/\tilde{\Phi}^3\,\Xi,$$
$$w_i = \partial_i\phi/\partial_4\phi = \partial_i\Phi/\partial_4\Phi, \qquad (64)$$

where ${}_1n_k(x^i)$ and ${}_2n_k(x^i)$, or ${}_2\widetilde{n}_k(x^i) = 8\ {}_2n_k(x^i)|\widetilde{\Lambda}|^{3/2}$, are integration functions. The quadratic line elements determined by the coefficients (63)–(64) are parameterized in the form

$$ds^2_{4dK} = g_{\alpha\beta}(x^k,y^4)du^\alpha du^\beta = \epsilon_i e^{\psi(x^k)}(dx^i)^2 + \frac{\tilde{\Phi}^2}{4\widetilde{\Lambda}}\left[dy^3 + \left({}_1n_k + {}_2\widetilde{n}_k\int dy^4\frac{(\partial_4\tilde{\Phi})^2}{\tilde{\Phi}^3\,\Xi}\right)dx^k\right]^2 + \frac{(\partial_4\tilde{\Phi})^2}{\Xi}\left[dy^4 + \frac{\partial_i\Phi}{\partial_4\Phi}dx^i\right]^2. \qquad (65)$$

This line element defines a family of generic off-diagonal solutions with Killing symmetry in $\partial/\partial y^3$ of the 4-d Einstein equation (46) for the canonical d-connection $\widehat{\mathbf{D}}$ (the label $4dK$ is for "nonholonomic 4-d Killing solutions). We can verify by straightforward computations of the corresponding anholonomy coefficients $W^{\gamma}_{\alpha\beta}$ in (9) that such values are not zero if an arbitrary generating function ϕ and integration ones $({}^0h_a, {}_1n_k$, and ${}_2n_k)$ are considered.

4-d vacuum configurations:

The limits to the off-diagonal solutions with $\Lambda = {}^{\upsilon}\Lambda = 0$ cannot be smooth because, for instance, we have multiples of $({}^{\upsilon}\Lambda)^{-1}$ in the coefficients of (65). For the ansatz (35), we can analyze solutions when the nontrivial coefficients of the Ricci d-tensor (38)–(45) are zero. The first equation is a typical example of a 2-d wave, or Laplace, equation. We can express such solutions in a similar form $g_i = \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2$.

There are three classes of off-diagonal metrics which result in zero coefficients (39)–(45).

- In the first case, we can impose the condition $\partial_4 h_3 = 0$, $h_3 \neq 0$, which results only in one nontrivial equation (derived from (40)),

$$\partial_{44}n_k + \partial_4 n_k\ \partial_4\ln|h_4| = 0,$$

where $h_4(x^i,y^4)\neq 0$ and $w_k(x^i,y^4)$ are arbitrary functions. If $\partial_4h_4 = 0$, we must take $\partial_{44}n_k = 0$. For $\partial_4h_4 \neq 0$, we get

$$n_k = {}_1n_k + {}_2n_k\int dy^4/h_4 \qquad (66)$$

with integration functions ${}_1n_k(x^i)$ and ${}_2n_k(x^i)$. The corresponding quadratic line element is of the type

$$ds^2_{v1} = \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2 + {}^0h_3(x^k)[dy^3 + ({}_1n_k(x^i) + {}_2n_k(x^i)\int dy^4/h_4)dx^i]^2 + h_4(x^i,y^4) \times[dy^4 + w_i(x^k,y^4)dx^i]^2. \qquad (67)$$

- In the second case, $\partial_4h_3 \neq 0$ and $\partial_4h_4 \neq 0$. We can solve (39) and/or (48) in a self-consistent form for ${}^{\upsilon}\Lambda = 0$ if $\partial_4\phi = 0$ for coefficients (55) and (56). For $\phi = \phi_0 = const$, we can consider arbitrary functions $w_i(x^k,y^4)$ because $\beta = \alpha_i = 0$ for such configurations. The condition (55) is satisfied by any

$$h_4 = {}^0h_4(x^k)(\partial_4\sqrt{|h_3|})^2, \qquad (68)$$

where ${}^0h_3(x^k)$ is an integration function and $h_3(x^k,y^4)$ is any generating function. The coefficients n_k can be found from (40); see (66). Such a family of vacuum metrics is described by

$$ds^2_{v2} = \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2 + h_3(x^i,y^4)[dy^3 + ({}_1n_k(x^i) + {}_2n_k(x^i)\int dy^4/h_4)dx^i]^2 + {}^0h_4(x^k)(\partial_4\sqrt{|h_3|})^2[dy^4 + w_i(x^k,y^4)dx^i]^2. \qquad (69)$$

- In the third case, $\partial_4h_3 \neq 0$ but $\partial_4h_4 = 0$. Equation (39) transforms into $\partial_{44}h_3 - \frac{(\partial_4h_3)^2}{2h_3} = 0$, when the general solution is $h_3(x^k,y^4) = \left[c_1(x^k) + c_2(x^k)y^4\right]^2$, with generating functions $c_1(x^k)$, $c_2(x^k)$, and $h_4 = {}^0h_4(x^k)$. For $\phi = \phi_0 = const$, we can take any values $w_i(x^k,y^4)$, because $\beta = \alpha_i = 0$. The coefficients n_i are found from (40) and/or, equivalently, from (49) with $\gamma = \frac{3}{2}\partial_4|h_3|$. We obtain

$$n_i = {}_1n_i(x^k) + {}_2n_i(x^k)\int dy^4|h_3|^{-3/2} = {}_1n_i(x^k) + {}_2\widetilde{n}_i(x^k)\left[c_1(x^k) + c_2(x^k)y^4\right]^{-2},$$

with integration functions ${}_1n_i(x^k)$ and ${}_2n_i(x^k)$, or the re-defined ${}_2\widetilde{n}_i = -{}_2n_i/2c_2$. The quadratic line element for this class of solutions for vacuum metrics is described by

$$\begin{aligned} ds^2_{v3} = {} & \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2 + \left[c_1(x^k) + c_2(x^k)y^4\right]^2 \\ & \times[dy^3 + ({}_1n_i(x^k) + {}_2\widetilde{n}_i(x^k) \\ & \times\left[c_1(x^k) + c_2(x^k)y^4\right]^{-2})dx^i]^2 \\ & + {}^0h_4(x^k)[dy^4 + w_i(x^k, y^4)dx^i]^2. \end{aligned} \quad (70)$$

Finally, we note that such solutions have nontrivial induced torsions (15).

Extra-dimensional non-vacuum solutions:

The solutions for higher dimensions can be constructed in a certain fashion, similar to the 4-d ones, using new classes of generating and integration functions with dependencies on extra-dimensional coordinates. For instance, we can generate solutions of the system (51)–(53) with coefficients (57) and (58) following a formal analogy when $\partial_4 \to \partial_6, \phi(x^k, y^4) \to {}^1\phi(u^\tau, y^6)$, ${}^v\Lambda(x^k, y^4) \to {}^v_1\Lambda(u^\tau, y^6)\ldots$ and associated values ${}^1\tilde{\Phi}(u^\tau, y^6)$ and ${}^1\widetilde{\Lambda}$ as we considered in the previous paragraph.

The extra-dimensional coefficients are computed by

$$h_5[{}^1\tilde{\Phi}, {}^1\widetilde{\Lambda}] = \frac{{}^1\tilde{\Phi}^2}{4\,{}^1\widetilde{\Lambda}} \text{ and } h_6[{}^1\tilde{\Phi}] = \frac{(\partial_6\,{}^1\tilde{\Phi})^2}{{}^1\Xi},$$

for ${}^1\Xi = \int dy^6({}^v_1\Lambda)\partial_6({}^1\tilde{\Phi}^2)$ and, for N coefficients,

$$\begin{aligned} {}^1n_\tau = {} & {}^1_1n_\tau + {}^1_2n_\tau \int dy^6 h_6/(\sqrt{|h_5|})^3 = {}^1_1n_k \\ & + {}^1_2\widetilde{n}_k \int dy^6(\partial_6\,{}^1\tilde{\Phi})^2/({}^1\tilde{\Phi})^3\,{}^1\Xi, \\ {}^1w_\tau = {} & \partial_\tau\,{}^1\phi/\partial_6\,{}^1\phi = \partial_\tau\,{}^1\Phi/\partial_6\,{}^1\Phi, \end{aligned}$$

where ${}^0h_{a_1} = {}^0h_{a_1}(u^\tau)$, ${}^1_1n_k(u^\tau)$ and ${}^1_2n_k(u^\tau)$, are integration functions.

A general class of quadratic line elements in 6-d spacetimes can be parameterized in the form

$$\begin{aligned} ds^2_{6dK} = {} & ds^2_{4dK} + \frac{{}^1\tilde{\Phi}^2}{4\,{}^1\widetilde{\Lambda}} \\ & \times\left[dy^5 + \left({}^1_1n_k + {}^1_2\widetilde{n}_k \int dy^6 \frac{(\partial_6\,{}^1\tilde{\Phi})^2}{({}^1\tilde{\Phi})^3\,{}^1\Xi}\right)du^\tau\right]^2 \\ & + \frac{(\partial_6\,{}^1\tilde{\Phi})^2}{{}^1\Xi}\left[dy^6 + \frac{\partial_\tau\,{}^1\Phi}{\partial_6\,{}^1\Phi}du^\tau\right]^2, \end{aligned} \quad (71)$$

where ds^2_{4dK} is given by (65) and $\tau = 1, 2, 3, 4$. This quadratic line element has a Killing symmetry in ∂_5 (in N adapted frames, the metric does not depend on y^5).

Extending the constructions to the shell $s = 2$ with $\partial_6 \to \partial_8$, ${}^1\phi(u^\tau, y^6) \to {}^2\phi(u^{\tau_1}, y^8)$, ${}^v_1\Lambda(u^\tau, y^6) \to {}^v_2\Lambda(u^{\tau_1}, y^8)\ldots$, ${}^2\tilde{\Phi}(u^{\tau_1}, y^8)$, ${}^2\widetilde{\Lambda}$, where $\tau_1 = 1, 2, \ldots, 5, 6$, we generate off-diagonal solutions in 8-d gravity,

$$\begin{aligned} ds^2_{8dK} = {} & ds^2_{6dK} + \frac{{}^2\tilde{\Phi}^2}{4\,{}^2\widetilde{\Lambda}} \\ & \times\left[dy^7 + \left({}^2_1n_k + {}^2_2\widetilde{n}_k \int dy^8 \frac{(\partial_8\,{}^2\tilde{\Phi})^2}{({}^2\tilde{\Phi})^3\,{}^2\Xi}\right)du^{\tau_1}\right]^2 \\ & + \frac{(\partial_8\,{}^2\tilde{\Phi})^2}{{}^2\Xi}\left[dy^8 + \frac{\partial_{\tau_1}\,{}^2\Phi}{\partial_8\,{}^2\Phi}du^{\tau_1}\right]^2, \end{aligned} \quad (72)$$

where ds^2_{6dK} is given by (71), ${}^2\Xi = \int dy^8({}^v_2\Lambda)\partial_8({}^2\tilde{\Phi}^2)$, and the corresponding integration/generating functions are ${}^0h_{a_2}(u^{\tau_1})$; $a_2 = 7, 8$; ${}_1n_{\tau_1}(u^{\tau_1})$, and ${}_2n_{\tau_1}(u^{\tau_1})$.

Using $2+2+\ldots$ symmetries of off-diagonal parameterizations (36), we can construct exact solutions for arbitrary finite dimension of the extra-dimensional spacetime ${}^s\mathbf{V}$.

Extra-dimensional vacuum solutions: The off-diagonal solutions (65), (71), (72),... have been constructed for nontrivial sources ${}^v\Lambda(x^k, y^4)$, ${}^v_1\Lambda(u^\tau, y^6)$, ${}^v_2\Lambda(u^\tau, y^8), \ldots$ In a similar manner, we can generate vacuum configurations with effective zero cosmological constants by extending to higher dimensions the 4-d vacuum metrics of type ds^2_{v1} (67), ds^2_{v2} (69), ds^2_{v3} (70) etc. It is possible to generate solutions when the sources for (46) are zero on some shells and nonzero for other ones.

We provide here an example of a quadratic line element for 6-d gravity derived as a $s = 1$ generalization of (69). For such solutions, $\partial_4 h_a \neq 0, \partial_6 h_{a_1} \neq 0, \ldots$ and $\phi = \phi_0 = \text{const}$, ${}^1\phi = {}^1\phi_0 = \text{const}, \ldots$

$$\begin{aligned} ds^2_{v2s3} = {} & \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2 + h_3(x^i, y^4)[dy^3 \\ & + \left({}_1n_k(x^i) + {}_2n_k(x^i)\int dy^4/h_4\right)dx^i]^2 \\ & + {}^0h_4(x^k)(\partial_4\sqrt{|h_3|})^2[dy^4 \\ & + w_i(x^k, y^4)dx^i]^2 + h_5(u^\tau, y^6)[dy^5 \\ & + \left({}^1_1n_\lambda(u^\tau) + {}^1_2n_\lambda(u^\tau)\int dy^6/h_6\right)du^\lambda]^2 \\ & + {}^0h_6(u^\tau)(\partial_6\sqrt{|h_5|})^2[dy^6 + {}^1w_\lambda(u^\tau, y^6)du^\lambda]^2, \end{aligned} \quad (73)$$

where ${}^0h_3(x^k)$, ${}^0h_5(u^\tau)$, ${}_1n_k(x^i)$, ${}_2n_k(x^i)$, ${}^1_1n_\lambda(u^\tau)$, ${}^1_2n_\lambda(u^\tau)$ are integration functions. The values $h_4(x^k, y^4)$ and $h_6(u^\tau, y^6)$ are any generating functions. We can consider arbitrary functions $w_i(x^k, y^4)$ and ${}^1w_\lambda(u^\tau, y^6)$ because, respectively, $\beta = \alpha_i = 0$ and ${}^1\beta = {}^1\alpha_\tau = 0$ for such configurations; see (55), (56) and (57), and (58).

3.1.5 Coefficients of metrics as generating functions

For nontrivial sources ${}^{v}\Lambda(x^k, y^4)$, ${}^{v}_{1}\Lambda(u^\tau, y^6)$, ${}^{v}_{2}\Lambda(u^\tau, y^8)$, $\ldots$, we can prescribe, respectively, h_3, h_5, and h_7 (with nonzero $\partial_4 h_3$, $\partial_6 h_5$, and $\partial_8 h_7$) as generating functions. Let us perform such constructions in explicit form for $s = 0$. Using (60), we find (up to an integration function depending on x^i) that

$$\Phi^2 = 2\varepsilon_\Phi \int dy^4\, {}^{v}\Lambda\ \partial_4 h_3, \tag{74}$$

where $\varepsilon_\Phi = \pm 1$ in order to have $\Phi^2 > 0$. Inserting this value into (61), we express h_4 in terms of ${}^{v}\Lambda$ and h_3,

$$h_4[\,{}^{v}\Lambda,\ h_3] = \varepsilon_4 (\partial_4 h_3)^2/2\,{}^{v}\Lambda h_3 \int dy^4({}^{v}\Lambda h_3),\ \varepsilon_4 = \pm 1.$$

The N connection coefficients are computed following the formulas in (64) with $\Phi[\,{}^{v}\Lambda,\ h_3]$ expressed in the form (74),

$$w_i[\,{}^{v}\Lambda,\ h_3] = \frac{\partial_i \Phi}{\partial_4 \Phi} = \frac{\partial_i \Phi^2}{\partial_4 \Phi^2} = \frac{\int dy^4 \partial_i |\,{}^{v}\Lambda \partial_4 h_3|}{|\,{}^{v}\Lambda \partial_4 h_3|},$$

$$n_k[\,{}^{v}\Lambda,\ h_3] = {}_1n_k + {}_2n_k \times \int dy^4 \frac{(\partial_4 h_3)^2}{{}^{v}\Lambda(\sqrt{|h_3|})^5 \int_0^{y^4} dy^{4'}({}^{v}\Lambda h_3)},$$

where $\varepsilon_4/2$ is included in n_2.

We can use for $s = 1$ and $s = 2$ certain formulas similar to (74),

$${}^1\Phi^2 = 2\varepsilon_{\,{}^1\Phi} \int dy^6\, {}^{v}_{1}\Lambda\ \partial_6 h_5 \text{ and } {}^2\Phi^2 = 2\varepsilon_{\,{}^2\Phi} \int dy^8\, {}^{v}_{2}\Lambda\ \partial_8 h_7,\ \varepsilon_{\,{}^1\Phi} = \pm 1, \varepsilon_{\,{}^2\Phi} = \pm 2.$$

The solutions (65), (71), and (72) are respectively re-parameterized as

$$ds^2_{4dK} = \epsilon_i e^{\psi(x^k)}(dx^i)^2 + h_3\left[dy^3 + \left({}_1n_k + {}_2n_k \int dy^4 \times \frac{(\partial_4 h_3)^2}{{}^{v}\Lambda(\sqrt{|h_3|})^5 \int_0^{y^4} dy^{4'}({}^{v}\Lambda h_3)}\right)dx^k\right]^2 + \varepsilon_4 \frac{(\partial_4 h_3)^2}{2\,{}^{v}\Lambda h_3 \int dy^4({}^{v}\Lambda h_3)} \times \left[dy^4 + \frac{\int dy^4 \partial_i |\,{}^{v}\Lambda \partial_4 h_3|}{|\,{}^{v}\Lambda \partial_4 h_3|}dx^i\right]^2,$$

$$ds^2_{6dK} = ds^2_{4dK} + h_5\left[dy^5 + \left({}^1_1n_\tau + {}^1_2n_\tau \int dy^6 \times \frac{(\partial_6 h_5)^2}{{}^{v}_{1}\Lambda(\sqrt{|h_5|})^5 \int_0^{y^6} dy^{6'}({}^{v}_{1}\Lambda h_5)}\right)du^\tau\right]^2 + \varepsilon_6 \frac{(\partial_6 h_5)^2}{2\,{}^{v}_{1}\Lambda h_5 \int dy^6({}^{v}_{1}\Lambda h_5)} \times \left[dy^6 + \frac{\int dy^6 \partial_\tau |\,{}^{v}_{1}\Lambda \partial_6 h_5|}{|\,{}^{v}_{1}\Lambda \partial_6 h_5|}du^\tau\right]^2,$$

and

$$ds^2_{8dK} = ds^2_{6dK} + h_7\left[dy^7 + \left({}^2_1n_{\tau_1} + {}^2_2n_{\tau_1} \int dy^8 \times \frac{(\partial_8 h_7)^2}{{}^{v}_{2}\Lambda(\sqrt{|h_7|})^5 \int_0^{y^8} dy^{8'}({}^{v}_{2}\Lambda h_7)}\right)du^{\tau_1}\right]^2 + \varepsilon_8 \frac{(\partial_8 h_7)^2}{2\,{}^{v}_{2}\Lambda h_7 \int dy^8({}^{v}_{2}\Lambda h_7)} \times \left[dy^8 + \frac{\int dy^8 \partial_{\tau_1} |\,{}^{v}_{2}\Lambda \partial_8 h_7|}{|\,{}^{v}_{2}\Lambda \partial_8 h_7|}du^{\tau_1}\right]^2.$$

We can introduce effective cosmological constants via a re-definition of the generating functions of the type (62) when $(\Phi,\ {}^{v}\Lambda) \to (\tilde{\Phi}, \widetilde{\Lambda})$, $({}^1\Phi,\ {}^{v}_{1}\Lambda) \to ({}^1\tilde{\Phi},\ {}_1\widetilde{\Lambda})$ and $({}^2\Phi,\ {}^{v}_{2}\Lambda) \to ({}^2\tilde{\Phi},\ {}_2\widetilde{\Lambda})$. For such parameterizations, the coefficients of the metrics depend explicitly on $\tilde{\Phi}$, ${}^1\tilde{\Phi}$ and ${}^2\tilde{\Phi}$. Finally, we note that such formulas can be similarly generalized for higher dimensions with shells $s = 3, 4, \ldots$.

3.1.6 The Levi–Civita conditions

All solutions constructed in previous sections define certain subclasses of generic off-diagonal metrics (35) for canonical d-connections ${}^s\widehat{\mathbf{D}}$ and nontrivial nonholonomically induced d-torsion coefficients $\widehat{\mathbf{T}}^{\gamma_s}_{\alpha_s\beta_s}$ (15). Such a torsion vanishes for a subclass of nonholonomic distributions with necessary types of parameterizations of the generating and integration functions and sources. In explicit form, we construct LC-configurations by imposing additional constraints, shell by shell, on the d-metric and N connection coefficients. By straightforward computations (see the details in Refs. [5–8] and Appendix 1), we can verify that if in N adapted frames

$$\begin{aligned}
\text{for } s = 0:\ & \partial_4 w_i = \mathbf{e}_i \ln\sqrt{|h_4|}, \mathbf{e}_i \ln\sqrt{|h_3|} = 0,\\
& \partial_i w_j = \partial_j w_i \text{ and } \partial_4 n_i = 0;\\
s = 1:\ & \partial_6\, {}^1w_\alpha = {}^1\mathbf{e}_\alpha \ln\sqrt{|h_6|},\ {}^1\mathbf{e}_\alpha \ln\sqrt{|h_5|} = 0,\\
& \partial_\alpha\, {}^1w_\beta = \partial_\beta\, {}^1w_\alpha \text{ and } \partial_6\, {}^1n_\gamma = 0;\\
s = 2:\ & \partial_8\, {}^2w_{\alpha_1} = {}^2\mathbf{e}_{\alpha_1} \ln\sqrt{|h_8|},\ {}^2\mathbf{e}_{\alpha_1} \ln\sqrt{|h_7|} = 0,\\
& \partial_{\alpha_1}\, {}^2w_{\beta_1} = \partial_{\beta_1}\, {}^2w_{\alpha_1} \text{ and } \partial_8\, {}^2n_{\gamma_1} = 0
\end{aligned} \tag{75}$$

(similar equations can be written recurrently for arbitrary finite extra dimensions), then the torsion coefficients become zero. For the n-coefficients, such conditions are satisfied if ${}_2n_k(x^i) = 0$ and $\partial_i\ {}_1n_j(x^k) = \partial_j\ {}_1n_i(x^k)$; ${}^1_2n_\alpha(u^\beta) = 0$ and $\partial_\gamma\ {}^1_1n_\tau(u^\beta) = \partial_\tau\ {}^1_1n_\gamma(u^\beta)$; ${}^2_2n_{\alpha_1}(u^{\beta_1}) = 0$ and

$\partial_{\gamma_1}\, {}^2_1 n_{\tau_1}(u^{\beta_1}) = \partial_{\tau_1}\, {}^2_1 n_{\gamma_1}(u^{\beta_1})$ etc. The explicit form of the solutions of the constraints on w_k derived from (75) depend on the class of vacuum or non-vacuum metrics we try to construct.

Let us show how we can satisfy the LC-conditions (75) for $s=0$. We note that such nonholonomic constraints cannot be solved in explicit form for arbitrary data $(\Phi,\ {}^v\Lambda)$, or $(\tilde{\Phi},\ \widetilde{\Lambda})$, and all types of nonzero integration functions ${}_1n_j(x^k)$ and ${}_2n_k(x^i) = 0$. Nevertheless, certain general classes of solutions can be written in explicit form if via coordinate and frame transformations we can fix ${}_2n_k(x^i) = 0$ and ${}_1n_j(x^k) = \partial_j n(x^k)$ for a function $n(x^k)$. Then we use the property that

$$\mathbf{e}_i\Phi = (\partial_i - w_i\partial_4)\Phi \equiv 0$$

for any Φ if $w_i = \partial_i\Phi/\partial_4\Phi$; see (64). For any functional $H[\Phi]$, one has the equality

$$\mathbf{e}_i H = (\partial_i - w_i\partial_4)H = \frac{\partial H}{\partial\Phi}(\partial_i - w_i\partial_4)\Phi \equiv 0.$$

We can restrict our construction to a subclass of generating data $(\Phi,\ {}^v\Lambda)$ and $(\tilde{\Phi},\ \widetilde{\Lambda})$, which are related via (62) when $H = \tilde{\Phi}[\Phi]$ is a functional which allows us to generate LC-configurations in explicit form. Using $h_3[\tilde{\Phi}] = \tilde{\Phi}^2/4\widetilde{\Lambda}$ (63) for $H = \tilde{\Phi} = \ln\sqrt{|\,h_3|}$, we satisfy the second condition, $\mathbf{e}_i \ln\sqrt{|\,h_3|} = 0$, in (75) for $s=0$.

In the second step, we solve firstly the condition in (75), for $s=0$. Taking the derivative ∂_4 of $w_i = \partial_i\Phi/\partial_4\Phi$ (64), we obtain

$$\partial_4 w_i = \frac{(\partial_4\partial_i\Phi)(\partial_4\Phi) - (\partial_i\Phi)\partial_4\partial_4\Phi}{(\partial_4\Phi)^2} = \frac{\partial_4\partial_i\Phi}{\partial_4\Phi} - \frac{\partial_i\Phi}{\partial_4\Phi}\frac{\partial_4\partial_4\Phi}{\partial_4\Phi}. \quad (76)$$

If $\Phi = \check{\Phi}$, for which

$$\partial_4\partial_i\check{\Phi} = \partial_i\partial_4\check{\Phi}, \quad (77)$$

and using (76), then we compute $\partial_4 w_i = \mathbf{e}_i \ln|\partial_4\Phi|$. For $h_4[\Phi,\ {}^v\Lambda]$ (61), $\mathbf{e}_i \ln\sqrt{|\,h_4|} = \mathbf{e}_i[\ln|\partial_4\Phi| - \ln\sqrt{|\,{}^v\Lambda|}]$, where we used the conditions (77) and the property $\mathbf{e}_i\check{\Phi} = 0$. Using the last two formulas, we obtain $\partial_4 w_i = \mathbf{e}_i \ln\sqrt{|\,h_4|}$ if $\mathbf{e}_i \ln\sqrt{|\,{}^v\Lambda|} = 0$. This is possible for ${}^v\Lambda = const$, or if ${}^v\Lambda$ can be expressed as a functional ${}^v\Lambda(x^i, y^4) = {}^v\Lambda[\check{\Phi}]$.

Finally, we note that the third condition for $s=0$, $\partial_i w_j = \partial_j w_i$, see (75), holds for any $\check{A} = \check{A}(x^k, y^4)$ for which $w_i = \check{w}_i = \partial_i\check{\Phi}/\partial_4\check{\Phi} = \partial_i\check{A}$.

Following similar considerations for other shells' generating functions,

$$\begin{aligned} s=1:\ & {}^1\Phi = {}^1\check{\Phi}(u^\tau, y^6),\ \partial_6\partial_\tau\, {}^1\check{\Phi} = \partial_\tau\partial_6\, {}^1\check{\Phi}; \\ & \partial_\alpha\, {}^1\check{\Phi}/\partial_6\, {}^1\check{\Phi} = \partial_\alpha\, {}^1\check{A};\ {}^1_1 n_\tau = \partial_\tau\, {}^1 n(u^\beta); \\ s=2:\ & {}^2\Phi = {}^2\check{\Phi}(u^{\tau_1}, y^8),\ \partial_8\partial_{\tau_1}\, {}^2\check{\Phi} = \partial_{\tau_1}\partial_8\, {}^2\check{\Phi}; \\ & \partial_{\alpha_1}\, {}^2\check{\Phi}/\partial_8\, {}^2\check{\Phi} = \partial_{\alpha_2}\, {}^2\check{A};\ {}^2_1 n_{\tau_1} = \partial_{\tau_1}\, {}^2 n(u^{\beta_1}); \end{aligned} \quad (78)$$

(similar formulas can be written recurrently for arbitrary extra shells), we can construct quadratic line elements for the LC-configurations

$$\begin{aligned} ds^2_{8dK} = {} & \epsilon_i e^{\psi(x^k)}(dx^i)^2 + \frac{(\tilde{\Phi}[\check{\Phi}])^2}{4\widetilde{\Lambda}}\left[dy^3 + (\partial_i\, n)dx^i\right]^2 \\ & + \frac{(\partial_4\tilde{\Phi}[\check{\Phi}])^2}{\Xi(\tilde{\Phi}[\check{\Phi}])}\left[dy^4 + (\partial_i\, \check{A})dx^i\right]^2 \\ & + \frac{({}^1\tilde{\Phi}[{}^1\check{\Phi}])^2}{4\,{}^1\widetilde{\Lambda}}\left[dy^5 + (\partial_\tau\, {}^1 n)du^\tau\right]^2 \\ & + \frac{(\partial_6\, {}^1\tilde{\Phi}[{}^1\check{\Phi}])^2}{{}^1\Xi({}^1\tilde{\Phi}[{}^1\check{\Phi}])}\left[dy^6 + (\partial_\tau\, {}^1\check{A})du^\tau\right]^2 \\ & + \frac{({}^2\tilde{\Phi}[{}^2\check{\Phi}])^2}{4\,{}^2\widetilde{\Lambda}}\left[dy^7 + (\partial_{\tau_1}\, {}^2 n)du^{\tau_1}\right]^2 \\ & + \frac{(\partial_8\, {}^2\tilde{\Phi}[{}^2\check{\Phi}])^2}{{}^2\Xi({}^2\tilde{\Phi}[{}^2\check{\Phi}])}\left[dy^8 + (\partial_{\tau_1}\, {}^2\check{A})du^{\tau_1}\right]^2. \end{aligned} \quad (79)$$

In these formulas, the generating functions are functionals of "inverse hat" values, when

$$\begin{aligned} \check{\Phi}^2 &= \widetilde{\Lambda}^{-1}\int dy^4({}^v\Lambda)\partial_4(\tilde{\Phi}^2) \text{and } \tilde{\Phi}^2 = \widetilde{\Lambda}\int dy^4({}^v\Lambda)^{-1}\partial_4(\check{\Phi}^2); \\ {}^1\check{\Phi}^2 &= ({}^1\widetilde{\Lambda})^{-1}\int dy^6({}^v_1\Lambda)\partial_6({}^1\tilde{\Phi}^2) \text{and } {}^1\tilde{\Phi}^2 = {}^1\widetilde{\Lambda}\int dy^6({}^v_1\Lambda)^{-1}\partial_6({}^1\check{\Phi}^2); \\ {}^2\check{\Phi}^2 &= ({}^2\widetilde{\Lambda})^{-1}\int dy^8({}^v_2\Lambda)\partial_8({}^2\tilde{\Phi}^2) \text{and } {}^2\tilde{\Phi}^2 = {}^2\widetilde{\Lambda}\int dy^8({}^v_2\Lambda)^{-1}\partial_8({}^2\check{\Phi}^2). \end{aligned}$$

We can compute the values $\Xi(\tilde{\Phi}[\check{\Phi}])$, ${}^1\Xi({}^1\tilde{\Phi}[{}^1\check{\Phi}])$, and ${}^2\Xi({}^2\tilde{\Phi}[{}^2\check{\Phi}])$ as in (72).

The torsions for such non-vacuum exact solutions (79) generated by the respective data $({}^s\boldsymbol{g},\ {}^s\boldsymbol{N},\ {}^s\boldsymbol{\nabla})$ are zero, which is different from the class of exact solutions (72) with nontrivial canonical d-torsions (15) and completely determined by arbitrary data $({}^s\mathbf{g},\ {}^s\mathbf{N},\ {}^s\widehat{\mathbf{D}})$ with Killing symmetry on ∂_7.

3.2 Non-Killing configurations

The off-diagonal integral varieties of the solutions of the gravitational field equations constructed in the previous section possess for any shell $s \geq 0$ at least one Killing vector symmetry on $\partial/\partial y^{a_s-1}$ when the metrics do not depend on the coordinate y^{a_s-1} in a class of N adapted frames. There are two general possibilities to generate "non-Killing" configurations: 1) performing a formal embedding into higher-dimensional vacuum spacetimes and/or via 2) "vertical" conformal nonholonomic deformations.

3.2.1 Embedding into a higher dimension vacuum

We analyze a subclass of off-diagonal metrics for 6-d spaces which via nonholonomic constraints and re-parameterizations transform into 4-d non-Killing vacuum solutions. Let us consider certain geometric data $\Lambda = {}^v\Lambda = {}^v_1\Lambda = 0$ and $h_3 = \epsilon_3, h_5 = \epsilon_5, n_k = 0$ and ${}^1n_\alpha = 0$ with a 2-d h -metric $\epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2$. The coefficients of the Ricci d-tensor are zero (see (38)–(41) and (42)–(44)). Here we note that one cannot use (47)–(53) derived for $\partial_4 h_3 \neq 0$, $\partial_6 h_5 \neq 0$ etc. which does not allow, for instance, the values $h_3 = \epsilon_3, h_5 = \epsilon_5$, for any nontrivial data $h_4(x^i, y^4)$, $w_k(x^i, y^4)$; $h_6(x^i, y^4, y^6)$, ${}^1w_k(x^i, y^4)$, ${}^1w_4(x^i, y^4, y^6)$. Such values can be considered as generating functions for the vacuum quadratic line elements

$$
\begin{aligned}
ds^2_{6\to 4} = {} & \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2 + \epsilon_3(dy^3)^2 \\
& + h_4(dy^4 + w_k dx^k)^2 + \epsilon_5(dy^5)^2 \\
& + h_6(dy^6 + {}^1w_k dx^k + {}^1 w_4 dy^4)^2. \qquad (80)
\end{aligned}
$$

In general, this class of vacuum 6-d metrics have a nonzero nonholonomically induced d-torsion (15). Such solutions do not consist necessarily of a subclass of vacuum solutions (73) when $h_3 \to \epsilon_3$ and $h_5 \to \epsilon_5$; the conditions $\partial_4 h_3 \neq 0$ and $\partial_6 h_5 \neq 0$ restrict the class of possible generating functions h_4 and h_6. If we fix from the very beginning certain configurations with $\partial_4 h_3 = 0$ and $\partial_6 h_5 = 0$, we can consider h_4, h_6 and w_k, ${}^1w_k$, ${}^1w_4$ as independent generating functions.

If the coefficients in (80) are subjected additionally to the constraints (75) for $s = 0$ and $s = 1$, we generate the LC-configurations. We can follow a formal procedure which is similar to that outlined in Sect. 3.1.6. The conditions $\mathbf{e}_i \ln\sqrt{|h_3|} = 0$ and ${}^1\mathbf{e}_\alpha \ln\sqrt{|h_5|} = 0$ are satisfied, respectively, for any constant $h_3 = \epsilon_3$ and $h_5 = \epsilon_5$. Let us show how we can restrict the class of generating functions in order to obtain solutions for which

$$
\begin{aligned}
& \partial_4 w_i(x^i, y^4) = \mathbf{e}_i \ln\sqrt{|h_4(x^i, y^4)|}, \partial_i w_j = \partial_j w_i, \text{ and} \\
& \partial_6\, {}^1w_\alpha(x^i, y^4, y^6) = {}^1\mathbf{e}_\alpha \ln\sqrt{|h_6(x^i, y^4, y^6)|}, \\
& \quad \partial_\alpha\, {}^1w_\beta = \partial_\beta\, {}^1w_\alpha. \qquad (81)
\end{aligned}
$$

We emphasize that the above N adapted formulas do not depend on y^3 and y^5. Prescribing any values of h_4 and h_6 we can find LC-admissible w-coefficients solving the respective systems of the first order partial derivative equations in (81). In general, such solutions are defined by nonholonomic configurations, i.e. in "non-explicit" form. If all values $h_4[\check{\Phi}]$, $h_6[{}^1\check{\Phi}]$, and $w_k[\check{\Phi}]$, ${}^1w_k[{}^1\check{\Phi}]$, ${}^1w_4[{}^1\check{\Phi}]$ are, respectively, determined by $\check{\Phi}(x^i, y^4)$ and ${}^1\check{\Phi}(x^i, y^4, y^6)$ satisfying conditions of type (77) and (78) (but h_3 and h_5 are not functionals of type (63)), we can solve (81) in explicit form. Let us choose any generating functions $\check{\Phi}$ and ${}^1\check{\Phi}$, consider any functionals $h_4[\check{\Phi}]$, $h_6[{}^1\check{\Phi}]$, and compute

$$
\begin{aligned}
& w_i = \check{w}_i = \partial_i\check{\Phi}/\partial_4\check{\Phi} = \partial_i\check{A} \text{ and} \qquad (82) \\
& {}^1w_i = {}^1\check{w}_i = \partial_i\, {}^1\check{\Phi}/\partial_6\, {}^1\check{\Phi} = \partial_i\, {}^1\check{A},\ {}^1w_4 = {}^1\check{w}_4 \\
& \quad = \partial_4\, {}^1\check{\Phi}/\partial_6\, {}^1\check{\Phi} = \partial_4\, {}^1\check{A},
\end{aligned}
$$

for some $\check{A}(x^i, y^4)$ and ${}^1\check{A}(x^i, y^4, y^6)$, which are necessary for $\partial_i w_j = \partial_j w_i$ and $\partial_\alpha\, {}^1w_\beta = \partial_\beta\, {}^1w_\alpha$. Considering functional derivatives of type (76) and N coefficients of the type in (82) when $H[\check{\Phi}] = \ln\sqrt{|h_4|}$ and ${}^1H[{}^1\check{\Phi}] = \ln\sqrt{|h_6|}$, we can satisfy the LC-conditions (81).

Putting together the above formulas, we construct a subclass of metrics of (80) determined by generic off-diagonal metrics as solutions of 6-d vacuum Einstein equations,

$$
\begin{aligned}
ds^2_{6\to 4} = {} & \epsilon_i e^{\psi(x^k,\Lambda=0)}(dx^i)^2 + \epsilon_3(dy^3)^2 + h_4[\check{\Phi}] \\
& \times(dy^4 + \partial_k\check{A}dx^k)^2 + \epsilon_5(dy^5)^2 + h_6[{}^1\check{\Phi}] \\
& \times(dy^6 + \partial_k\, {}^1\check{A}\, dx^k + \partial_4\, {}^1\check{A}\, dy^4)^2. \qquad (83)
\end{aligned}
$$

We note that in this quadratic line element the terms $\epsilon_3(dy^3)^2$ and $\epsilon_5(dy^5)^2$ are used for trivial extensions from 4-d to 6-d. Re-defining the coordinate $y^6 \to y^3$, we generate vacuum solutions in 4-d gravity with metrics (83) depending on all four coordinates x^i, y^3 and y^4. The anholonomy coefficients (9) are not zero and such metrics cannot be diagonalized by coordinate transformations. This class of 4-d vacuum spacetimes do not possess, in general, Killing symmetries.

3.2.2 "Vertical" conformal nonholonomic deformations

There is another possibility to generate off-diagonal solutions depending on all spacetime coordinates and, in general, with nontrivial sources of the type in (46); see details and proofs in Ref. [8]. By straightforward computations, we can check that any metric

$$
\begin{aligned}
\mathbf{g} = {} & g_i(x^k)dx^i \otimes dx^i + \omega^2(u^\alpha)h_a(x^k, y^4)\mathbf{e}^a \otimes \mathbf{e}^a \\
& + {}^1\omega^2(u^{\alpha_1})h_{a_1}(u^\alpha, y^6)\mathbf{e}^{a_1} \otimes \mathbf{e}^{a_1} + {}^2\omega^2(u^{\alpha_2}) \\
& \times h_{a_2}(u^{\alpha_1}, y^8)\mathbf{e}^{a_2} \otimes \mathbf{e}^{a_2} + \ldots, \qquad (84)
\end{aligned}
$$

with the conformal v-factors subjected to the conditions

$$
\begin{aligned}
& \mathbf{e}_k\omega = \partial_k\omega + n_k\partial_3\omega + w_k\partial_4\omega = 0, \\
& {}^1\mathbf{e}_\beta\, {}^1\omega = \partial_\beta\, {}^1\omega + {}^1n_\beta\partial_5\, {}^1\omega + {}^1w_\beta\partial_6\, {}^1\omega = 0, \qquad (85) \\
& {}^2\mathbf{e}_{\beta_1}\, {}^2\omega = \partial_{\beta_1}\, {}^2\omega + {}^2n_{\beta_1}\partial_7\, {}^2\omega + {}^2w_{\beta_1}\partial_8\, {}^2\omega = 0
\end{aligned}
$$

(similar equations can be written recurrently for arbitrary finite extra dimensions), does not change the Ricci d-tensor (38)–(45). Any class of solutions considered in this section can be generalized to non-Killing configurations using nonholonomic "vertical" conformal transformations.

In 4-d, the ansatz (84) can be parameterized with respect to coordinate frames in a form with nontrivial $\omega^2(u^\alpha)$ which is different from that given in Fig. 1,

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} g_1+\omega^2(n_1{}^2h_3+w_1{}^2h_4) & \omega^2(n_1n_2h_3+w_1w_2h_4) & \omega^2 n_1h_3 & \omega_1^2 w_1h_4 \\ \omega^2(n_1n_2h_3+w_1w_2h_4) & g_2+\omega^2(n_2^2h_3+w_2^2h_4) & \omega^2 n_2h_3 & \omega^2 w_2h_4 \\ \omega^2 n_1h_3 & \omega^2 n_2h_3 & \omega^2 h_3 & 0 \\ \omega^2 w_1h_4 & \omega^2 w_2h_4 & 0 & \omega^2 h_4 \end{bmatrix}. \quad (86)$$

A general metric $g_{\alpha\beta}(u^\gamma)$ can be parameterized in the form (86) if there are any geometrically and physically well-defined frame transformations $g_{\alpha\beta} = e^{\underline{\alpha}}{}_{\alpha} e^{\underline{\beta}}{}_{\beta} g_{\underline{\alpha}\underline{\beta}}$. For certain given values $g_{\alpha\beta}$ and $g_{\underline{\alpha}\underline{\beta}}$ (in GR, there are $6+6$ independent components), we have to solve a system of quadratic algebraic equation in order to determine 16 coefficients $e^{\underline{\alpha}}{}_{\alpha}$, up to a fixed coordinate system. We have to fix such nonholonomic 2+2 splitting and partitions on manifolds when the algebraic equations have real nondegenerate solutions.

Finally, we note that we can consider generic off-diagonal coordinate decompositions which are similar to (86) but with dependencies on all coordinates for higher order shells.

4 Nonholonomic deformations and the Kerr metric

In this section, we show how, using the AFDM formalism, the Kerr solution can be constructed as a particular case when corresponding types of generating and integration functions are prescribed. We provide a series of new classes of solutions when the metrics are nonholonomically deformed into general or ellipsoidal stationary configurations in four-dimensional gravity and/or extra dimensions. Explicit examples are studied of generic off-diagonal metrics encoding interactions in massive gravity, f-modifications and nonholonomically induced torsion effects. We find such nonholonomic constraints when modified massive, and zero mass, gravitational effects can be modeled by nonlinear off-diagonal interactions in GR.

4.1 Generating the Kerr vacuum solution

Let us consider the ansatz

$$ds^2_{[0]} = Y^{-1}e^{2h}(d\rho^2+dz^2) - \rho^2 Y^{-1}dt^2 + Y(d\varphi + A dt)^2$$

parameterized in terms of three functions (h, Y, A) on coordinates (ρ, z). We obtain the Kerr solution of the vacuum Einstein equations in 4-d, for rotating black holes, if we choose

$$Y = \frac{1-(p\widehat{x}_1)^2-(q\widehat{x}_2)^2}{(1+p\widehat{x}_1)^2+(q\widehat{x}_2)^2},\ A = 2M\frac{q}{p}\frac{(1-\widehat{x}_2)(1+p\widehat{x}_1)}{1-(p\widehat{x}_1)-(q\widehat{x}_2)},$$
$$e^{2h} = \frac{1-(p\widehat{x}_1)^2-(q\widehat{x}_2)^2}{p^2[(\widehat{x}_1)^2+(\widehat{x}_2)^2]},\ \rho^2 = M^2(\widehat{x}_1^2-1)(1-\widehat{x}_2^2),$$
$$z = M\widehat{x}_1\widehat{x}_2,$$

where $M = const$ and $\rho = 0$ consists of the horizon $\widehat{x}_1 = 0$ and the "north/south" segments of the rotation axis, $\widehat{x}_2 = +1/-1$. Such a metric can be written in the form (36),

$$ds^2_{[0]} = (dx^1)^2 + (dx^2)^2 - \rho^2 Y^{-1}(\mathbf{e}^3)^2 + Y(\mathbf{e}^4)^2, \quad (87)$$

if the coordinates $x^1(\widehat{x}_1, \widehat{x}_2)$ and $x^2(\widehat{x}_1, \widehat{x}_2)$ are defined for any

$$(dx^1)^2 + (dx^2)^2 = M^2 e^{2h}(\widehat{x}_1^2 - \widehat{x}_2^2)Y^{-1} \times \left(\frac{d\widehat{x}_1^2}{\widehat{x}_1^2-1} + \frac{d\widehat{x}_2^2}{1-\widehat{x}_2^2}\right)$$

and $y^3 = t + \widehat{y}^3(x^1, x^2)$, $y^4 = \varphi + \widehat{y}^4(x^1, x^2, t)$, when

$$\mathbf{e}^3 = dt + (\partial_i \widehat{y}^3)dx^i,\ \mathbf{e}^4 = dy^4 + (\partial_i \widehat{y}^4)dx^i,$$

for some functions $\widehat{y}^a$, $a = 3, 4$, with $\partial_t \widehat{y}^4 = -A(x^k)$.

For many purposes, the Kerr metric was written in the so-called Boyer–Linquist coordinates $(r, \vartheta, \varphi, t)$, for $r = m_0(1+p\widehat{x}_1)$, $\widehat{x}_2 = \cos\vartheta$. The parameters p, q are related to the total black hole mass, m_0 (it should be not confused with the parameter μ_g in massive gravity) and the total angular momentum, am_0, for the asymptotically flat, stationary, and axisymmetric Kerr spacetime. The formulas $m_0 = Mp^{-1}$ and $a = Mqp^{-1}$ when $p^2+q^2 = 1$ imply $m_0^2 - a^2 = M^2$ (see the monographs [1,27,33] for the standard methods and bibliography on stationary black hole solutions; we note here that the coordinates $\widehat{x}_1, \widehat{x}_2$ correspond, respectively, to x, y from chapter 4 of the first book). In such variables, the vacuum solution (87) can be written

$$ds^2_{[0]} = (dx^{1'})^2 + (dx^{2'})^2 + \overline{A}(\mathbf{e}^{3'})^2 + (\overline{C} - \overline{B}^2/\overline{A})(\mathbf{e}^{4'})^2,$$
$$\mathbf{e}^{3'} = dt + d\varphi\overline{B}/\overline{A} = dy^{3'} - \partial_{i'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})dx^{i'},$$
$$\mathbf{e}^{4'} = dy^{4'} = d\varphi, \quad (88)$$

for any coordinate functions

$$x^{1'}(r,\vartheta),\ x^{2'}(r,\vartheta),\ y^{3'} = t + \widehat{y}^{3'}(r,\vartheta,\varphi) + \varphi\overline{B}/\overline{A},\\ y^{4'} = \varphi,\ \partial_\varphi \widehat{y}^{3'} = -\overline{B}/\overline{A},$$

for which $(\mathrm{d}x^{1'})^2 + (\mathrm{d}x^{2'})^2 = \Xi\left(\Delta^{-1}dr^2 + d\vartheta^2\right)$, and the coefficients are

$$\overline{A} = -\Xi^{-1}(\Delta - a^2\sin^2\vartheta),\\ \overline{B} = \Xi^{-1}a\sin^2\vartheta\left[\Delta - (r^2+a^2)\right],\\ \overline{C} = \Xi^{-1}\sin^2\vartheta\left[(r^2+a^2)^2 - \Delta a^2\sin^2\vartheta\right],\ \text{and}\\ \Delta = r^2 - 2m_0 + a^2,\ \Xi = r^2 + a^2\cos^2\vartheta. \quad (89)$$

The quadratic linear elements (87) (or (88)) with prime data

$$\mathring{g}_1 = 1, \mathring{g}_2 = 1, \mathring{h}_3 = -\rho^2 Y^{-1}, \mathring{h}_4 = Y, \mathring{N}_i^a = \partial_i \widehat{y}^a,\\ (\text{ or } \mathring{g}_{1'} = 1, \mathring{g}_{2'} = 1, \mathring{h}_{3'} = \overline{A}, \mathring{h}_{4'} = \overline{C} - \overline{B}^2/\overline{A},\\ \mathring{N}_{i'}^3 = \mathring{n}_{i'} = -\partial_{i'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A}), \mathring{N}_{i'}^4 = \mathring{w}_{i'} = 0) \quad (90)$$

define solutions of the vacuum Einstein equations parameterized in the form (21) and (22) with zero sources. Here we note that we have to consider a correspondingly N adapted system of coordinates instead of the "standard" prolate spherical, or Boyer–Linquist ones because parameterizations with the data (90) are most convenient for a straightforward application of the AFDM. Following such an approach, we can generalize the solutions in order to get dependencies of the coefficients on more than two coordinates, with non-Killing configurations and/or extra dimensions.

In some sense, the Kerr vacuum solution in GR consists of a "degenerate" case of the 4-d off-diagonal vacuum solutions determined by primary metrics with the data (90) when the diagonal coefficients depend only on two "horizontal" N adapted coordinates and the off-diagonal terms are induced by rotation frames.

4.2 Deformations of Kerr metrics in 4-d massive gravity

Let us consider the coefficients (90) for the Kerr metric as the data for a prime metric $\mathring{\mathbf{g}}$ (in general, it may or may not be an exact solution of the Einstein or other modified gravitational equations, or any fiducial metric). Our goal is to construct nonholonomic deformations,

$$(\mathring{\mathbf{g}}, \mathring{\mathbf{N}},\ {}^{v}\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \to (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}},\ {}^{v}\widetilde{\Upsilon} = \widetilde{\lambda}, \widetilde{\Upsilon} = \widetilde{\lambda}),\\ \widetilde{\lambda} = \text{const} \neq 0;$$

see the sources (34) for the shell $s = 0$ and (33). The main condition is that the target metric $\mathbf{g}$ positively defines a generic off-diagonal solution offield equations in 4-d massive gravity. The N adapted deformations of coefficients of the metrics, frames, and sources are parameterized in the form

$$[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i] \to [\widetilde{g}_i = \widetilde{\eta}_i\mathring{g}_i, \widetilde{h}_3 = \widetilde{\eta}_3\mathring{h}_3, \widetilde{h}_4 = \widetilde{\eta}_4\mathring{h}_4,\\ \widetilde{w}_i = \mathring{w}_i + {}^{\eta}w_i, n_i = \mathring{n}_i + {}^{\eta}n_i],\\ \widetilde{\Upsilon} = \widetilde{\lambda},\ {}^{v}\hat{\Upsilon}(x^{k'}) = {}^{v}\Lambda = \mu_g^2\,\lambda(x^{k'})\mathring{h}_4^{-1},\\ \widetilde{\Lambda} = \mu_g^2\,\widetilde{\lambda},\ \tilde{\Phi}^2 = \exp[2\varpi(x^{k'}, y^4)]\,\mathring{h}_3, \quad (91)$$

where the values $\widetilde{\eta}_a, \widetilde{w}_i, \widetilde{n}_i$, and ϖ are functions of three coordinates $(x^{k'}, y^4)$ and $\widetilde{\eta}_i(x^k)$ depend only on h-coordinates. The prime data $\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i$ are given by coefficients depending only on (x^k).

In terms of the η-functions (37) resulting in $h_a^* \neq 0$ and $g_i = c_i \mathrm{e}^{\psi(x^k)}$, the solutions of type (65) with $\widetilde{\Lambda} \to \widetilde{\lambda}$ and ${}_2n_{k'} = 0$ (we use "primed" coordinates and prime Kerr data (88) and (90)) can be re-written in the form

$$\mathrm{d}s^2 = \mathrm{e}^{\psi(x^{k'})}[(\mathrm{d}x^{1'})^2 + (\mathrm{d}x^{2'})^2]\\ -\frac{\mathrm{e}^{2\varpi}}{4\mu_g^2\,|\widetilde{\lambda}|}\overline{A}\left[\mathrm{d}y^{3'} + \left(\partial_{k'}\,{}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)\mathrm{d}x^{k'}\right]^2\\ +\frac{(\varpi^*)^2}{\mu_g^2\,\lambda(x^{k'})}(\overline{C} - \overline{B}^2/\overline{A})[\mathrm{d}\varphi + (\partial_{i'}\,{}^{\eta}\widetilde{A})\mathrm{d}x^{i'}]^2, \quad (92)$$

for

$$\Xi = \int \mathrm{d}y^4({}^{v}\Lambda)\partial_4(\tilde{\Phi}^2) = \mu_g^2\,\lambda(x^{k'})\mathring{h}_4^{-1}\tilde{\Phi}^2,$$

with $\tilde{\Phi}^2/\mathring{h}_4$ parameterized using (91).[10] The gravitational polarizations (η_i, η_a) and N coefficients (n_i, w_i) are computed as follows:

$$\mathrm{e}^{\psi(x^k)} = \widetilde{\eta}_{1'} = \widetilde{\eta}_{2'},\ \widetilde{\eta}_{3'} = \frac{\mathrm{e}^{2\varpi}}{4\mu_g^2\,|\widetilde{\lambda}|},\ \widetilde{\eta}_{4'} = \frac{(\varpi^*)^2}{\mu_g^2\,\lambda(x^{k'})},\\ w_{i'} = \mathring{w}_{i'} + {}^{\eta}w_{i'} = \partial_{i'}({}^{\eta}\widetilde{A}[\varpi]),\ n_{k'} = \mathring{n}_{k'} + {}^{\eta}n_{k'}\\ = \partial_{k'}(-\widehat{y}^{3'} + \varphi\overline{B}/\overline{A} + {}^{\eta}n),$$

where ${}^{\eta}\widetilde{A}(x^k, y^4)$ is introduced via formulas and assumptions similar to (78), for $s = 1$, and $\psi^{\bullet\bullet} + \psi'' = 2\,\mu_g^2\,\lambda(x^{k'})$. For the N coefficients, the parameterizations are used (64) with $\check{\Phi} = \exp[\varpi(x^{k'}, y^4)]\sqrt{|\mathring{h}_{3'}|}$, when $\mathring{h}_{3'}\mathring{h}_{4'} = \overline{AC} - \overline{B}^2$ and

$$w_{i'} = \mathring{w}_{i'} + {}^{\eta}w_{i'} = \partial_{i'}(\,\mathrm{e}^{\varpi}\sqrt{|\overline{AC} - \overline{B}^2|})/\\ \varpi^*\mathrm{e}^{\varpi}\sqrt{|\overline{AC} - \overline{B}^2|} = \partial_{i'}\,{}^{\eta}\widetilde{A}.$$

We can take any function ${}^{\eta}n(x^k)$ and put $\lambda = const \neq 0$ using the corresponding re-definitions of the coordinates and generating functions.

[10] Hereafter we shall consider that we can approximate $\lambda(x^{k'}) \simeq \widetilde{\lambda} = const$.

The solutions (92) are valid for stationary LC-configurations determined by off-diagonal massive gravity effects on Kerr black holes when the new class of spacetimes have a Killing symmetry in $\partial/\partial y^{3'}$ and a generic dependence on three (from maximally four) coordinates, $(x^{i'}(r,\vartheta),\varphi)$. Off-diagonal modifications are possible even for very small values of the mass parameter μ_g. The solutions depend on the type of generating function $\varpi(x^{i'},\varphi)$ we have to fix in order to satisfy certain experimental/observational data in certain fixed systems of reference/coordinates. Various data can be re-parameterized for an effective $\lambda = \mathrm{const} \neq 0$. In such variables, we can mimic stationary massive gravity effects by off-diagonal configurations in GR with integration parameters which should also be fixed by imposing additional assumptions on the symmetries of the interactions (for instance, to have an ellipsoid configuration; see Sect. 4.3, and the details and the discussion of parametric Killing symmetries in Refs. [3–5]).

4.2.1 Nonholonomically induced torsion and massive gravity

If we do not impose the LC-conditions (22), a nontrivial source ${}^{\mu}\widetilde{\Lambda} = \mu_g^2\,\tilde{\lambda}$ from massive gravity induces a stationary configuration with nontrivial d-torsion (15). The torsion coefficients are determined by metrics of the type (65) with $\widetilde{\Lambda} \to \tilde{\lambda}$ and parameterizations of coefficients and coordinates distinguishing the prime data for a Kerr metric (90). Such solutions can be written in the form

$$\begin{aligned} ds^2 &= e^{\psi(x^{k'})}\Big[(dx^{1'})^2 + (dx^{2'})^2\Big] - \frac{\Phi^2}{4\mu_g^2\,|\tilde{\lambda}|}\overline{A}\Big[dy^{3'} \\ &\quad + \Big({}_1n_{k'}(x^{i'}) + {}_2n_{k'}(x^{i'})\frac{4\mu_g(\Phi^*)^2}{\Phi^5} \\ &\quad - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\Big)dx^{k'}\Big]^2 + \frac{(\partial_\varphi\Phi)^2}{\mu_g^2\,\lambda(x^{k'})\Phi^2} \\ &\quad \times(\overline{C} - \overline{B}^2/\overline{A})\Big[d\varphi + \frac{\partial_{i'}\Phi}{\partial_\varphi\Phi}dx^{i'}\Big]^2, \end{aligned} \tag{93}$$

where we use a generating function $\Phi(x^{i'},\varphi)$ instead of e^{ϖ} and consider nonzero values of ${}_2n_k(x^{i'})$. We can see that nontrivial stationary off-diagonal torsion effects may result in additional effective rotations proportional to μ_g if the integration function ${}_2n_k \neq 0$. Considering two different classes of off-diagonal solutions (93) and (92), we can study if a massive gravity theory is described in terms of an induced torsion or characterized by additional nonholonomic constraints as in GR (with zero torsion).

It should be noted that configurations of the type (93) can be constructed in various theories with noncommutative, brane, extra-dimension, warped, and trapped brane type variables in the string, or Finsler-like and/or Hořava–Lifshits theories [6,10,11,13,25] when nonholonomically induced torsion effects play a substantial role. Those classes of solutions were constructed for different sets of interactions constants and, for instance, for propagating Schwarzschild and/or ellipsoid type configurations on Tau NUT backgrounds etc. The off-diagonal deformations and effective polarizations of the coefficients of the metrics correspond to a prime Kerr metric and are related to the target configuration in massive gravity.

4.2.2 Small f-modifications of Kerr metrics and massive gravity

Using the AFDM, we can construct off-diagonal solutions for a superposition of f-modified and massive gravity interactions. Such nonlinear effects can be distinguished in explicit form if we consider for additional f-deformations, for instance, a "prime" solution for massive gravity/effectively modeled in GR with source ${}^{\mu}\Lambda = \mu_g^2\,\lambda(x^{k'})$, or re-defined to ${}^{\mu}\tilde{\Lambda} = \mu_g^2\,\tilde{\lambda} = const$. Adding a "small" value $\widetilde{\Lambda}$ determined by f-modifications, we work in N adapted frames with an effective source $\Upsilon = \widetilde{\Lambda} + \tilde{\lambda}$ (see formulas (33) and (34)). As a result, we construct a class of off-diagonal solutions in modified f-gravity generated from the Kerr black hole solution as a result of two nonholonomic deformations,

$$\begin{aligned} &(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{v}\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \to (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon} = \widetilde{\lambda}, \widetilde{\Upsilon} = \widetilde{\lambda}) \\ &\quad \to ({}^{\varepsilon}\mathbf{g}, {}^{\varepsilon}\mathbf{N}, \Upsilon = \varepsilon\,\widetilde{\Lambda} + {}^{\mu}\tilde{\Lambda}, \Upsilon = \varepsilon\,\widetilde{\Lambda} + {}^{\mu}\tilde{\Lambda}), \end{aligned}$$

when the target data $\mathbf{g} = {}^{\varepsilon}\mathbf{g}$ and $\mathbf{N} = {}^{\varepsilon}\mathbf{N}$ depend on a small parameter ε, $0 < \varepsilon \ll 1$. For simplicity, we restrict our considerations for solutions when $|\varepsilon\,\widetilde{\Lambda}| \ll |{}^{\mu}\tilde{\Lambda}|$, i.e. consider that f-modifications in N adapted frames are much smaller than massive gravity effects (in a similar from, we can analyze nonlinear interactions with $|\varepsilon\,\widetilde{\Lambda}| \gg |{}^{\mu}\tilde{\Lambda}|$). The corresponding N adapted transformations are parameterized as

$$\begin{aligned} &[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i] \to [g_i = (1+\varepsilon\chi_i)\widetilde{\eta}_i\mathring{g}_i, h_3 \\ &\quad = (1+\varepsilon\chi_3)\widetilde{\eta}_3\mathring{h}_3, h_4 = (1+\varepsilon\chi_4) \\ &\widetilde{\eta}_4\mathring{h}_4,\ {}^{\varepsilon}w_i = \mathring{w}_i + \widetilde{w}_i + \varepsilon\overline{w}_i,\ {}^{\varepsilon}n_i = \mathring{n}_i + \tilde{n}_i + \varepsilon\overline{n}_i]; \\ &\Upsilon = {}^{\mu}\tilde{\Lambda}(1 + \varepsilon\,\widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda});\quad {}^{\varepsilon}\tilde{\Phi} = \tilde{\Phi}(x^k,\varphi) \\ &[1 + \varepsilon\ {}^{1}\tilde{\Phi}(x^k,\varphi)/\tilde{\Phi}(x^k,\varphi)] = \exp[\ {}^{\varepsilon}\varpi(x^k,\varphi)], \end{aligned} \tag{94}$$

leading to a 4-d LC-configuration with d-metric,

$$\begin{aligned} ds^2_{4\varepsilon dK} &= \epsilon_i(1+\varepsilon\chi_i)e^{\psi(x^k)}(dx^i)^2 \\ &\quad + \frac{{}^{\varepsilon}\tilde{\Phi}^2}{4\,\Upsilon}\Big[dy^3 + (\partial_i\, n)dx^i\Big]^2 \\ &\quad + \frac{(\partial_\varphi\ {}^{\varepsilon}\tilde{\Phi})^2}{\Upsilon\ {}^{\varepsilon}\tilde{\Phi}^2}\Big[dy^4 + (\partial_i\ {}^{\varepsilon}\check{A})dx^i\Big]^2, \end{aligned}$$

for $\partial_i\ {}^{\varepsilon}\check{A} = \partial_i\ {}^{\varepsilon}\check{A} + \varepsilon\partial_i\ {}^{1}\check{A}$ determined by ${}^{\varepsilon}\tilde{\Phi} = \tilde{\Phi} + \varepsilon\ {}^{1}\tilde{\Phi}$ following the conditions in (82). The values labeled by "$\circ$" and "$\frown$" are taken from (91) (for simplicity, we omit priming of indices). The χ- and w-values (corresponding to a re-definition of the coefficients; for simplicity, we consider $\varepsilon\overline{n}_i = 0$) have to be computed to define ε -deformed LC-configurations; see (75) for $s = 0$, as solutions of the system (46) in the form (47)–(50) for a source $\Upsilon = {}^{\mu}\tilde{\mathbf{\Lambda}} + \varepsilon\widetilde{\Lambda}$.

The deformations (94) of the off-diagonal solutions (92) result in a new class of ε-deformed solutions with

$$\begin{aligned}
&\chi_1 = \chi_2 = \chi, \text{ for } \partial_{11}\chi + \epsilon_2\partial_{22}\chi = 2\widetilde{\Lambda};\\
&\chi_3 = 2\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda},\\
&\chi_4 = 2\partial_4\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - 2\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda},\\
&\overline{w}_i = (\frac{\partial_i\,{}^{1}\tilde{\Phi}}{\partial_i\tilde{\Phi}} - \frac{\partial_4\,{}^{1}\tilde{\Phi}}{\partial_4\tilde{\Phi}})\frac{\partial_i\tilde{\Phi}}{\partial_4\tilde{\Phi}} = \partial_i\ {}^{1}\check{A}, \overline{n}_i = 0,
\end{aligned} \quad (95)$$

where there is not summation on index "i''" in the last formula and $\mathring{h}_{3'}\mathring{h}_{4'} = \overline{AC} - \overline{B}^2$. Such nonholonomic deformations are determined, respectively, by two generating functions $\tilde{\Phi} = \mathrm{e}^{\varpi}$ and ${}^{1}\tilde{\Phi}$ and two sources ${}^{\mu}\tilde{\Lambda}$ and $\widetilde{\Lambda}$. Putting all this together, we construct an off-diagonal generalization of the Kerr metric via "main" massive gravity terms and additional ε-parametric f-modifications,

$$\begin{aligned}
\mathrm{d}s^2 = &\, \mathrm{e}^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(\mathrm{d}x^{1'})^2 + (\mathrm{d}x^{2'})^2]\\
&-\frac{\mathrm{e}^{2\varpi}}{4|\,{}^{\mu}\tilde{\Lambda}|}\overline{A}[1 + \varepsilon(2\mathrm{e}^{-\varpi}\,{}^{1}\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda})][\mathrm{d}y^{3'}\\
&+\left(\partial_{k'}\,{}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)\mathrm{d}x^{k'}]^2\\
&+\frac{(\varpi^*)^2}{{}^{\mu}\tilde{\Lambda}}(\overline{C} - \overline{B}^2/\overline{A})[1 + \varepsilon(2\mathrm{e}^{-\varpi}\partial_4\,{}^{1}\tilde{\Phi} - 2\mathrm{e}^{-\varpi}\,{}^{1}\tilde{\Phi}\\
&-\widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda})][\mathrm{d}\varphi + (\partial_{i'}\widetilde{A} + \varepsilon\partial_{i'}\ {}^{1}\check{A})\mathrm{d}x^{i'}]^2.
\end{aligned} \quad (96)$$

We can consider ε-deformations of the type (94) for (93), which allows us to generate new classes of off-diagonal solutions with nonholonomically induced torsion determined both by massive and f-modifications of GR. Such a spacetime cannot be modeled as an effective one with anisotropic polarizations in GR.

4.3 Ellipsoidal 4-d deformations of the Kerr metric

We provide some examples of how the Kerr primary data (90) is nonholonomically deformed into target generic off-diagonal solutions of vacuum and non-vacuum Einstein equations for the canonical d-connection and/or the Levi–Civita connection.

4.3.1 Vacuum ellipsoidal configurations

Let us construct a class of parametric solutions with such nonholonomic constraints on the coefficients given by (93) which transform the metrics into effective 4-d vacuum LC-configurations of the type (69). This defines a model when f-modifications compensate massive gravity deformations of a Kerr solution, with $\Upsilon = {}^{\mu}\tilde{\mathbf{\Lambda}} + \varepsilon\widetilde{\Lambda} = 0$, and result in ellipsoidal off-diagonal configurations in GR, where $\varepsilon = -{}^{\mu}\tilde{\mathbf{\Lambda}}/\widetilde{\mathbf{\Lambda}} \ll 1$ can be considered as an eccentricity parameter. We find solutions for ε-deformations to vacuum solutions. The ansatz for the target metrics is of the type

$$\begin{aligned}
\mathrm{d}s^2 = &\, \mathrm{e}^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(\mathrm{d}x^{1'})^2 + (\mathrm{d}x^{2'})^2]\\
&-\frac{\mathrm{e}^{2\varpi}}{4\mu_g^2|\,\widetilde{\lambda}|}\overline{A}[1 + \varepsilon\chi_{3'}][\mathrm{d}y^{3'}\\
&+\left(\partial_{k'}\,{}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)\mathrm{d}x^{k'}]^2\\
&+\frac{(\partial_4\varpi)^2\eta_{4'}}{\mu_g^2\,\widetilde{\lambda}}(\overline{C} - \overline{B}^2/\overline{A})[1 + \varepsilon\chi_{4'}]\\
&\times[\mathrm{d}\varphi + (\partial_{i'}\widetilde{A} + \varepsilon\partial_{i'}\ {}^{1}\check{A})\mathrm{d}x^{i'}]^2,
\end{aligned} \quad (97)$$

when the prime metrics (92) are obtained for $\eta_{4'} = 1$. The condition (55) for $\phi = const$, i.e. for vacuum off-diagonal configurations, when $h_{4'} = {}^{0}h_{4'}(\partial_4\sqrt{|h_{3'}|})^2$, see (68), is satisfied for $\eta_{4'} = \overline{A}\sqrt{|\overline{B}^2 - \overline{CA}|}\mathrm{e}^{2\varpi}$. For terms proportional to ε), we compute $\chi_{4'} = (\partial_4\varpi)^{-1}(1 + \mathrm{e}^{-\varpi}\chi_{3'})$, where $\varpi(r,\vartheta,\varphi)$ and $\chi_{3'}(r,\vartheta,\varphi)$ are generating functions. We can consider as generating functions for N coefficients any $\widetilde{A}(r,\vartheta,\varphi)$ and ${}^{1}\check{A}(r,\vartheta,\varphi)$, which for $w_{i'} = \partial_{i'}(\widetilde{A} + \varepsilon\,{}^{1}\check{A})$ solve the LC-conditions. The LC-conditions $\mathbf{e}_i \ln\sqrt{|\,h_3|} = 0$, $\partial_i w_j = \partial_j w_i$ for $s = 0$, see (75), can be satisfied if we parameterize

$$w_{i'} = \partial_{i'}\,{}^{\varepsilon}\Phi/\partial_{\varphi}\,{}^{\varepsilon}\Phi = \partial_{i'}(\widetilde{A} + \varepsilon\,{}^{1}\check{A}),$$

for ${}^{\varepsilon}\Phi = \exp(\varpi + \varepsilon\chi_{3'})$; see the discussions related to (81) and (82). Because $h_{4'}$ for (97) can be approximated up to ε^2 to be a functional on ${}^{\varepsilon}\Phi$, we can satisfy for certain classes of generating functions ${}^{\varepsilon}\Phi = {}^{\varepsilon}\tilde{\Phi} = {}^{\varepsilon}\check{\Phi}$, see (77), the conditions $\partial_{\varphi}w_{i'} = \mathbf{e}_{i'}\ln\sqrt{|\,h_4|}$.

We can choose such a generating function $\chi_{3'}$, when the constraint $h_{3'} = 0$ defines a stationary rotoid configuration (different from to the ergo sphere for the Kerr solutions). Prescribing

$$\chi_{3'} = 2\underline{\zeta}\sin(\omega_0\varphi + \varphi_0), \quad (98)$$

for constant parameters $\underline{\zeta}, \omega_0$ and φ_0, and introducing the values

$$\begin{aligned}
&\overline{A}(r,\vartheta)[1 + \varepsilon\chi_{3'}(r,\vartheta,\varphi)]\\
&\quad \widehat{A}(r,\vartheta,\varphi) = -\Xi^{-1}(\widehat{\Delta} - a^2\sin^2\vartheta),\\
&\quad \widehat{\Delta}(r,\varphi) = r^2 - 2m(\varphi) + a^2,
\end{aligned}$$

as ε-deformations of Kerr coefficients (89), we get an effective "anisotropically polarized" mass

$$m(\varphi) = m_0/\left(1 + \varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)\right). \quad (99)$$

The condition $h_3 = 0$, i.e. ${}^{\varphi}\Delta(r, \varphi, \varepsilon) = a^2 \sin^2 \vartheta$, results in an ellipsoidal "deformed horizon" $r(\vartheta, \varphi) = m(\varphi) + \left(m^2(\varphi) - a^2 \sin^2 \vartheta\right)^{1/2}$. For $a = 0$, this is just the parametric formula for an ellipse with eccentricity ε,

$$r_+ = \frac{2m_0}{1 + \varepsilon \underline{\zeta} \sin(\omega_0 \varphi + \varphi_0)}.$$

If the anholonomy coefficients (9) computed for (97) are not trivial for such w_i and $n_k = {}_1n_k$, the generated solutions cannot be diagonalized via coordinate transformations. The corresponding 4-d spacetimes have one Killing symmetry with respect to $\partial/\partial y^{3'}$. For small ε, the singularity at $\Xi = 0$ is "hidden" under ellipsoidal deformed horizons if $m_0 \geq a$. Both the event horizon, $r_+ = m(\varphi) + (m^2(\varphi) - a^2 \sin^2 \vartheta)^{1/2}$, and the Cauchy horizon, $r_- = m(\varphi) - (m^2(\varphi) - a^2 \sin^2 \vartheta)^{1/2}$, are φ-deformed and are effectively embedded into an off-diagonal background determined by the N coefficients. In some sense, such configurations determine Kerr-like black hole solutions with additional dependencies on the variable φ of certain diagonal and off-diagonal coefficients of the metric. For $a = 0$, but with $\varepsilon \neq 0$, we get ellipsoidal deformations of the Schwarzschild black holes (see [6] and references therein on the stability and interpretation of such solutions with both commutative and/or noncommutative parameters). Such an interpretation is not possible for "non-small" N-deformations of the Kerr metric. In general, it is not clear what physical importance such target exact solutions may have, even if they may be defined to preserve the Levi–Civita configurations.

The eccentricity $\varepsilon = -\widetilde{\lambda}/\widetilde{\Lambda} \ll 1$ depends both on massive gravity and f-modifications encoded into effective cosmological constants. We proved that via nonholonomic deformations it is possible to transform non-vacuum solutions with an effective locally anisotropically cosmological constant into effective off-diagonal vacuum configurations in GR. If the generating functions are prescribed to possess necessarily smooth conditions of a certain type, the solutions are similar to certain Kerr black holes with ellipsoidal ε-deformed horizons and embedded self-consistently into nontrivial off-diagonal vacuum configurations. Polarizations of such vacuums encode massive gravity contributions and f-modifications.

4.3.2 *Ellipsoid Kerr–de Sitter configurations*

We construct a subclass of solutions (96) with rotoid configurations if we constrain χ_3 appearing in the ε-deformations in (95) to be of the form

$$\chi_3 = 2\, {}^1\tilde{\Phi}/\tilde{\Phi} - \widetilde{\Lambda}/\, {}^{\mu}\tilde{\Lambda} = 2\underline{\zeta} \sin(\omega_0 \varphi + \varphi_0),$$

which is similar to (98). Expressing ${}^1\tilde{\Phi} = e^{\varpi}[\,\widetilde{\Lambda}/2\, {}^{\mu}\tilde{\Lambda} + \underline{\zeta} \sin(\omega_0\varphi + \varphi_0)]$, for $\tilde{\Phi} = e^{\varpi}$, we generate a class of generic off-diagonal metrics associated with the ellipsoid Kerr–de Sitter configurations,

$$\begin{aligned} ds^2 &= e^{\psi(x^{k'})}(1 + \varepsilon \chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] \\ &- \frac{e^{2\varpi}}{4|\, {}^{\mu}\tilde{\Lambda}|}\overline{A}[1 + 2\varepsilon \underline{\zeta} \sin(\omega_0\varphi + \varphi_0)][dy^{3'} \\ &+ \left(\partial_{k'}\, {}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi \overline{B}/\overline{A})\right) dx^{k'}]^2 \\ &+ \frac{(\varpi^*)^2}{{}^{\mu}\tilde{\Lambda}}(\overline{C} - \overline{B}^2/\overline{A})[1 + \varepsilon(\partial_4 \varpi\ \widetilde{\Lambda}/\widetilde{\lambda} \\ &\times + 2\partial_4 \varpi\, \underline{\zeta} \sin(\omega_0\varphi + \varphi_0) \\ &+ 2\omega_0\ \underline{\zeta} \cos(\omega_0\varphi + \varphi_0))] \\ &[d\varphi + (\partial_{i'}\ \widetilde{A} + \varepsilon \partial_{i'}\ {}^1\check{A})dx^{i'}]^2. \end{aligned} \tag{100}$$

Such metrics have a Killing symmetry in $\partial/\partial y^3$ and are completely defined by a generating function $\varpi(x^{k'}, \varphi)$ and the sources ${}^{\mu}\tilde{\Lambda} = \mu_g^2\, \lambda$ and $\widetilde{\Lambda}$. They define ε-deformations of Kerr–de Sitter black holes into ellipsoid configurations with effective (polarized) cosmological constants determined by the constants in massive gravity and f-modifications. If the LC-conditions are satisfied, such metrics can be modeled in GR.

4.4 Extra-dimensional off-diagonal (non-) massive modifications of the Kerr solutions

Various classes of generic off-diagonal deformations of the Kerr metric into higher-dimensional exact solutions can be constructed. The explicit geometric and physical properties depend on the type of additional generating and integration functions and (non-) vacuum configurations and (non-) zero sources we consider. Let us analyze a series of 6-d and 8-d solutions encoding possible higher dimensional interactions with effective cosmological constants, nontrivial massive gravity contributions, f-modifications, and certain analogies to Finsler gravity models.

4.4.1 *6-d deformations with nontrivial cosmological constant*

Off-diagonal extra-dimensional gravitational interactions modify a Kerr metric for any nontrivial cosmological constant in 6-d.[11] Such higher-dimensional Kerr–de Sitter configurations can be generated by nonholonomic deformations $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{v}\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon} = \Lambda, \widetilde{\Upsilon} = \Lambda, {}^{v_1}\widetilde{\Upsilon} = \Lambda)$. The solutions are not stationary; they are characterized by a Killing symmetry in $\partial/\partial y^5$ and can be parameterized in the form

[11] In a similar form we can generalize the constructions in 8-d gravity.

$$
\begin{aligned}
ds^2 &= e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{e^{2\varpi}}{4\Lambda}\overline{A}[dy^{3'} \\
&+ \left(\partial_{k'}\ {}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi \overline{B}/\overline{A})\right) dx^{k'}]^2 \\
&+ \frac{(\partial_\varphi \varpi)^2}{\Lambda}(\overline{C} - \overline{B}^2/\overline{A})[d\varphi + (\partial_{i'}\ {}^{\eta}\widetilde{A})dx^{i'}]^2 \\
&+ \frac{{}^1\tilde{\Phi}^2}{4\ \Lambda}\left[dy^5 + (\partial_\tau\ {}^1 n)du^\tau\right]^2 \\
&+ \frac{(\partial_6\ {}^1\tilde{\Phi})^2}{\Lambda\ {}^1\tilde{\Phi}^2}\left[dy^6 + (\partial_\tau\ {}^1\check{A})du^\tau\right]^2 .
\end{aligned} \tag{101}
$$

The "primary" data $\overline{A}, \overline{B}, \overline{C}$ are described by (89) and the generating functions

$$
\begin{aligned}
&\varpi = \varpi(x^{k'}, \varphi),\ {}^1\tilde{\Phi}(u^\beta, y^6) = {}^1\tilde{\Phi}(x^{k'}, t, \varphi, y^6); \\
&{}^{\eta}n = {}^{\eta}n(x^{i'}),\ {}^1 n = {}^1 n(u^\beta, y^6);\ {}^{\eta}\widetilde{A} = {}^{\eta}\widetilde{A}(x^{k'}, \varphi), \\
&{}^1\check{A} = {}^1\check{A}(u^\beta, y^6),
\end{aligned}
$$

subjected to the LC-conditions and integrability conditions.

We can "extract" ellipsoid configurations for a subclass of metrics with "additional" ε-deformations,

$$
\begin{aligned}
ds^2 &= e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] \\
&- \frac{e^{2\varpi}}{4\Lambda}\overline{A}\left[1 + 2\varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)\right]\left[dy^{3'}\right. \\
&+ \left.\left(\partial_{k'}\ {}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right) dx^{k'}\right]^2 \\
&+ \frac{(\partial_\varphi\varpi)^2}{\Lambda}(\overline{C} - \overline{B}^2/\overline{A})\left[1 + \varepsilon(2\partial_4\varpi\,\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)\right. \\
&+ \left. 2\omega_0\underline{\zeta}\cos(\omega_0\varphi + \varphi_0))\right]\left[d\varphi + (\partial_{i'}\ {}^{\eta}\widetilde{A})dx^{i'}\right]^2 \\
&+ \frac{{}^1\tilde{\Phi}^2}{4\ \Lambda}\left[dy^5 + (\partial_\tau\ {}^1 n)du^\tau\right]^2 + \frac{(\partial_6\ {}^1\tilde{\Phi})^2}{\Lambda\ {}^1\tilde{\Phi}^2} \\
&\times\left[dy^6 + (\partial_\tau\ {}^1\check{A})du^\tau\right]^2 .
\end{aligned}
$$

For small values of the eccentricity ε, such metrics describe "slightly" deformed Kerr black holes embedded self-consistently into a generic off-diagonal 6-d spacetime. In general, extra dimensions are not compactified. Nevertheless, imposing additional constraints on the generating functions ${}^1\tilde{\Phi},\ {}^1 n, {}^1\check{A}$, we can construct warped/trapped configurations as in brane gravity models and generalizations; see similar examples in [8,12,13,25].

4.4.2 8-d deformations and Finsler-like configurations

Next, we generate a 8-d metric with nontrivial induced torsion describing nonholonomic deformations, $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{v}\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon} = \Lambda, \widetilde{\Upsilon} = \Lambda, {}^{v_1}\widetilde{\Upsilon} = \Lambda, {}^{v_2}\widetilde{\Upsilon} = \Lambda)$. A similar 4-d example is given by (93) but here we use a different source (in this subsection, we take the source as a cosmological constant Λ in 8-d). This class of solutions is parameterized in the form

$$
\begin{aligned}
ds^2 &= e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{\Phi^2}{4\Lambda}\overline{A}[dy^{3'} \\
&+ \left(\ {}_1 n_{k'}(x^{i'}) + {}_2 n_{k'}(x^{i'})\frac{4\mu_g(\Phi^*)^2}{\Phi^5}\right. \\
&- \left.\partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right) dx^{k'}]^2 \\
&+ \frac{(\partial_\varphi\Phi)^2}{\Lambda\Phi^2}(\overline{C} - \overline{B}^2/\overline{A})[d\varphi + \frac{\partial_{i'}\Phi}{\partial_\varphi\Phi}dx^{i'}]^2 \\
&+ \frac{{}^1\tilde{\Phi}^2}{4\ \Lambda}\left[dy^5 + (\partial_\tau\ {}^1 n)du^\tau\right]^2 + \frac{(\partial_6\ {}^1\tilde{\Phi})^2}{\Lambda\ {}^1\tilde{\Phi}^2} \\
&\times\left[dy^6 + (\partial_\tau\ {}^1\check{A})du^\tau\right]^2 \\
&+ \frac{{}^2\tilde{\Phi}^2}{4\ \Lambda}\left[dy^7 + (\partial_{\tau_1}\ {}^2 n)du^{\tau_1}\right]^2 \\
&+ \frac{(\partial_8\ {}^2\tilde{\Phi})^2}{\Lambda\ {}^2\tilde{\Phi}^2}\left[dy^8 + (\partial_{\tau_1}\ {}^2\check{A})du^{\tau_1}\right]^2 ,
\end{aligned} \tag{102}
$$

where the generating functions are chosen

$$
\begin{aligned}
&\Phi = \Phi(x^{k'}, \varphi),\ {}^1\tilde{\Phi}(u^\beta, y^6) = {}^1\tilde{\Phi}(x^{k'}, t, \varphi, y^6), \\
&{}^2\tilde{\Phi}(u^{\beta_1}, y^8) = {}^2\tilde{\Phi}(x^{k'}, t, \varphi, y^5, y^6, y^8); \\
&{}^1 n = {}^1 n(u^\beta, y^6),\ {}^2 n = {}^2 n(u^{\beta_1}, y^8), \\
&{}^{\eta}\widetilde{A} = {}^{\eta}\widetilde{A}(x^{k'}, \varphi),\ {}^1\check{A} = {}^1\check{A}(x^{k'}, t, \varphi, y^6), \\
&{}^2\check{A} = {}^2\check{A}(x^{k'}, t, \varphi, y^5, y^6, y^8).
\end{aligned} \tag{103}
$$

The generating functions for the class of solutions (102) are chosen in a form where the nonholonomically induced torsion (15) is effectively modeled on a 4-d pseudo-Riemannian spacetime, but on the higher shells $s = 1$ and $s = 2$ the torsion fields are zero. We can generate extra-dimensional torsion N adapted coefficients if nontrivial integration functions of the type ${}_2 n_{k'}(x^{i'})$ are considered for the higher dimensions.

Metrics of type (102) can be re-parameterized to define exact solutions in the so-called Einstein–Finsler gravity and fractional derivative modifications constructed on tangent bundles to Lorentz manifolds; see details in Refs. [13,25,28,29] and the following different Finsler or fractional models [30–32,34,35]. For Finsler-like theories, we have to consider y^5, y^6, y^7, y^8 as fiber coordinates for a tangent bundle with local coordinates $x^{i'}, y^{3'}, \varphi$ when the ${}^1 v + {}^2 v$ coefficients of the metric and other geometric/physical objects can be transformed into standard ones in Finsler geometry via frame and coordinate transformations. In some sense, Finsler-like theories with small corrections to GR are extra-dimensional ones with "velocity/momentum" coordinates and with low "speed/energy" nonlinear corrections.

Finally we note that the class of metrics (102) contains a subclass of the 6-d→8-d generalization of (101) to those configurations with zero torsion if we choose $\Phi = e^{2\varpi}$ and impose on the N coefficients respective constraints which are necessary for selecting LC-configurations.

4.4.3 Kerr massive deformations and vacuum extra dimensions

In this subsection, we momentarily return to the vacuum ellipsoid solutions (97) and extend the metric to extra dimensions when the source is of type $\Upsilon = \widetilde{\lambda} + \varepsilon(\widetilde{\Lambda} + \Lambda) = 0$, ${}^{\mu}\tilde{\Lambda} = \mu_g^2 |\lambda|$, and result in ellipsoidal off-diagonal configurations in GR, where $\varepsilon = -{}^{\mu}\tilde{\Lambda}/(\widetilde{\Lambda} + \Lambda) \ll 1$ can be considered as an eccentricity parameter. We can construct models of off-diagonal extra-dimensional interactions when the f-modifications $\widetilde{\Lambda}$ compensate an extra-dimensional contribution via the effective constant $\widetilde{\Lambda}$ and which are related to the configurations of massive gravity deformations of a Kerr solution. We select a subclass of solutions for ε-deformations of the vacuum solutions described by the ansatz for the target metrics

$$
\begin{aligned}
ds^2 =\; & e^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))\left[(dx^{1'})^2 + (dx^{2'})^2\right] \\
& - \frac{e^{2\varpi}}{4\,{}^{\mu}\tilde{\Lambda}}\overline{A}\left[1 + \varepsilon\chi_{3'}\right]\left[dy^{3'} + \left(\partial_{k'}\,{}^{\eta}n(x^{i'})\right.\right. \\
& \left.\left. - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)dx^{k'}\right]^2 + \frac{(\partial_4\varpi)^2\eta_{4'}}{{}^{\mu}\tilde{\Lambda}}(\overline{C} - \overline{B}^2/\overline{A}) \\
& \times\left[1 + \varepsilon\chi_{4'}\right]\left[d\varphi + (\partial_{i'}\,\widetilde{A} + \varepsilon\partial_{i'}\,{}^{1}\check{A})dx^{i'}\right]^2 \\
& + \frac{{}^{1}\tilde{\Phi}^2}{4(\widetilde{\Lambda} + \Lambda)}\left[dy^5 + (\partial_\tau\,{}^{1}n)du^\tau\right]^2 \\
& + \frac{(\partial_6\,{}^{1}\tilde{\Phi})^2}{(\widetilde{\Lambda} + \Lambda)\,{}^{1}\tilde{\Phi}^2}\left[dy^6 + (\partial_\tau\,{}^{1}\check{A})du^\tau\right]^2 \\
& + \frac{{}^{2}\tilde{\Phi}^2}{4(\widetilde{\Lambda} + \Lambda)}\left[dy^7 + (\partial_{\tau_1}\,{}^{2}n)du^{\tau_1}\right]^2 \\
& + \frac{(\partial_8\,{}^{2}\tilde{\Phi})^2}{(\widetilde{\Lambda} + \Lambda)\,{}^{2}\tilde{\Phi}^2}\left[dy^8 + (\partial_{\tau_1}\,{}^{2}\check{A})du^{\tau_1}\right]^2. \qquad (104)
\end{aligned}
$$

The extra-dimension components of this metric are generated by the functions ${}^{1}\tilde{\Phi}$, ${}^{2}\tilde{\Phi}$ and the N coefficients similarly to (102) but with modified effective sources in the extra dimensions, $\Lambda \to \widetilde{\Lambda} + \Lambda$. This result shows that extra dimensions can mimic the ε-deformations in order to compensate contributions from the f-modifications and even effective vacuum configurations of the 4-d horizontal part. In general, vacuum metrics (104) encode extra-dimension modifications/polarizations of the physical constants and coefficients of the metrics under nonlinear polarizations of an effective 8-d vacuum distinguishing the 4-d nonholonomic configurations and massive gravity contributions. Extra-dimensional and f-modified contributions are described by terms proportional to the eccentricity ε.

4.4.4 Extra-dimensional massive ellipsoid Kerr–de Sitter configurations

Combining the solutions (100) and (102), we construct a class of non-vacuum 8-d solutions with rotoid configurations if we constrain χ_3 in the ε-deformations (for 4-d, see the similar formula (95)) to be of the form

$$
\chi_3 = 2\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - (\widetilde{\Lambda} + \Lambda)/{}^{\mu}\tilde{\Lambda} = 2\underline{\zeta}\sin(\omega_0\varphi + \varphi_0).
$$

We reexpress ${}^{1}\tilde{\Phi} = e^{\varpi}[(\widetilde{\Lambda} + \Lambda)/2\,{}^{\mu}\tilde{\Lambda} + \underline{\zeta}\sin(\omega_0\varphi + \varphi_0)]$, for $\tilde{\Phi} = e^{\varpi}$ and (103), and generate a class of generic off-diagonal extra-dimensional metrics for ellipsoid Kerr–de Sitter configurations

$$
\begin{aligned}
ds^2 =\; & e^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] \\
& - \frac{e^{2\varpi}}{4|\,{}^{\mu}\tilde{\Lambda}|}\overline{A}[1 + 2\varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)][dy^{3'} \\
& + \left(\partial_{k'}\,{}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)dx^{k'}]^2 \\
& + \frac{(\varpi^*)^2}{{}^{\mu}\tilde{\Lambda}}(\overline{C} - \overline{B}^2/\overline{A})\left[1 + \varepsilon\left(\partial_4\varpi\frac{\widetilde{\Lambda} + \Lambda}{{}^{\mu}\tilde{\Lambda}}\right.\right. \\
& \left.\left. + 2\partial_4\varpi\,\underline{\zeta}\sin(\omega_0\varphi + \varphi_0) + 2\omega_0\,\underline{\zeta}\cos(\omega_0\varphi + \varphi_0)\right)\right] \\
& \times[d\varphi + (\partial_{i'}\,\widetilde{A} + \varepsilon\partial_{i'}\,{}^{1}\check{A})dx^{i'}]^2 \\
& + \frac{{}^{1}\tilde{\Phi}^2}{4(\widetilde{\Lambda} + \Lambda)}\left[dy^5 + (\partial_\tau\,{}^{1}n)du^\tau\right]^2 \\
& + \frac{(\partial_6\,{}^{1}\tilde{\Phi})^2}{(\widetilde{\Lambda} + \Lambda)\,{}^{1}\tilde{\Phi}^2}\left[dy^6 + (\partial_\tau\,{}^{1}\check{A})du^\tau\right]^2 \\
& + \frac{{}^{2}\tilde{\Phi}^2}{4(\widetilde{\Lambda} + \Lambda)}\left[dy^7 + (\partial_{\tau_1}\,{}^{2}n)du^{\tau_1}\right]^2 \\
& + \frac{(\partial_8\,{}^{2}\tilde{\Phi})^2}{(\widetilde{\Lambda} + \Lambda)\,{}^{2}\tilde{\Phi}^2}\left[dy^8 + (\partial_{\tau_1}\,{}^{2}\check{A})du^{\tau_1}\right]^2.
\end{aligned}
$$

Such non-vacuum solutions can also be modeled for Einstein–Finsler spaces if the extra-dimension coordinates are treated as velocity/momentum ones. The metrics possess a respective Killing symmetry in $\partial/\partial y^7$. They define ε-deformations of Kerr–de Sitter black holes into ellipsoid configurations with effective cosmological constants determined by the constants in massive gravity, f-modifications and extra-dimensional contributions.

5 Concluding remarks

In this work, we have elaborated the anholonomic frame deformation method, AFDM, in constructing exact solutions in gravity theories, which we formulated and developed in

[5–9]; see also the references therein. The method is based on a general decoupling property of the gravitational field equations, which is possible for certain classes of nonholonomic $2+2+\ldots$ splittings of the spacetime dimensions. Such solutions are generic off-diagonal, with zero or nonzero torsion structure, and they may depend on all (higher dimensions, or 4-d) spacetime coordinates. In the simplest form, the constructions can be performed by using an "auxiliary" metric-compatible connection which is constructed along with the "standard" Levi–Civita connection and from the same metric structure. Both connections are related via a distortion tensor which is completely determined by the coefficients of the metric and the frame splitting. After a class of off-diagonal solutions are constructed in general, we can impose certain conditions on the structure of the nonholonomic frames, when the coefficients of both the auxiliary and the standard connections are the same, and we can extract solutions with zero torsion, for instance, in general relativity theory.

In general form, the off-diagonal metrics and nonlinear and linear connections constructed following the AFDM method depend on various classes of generating and integration functions, certain symmetry parameters, and on possible nonzero sources and/or (polarized) cosmological constants. This is possible because in our approach the (generalized/modified) Einstein equations are transformed (after choosing the corresponding ansatz for the metrics) into systems of nonlinear partial differential equations which can be integrated in a very general form and depending on certain classes of generating/integration functions. This is different from the case of a diagonal ansatz, for instance, for the Schwarzschild metric when the gravitational field equations transform into a system of nonlinear ordinary differential equations depending on certain integration constants. We can construct chains of nonholonomic frame deformations in order to transform a given primary metric (it may be an exact solution, or not, in a gravity theory) into other classes of target metrics and which can be fixed to be exact solutions in a "metric compatible" gravity theory. From a formal point of view, the chains' metrics can correspond to spaces with nontrivial topology, have a singular/stochastic/evolution etc. behavior and various types of horizons, symmetries, and boundary conditions. In general, it is not possible to formulate some uniqueness property or limiting/asymptotic conditions. Certain geometric data and physical information of "intermediary" metrics are encoded, step by step, into the target metrics. We can impose certain nonholonomic constraints on such integral varieties in order to relate a new class of target metric solutions to some well-defined primary metrics. However, it is not clear what physical importance these "very general" classes of target metric exact solutions may have.

In a series of works [10–13] (see details and references in [9]) we studied various examples. When using the AFDM we can construct locally anisotropic black ellipsoid/hole, spinning and/or solitonic spaces etc. Certain configurations seem to be stable [6] and maintain, for instance, the main properties of the Schwarzschild metric but for small rotoid deformations.

The goal of this article was fourfold:

1. to elaborate the AFDM in a form which allows us to construct generic off-diagonal solutions with Killing symmetries and the generalizations to non-Killing configurations using extensions to higher dimensions and so-called "vertical" conformal factors;
2. to study off-diagonal modifications of the Kerr metric under massive gravity and f-modified nonlinear interactions, via higher dimensions, and state the conditions when such configurations can be modeled as effective ones in general gravity, or via nonholonomically induced torsion fields etc.;
3. to show how the well-known and physically important exact solution for the Kerr black hole can be constructed, for some special class-types of integration functions, following the AFDM, and
4. to provide certain examples when the solutions in point 2 can be generalized to various vacuum and non-vacuum configurations with ellipsoidal symmetries.

In some cases of rotoidal deformations with small eccentricity parameter, we have been able to prove that the physical properties of the primeval metrics are preserved but with certain effective polarizations of the physical constants and deformation to ellipsoidal configurations. It is possible to construct exact solutions for very general off-diagonal deformations (not depending of small parameters) but the physical properties are not clear if, for instance, additional smooth, symmetry, Cauchy, and/or boundary conditions are not imposed. A very important property is that off-diagonal nonlinear gravitational interactions can mimic effective modified gravity theories, with anisotropies and rescalings, which can find applications in modern cosmology and/or elaborate new models of quantum gravity [13,14,26].

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Appendix A: The conditions for zero torsion

We can consider frame transformations to the N coefficients and the ansatz (35) when all coefficients of a nonholonomically induced torsion (15) are zero and ${}_{\shortmid}\Gamma^{\gamma_s}_{\ \alpha_s\beta_s} = \widehat{\Gamma}^{\gamma_s}_{\ \alpha_s\beta_s}$. For simplicity, we analyze such conditions for the shell $s = 0$, i.e. for 4-d spacetimes.

In N adapted frames, the coefficients of the d-torsion (15) are $\widehat{T}^i_{\ jk} = \widehat{L}^i_{\ jk} - \widehat{L}^i_{\ kj} = 0$, $\widehat{T}^i_{\ ja} = \widehat{C}^i_{\ jb} = 0$, $\widehat{T}^a_{\ bc} = \widehat{C}^a_{\ bc} - \widehat{C}^a_{\ cb} = 0$ for any ansatz (35). The nontrivial coefficients are $\widehat{T}^c_{\ aj} = \widehat{L}^c_{\ aj} - e_a(N^c_j)$ and $\widehat{T}^a_{\ ji} = -\Omega^a_{\ ji}$. The values

$$\widehat{L}^a_{\ bi} = \partial_b N^a_i + \frac{1}{2}h^{ac}(\partial_i h_{bc} - N^e_i \partial_e h_{bc} - h_{dc}\partial_b N^d_i - h_{db}\partial_c N^d_i),$$

$$\widehat{T}^c_{\ aj} = \frac{1}{2}h^{ac}(\partial_i h_{bc} - N^e_i \partial_e h_{bc} - h_{dc}\partial_b N^d_i - h_{db}\partial_c N^d_i)$$

are computed for $N^3_i = n_i(x^k, y^4)$, $N^4_i = w_i(x^k, y^4)$; $h_{bc} = diag[h_3(x^k, y^4), h_4(x^k, y^4)]$; $h^{ac} = diag[(h_3)^{-1}, (h_4)^{-1}]$. We have

$$\widehat{T}^3_{bi} = \frac{1}{2}h^{3c}(\partial_i h_{bc} - N^e_i \partial_e h_{bc} - h_{dc}\partial_b N^d_i - h_{db}\partial_c N^d_i)$$
$$= \frac{1}{2h_3}(\partial_i h_{b3} - w_i \partial_4 h_{b3} - h_3 \partial_b n_i),$$

i.e. $\widehat{T}^3_{3i} = \frac{1}{2h_3}(\partial_i h_3 - w_i \partial_4 h_3)$, $\widehat{T}^3_{4i} = \frac{1}{2}\partial_4 n_i$.

Similarly, we get

$$\widehat{T}^4_{bi} = \frac{1}{2}h^{4c}(\partial_i h_{bc} - N^e_i \partial_e h_{bc} - h_{dc}\partial_b N^d_i - h_{db}\partial_c N^d_i)$$
$$= \frac{1}{2h_4}(\partial_i h_{b4} - w_i \partial_4 h_{b4} - h_4 \partial_b w_i - h_{3b}\partial_4 n_i - h_{4b}\partial_4 w_i),$$

i.e. $\widehat{T}^4_{3i} = -\frac{h_3}{2h_4}\partial_4 n_i$, $\widehat{T}^4_{4i} = \frac{1}{2h_4}(\partial_i h_4 - w_i \partial_4 h_4) - \partial_4 w_i$.

The coefficients $\Omega^a_{ij} = \mathbf{e}_j\left(N^a_i\right) - \mathbf{e}_i(N^a_j)$ are computed thus:

$$\Omega^a_{ij} = \partial_j\left(N^a_i\right) - \partial_i(N^a_j) - N^b_j \partial_b N^a_i + N^b_i \partial_b N^a_j$$
$$= \partial_j\left(N^a_i\right) - \partial_i(N^a_j) - w_j \partial_4 N^a_i + w_i \partial_4 N^a_j.$$

We obtain such nontrivial values as

$$\Omega^3_{12} = -\Omega^3_{21} = \partial_2 n_1 - \partial_1 n_2 - w_2 \partial_4 n_1 + w_1 \partial_4 n_2,$$
$$\Omega^4_{12} = -\Omega^4_{21} = \partial_2 w_1 - \partial_1 w_2 - w_2 \partial_4 w_1 + w_1 \partial_4 w_2. \quad (6.1)$$

Summarizing the above formulas for $\partial_4 n_i = 0$ and $\partial_2 n_1 - \partial_1 n_2 = 0$, we get the condition of zero torsion for the ansatz in (35) with $n_k = \partial_k n(x^i)$,

$$\frac{1}{2h_3}(\partial_i h_3 - w_i \partial_4 h_3) = 0, \quad (6.2)$$

$$\frac{1}{2h_4}(\partial_i h_4 - w_i \partial_4 h_4) = \partial_4 w_i, \quad (6.3)$$

$$\partial_2 w_1 - \partial_1 w_2 - w_2 \partial_4 w_1 + w_1 \partial_4 w_2 = 0. \quad (6.4)$$

In this form we can define a LC-configuration. The final step is to impose the condition that the coefficients n_k do not depend on y^4. This can be fixed in the form ${}_1n_k(x^i) = \partial_k n(x^i)$ and ${}_2n_k = 0$, i.e. $n_k = \partial_k n(x^i)$.

Finally, we note that the LC-conditions can be formulated recurrently, in similar forms, for higher order shells both for zero and nonzero sources.

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