

The Higgs boson inclusive decay channels $H \rightarrow b\bar{b}$ and $H \rightarrow gg$ up to four-loop level

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Abstract The principle of maximum conformality (PMC) has been suggested to eliminate the renormalization scheme and renormalization scale uncertainties, which are unavoidable for the conventional scale setting and are usually important errors for theoretical estimations. In this paper, by applying PMC scale setting, we analyze two important inclusive Standard Model Higgs decay channels, $H \rightarrow b\bar{b}$ and $H \rightarrow gg$, up to four-loop and three-loop levels, respectively. After PMC scale setting, it is found that the conventional scale uncertainty for these two channels can be eliminated to a high degree. There is small residual initial scale dependence for the Higgs decay widths due to unknown higher-order $\{\beta_i\}$ terms. Up to four-loop level, we obtain $\Gamma(H \rightarrow b\bar{b}) = 2.389 \pm 0.073 \pm 0.041$ MeV and up to three-loop level, we obtain $\Gamma(H \rightarrow gg) = 0.373 \pm 0.030$ MeV, where the first error is caused by varying $M_H = 126 \pm 4$ GeV and the second error for $H \rightarrow b\bar{b}$ is caused by varying the $\overline{\text{MS}}$ -running mass $m_b(m_b) = 4.18 \pm 0.03$ GeV. Taking $H \rightarrow b\bar{b}$ as an example, we present a comparison of three BLM-based scale-setting approaches, e.g. the PMC-I approach based on the PMC–BLM correspondence, the R_δ -scheme and the seBLM approach, all of which are designed to provide effective ways to identify non-conformal $\{\beta_i\}$ -series at each perturbative order. At four-loop level, all those approaches lead to good pQCD convergence, they have almost the same pQCD series, and their predictions are almost independent on the initial renormalization scale. In this sense, those approaches are equivalent to each other.

1 Introduction

In 2012 a new boson has been discovered by CMS and ATLAS experiments at the large hadron collider (LHC) [1, 2], whose properties are remarkably similar to the standard

model (SM) Higgs [3–9]. For example, its mass is determined to be $125.5 \pm 0.2^{+0.5}_{-0.6}$ GeV by the ATLAS collaboration [9] or $125.7 \pm 0.3 \pm 0.3$ GeV by the CMS collaboration [4], where the first error stands for the statistic error and the second one stands for the systematic error.

If the SM Higgs has a mass around 126 GeV, then its decay width shall be dominated by $H \rightarrow b\bar{b}$ [10, 11]. Theoretically, many efforts have been made in studying the Higgs decays into a bottom pair [12–25]. As a reference, the pure QCD corrections at the two-loop or the three-loop level have been reported in Refs. [26–28], and the explicit expressions up to $\mathcal{O}(\alpha_s^5)$ have been given by Ref. [29]. In addition, the Higgs decay channel $H \rightarrow gg$ also plays a crucial role in studying the properties of Higgs boson. The coupling of Higgs to a pair of gluons, which is mediated at one loop by virtual massive quarks (essentially generated by the top quark alone), becomes independent of the top-quark mass m_t in the limit $M_H \ll 2m_t$. An important feature of $H \rightarrow gg$ is its affinity to the gluon–gluon fusion mechanism for Higgs production. The effective coupling ggH eventually may provide a way to count the number of heavy quarks beyond SM [30]. Its next-to-leading order (NLO) QCD corrections are quite large and amount to about 70 % in comparison to the leading order contribution [31–36]. Later on, the QCD corrections for $H \rightarrow gg$ up to three-loop level have been calculated in the limit of an infinitely heavy top-quark mass [37–40]. Those great improvements on loop calculations provide us with opportunities for deriving more accurate estimations of the Higgs properties.

The physical predictions of the theory, calculated up to all orders, are surely independent of the choice of renormalization scale and renormalization scheme due to the renormalization group invariance [41–45]. As is well known, there are renormalization scale and renormalization scheme ambiguities at any finite order. It is helpful to find an optimal scale setting so that there is no (or greatly suppressed) scale or scheme ambiguity at any fixed order and one can achieve the

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most accurate estimation based on the known perturbative calculations.

As an estimation of the physical observable, one can first take an arbitrary initial renormalization scale $\mu_r = \mu_r^{\text{init}}$ and apply some scale-setting method to improve the pQCD estimation. For the conventional scale setting, once the renormalization scale is set to an initial value, it will always be fixed during the whole analysis. That is, for the processes involving Higgs, in the literature, one usually take $\mu_r \equiv \mu_r^{\text{init}} = M_H$ as the central value, which eliminates the large logs in a form as $\ln(\mu_r/M_H)$. Because there is no strong reasons for such a choice, as a compensation, one will vary the scale within a certain region, e.g. $\mu_r^{\text{init}} \in [M_H/2, 2M_H]$, to ascertain the scale uncertainty. It is often argued that by setting and varying the scale in such a way, one can estimate contributions from higher-order terms; i.e. changing in scale will affect how much of a result comes from Feynman diagrams without loops, and how much it comes from the leftover finite parts of the loop diagrams. Because of its perturbative nature, it is a common belief that those scale uncertainties can be reduced after finishing a higher-and-higher-order calculation. However, this ad hoc assignment of scale and its range usually constitutes an important systematic error in the theoretical and experimental analysis. More explicitly, the conventional scale setting cannot answer the questions: why is it M_H and not $M_H/2$ or others that provides the central estimation? If there are several typical energy scales for the process, then which one provides the central value?

Several scale-setting methods have been suggested, e.g. the renormalization group improved effective coupling method (FAC) [46–49], the principle of minimum sensitivity (PMS) [50–52], the Brodsky–Lepage–Mackenzie method (BLM) [53] and its underlying principle of maximum conformality (PMC) [54–61]. The FAC is to improve the perturbative series by requiring all higher-order terms vanish and the PMS is to force the fixed-order series to satisfy the renormalization group invariance at the renormalization point. The BLM improves the perturbative series by requiring the n_f terms at each perturbative order vanish. The PMC provides the principle underlying the BLM, and it suggests a principle to set the optimal renormalization scales up to all orders, they are equivalent to each other through the PMC–BLM correspondence principle [55]. Those methods, being designed to eliminate the scale ambiguity, have quite different consequences and may or may not achieve their goals. A detailed introduction and comparison of these methods can be found in a recent review [61]. In the present paper, we adopt PMC for analyzing the Higgs decays.

The main idea of PMC lies in that the PMC scales at each perturbative order are formed by absorbing all non-conformal terms that govern the running behavior of the coupling constant into the coupling constant. At each perturbative order, new types of $\{\beta_i\}$ terms will occur, so the PMC scale for

each perturbative order is generally different. Even though one can choose any value to be μ_r^{init} , the optimal PMC scales and the resulting finite-order PMC prediction are both to high accuracy independent of such arbitrariness. After PMC scale setting, the divergent renormalon series does not appear and the convergence of the pQCD series can be greatly improved. Because of these advantages, the PMC method can be widely applied to high-energy physics processes, some examples of which can be found in Refs. [55–58, 60, 62, 63].

Because the PMC provides the underlying principle for BLM, the previous features or properties derived by using BLM can also be understood by using PMC. Before applying PMC or BLM to high-energy processes, one needs to use the expression with full initial renormalization scale dependence. That is, those terms that have been eliminated by setting the renormalization scale to be equal to the factorization scale or by setting the initial renormalization to be the typical momentum transfer should be retrieved. So, the previous scale dependence analysis or conclusions drawn under the BLM should be adopted with great care, since there is misuse of BLM in the literature. It is interesting to show whether the PMC can work well for the inclusive Higgs decays and whether the accuracy of the estimations can be improved. In the present paper, we show the newly suggested PMC procedure, the so-called R_δ -scheme [60, 64], with much more detail. A comparison with some other suggestions to extend the BLM scale setting up to any perturbative orders shall also be presented.

The remaining parts of the paper are organized as follows. In Sect. 2, we present the calculation technology for applying PMC to Higgs decay processes $H \rightarrow b\bar{b}$ and $H \rightarrow g\bar{g}$ up to four-loop level. In Sect. 3, we present the numerical results and discussions. The final section is reserved for a summary.

2 Calculation technology

In this section, we present an improved analysis for the Higgs decay channels $H \rightarrow b\bar{b}$ and $H \rightarrow g\bar{g}$ by using the PMC R_δ -scheme. For this purpose, we do the following.

- We shall first rearrange the four-loop expressions [29, 39] that have been derived under the conventional scale setting in a more general form. That is, the n_f terms in those expressions that are coming from the light-quark loops and are responsible for controlling the running behavior of the coupling constant shall be transformed into $\{\beta_i\}$ -series. Those $\{\beta_i\}$ -series via the renormalization group equation rightly control the running behavior of the strong coupling constant. Every process has its own $\{\beta_i\}$ -series and its own optimal (PMC) scales. Thus, after absorbing all those $\{\beta_i\}$ -series into the strong coupling constant via an order-by-order way, we can obtain

the optimal running coupling constant for the specific process.

- As stated in the Introduction, before applying the PMC scale setting, we need to obtain the expressions with full (initial) renormalization scale dependence. For the purpose, at present, we need to transform the four-loop results of $H \rightarrow b\bar{b}$ or the three-loop results of $H \rightarrow gg$ derived under the conventional assumption of $\mu_r^{\text{init}} \equiv M_H$ to a more general form that explicitly contains the initial renormalization scale μ_r^{init} , which may or may not equal to M_H . This can be achieved by using the strong coupling constant's scale transformation equation up to four-loop level, i.e.

$$\begin{aligned} a_s(Q^*) &= a_s(Q) - \beta_0 \ln\left(\frac{Q^{*2}}{Q^2}\right) a_s^2(Q) \\ &+ \left[\beta_0^2 \ln^2\left(\frac{Q^{*2}}{Q^2}\right) - \beta_1 \ln\left(\frac{Q^{*2}}{Q^2}\right)\right] a_s^3(Q) \\ &+ \left[-\beta_0^3 \ln^3\left(\frac{Q^{*2}}{Q^2}\right) + \frac{5}{2}\beta_0\beta_1 \ln^2\left(\frac{Q^{*2}}{Q^2}\right) \right. \\ &\quad \left. - \beta_2 \ln\left(\frac{Q^{*2}}{Q^2}\right)\right] a_s^4(Q) + \mathcal{O}(a_s^5), \end{aligned} \quad (1)$$

where $a_s = \alpha_s/4\pi$, and Q^* and Q are two arbitrary renormalization scales.

- We shall set the PMC scales in an order-by-order manner according to R_δ -scheme. By doing the loop calculations, the b -quark mass is treated as massless, cf. the review on Higgs properties [11], there is only an overall m_b^2 -factor in the decay width. Since its value is irrelevant to the PMC procedures and should be kept separate during the PMC scale setting, either the choice of $\overline{\text{MS}}$ -running mass or the pole mass is reasonable. In the formulas, we fix its value to be $m_b(M_H)$ within the $\overline{\text{MS}}$ -scheme. A detailed discussion of this point is in preparation, which shows, by applying PMC properly, that either the choice of pole mass or $\overline{\text{MS}}$ -running mass can lead to a consistent estimation. Recently, a discussion of $H \rightarrow \gamma\gamma$ has presented such an example [65].

In the following, we subsequently present the results before and after PMC scale setting for the two channels $H \rightarrow b\bar{b}$ and $H \rightarrow gg$.

2.1 The general form for $H \rightarrow b\bar{b}$ under the conventional scale setting

By taking the initial renormalization scale $\mu_r^{\text{init}} = M_H$, the analytic decay width with explicit n_f dependence for the channel $H \rightarrow b\bar{b}$ can be formally written as

$$\begin{aligned} \Gamma(H \rightarrow b\bar{b}) &= \frac{3G_F M_H m_b^2(M_H)}{4\sqrt{2}\pi} [1 + c_{1,0} a_s(M_H) \\ &+ (c_{2,0} + c_{2,1} n_f) a_s^2(M_H) + (c_{3,0} + c_{3,1} n_f \\ &+ c_{3,2} n_f^2) a_s^3(M_H) + (c_{4,0} + c_{4,1} n_f \\ &+ c_{4,2} n_f^2 + c_{4,3} n_f^3) a_s^4(M_H) + \mathcal{O}(a_s^5)] \quad (2) \\ &= \frac{3G_F M_H m_b^2(M_H)}{4\sqrt{2}\pi} [1 + 1.804 \alpha_s(M_H) \\ &+ 2.953 \alpha_s^2(M_H) + 1.347 \alpha_s^3(M_H) \\ &- 8.475 \alpha_s^4(M_H) + \mathcal{O}(\alpha_s^5)] \quad (3) \end{aligned}$$

where G_F is the Fermi constant and $m_b(M_H)$ is the $\overline{\text{MS}}$ running mass at the scale M_H . For convenience, in the third line, we present the values for the coefficients over the α_s -expansion by setting $n_f = 5$, which explicitly show the relative importance of the perturbative series. The coefficients $c_{i,j}$ up to four-loop levels are [29]

$$\begin{aligned} c_{1,0} &= 22.667, \quad c_{2,0} = 575.04, \quad c_{2,1} = -21.738, \\ c_{3,0} &= 10,504.9, \quad c_{3,1} = -1,649.4, \quad c_{3,2} = 16.574, \\ c_{4,0} &= 10,071, \quad c_{4,1} = -56,550, \quad c_{4,2} = 2,479.4, \\ c_{4,3} &= -5.248. \end{aligned} \quad (4)$$

The R_δ -scheme [60,64] not only illuminates the $\{\beta_i\}$ -pattern of the process but also exposes a special degeneracy of the coefficients at different perturbative orders. Such degeneracy is necessary, which, similar to the PMC-BLM correspondence principle [55], ensures the one-to-one correspondence between the n_f -series and the $\{\beta_i\}$ -series at each perturbative order.

Applying the R_δ -scheme [60,64] to Eq. (2), one can derive the general form of $H \rightarrow b\bar{b}$ for $\mu_r^{\text{init}} \neq M_H$ with the help of Eq. (1), which can be written as

$$\begin{aligned} \Gamma(H \rightarrow b\bar{b}) &= \frac{3G_F M_H m_b^2(M_H)}{4\sqrt{2}\pi} \{1 + r_{1,0}(\mu_r^{\text{init}}) a_s(\mu_r^{\text{init}}) \\ &+ [r_{2,0}(\mu_r^{\text{init}}) + \beta_0 r_{2,1}(\mu_r^{\text{init}})] a_s^2(\mu_r^{\text{init}}) \\ &+ [r_{3,0}(\mu_r^{\text{init}}) + \beta_1 r_{2,1}(\mu_r^{\text{init}}) + 2\beta_0 r_{3,1}(\mu_r^{\text{init}}) \\ &+ \beta_0^2 r_{3,2}(\mu_r^{\text{init}})] a_s^3(\mu_r^{\text{init}}) + [r_{4,0}(\mu_r^{\text{init}}) \\ &+ \beta_2 r_{2,1}(\mu_r^{\text{init}}) + 2\beta_1 r_{3,1}(\mu_r^{\text{init}}) \\ &+ \frac{5}{2}\beta_1 \beta_0 r_{3,2}(\mu_r^{\text{init}}) + 3\beta_0 r_{4,1}(\mu_r^{\text{init}}) \\ &+ 3\beta_0^2 r_{4,2}(\mu_r^{\text{init}}) + \beta_0^3 r_{4,3}(\mu_r^{\text{init}})] \\ &\times a_s^4(\mu_r^{\text{init}}) + \mathcal{O}(a_s^5)\}. \end{aligned} \quad (5)$$

Here, for later convenience, we have transformed the n_f -series into the required $\{\beta_i\}$ -series. Explicit expressions for β_0, β_1 , and β_2 in the n_f -series can be found in Refs. [66–68]. The $r_{i,0}$ with $i = (1, \dots, 4)$ are conformal coefficients, and

the $r_{i,j}$ with $1 \leq j < i \leq 4$ are non-conformal coefficients that should be absorbed into the coupling constant. Those coefficients can be obtained by comparing Eq. (2) with Eq. (5), and they are

$$r_{1,0}(\mu_r^{\text{init}}) = c_{1,0}, \quad (6)$$

$$r_{2,0}(\mu_r^{\text{init}}) = c_{2,0} + \frac{33}{2}c_{2,1}, \quad (7)$$

$$r_{2,1}(\mu_r^{\text{init}}) = -\frac{3}{2}c_{2,1} - c_{1,0} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2}, \quad (8)$$

$$r_{3,0}(\mu_r^{\text{init}}) = \frac{1}{4}(-642c_{2,1} + 4c_{3,0} + 66c_{3,1} + 1,089c_{3,2}), \quad (9)$$

$$r_{3,1}(\mu_r^{\text{init}}) = \frac{1}{16} \left(228c_{2,1} - 12c_{3,1} - 396c_{3,2} - 16c_{2,0} \right. \\ \left. \times \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} - 264c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right), \quad (10)$$

$$r_{3,2}(\mu_r^{\text{init}}) = \frac{9}{4}c_{3,2} + c_{1,0} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} + 3c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2}, \quad (11)$$

$$r_{4,0}(\mu_r^{\text{init}}) = \frac{1}{16} (11,675c_{2,1} - 2,568c_{3,1} \\ - 84,744c_{3,2} + 16c_{4,0} + 264c_{4,1} \\ + 4,356c_{4,2} + 71,874c_{4,3}), \quad (12)$$

$$r_{4,1}(\mu_r^{\text{init}}) = \frac{1}{16} \left[-1,916c_{2,1} + 152c_{3,1} + 8,226c_{3,2} \right. \\ \left. - 8c_{4,1} - 264c_{4,2} - 6,534c_{4,3} \right. \\ \left. + 2,568c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} - 16c_{3,0} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right. \\ \left. - 264c_{3,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} - 4,356c_{3,2} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right], \quad (13)$$

$$-456c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} + 24c_{3,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \\ + 792c_{3,2} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \Big), \quad (14)$$

$$r_{4,3}(\mu_r^{\text{init}}) = \frac{1}{6} \left(-\frac{81}{4}c_{4,3} - 6c_{1,0} \ln^3 \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right. \\ \left. - 27c_{2,1} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} - \frac{81}{2}c_{3,2} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right). \quad (15)$$

It is noted that, as expected, the conformal coefficients $r_{i,0}$ are independent of μ_r^{init} , and we can omit the variable μ_r^{init} in $r_{i,0}$.

2.2 Results for $H \rightarrow b\bar{b}$ after PMC scale setting

Following the standard procedure of the R_δ -scheme, by absorbing all non-conformal $\{\beta_i\}$ -series that control the running behavior of the coupling constant into the coupling constant, we can reduce Eq. (5) to the following conformal series:

$$\Gamma(H \rightarrow b\bar{b}) = \frac{3G_F M_H m_b^2(M_H)}{4\sqrt{2}\pi} [1 + r_{1,0}a_s(Q_1) \\ + r_{2,0}a_s^2(Q_2) + r_{3,0}a_s^3(Q_3) + r_{4,0}a_s^4(Q_4)] \\ = \frac{3G_F M_H m_b^2(M_H)}{4\sqrt{2}\pi} [1 + 1.804\alpha_s(Q_1) \\ + 1.370\alpha_s^2(Q_2) - 4.389\alpha_s^3(Q_3) \\ - 4.430\alpha_s^4(Q_4)], \quad (16)$$

where in the second line we present the values for the conformal coefficients over the α_s expansion, which explicitly show the relative importance of the perturbative series. Here Q_i with $i = (1, \dots, 4)$ are PMC scales, which can be obtained through the following formulas:

$$Q_1 = \mu_r^{\text{init}} \exp \left\{ \frac{1}{2} \frac{-r_{2,1}(\mu_r^{\text{init}}) + \frac{r_{3,2}(\mu_r^{\text{init}})}{2} \frac{\partial \beta}{\partial a_s} - \frac{r_{4,3}(\mu_r^{\text{init}})}{3!} \left[\beta \frac{\partial^2 \beta}{\partial a_s^2} + \left(\frac{\partial \beta}{\partial a_s} \right)^2 \right]}{r_{1,0}(\mu_r^{\text{init}}) - \frac{r_{2,1}(\mu_r^{\text{init}})}{2} \left(\frac{\partial \beta}{\partial a_s} \right) + \frac{r_{3,2}(\mu_r^{\text{init}})}{4} \left(\frac{\partial \beta}{\partial a_s} \right)^2 + \frac{1}{3!} \left[\beta \frac{\partial^2 \beta}{\partial a_s^2} - \frac{1}{2} \left(\frac{\partial \beta}{\partial a_s} \right)^2 \right] \frac{r_{2,1}^2(\mu_r^{\text{init}})}{r_{1,0}(\mu_r^{\text{init}})}} \right\}, \quad (17)$$

$$r_{4,2}(\mu_r^{\text{init}}) = \frac{1}{16} \left(\frac{325}{3}c_{2,1} - 570c_{3,2} + 12c_{4,2} + 594c_{4,3} \right. \\ \left. + 16c_{2,0} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} + 264c_{2,1} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right.$$

$$Q_2 = \mu_r^{\text{init}} \exp \left\{ \frac{1}{2} \frac{-r_{3,1}(\mu_r^{\text{init}}) + \frac{r_{4,2}(\mu_r^{\text{init}})}{2} \left[\frac{\partial \beta}{\partial a_s} + \frac{\beta}{a_s} \right]}{r_{2,0}(\mu_r^{\text{init}}) - \frac{r_{3,1}(\mu_r^{\text{init}})}{2} \left[\frac{\partial \beta}{\partial a_s} + \frac{\beta}{a_s} \right]} \right\}, \quad (18)$$

$$Q_3 = \mu_r^{\text{init}} \exp \left\{ \frac{1}{2} \frac{-r_{4,1}(\mu_r^{\text{init}})}{r_{3,0}(\mu_r^{\text{init}})} \right\}, \quad (19)$$

where $\beta = -a_s^2 \sum_{i=0}^{\infty} \beta_i a_s^i$ being the conventional QCD renormalization group β -function. We note that the last scale leaves some ambiguity in the PMC scale setting, since there is no $\{\beta_i\}$ terms that can determine its optimal value, we set $Q_4 = \mu_r^{\text{init}}$. Because the PMC scales (Q_1 , Q_2 , and Q_3) themselves are in perturbative series, the residual scale dependence due to unknown higher-order $\{\beta_i\}$ terms shall be highly suppressed.

2.3 The decay width of the $H \rightarrow gg$

By taking the initial renormalization scale $\mu_r^{\text{init}} = M_H$, the analytic expression for the decay width of $H \rightarrow gg$ can be written as

$$\Gamma(H \rightarrow gg) = \frac{4G_F M_H^3}{9\sqrt{2}\pi} [c_{1,0} a_s^2(M_H) + (c_{2,0} + c_{2,1} n_f) a_s^3(M_H) + (c_{3,0} + c_{3,1} n_f + c_{3,2} n_f^2) a_s^4(M_H) + (c_{4,0} + c_{4,1} n_f + c_{4,2} n_f^2 + c_{4,3} n_f^3) \times a_s^5(M_H) + \mathcal{O}(a_s^6)] \quad (20)$$

$$= \frac{4G_F M_H^3}{9\sqrt{2}\pi} \times 10^{-3} [6.333 \alpha_s^2(M_H) + 36.114 \alpha_s^3(M_H) - 98.267 \alpha_s^4(M_H) + 80.443 \alpha_s^5(M_H) + \mathcal{O}(\alpha_s^6)]. \quad (21)$$

For convenience, in the third line we present the values for the coefficients over the α_s -expansion by setting $n_f = 5$, which explicitly show the relative importance of the perturbative series. The coefficients $c_{i,j}$ are [39]

$$\begin{aligned} c_{1,0} &= 1; \quad c_{2,0} = 95.0, \quad c_{2,1} = -4.667; \\ c_{3,0} &= 5898.8, \quad c_{3,1} = -761.81, \quad c_{3,2} = 14.428; \\ c_{4,0} &= 287583, \quad c_{4,1} = -68580, \quad c_{4,2} = 3393.1, \\ c_{4,3} &= -34.42. \end{aligned} \quad (22)$$

From Eq. (20), the general form for the Higgs decay process $H \rightarrow gg$ with $\mu_r^{\text{init}} \neq M_H$ can be written in the following form:

$$\begin{aligned} \Gamma(H \rightarrow gg) &= \frac{4G_F M_H^3}{9\sqrt{2}\pi} \{ r_{1,0}(\mu_r^{\text{init}}) a_s^2(\mu_r^{\text{init}}) \\ &+ [r_{2,0}(\mu_r^{\text{init}}) + 2\beta_0 r_{2,1}(\mu_r^{\text{init}})] a_s^3(\mu_r^{\text{init}}) \\ &+ [r_{3,0}(\mu_r^{\text{init}}) + 2\beta_1 r_{2,1}(\mu_r^{\text{init}}) \\ &+ 3\beta_0 r_{3,1}(\mu_r^{\text{init}}) + 3\beta_0^2 r_{3,2}(\mu_r^{\text{init}})] \\ &\times a_s^4(\mu_r^{\text{init}}) + [r_{4,0}(\mu_r^{\text{init}}) + 2\beta_2 r_{2,1}(\mu_r^{\text{init}}) \\ &+ 3\beta_1 r_{3,1}(\mu_r^{\text{init}}) + 7\beta_1 \beta_0 r_{3,2}(\mu_r^{\text{init}}) \\ &+ 4\beta_0 r_{4,1}(\mu_r^{\text{init}}) + 6\beta_0^2 r_{4,2}(\mu_r^{\text{init}}) \\ &+ 4\beta_0^3 r_{4,3}(\mu_r^{\text{init}})] a_s^5(\mu_r^{\text{init}}) + \mathcal{O}(a_s^6) \}. \end{aligned} \quad (23)$$

Following the same procedures of R_δ -scheme, the conformal or non-conformal coefficients $r_{i,j}(\mu_r^{\text{init}})$ can be written as

$$r_{1,0}(\mu_r^{\text{init}}) = c_{1,0}. \quad (24)$$

$$r_{2,0}(\mu_r^{\text{init}}) = c_{2,0} + \frac{33}{2} c_{2,1}, \quad (25)$$

$$r_{2,1}(\mu_r^{\text{init}}) = -\frac{3}{4} c_{2,1} - c_{1,0} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2}. \quad (26)$$

$$r_{3,0}(\mu_r^{\text{init}}) = \frac{1}{4} (-642 c_{2,1} + 4 c_{3,0} + 66 c_{3,1} + 1,089 c_{3,2}), \quad (27)$$

$$\begin{aligned} r_{3,1}(\mu_r^{\text{init}}) &= \frac{1}{24} \left(228 c_{2,1} - 12 c_{3,1} - 396 c_{3,2} - 396 c_{2,1} \right. \\ &\times \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} - 24 c_{2,0} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \left. \right), \end{aligned} \quad (28)$$

$$r_{3,2}(\mu_r^{\text{init}}) = \frac{3}{4} c_{3,2} + c_{1,0} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} + \frac{3}{2} c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2}. \quad (29)$$

$$\begin{aligned} r_{4,0}(\mu_r^{\text{init}}) &= \frac{1}{8} (-12,721 c_{2,1} - 1,284 c_{3,1} \\ &- 42,372 c_{3,2} + 8 c_{4,0} + 132 c_{4,1} \\ &+ 2,178 c_{4,2} + 35,937 c_{4,3}), \end{aligned} \quad (30)$$

$$\begin{aligned} r_{4,1}(\mu_r^{\text{init}}) &= \frac{1}{32} \left(-1,416 c_{2,1} + 228 c_{3,1} \right. \\ &+ 12,018 c_{3,2} - 12 c_{4,1} - 396 c_{4,2} - 9,801 c_{4,3} \\ &+ 5,136 c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} - 32 c_{3,0} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \\ &- 528 c_{3,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} - 8,712 c_{3,2} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \left. \right), \end{aligned} \quad (31)$$

$$\begin{aligned} r_{4,2}(\mu_r^{\text{init}}) &= \frac{1}{48} \left(325 c_{2,1} - 798 c_{3,2} + 18 c_{4,2} + 891 c_{4,3} \right. \\ &+ 48 c_{2,0} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} + 792 c_{2,1} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} \\ &- 912 c_{2,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} + 48 c_{3,1} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \\ &+ 1,584 c_{3,2} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \left. \right), \end{aligned} \quad (32)$$

$$\begin{aligned} r_{4,3}(\mu_r^{\text{init}}) &= -\frac{1}{32} \left(27 c_{4,3} + 32 c_{1,0} \ln^3 \frac{M_H^2}{(\mu_r^{\text{init}})^2} \right. \\ &+ 72 c_{2,1} \ln^2 \frac{M_H^2}{(\mu_r^{\text{init}})^2} + 72 c_{3,2} \ln \frac{M_H^2}{(\mu_r^{\text{init}})^2} \left. \right). \end{aligned} \quad (33)$$

Table 1 Decay width for $H \rightarrow b\bar{b}$ up to four-loop level. For conventional scale setting, we set the renormalization scale $\mu_r \equiv M_H$. For PMC scale setting, we set the initial renormalization scale $\mu_r^{\text{init}} = M_H$. Here Γ_i stands for the decay width at each perturbative order with $i = \text{LO, NLO, etc.}$, Γ_{tot} stands for the total decay width. $M_H = 126 \text{ GeV}$

	Conventional scale setting						PMC scale setting					
	LO	NLO	N ² LO	N ³ LO	N ⁴ LO	Total	LO	NLO	N ² LO	N ³ LO	N ⁴ LO	Total
Γ_i (KeV)	1,924.28	391.74	72.38	3.73	−2.65	2,389.48	1,924.28	436.23	48.12	−18.12	−1.38	2,389.13
$\Gamma_i / \Gamma_{\text{tot}}$ (%)	80.53	16.39	3.03	0.16	−0.11		80.54	18.26	2.01	−0.76	−0.06	

It is noted that, as required, the conformal coefficients $r_{i,0}$ are independent of μ_r^{init} and we can omit the argument (μ_r^{init}) in $r_{i,0}$.

By absorbing all non-conformal $\{\beta_i\}$ -series that control the running behavior of the coupling constant into the coupling constant, we can reduce Eq. (23) into the following conformal series:

$$\Gamma(H \rightarrow gg) = \frac{4G_F M_H^3}{9\sqrt{2}\pi} [r_{1,0}a_s^2(Q_1) + r_{2,0}a_s^3(Q_2) + r_{3,0}a_s^4(Q_3) + r_{4,0}a_s^5(Q_4) + \mathcal{O}(a_s^6)] \quad (34)$$

$$= \frac{4G_F M_H^3}{9\sqrt{2}\pi} \times 10^{-3} [6.333 \alpha_s^2(Q_1) + 9.068 \alpha_s^3(Q_2) - 79.962 \alpha_s^4(Q_3) - 68.804 \alpha_s^5(Q_4) + \mathcal{O}(\alpha_s^6)], \quad (35)$$

where in the second line, we present the values for the conformal coefficients over the α_s expansion. Here $r_{i,0}(\mu_r^{\text{init}})$ are conformal coefficients. The PMC scales Q_i with $i = (1, 2, 3)$ can be obtained in the same way as that of Eqs. (17), (18), and (19), Q_4 is also the undetermined scale due to the unknown higher-order $\{\beta_i\}$ terms and we also fix it to be μ_r^{init} .

3 Numerical results and discussions

To do numerical calculation, we take $G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2}$, the Higgs mass $M_H = 126 \text{ GeV}$ and the top-quark mass $m_t = 173.5 \text{ GeV}$. We adopt the four-loop α_s running with its asymptotic scale determined by the fixed point $\alpha_s(M_Z) = 0.1184$ [69]: $\Lambda_{\text{QCD}}^{n_f=3} = 0.339 \text{ GeV}$, $\Lambda_{\text{QCD}}^{n_f=4} = 0.296 \text{ GeV}$ and $\Lambda_{\text{QCD}}^{n_f=5} = 0.213 \text{ GeV}$. The $\overline{\text{MS}}$ -running quark mass $m_b(m_b) = 4.18 \pm 0.03 \text{ GeV}$ [69], and by using the quark mass anomalous dimension expressions listed in Ref. [70, 71], we obtain $m_b(M_H) = 2.78 \pm 0.02 \text{ GeV}$.

3.1 Basic results for $H \rightarrow b\bar{b}$

The decay widths of $H \rightarrow b\bar{b}$ before and after PMC scale setting are presented in Table 1, where Γ_i stands for the decay width at each perturbative order with $i = \text{LO, NLO, etc.}$, and Γ_{tot} stands for the total decay width. We set the

renormalization scale $\mu_r \equiv \mu_r^{\text{init}} = M_H$ for the conventional scale setting, and we take the initial renormalization scale $\mu_r^{\text{init}} = M_H$ to initialize the PMC scale setting.

Table 1 shows that either before or after PMC scale setting about 80 % of the contribution comes from the LO terms, which is exact and free from the strong interactions. The total decay width for $H \rightarrow b\bar{b}$ remains almost unchanged, $\Gamma(H \rightarrow b\bar{b}) \simeq 2.39 \text{ MeV}$. This shows that the choice of $\mu_r \equiv M_H$ is a lucky guess for the conventional scale setting. Because of the elimination of the renormalon terms, one could expect a better pQCD convergence after PMC scale setting. This is clearly shown in Table 1, e.g. the four-loop terms only give $\sim 0.1 \%$ contributions to the total decay width.

For the $H \rightarrow b\bar{b}$ process at $\mathcal{O}(\alpha_s^4)$, we need to introduce four PMC scales, i.e. the LO PMC scale Q_1 , the NLO PMC scale Q_2 , the N²LO PMC scale Q_3 and the N³LO PMC scale Q_4 . As stated in the last section, since there is no β terms that can determine its optimal value, we set $Q_4 \equiv \mu_r^{\text{init}}$.¹ Using the formulas (17), (18), and (19), we show how the PMC scales depend on the initial renormalization scale, which are presented in Fig. 1. Figure 1 shows that the PMC scales $Q_{1,2,3}$ are highly independent on the choice of initial renormalization scale. This indicates that the PMC scale setting does provide a principle for setting the optimal (solitary) renormalization scale of high-energy processes. For example, setting $\mu_r^{\text{init}} = M_H$, we obtain

$$Q_1 = 62.3 \text{ GeV}, \quad Q_2 = 40.5 \text{ GeV}, \quad Q_3 = 53.1 \text{ GeV}. \quad (36)$$

These PMC scales are smaller than M_H to a certain degree due to the exponential suppressions from the absorption of higher-order $\{\beta_i\}$ terms. These PMC scales are different, which shows that they are controlled by different $\{\beta_i\}$ -series at each perturbative order.

As a further comparison, we show the total decay width $\Gamma(H \rightarrow b\bar{b})$ versus the initial renormalization scale μ_r^{init} before and after PMC scale setting in Figs. 2 and 3. In these two figures, the dash-dot, dashed, dotted, and solid lines are

¹ This corresponds to the second type of residual scale dependence after PMC scale setting [63], which, as is the present case, can also be highly suppressed when the pQCD convergence is under well control

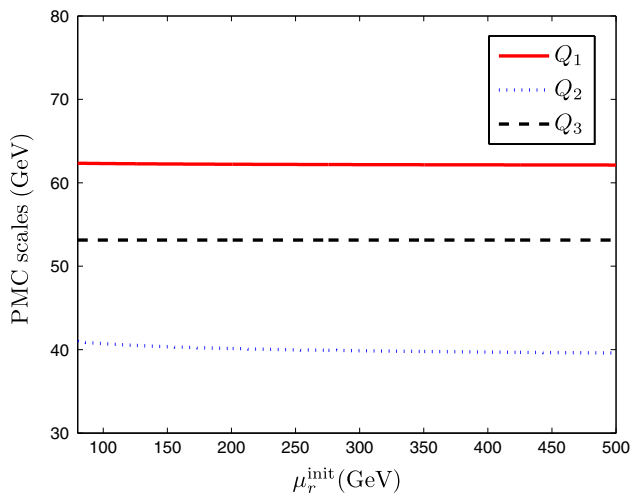


Fig. 1 The LO, NLO, and NNLO PMC scales Q_1 , Q_2 and Q_3 versus the initial renormalization scale μ_r^{init} for the $H \rightarrow b\bar{b}$ process, which are shown by *solid*, *dotted*, and *dashed* lines, respectively. $M_H = 126$ GeV

for NLO, $N^2\text{LO}$, $N^3\text{LO}$, and $N^4\text{LO}$ estimations, respectively. From these two figures, we observe that

- Figure 2 interprets the idea of the conventional scale setting. Conventionally, the renormalization scale is taken as the typical momentum transfer of the process or a value which minimizes the contributions of the loop diagrams. For the present process, $\mu_r \equiv \mu_r^{\text{init}} = M_H$. The total decay width $\Gamma(H \rightarrow b\bar{b})$ shows a relatively strong dependence on the value of μ_r^{init} at the NLO level, e.g.

$$\begin{aligned} \mu_r^{\text{init}} &= M_H/2 \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.36 \text{ MeV}, \\ \mu_r^{\text{init}} &= M_H \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.32 \text{ MeV}, \\ \mu_r^{\text{init}} &= 2M_H \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.28 \text{ MeV}, \\ \mu_r^{\text{init}} &= 4M_H \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.25 \text{ MeV}. \end{aligned} \quad (37)$$

This shows that the NLO scale error is $\pm 2\%$ for $\mu_r^{\text{init}} \in [M_H/2, 2M_H]$ and $\left(\begin{smallmatrix} +2\% \\ -3\% \end{smallmatrix}\right)$ for $\mu_r^{\text{init}} \in [M_H/2, 4M_H]$. As one includes higher-and-higher orders, the guessed scale will lead to a better estimate. For example, at the NNLO level, we have

$$\begin{aligned} \mu_r^{\text{init}} &= M_H/2 \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.40 \text{ MeV}, \\ \mu_r^{\text{init}} &= M_H \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.39 \text{ MeV}, \\ \mu_r^{\text{init}} &= 2M_H \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.37 \text{ MeV}, \\ \mu_r^{\text{init}} &= 4M_H \rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 2.35 \text{ MeV}. \end{aligned} \quad (38)$$

This shows that the NNLO scale error reduces to $\left(\begin{smallmatrix} +0.5\% \\ -0.8\% \end{smallmatrix}\right)$ for $\mu_r^{\text{init}} \in [M_H/2, 2M_H]$ and $\left(\begin{smallmatrix} +0.5\% \\ -1.7\% \end{smallmatrix}\right)$ for $\mu_r^{\text{init}} \in [M_H/2, 4M_H]$. When considering up to three-loop level or four-loop level, the decay width becomes almost invariant within the present considered region of

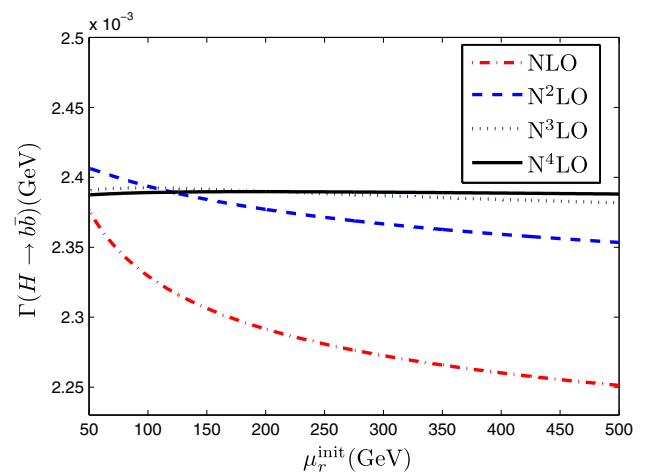


Fig. 2 Total decay width $\Gamma(H \rightarrow b\bar{b})$ up to four-loop level under conventional scale setting versus the scale $\mu_r \equiv \mu_r^{\text{init}}$. The *dash-dot*, *dashed*, *dotted*, and *solid* lines are for NLO, $N^2\text{LO}$, $N^3\text{LO}$, and $N^4\text{LO}$ estimations, respectively

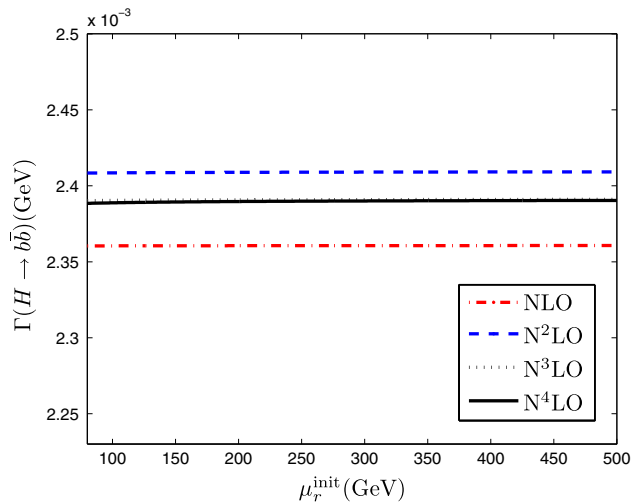


Fig. 3 Total decay width $\Gamma(H \rightarrow b\bar{b})$ up to four-loop level after PMC scale setting versus the initial renormalization scale μ_r^{init} . The *dash-dot*, *dashed*, *dotted* and *solid* lines are for NLO, $N^2\text{LO}$, $N^3\text{LO}$, and $N^4\text{LO}$ estimations, respectively

$[M_H/2, 4M_H]$. This agrees with the conventional wisdom that by finishing a higher enough calculation, one can get desirable convergent and scale-invariant estimations.

We would like to stress that even if a proper choice of μ_r^{init} may lead to a value close to the experimental data by using conventional scale setting, this may not be the correct answer for a fixed-order estimation. Especially, if a process does not converge enough, one has to finish a more and more complex loop calculations so as to achieve the precision goal.

- Figure 3 shows that after PMC scale setting, the total decay width of $H \rightarrow b\bar{b}$ are almost flat versus the choice

of renormalization scale even at the NLO level. This is due to the fact that the PMC scales Q_1 , Q_2 and Q_3 themselves are highly independent on the choice of μ_r^{init} , as shown by Fig. 1. The residual scale dependence due to unknown higher-order $\{\beta_i\}$ -series has been highly and exponentially suppressed.

It is noted that there is slight difference for the decay widths at different perturbative orders, e.g.

$$\Gamma(H \rightarrow b\bar{b})|_{\text{NLO}} \simeq 2.36 \text{ MeV},$$

$$\Gamma(H \rightarrow b\bar{b})|_{\text{N}^2\text{LO}} \simeq 2.41 \text{ MeV},$$

$$\Gamma(H \rightarrow b\bar{b})|_{\text{N}^3\text{LO}} \simeq 2.39 \text{ MeV},$$

$$\Gamma(H \rightarrow b\bar{b})|_{\text{N}^4\text{LO}} \simeq 2.39 \text{ MeV}.$$

Such difference shows that even though by absorbing the non-conformal terms into the coupling constant, one can greatly improve the pQCD convergence and simultaneously eliminate the scale dependence at lower perturbative orders, one may still need to know higher-order conformal contributions if one wants to achieve even higher precision. For example, the N^2LO improves NLO estimation by about 2 % and the N^3LO improves N^2LO estimation by about 1 %. Moreover, the unknown higher-order non-conformal contributions can be roughly estimated by varying the final undetermined PMC scale as Q_4 via the conventional way, e.g. $[Q_4/2, 2Q_4]$.

3.2 Basic results for $H \rightarrow gg$

We can estimate the properties of $H \rightarrow gg$ in a similar way as that of $H \rightarrow b\bar{b}$.

We put the decay width for $H \rightarrow gg$ before and after PMC scale setting in Table 2, where Γ_i stands for the decay width at each perturbative order with $i = \text{LO}, \text{NLO}, \text{etc.}$, and Γ_{tot} stands for the total decay width. The total decay width Γ_{tot} is $\sim 350 \text{ KeV}$ under the conventional scale setting, which improves to be $\sim 373 \text{ KeV}$ after PMC scale setting. Such a small increment in some sense means the choice of $\mu_r^{\text{init}} = M_H$ is a viable choice for the conventional scale

setting up to three-loop level. Under the conventional scale setting, we have

$$\frac{\Gamma_{\text{LO}}}{\Gamma_{\text{tot}}} : \frac{\Gamma_{\text{NLO}}}{\Gamma_{\text{tot}}} : \frac{\Gamma_{\text{N}^2\text{LO}}}{\Gamma_{\text{tot}}} : \frac{\Gamma_{\text{N}^3\text{LO}}}{\Gamma_{\text{tot}}} \approx 54 \% : 35 \% : 11 \% : 0.9 \% \quad (39)$$

and after PMC scale setting, we have

$$\frac{\Gamma_{\text{LO}}}{\Gamma_{\text{tot}}} : \frac{\Gamma_{\text{NLO}}}{\Gamma_{\text{tot}}} : \frac{\Gamma_{\text{N}^2\text{LO}}}{\Gamma_{\text{tot}}} : \frac{\Gamma_{\text{N}^3\text{LO}}}{\Gamma_{\text{tot}}} \approx 89 \% : 32 \% : (-20 \%) : 0.8 \%. \quad (40)$$

This shows that for the decay of $H \rightarrow gg$, only after a three-loop correction, one can obtain a desirable convergent estimation. Note the pQCD convergence after PMC is weaker than the case for the conventional scale setting for $H \rightarrow gg$, since the N^2LO part becomes (-20%) . This could mean that we need to know a more accurate $\{\beta_i\}$ series so as to determine the N^2LO PMC scale.

For the $H \rightarrow gg$ process at $\mathcal{O}(\alpha_s^5)$, we need to introduce four PMC scales, i.e. the LO PMC scale Q_1 , the NLO PMC scale Q_2 , the N^2LO PMC scale Q_3 and the N^3LO PMC scale Q_4 . As stated in the last section, since there are no β terms that can determine its optimal value, and we set $Q_4 \equiv \mu_r^{\text{init}}$. Figure 4 shows that the PMC scales $Q_{1,2,3}$ are highly independent of the choice of initial renormalization scale. This indicates that the PMC scale setting do provide a principle for setting the optimal (solitary) renormalization scale of high-energy processes. By setting $\mu_r^{\text{init}} = M_H \text{ GeV}$, we obtain

$$Q_1 = 23.0 \text{ GeV}, \quad Q_2 = 10.5 \text{ GeV}, \quad Q_3 = 31.5 \text{ GeV}. \quad (41)$$

These PMC scales are also smaller than M_H to a certain degree due to exponential suppressions from the absorption of higher-order $\{\beta_i\}$ terms. We show the total decay width $\Gamma(H \rightarrow gg)$ versus the initial renormalization scale μ_r^{init} before and after PMC scale setting in Figs. 5 and 6. In these two figures, the dash-dot, dashed, dotted, and solid lines are for LO, NLO, N^2LO and N^3LO estimations, respectively.

These results show that for the decay channel of $H \rightarrow gg$, we can obtain similar conclusions as those of $H \rightarrow b\bar{b}$. More explicitly,

Table 2 Decay width for the process $H \rightarrow gg$ up to three-loop level. For conventional scale setting, we set the renormalization scale $\mu_r \equiv M_H$. For PMC scale setting, we set the initial renormalization scale $\mu_r^{\text{init}} = M_H$. Here Γ_i stands for the decay width at each perturbative order with $i = \text{LO}, \text{NLO}, \text{etc.}$, Γ_{tot} stands for the total decay width. $M_H = 126 \text{ GeV}$

	Conventional scale setting					PMC scale setting				
	LO	NLO	N^2LO	N^3LO	Total	LO	NLO	N^2LO	N^3LO	Total
$\Gamma_i \text{ (KeV)}$	188.27	121.18	37.21	3.26	349.92	332.36	117.84	-74.45	-2.94	372.81
$\Gamma_i / \Gamma_{\text{tot}} \text{ (%)}$	53.80	34.63	10.63	0.93		89.15	31.61	-20.00	-0.79	

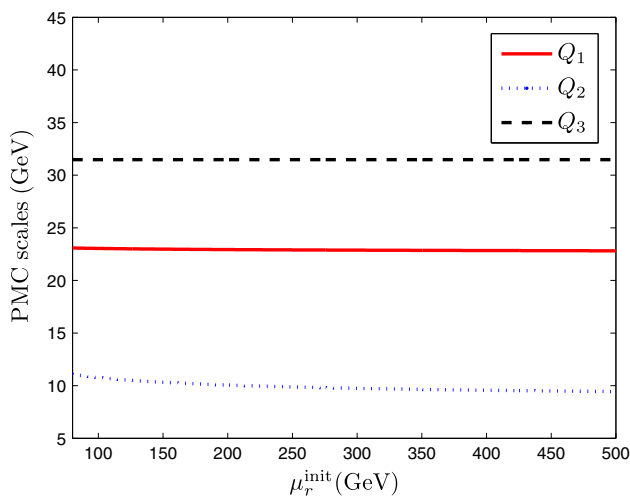


Fig. 4 The LO, NLO, and NNLO PMC scales Q_1 , Q_2 and Q_3 versus the initial renormalization scale μ_r^{init} for the $H \rightarrow gg$ process, which are shown by solid, dotted, and dashed lines, respectively. $M_H = 126$ GeV

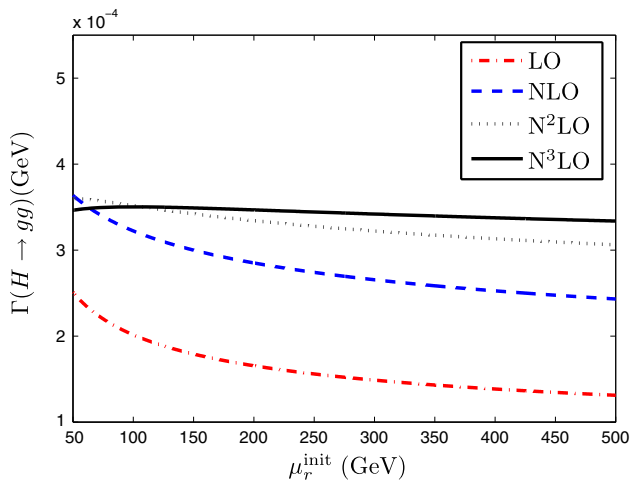


Fig. 5 Total decay width $\Gamma(H \rightarrow gg)$ up to three-loop level under conventional scale setting versus the renormalization scale $\mu_r \equiv \mu_r^{\text{init}}$. The dash-dot, dashed, dotted and solid lines are for LO, NLO, $N^2\text{LO}$, and $N^3\text{LO}$ estimations, respectively

- Figure 5 indicates that, under the conventional scale setting, the total decay width $\Gamma(H \rightarrow gg)$ shows a strong dependence on μ_r^{init} at the LO level, e.g.

$$\begin{aligned} \mu_r^{\text{init}} = M_H/2 &\rightarrow \Gamma(H \rightarrow gg) \simeq 232 \text{ KeV}, \\ \mu_r^{\text{init}} = M_H &\rightarrow \Gamma(H \rightarrow gg) \simeq 188 \text{ KeV}, \\ \mu_r^{\text{init}} = 2M_H &\rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 156 \text{ KeV}, \\ \mu_r^{\text{init}} = 4M_H &\rightarrow \Gamma(H \rightarrow b\bar{b}) \simeq 131 \text{ KeV}. \end{aligned} \quad (42)$$

This shows that the LO scale error is 40 % for $\mu_r^{\text{init}} \in [M_H/2, 2M_H]$ and 54 % for $\mu_r^{\text{init}} \in [M_H/2, 4M_H]$. Moreover, the scale error for $\mu_r^{\text{init}} \in [M_H/2, 4M_H]$ shall change down to 35, 15 and 4 % for NLO, $N^2\text{LO}$, and $N^3\text{LO}$ estimations, respectively. This shows that as one

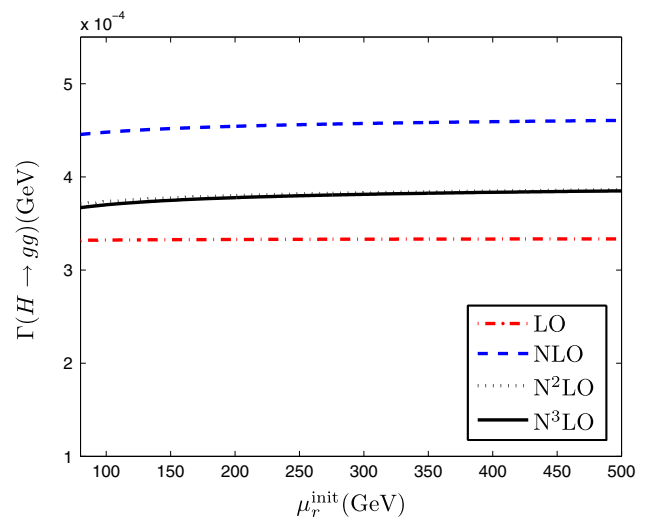


Fig. 6 Total decay width $\Gamma(H \rightarrow gg)$ up to three-loop level after PMC scale setting versus the initial renormalization scale μ_r^{init} . The dash-dot, dashed, dotted, and solid lines are for LO, NLO, $N^2\text{LO}$, and $N^3\text{LO}$ estimations, respectively

includes higher-and-higher orders, the guessed scale will lead to a better estimation. For the $H \rightarrow gg$ decay, only after a three-loop correction, one can obtain a desirable small about several percent scale error.

- Figure 6 shows that, after PMC scale setting, the total decay widths of $H \rightarrow gg$ up to the mentioned perturbative orders are almost flat versus the choice of initial renormalization scale μ_r^{init} . The residual scale dependence due to unknown higher-order $\{\beta_i\}$ -series has been highly suppressed. Similar to the case of $H \rightarrow b\bar{b}$, this is due to the fact that the PMC scales Q_1 , Q_2 and Q_3 themselves are highly independent on the choice of μ_r^{init} , which are shown clearly by Fig. 4.

3.3 A comparison of the approaches underlying BLM scale setting

The BLM scale setting is designed to improve the pQCD predictions by absorbing the n_f terms via a proper way into the coupling constant [53]. Since its invention by Brodsky–Lepage–Mackenzie in 1983, the BLM has been widely accepted in the literature for dealing with high-energy processes, such as the $e^+e^- \rightarrow$ hadrons, the deep inelastic scattering, the heavy meson or baryon productions or decays, the exclusive processes such as the pion–photon transition form factors, the QCD lattice perturbative theory, etc. Encouraged by its great successes, several approaches have been tried to extend BLM to any perturbative orders or put it in a more solid background, such as the PMC-I approach (first approach to achieve the goal of PMC via the PMC–BLM correspondence principle) [54,55], the R_δ -scheme (second

approach to achieve the goal of PMC) [60, 64] and the seBLM approach [72, 73].

It is noted that the role of the running coupling in any gauge theory is to absorb the physics of the β function, which governs its running behavior via the renormalization group equations. Any approach that properly identifies the $\{\beta_i\}$ -series for a physical observable will lead to equivalently the same result. Practically, one usually calculate the n_f terms by considering the vacuum polarization contributions. However, different ways of identifying n_f -series to $\{\beta_i\}$ -series may lead to: I) different effective $\{\beta_i\}$ -series at each known perturbative order; II) a different residual $\{\beta_i\}$ -dependence because of unknown perturbative orders; III) different conformal terms leaving at each perturbative order, and IV) a different pQCD convergence. If one can do the perturbative corrections to a high enough perturbative order, different effective schemes may result in consistent physical predictions. The equivalence of the PMC-I approach and the R_δ -scheme has already been shown in Refs. [60, 61, 64]. In the following, we take the Higgs decay channel $H \rightarrow b\bar{b}$ as an explicit example to show that the seBLM approach is also consistent with the PMC approaches. One subtle point of such a comparison (and also any applications of those approaches) lies in that we should first transform the estimations with full renormalization scale dependence with the help of the transformation formulas (1).

In the PMC-I approach, by introducing a PMC–BLM correspondence principle in which the $\{\beta_i\}$ -series for a physical observable has the same parton as the running coupling itself [55], the number of the effective independent $\{\beta_i\}$ terms exactly corresponds to the number of n_f terms at each perturbative order. In R_δ -scheme, by introducing the “degeneracy” properties of the $\{\beta_i\}$ -series observed by a generalization of the conventional dimensional regularization scheme to any dimensional-like ones [60], one can also obtain a one-to-one correspondence between the $\{\beta_i\}$ -series and the n_f -series. The calculation technologies for those two self-consistent approaches can be found in the corresponding references, the interesting readers may refer to those references or to the very recent review [61] for details. In Sect. 2, we have presented our analysis under the R_δ -scheme.

While the seBLM scheme provides quite a different way from those two PMC approaches, in which a general $\{\beta_i\}$ -series at each perturbative order have been introduced, and in order to get an one-to-one correspondence between the n_f -series with the $\{\beta_i\}$ -series, some extra approximations (or equivalently some extra degrees of freedom) have to be introduced [72]. More explicitly, the seBLM scheme transforms the standard power series $a_s^n(\mu_r^{\text{init}})$ to the series of the products $\prod_{i=1}^n a_s(Q_i)$. After applying the seBLM scheme to Eq. (2), the decay width of the process $H \rightarrow b\bar{b}$ can be expressed as follows:

$$\Gamma(H \rightarrow b\bar{b}) = \frac{3G_F M_H}{4\sqrt{2}\pi} m_b^2(M_H) \times \left[1 + \frac{r'_{1,0}(\mu_r^{\text{init}})}{\beta_0} A_1 \left(1 + \frac{A_2}{\beta_0} (r'_{2,0}(\mu_r^{\text{init}}) + \frac{A_3}{\beta_0} (r'_{3,0}(\mu_r^{\text{init}}) + \frac{A_4}{\beta_0} (r'_{4,0}(\mu_r^{\text{init}}) + \dots)) \right) \right],$$

where r'_i stands for the conformal coefficients of seBLM leaving at each perturbative order, $A_i = \beta_0 a_s(Q_i)$ stand for the redefined coupling constant. Because we have no higher-order $\{\beta_i\}$ terms to determine the scale for A_4 , we set $Q_4 = Q_3$ as suggested by seBLM. Three effective seBLM scales are

$$\ln \left(\frac{(\mu_r^{\text{init}})^2}{Q_1^2} \right) = \Delta_{1,0} + \Delta_{1,1} A_1 + \Delta_{1,2} A_1^2, \quad (43)$$

$$\ln \left(\frac{Q_1^2}{Q_2^2} \right) = \Delta_{2,0} + \Delta_{2,1} A_2, \quad (44)$$

$$\ln \left(\frac{Q_2^2}{Q_3^2} \right) = \Delta_{3,0}. \quad (45)$$

where the explicit expressions of the coefficients $\Delta_{i,j}$ can be found in Ref. [72].² Then we obtain

$$Q_1 = 62.5 \text{ GeV}, \quad Q_2 = 29.1 \text{ GeV}, \quad Q_3 = 127.0 \text{ GeV} \quad (46)$$

for $\mu_r^{\text{init}} = M_H = 126 \text{ GeV}$. It is noted that the expressions for Q_1 in Eqs. (17) and (44) are equal at the LO and NLO level, we obtain almost the same value for Q_1 under the R_δ -scheme and the seBLM approach.

We present a comparison of those three approaches in Table 3, which clearly shows the pQCD convergence of the perturbative series. Here, as a comparison, the estimations for conventional scale setting and the results for the BKM scheme [74] and the fractional analytic perturbation theory (FAPT) scheme [75] are also presented.

At the present considered estimation up to four-loop levels all of those schemes including the conventional scale setting lead to good pQCD convergence. Especially, the PMC-I scheme, the R_δ -scheme and the seBLM scheme have almost the same pQCD series. This is reasonable, since those three scale-setting schemes are designed to deal with the $\{\beta_i\}$ -series of the process.

In addition, one will observe that after eliminating the non-conformal $\{\beta_i\}$ -series, one may also derive the (initial)

² Because for $H \rightarrow b\bar{b}$, we have no extra constraints or degrees of freedom to set an one-to-one correspondence between the n_f -series and the $\{\beta_i\}$ -series, as an estimation, we directly adopt the same pattern of $\{\beta_i\}$ -series in each perturbative order that has been derived for the Adler D-function for the present process [72].

Table 3 A comparison of several approaches for calculating the perturbative coefficients of $H \rightarrow b\bar{b}$, where the predictions of the PMC-I scheme, the R_δ -scheme and the seBLM scheme, together with the ones derived under conventional scale setting, are presented. Here Γ_i stands for the decay width at each perturbative order with $i = \text{LO, NLO, etc.}$, Γ_{tot} stands for the total decay width. The initial renormalization scale is taken as $M_H = 126 \text{ GeV}$. To be a useful reference, the results of Refs. [74, 75] for the FAPT scheme and the BKM scheme are also presented

	Γ_{tot} (MeV)	$\Gamma_{\text{LO}}/\Gamma_{\text{tot}}$ (%)	$\Gamma_{\text{NLO}}/\Gamma_{\text{tot}}$ (%)	$\Gamma_{\text{N}^2\text{LO}}/\Gamma_{\text{tot}}$ (%)	$\Gamma_{\text{N}^3\text{LO}}/\Gamma_{\text{tot}}$ (%)	$\Gamma_{\text{N}^4\text{LO}}/\Gamma_{\text{tot}}$ (%)
Conventional scale setting	2.389	80.53	16.39	3.03	0.16	-0.11
seBLM	2.389	80.56	18.25	1.99	-0.72	-0.08
PMC-I	2.388	80.5	18.26	2.03	-0.76	-0.10
R_δ -scheme	2.389	80.54	18.26	2.01	-0.76	-0.06
BKM [74]	2.75	74.5	17.7	5.3	1.8	0.7
FAPT with $l = 2$ [75]	2.38	79.5	16.2	4.3	—	—
FAPT with $l = 3$ [75]	2.44	78.5	16.1	4.2	1.2	—

Table 4 Initial scale dependence for Γ_{NLO} of $H \rightarrow b\bar{b}$. Here Γ_i stands for the decay width at each perturbative order with $i = \text{LO, NLO, etc.}$. The predictions of the PMC-I, R_δ and seBLM schemes are almost independent of μ_r^{init} . The cases for higher-order decay widths $\Gamma_{\text{N}^2\text{LO}}$, $\Gamma_{\text{N}^3\text{LO}}$ and $\Gamma_{\text{N}^4\text{LO}}$ are the similar. $M_H = 126 \text{ GeV}$

μ_r^{init}	Γ_{NLO} (KeV)		
	$M_H/2$	M_H	$2M_H$
Conventional scale setting	435.42	391.73	356.18
seBLM [72]	435.95	435.95	435.95
PMC-I [55]	435.03	435.99	436.06
R_δ -scheme [60]	436.12	436.23	436.32

renormalization scale independence for a fixed-order prediction for those approaches. The initial scale dependence for the PMC-I, R_δ -scheme and seBLM approaches on the NLO decay widths Γ_{NLO} are presented in Table 4, where there typical initial scales $\mu_r^{\text{init}} = M_H/2, M_H$ and $2M_H$ are adopted. It shows clearly that the value of Γ_{NLO} are almost unchanged with μ_r^{init} . The higher-order terms have similar properties.

4 Summary

The conventional scale-setting procedure assigns an arbitrary range and an arbitrary systematic error to fixed-order pQCD predictions. And its error analysis can only get a rough estimation of the β -dependent non-conformal terms, not the entire perturbative series. As a possible solution, the PMC provides a systematic way to set the optimized renormalization scales for high-energy processes. In principle, the PMC needs an initial value to initialize renormalization scale and renormalization procedures. It is found that the estimates of PMC are to high accuracy independent of the initial renormalization scale; even the PMC scales themselves are in effect independent of the initial renormalization scale and are ‘physical’ at any fixed order. This is because the PMC scale

itself is a perturbative series and those unknown higher-order $\{\beta_i\}$ terms will be absorbed into the higher-order terms of the PMC scale, which is strongly exponentially suppressed. Since the renormalization scale and scheme ambiguities are removed, the PMC can improve the precision of tests of the Standard Model and enhances the sensitivity to new phenomena. It is noted that the PMC applies the known non-conformal $\{\beta_i\}$ terms in a strict and systematic way to determine the behavior of the coupling constant at each perturbative order. It provides an accurate estimation for the known perturbative series, and one may still need higher-order calculations so as to know even higher-order conformal contributions, especially when the perturbative series does not converge enough.

The PMC can be applied to a wide variety of perturbatively calculable collider and other processes. In addition to previous examples given in the literature, following its standard R_δ -scheme procedures, we have done a through analysis of these two processes up to four-loop and three-loop levels. A comparison of the estimations under three approaches, i.e. the PMC-I approach, the R_δ -scheme and the seBLM approach, have also been presented. We observe,

- Under conventional scale setting, it is often argued that by finishing a higher enough perturbative calculation, one can get desirable convergent and scale-invariant estimations. For the present considered channels, when considering up to three-loop level or four-loop level, as shown by Figs. 2 and 5, the decay width becomes almost invariant within the region of $\mu_r^{\text{init}} \in [M_H/2, 4M_H]$. However, even if a proper choice of μ_r^{init} may lead to a value close to the experimental data by using conventional scale setting, this is guess work and may not be the correct answer for a fixed-order estimation. Especially, if a process does not converge enough, one has to finish a more and more complex loop calculations so as to achieve the precision goal.

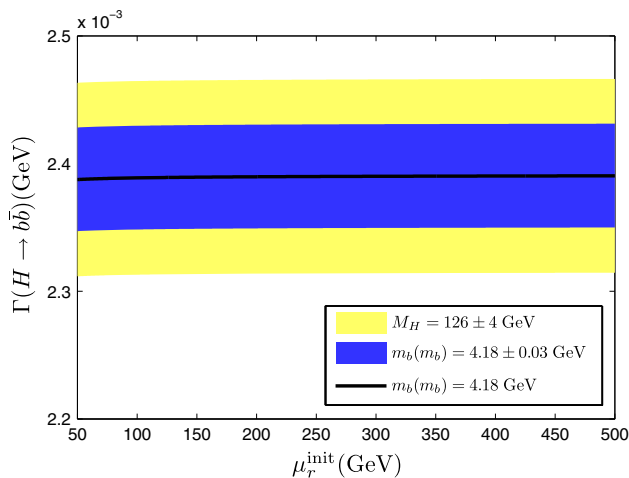


Fig. 7 Uncertainties for the decay width of $H \rightarrow b\bar{b}$ up to four-loop level. Two shaded bands show the uncertainties from $M_H = 126 \pm 4$ GeV and the $\overline{\text{MS}}$ b -quark mass $m_b(m_b) = 4.18 \pm 0.03$ GeV, respectively. The solid line is obtained by setting the parameters to be their central values

- As shown by Figs. 3 and 6, after PMC scale setting, the total decay widths of $H \rightarrow b\bar{b}$ and $H \rightarrow gg$ show a fast trend of stabilization versus the change of initial renormalization scale, which are almost flat even at the NLO level. The residual scale dependence due to unknown higher-order $\{\beta_i\}$ -series has been greatly suppressed. This indicates that the PMC scale setting does provide a principle for setting the optimal renormalization scale of high-energy processes.
- In comparison to the previous estimations, e.g. Refs. [76, 77], a more accurate prediction for those two Higgs decay processes can be obtained. After PMC scale setting, we obtain the total decay widths for those two channels:

$$\Gamma(H \rightarrow b\bar{b}) = 2.389 \pm 0.073 \pm 0.041 \text{ MeV}, \quad (47)$$

$$\Gamma(H \rightarrow gg) = 0.373 \pm 0.030 \text{ MeV}. \quad (48)$$

where the first error is caused by varying M_H with the region of [122, 130] GeV, and the second error for the $H \rightarrow b\bar{b}$ channel is caused by varying the $\overline{\text{MS}}$ running mass $m_b(m_b)$ with the region of [4.15, 4.21] GeV. For clarity, we put the uncertainty analysis for the $H \rightarrow b\bar{b}$ channel in Fig. 7.

- The β function governs the running behavior of the coupling constant via the renormalization group equations, thus any approach that can properly identify the $\{\beta_i\}$ -series for a physical observable will surely lead to equivalently the same predictions. Practically, one usually calculates the n_f terms by considering the vacuum polarization contributions. After a certain scale setting, different ways of identifying n_f -series to $\{\beta_i\}$ -series may lead to

different effective $\{\beta_i\}$ -series at each known perturbative order, different residual $\{\beta_i\}$ -dependence because of unknown perturbative orders, or a different pQCD convergence. A comparison of $H \rightarrow b\bar{b}$ for the PMC-I approach, the R_δ -scheme, and the seBLM approach has been presented in Table 3. At the four-loop level all those approaches lead to good pQCD convergence, they have almost the same pQCD expansion series, and all of them are almost independent of the wide choice of the initial renormalization scale. This shows that these three approaches are equivalent. The residual differences of these approaches are caused by the unknown $\{\beta_i\}$ terms that could be suppressed to a required accuracy by finishing a higher-order calculation.

- As one subtle point, one may meet the problem of quite small (or near the fixed point) effective scales for a specific scale-setting method. For example, we have noted that for the case of $R(e^+e^- \rightarrow \text{hadron})$ at the measured scale Q , we can obtain a convergent and precise conformal series up to four-loop level by applying the R_δ -scheme, whose LO, NLO, and NNLO PMC scales are [60] $Q_1 = 1.3Q$, $Q_2 = 1.2Q$, and $Q_3 = 5.3Q$, respectively. In contrast, by using the seBLM scheme, we shall obtain $\ln(Q^2/Q_2^2) \sim 167$ [72], which leads to quite small Q_2 out of pQCD domain. If, as suggested by PMC, we only deal with the n_f series that rightly controls the running behavior of the coupling constant into the coupling constant via the standard way of seBLM, then we shall obtain more moderate seBLM scales, $Q_1 = 1.3Q$, $Q_2 = 1.1Q$, and $Q_3 = 228.9Q$. Similar to the present Higgs decays, we can obtain consistent results for $R(e^+e^-)$ under both the seBLM and the PMC scale settings. Moreover, it is noted that such a situation could be softened to a certain degree by using the commensurate scale relation [78], or one may solve it by using proper running behavior of the coupling constant in lower scale region [79].
- As another subtle point, even if one can eliminate the scale dependence at lower perturbative order as NLO, it may be necessary to know the higher-order conformal contributions if we want to achieve even higher precision. Taking the case of $H \rightarrow b\bar{b}$ as an example, its N²LO terms improve the NLO estimation by about 2 % and the N³LO terms improve the N²LO estimation by about 1 %. The unknown higher-order non-conformal contributions can be roughly estimated by varying the final undetermined PMC scale as Q_4 via the conventional way, e.g. $[Q_4/2, 2Q_4]$. If after PMC scale setting the final terms at a certain fixed order give a negligible contribution, then we shall obtain quite accurate estimations at such a fixed order. As shown by Table 4, by setting $\mu_r^{\text{init}} = M_H/2$, the $H \rightarrow b\bar{b}$ NLO decay width Γ_{NLO} under the conventional scale setting is close to the PMC estimations.

In this sense, a choice of $\mu_r \equiv M_H/2$ is better than the choice of $\mu_r \equiv M_H$ for the conventional scale setting. In fact, under such a choice, one can also obtain a more convergent pQCD series for the conventional scale setting.

- As a minor point, taking $H \rightarrow b\bar{b}$ as an example, we point out a wrong way of estimating the conventional scale error. The correct way is to set an initial scale μ_r^{init} and get the full μ_r^{init} -dependent expression (5), e.g. those terms proportional to $\ln(\mu_r^{\text{init}}/M_H)$ are kept, and then to vary $\mu_r^{\text{init}} \in [M_H/2, 2M_H]$ to discuss its scale error. In this way, we have shown that the conventional scale error up to four-loop level is almost eliminated, as shown in Fig. 2. The wrong way is to adopt the expression (2) as the starting point, and to directly vary the scale of the coupling constant from M_H to $M_H/2$ or $2M_H$ to discuss the scale error. In this wrong treatment, the log terms involving $\mu_r^{\text{init}} \neq M_H$ disappear, which, however, may have sizable contributions. In fact, such a naive treatment shows the conventional scale error is still about $\pm 2\%$ [76,77] for varying the scale within the region of $[M_H/2, 2M_H]$ even at the four-loop level.

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References

1. G. Aad et al., ATLAS Collaboration, Phys. Lett. B **716**, 1 (2012)
2. S. Chatrchyan et al., CMS Collaboration, Phys. Lett. B **716**, 30 (2012)
3. [ATLAS Collaboration], ATLAS-CONF-2013-034 (2013)
4. [CMS Collaboration], CMS-PAS-HIG-13-005 (2013)
5. [CMS Collaboration], CMS-PAS-HIG-13-001 (2013)
6. [ATLAS Collaboration], ATLAS-CONF-2013-012 (2013)
7. [ATLAS Collaboration], ATLAS-CONF-2013-013 (2013)
8. [CMS Collaboration], CMS-PAS-HIG-13-002 (2013)
9. [ATLAS Collaboration], ATLAS-CONF-2013-014 (2013)
10. R. Walsh, CMS Collaboration, EPJ Web Conf. **49**, 12012 (2013)
11. A. Djouadi, Phys. Rept. **457**, 1 (2008)
12. S.G. Gorishnii, A.L. Kataev, S.A. Larin, Sov. J. Nucl. Phys. **40**, 329 (1984)
13. S.G. Gorishnii, A.L. Kataev, S.A. Larin, L.R. Surguladze, Mod. Phys. Lett. A **5**, 2703 (1990)
14. S.G. Gorishnii, A.L. Kataev, S.A. Larin, L.R. Surguladze, Phys. Rev. D **43**, 1633 (1991)
15. C. Becchi, S. Narison, E. de Rafael, F.J. Yndurain, Z. Phys. C **8**, 335 (1981)
16. N. Sakai, Phys. Rev. D **22**, 2220 (1980)
17. T. Inami, T. Kubota, Nucl. Phys. B **179**, 171 (1981)
18. K.G. Chetyrkin, B.A. Kniehl, M. Steinhauser, Phys. Rev. Lett. **78**, 594 (1997)
19. J. Fleischer, F. Jegerlehner, Phys. Rev. D **23**, 2001 (1981)
20. B.A. Kniehl, Nucl. Phys. B **376**, 3 (1992)
21. A. Dabelstein, W. Hollik, Z. Phys. C **53**, 507 (1992)
22. A. Djouadi, P. Gambino, B.A. Kniehl, Nucl. Phys. B **523**, 17 (1998)
23. A.L. Kataev, V.T. Kim, Mod. Phys. Lett. A **9**, 1309 (1994)
24. A.L. Kataev, V.T. Kim, [arXiv:0804.3992](https://arxiv.org/abs/0804.3992); [arXiv:0902.1442](https://arxiv.org/abs/0902.1442)
25. A.P. Bakulev, S.V. Mikhailov, N.G. Stefanis, JHEP **1006**, 085 (2010)
26. K.G. Chetyrkin, Phys. Lett. B **390**, 309 (1997)
27. K.G. Chetyrkin, M. Steinhauser, Phys. Lett. B **408**, 320 (1997)
28. C. Anastasiou, F. Herzog, A. Lazopoulos, [arXiv:1110.2368](https://arxiv.org/abs/1110.2368)
29. P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, Phys. Rev. Lett. **96**, 012003 (2006)
30. F. Wilczek, Phys. Rev. Lett. **39**, 1304 (1977)
31. T. Inami, T. Kubota, Y. Okada, Z. Phys. C **18**, 69 (1983)
32. S. Dawson, R.P. Kauffman, Phys. Rev. Lett. **68**, 2273 (1992)
33. A. Djouadi, M. Spira, P.M. Zerwas, Phys. Lett. B **264**, 440 (1991)
34. D. Graudenz, M. Spira, P.M. Zerwas, Phys. Rev. Lett. **70**, 1372 (1993)
35. S. Dawson, R.P. Kauffman, Phys. Rev. D **49**, 2298 (1994)
36. M. Spira, A. Djouadi, D. Graudenz, P.M. Zerwas, Nucl. Phys. B **453**, 17 (1995)
37. K.G. Chetyrkin, B.A. Kniehl, M. Steinhauser, Phys. Rev. Lett. **79**, 353 (1997)
38. M. Steinhauser, Phys. Rept. **364**, 247 (2002)
39. P.A. Baikov, K.G. Chetyrkin, Phys. Rev. Lett. **97**, 061803 (2006)
40. M. Schreck, M. Steinhauser, Phys. Lett. B **655**, 148 (2007)
41. E.C.G. Stueckelberg, A. Peterman, Helv. Phys. Acta **26**, 499 (1953)
42. N.N. Bogoliubov, D.V. Shirkov, Dok. Akad. Nauk SSSR **103**, 391 (1955)
43. A. Peterman, Phys. Rept. **53**, 157 (1979)
44. C.G. Callan, Phys. Rev. D **2**, 1541 (1970)
45. K. Symanzik, Commun. Math. Phys. **18**, 227 (1970)
46. G. Grunberg, Phys. Lett. B **95**, 70 (1980)
47. G. Grunberg, Phys. Lett. B **110**, 501 (1982)
48. G. Grunberg, Phys. Rev. D **29**, 2315 (1984)
49. G. Grunberg, Phys. Rev. D **40**, 680 (1989)
50. P.M. Stevenson, Phys. Lett. B **100**, 61 (1981)
51. Phys. Rev. D **23**, 2916 (1981)
52. Nucl. Phys. B **231**, 65 (1984)
53. S.J. Brodsky, G.P. Lepage, P.B. Mackenzie, Phys. Rev. D **28**, 228 (1983)
54. S.J. Brodsky, X.G. Wu, Phys. Rev. Lett. **109**, 042002 (2012)
55. S.J. Brodsky, X.G. Wu, Phys. Rev. D **85**, 034038 (2012)
56. S.J. Brodsky, X.G. Wu, Phys. Rev. D **85**, 114040 (2012)
57. S.J. Brodsky, X.G. Wu, Phys. Rev. D **86**, 014021 (2012)
58. S.J. Brodsky, L. Di Giustino, Phys. Rev. D **86**, 085026 (2012)
59. S.J. Brodsky, X.G. Wu, Phys. Rev. D **86**, 054018 (2012)
60. M. Mojaza, S.J. Brodsky, X.G. Wu, Phys. Rev. Lett. **110**, 192001 (2013)
61. X.G. Wu, S.J. Brodsky, M. Mojaza, Prog. Part. Nucl. Phys. **72**, 44 (2013)
62. S.Q. Wang, X.G. Wu, X.C. Zheng, J.M. Shen, Q.L. Zhang, Nucl. Phys. B **876**, 731 (2013)
63. X.C. Zheng, X.G. Wu, S.Q. Wang, J.M. Shen, Q.L. Zhang, JHEP **1310**, 117 (2013)
64. S.J. Brodsky, M. Mojaza, X.G. Wu, Phys. Rev. D **89**, 014027 (2014)
65. S.Q. Wang et al. [arXiv:1311.5106](https://arxiv.org/abs/1311.5106)
66. O.V. Tarasov, A.A. Vladimirov, A.Y. Zharkov, Phys. Lett. B **93**, 429 (1980)

67. S.A. Larin, J.A.M. Vermaseren, Phys. Lett. B **303**, 334 (1993)
68. T. van Ritbergen, J.A.M. Vermaseren, S.A. Larin, Phys. Lett. B **400**, 379 (1997)
69. J. Beringer et al., Particle Data Group. Phys. Rev. D **86**, 010001 (2012)
70. K.G. Chetyrkin, Phys. Lett. B **404**, 161 (1997)
71. K. Melnikov, T. van Ritbergen, Phys. Lett. B **482**, 99 (2000)
72. S.V. Mikhailov, JHEP **0706**, 009 (2007)
73. A.L. Kataev, S.V. Mikhailov, Theor. Math. Phys. **170**, 139 (2012)
74. D.J. Broadhurst, A.L. Kataev, C.J. Maxwell, Nucl. Phys. B **592**, 247 (2001)
75. A.P. Bakulev, S.V. Mikhailov, N.G. Stefanis, Phys. Rev. D **75**, 056005 (2007). Erratum-ibid. D **77**, 079901 (2008)
76. S. Dittmaier et al., [LHC Higgs Cross Section Working Group]. [arXiv:1201.3084](#)
77. S. Heinemeyer et al., [LHC Higgs Cross Section Working Group]. [arXiv:1307.1347](#)
78. S.J. Brodsky, H.J. Lu, Phys. Rev. D **51**, 3652 (1995)
79. Q.L. Zhang, X.G. Wu, X.C. Zheng, S.Q. Wang, H.B. Fu, Z.Y. Fang. [arXiv:1401.4268](#)