Eur. Phys. J. C (2014) 74:2819
DOI 10.1140/epjc/s10052-014-2819-1

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

The thermodynamics and thermodynamic geometry of the Park black hole

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Received: 9 January 2014 / Accepted: 8 March 2014 / Published online: 27 March 2014

Abstract We study the thermodynamics and thermodynamic geometry of the Park black hole in Hořava gravity. By incorporating the ideas of differential geometry, we investigate the thermodynamics by using the Weinhold and the Ruppeiner geometry. We have also analyzed it in the context of the newly developed geometrothermodynamics (GTD). The divergence of the specific heat is associated with a second-order phase transition. Here in the context of the Park black hole, both Weinhold's metric and Ruppeiner's metric well explain this phase transition. But these explanations depend on the choice of the potential. Hence the Legendre invariant GTD is used, and with the true singularities in the curvature scalar, they well explain the second-order phase transition. All these methods together give an exact idea of all the behaviors of the Park black hole thermodynamics.

1 Introduction

Over the past decade, a lot of interest has been given to various black holes in anti-de Sitter (AdS) space as well as in de Sitter (dS) space, due to the success of the AdS/CFT correspondence [1] and hence the proposal of the dS/CFT correspondence [2–4]. So by studying the thermodynamics of these black holes, one may obtain a real connection between gravity and quantum mechanics. Hořava proposed [5–7] a field theoretical model in 2009, which can be considered as a UV complete theory of gravity without full diffeomorphism invariance. It can be reduced to Einstein's theory in the IR regime and is non-relativistic in the UV regime. Recently its black hole solutions and thermodynamics have been intensively investigated [8–20]. Later, Park has obtained a $\lambda = 1$ black solution, known as Park black hole [21]. By introducing two parameters ω and Λ_W, Park found that both dS and AdS solutions exist.

In almost all macroscopic systems, usual thermodynamics entirely depends on the empirical results under certain constraints. But when we incorporate geometrical concepts into thermodynamics, it will further illuminate the hope toward the quantization of gravity. Gibbs [22] and Caratheodory [23] put forward the idea of applying differential geometry in thermodynamics. Later Hermann [24] and Mrugala [25,26] developed the idea of introducing contact geometry into the thermodynamic phase space. In 1976 Weinhold [27,28] proposed an alternative approach with a metric, known as Weinhold's metric defined ad hoc as the Hessian of the internal energy. The Weinhold's metric is given by

$$g_{ij}^W = \partial_i \partial_j U(S, N^r), \tag{1}$$

where N^r denotes the other extensive variables of the system. Many studies have been done using this metric. But later it was understood that the geometry based on this metric seems to be physically meaningless in the context of purely equilibrium thermodynamics. In 1979 Ruppeiner [29] introduced another metric in an attempt to formulate the concept of thermodynamic length. Ruppeiner's metric is defined as

$$g_{ij}^R = -\partial_i \partial_j S(M, N^r), \tag{2}$$

where this metric is conformally equivalent to Weinhold's metric and the geometry that can be obtained from these two are related through a line element relationship [30,31],

$$ds_R^2 = \frac{1}{T} ds_W^2, \tag{3}$$

where T denotes the temperature. For systems like the ideal classical gas, the multicomponent ideal gas, the ideal quantum gas, the 1-dimensional Ising model, the van der Waals model etc., the results obtained with the above two metrics are found to be consistent [32–42]. But when we consider black hole systems, it is found that these two metrics fail in explaining the properties as well as they lead to many puzzling situations. Among these inconsistencies, the dependence of the

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metric on the thermodynamic potential is the main problem [46,47]. Geometrothermodynamics (GTD) [43–45] is the newest approach among the geometric methods. The puzzling properties occurred in the previous methods is due to the fact that the system possesses different properties, when different thermodynamic potentials are used. But in the frame work of GTD, the metric we are considering is invariant with respect to Legendre transformation, hence they are independent of the choice of the thermodynamic potential of the system.

To investigate the mathematical structure of thermodynamics, it is necessary to use contact geometry. In GTD, to introduce the language of differential geometry in thermodynamics, we will consider $(2n+1)$-dimensional thermodynamic phase space $\mathcal{T}$. The coordinates of this phase space are defined by the set where $Z^A = \{\Phi, E^a, I^a\}$, where Φ is the thermodynamic potential, E^a represents a set of n extensive variables, and I^a is the corresponding dual intensive variables, with $a = 1, 2, \ldots, n$. Now the contact one form can be written as

$$\Theta = d\Phi - \delta_{ab} I^a dE^b; \qquad \delta_{ab} = \mathrm{diag}(1, 1, \ldots, 1) \tag{4}$$

The pair $(\mathcal{T}, \Theta)$ defines a contact manifold [24] if $\mathcal{T}$ is differentiable and Θ satisfies the condition $\Theta \wedge (d\Theta)^n \neq 0$. Consider G as a non-degenerate metric on $\mathcal{T}$. Then the set $(\mathcal{T}, \Theta, G)$ defines a Riemannian contact manifold [24,48] or the phase manifold. An n-dimensional Riemannian submanifold $\mathcal{E} \subset \mathcal{T}$ is defined as the equilibrium manifold by a smooth map $\varphi : \mathcal{E} \to \mathcal{T}$ which satisfies the pullback condition $\varphi^*(\Theta) = 0$. Then the metric induced on this equilibrium manifold $\mathcal{E}$, known as the Quevedo metric, plays the same role as that of Weinhold's and Ruppeiner's metric. This metric can be written as follows,

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\eta_{cd} dE^c dI^d) \tag{5}$$

and

$$g^Q = \varphi^*(G) = \left(E^c \frac{\partial \Phi}{\partial E^c}\right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d\right) \tag{6}$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, \ldots, 1)$ and this metric is Legendre invariant because of the invariance of the Gibbs one form.

This paper is organized as follows. In Sect. 2, we review the Park solution in Hořava gravity and its usual thermodynamics in detail. In Sect. 3, different thermodynamic geometry methods including GTD are studied in the case of the Park black hole by a detailed analysis of both the dS and the AdS cases. And paper concludes in Sect. 4 with a discussion regarding the results obtained from the present work.

2 Park solution in Hořava gravity and its thermodynamics

Let us consider the ADM decomposition of the metric,

$$ds_4^2 = -N^2 c^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{7}$$

and the IR modified Hořava action can be written as

$$\begin{aligned} S = \int dt d^3x \sqrt{g} N \Bigg[& \frac{2}{\kappa}\left(K_{ij}K^{ij} - \lambda K^2\right) - \frac{\kappa^2}{2\nu^4} C_{ij} C^{ij} \\ & + \frac{\kappa^2 \mu}{2\nu^2} \epsilon^{ijk} R^{(3)}_{il} \nabla_j R^{(3)l}_{k} - \frac{\kappa^2\mu^2}{8} R^{(3)}_{ij} R^{(3)ij} \\ & + \frac{\kappa^2\mu^2\omega}{8(3\lambda-1)} R^{(3)} + \frac{\kappa^2\mu^2}{8(3\lambda-1)} \\ & \times \left(\frac{4\lambda-1}{4}(R^{(3)})^2 - \Lambda_W R^{(3)} + 3\Lambda_W^2\right)\Bigg], \end{aligned} \tag{8}$$

where K_{ij} and C^{ij} are the extrinsic curvature and the Cotton tensor, respectively and $\kappa, \nu, \mu, \lambda, \Lambda_W, \omega$ are constant parameters. Among them Λ_W is related to the cosmological constant by the relation,

$$\Lambda_W = \frac{3}{2}\Lambda. \tag{9}$$

The last term in (8) represents a soft violation of the detailed balance condition [5]. For static and spherically symmetric solution, substituting the metric ansatz as

$$ds^2 = -N(r)^2 c^2 dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{10}$$

in the action (8) and after angular integration, we obtain the Lagrangian as

$$\begin{aligned} \mathcal{L} = \frac{\kappa^2\mu^2}{8(1-3\lambda)} \frac{N}{\sqrt{f}} \Bigg[& (2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda \frac{f-1}{r} f' \\ & + \frac{\lambda-1}{2} f'^2 - 2(\omega - \Lambda_W)(1 - f - rf') - 3\Lambda_W^2 r^2 \Bigg]. \end{aligned} \tag{11}$$

Kehagias and Sfetsos [10] obtained only the asymptotically flat solution (with $\Lambda_W = 0$) while Mu-In Park [21] considered an arbitrary Λ_W and obtained a general solution. Now the variations with respect to N and f give the equations of motion

$$\begin{aligned} & (2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r} f' + \frac{\lambda-1}{2} f'^2 2(\omega - \Lambda_W) \\ & \times (1 - f - rf') - 3\Lambda_W^2 r^2 = 0, \end{aligned} \tag{12}$$

and

$$\left(\frac{N}{\sqrt{f}}\right)'\left((\lambda-1)f'-2\lambda\frac{f-1}{r}+2(\omega-\Lambda_W)r\right)$$
$$+(\lambda-1)\frac{N}{\sqrt{f}}\left(f''-\frac{2(f-1)}{r^2}\right)=0. \tag{13}$$

By giving $\lambda=1$ and solving the field equations, we arrive at the Park solution [21],

$$N^2=f_{\mathrm{Park}}=1+(\omega-\Lambda_W)r^2$$
$$-\sqrt{r[\omega(\omega-2\Lambda_W)r^3+\beta]}, \tag{14}$$

where β is an integration constant related to the black hole mass. Park's solution can easily be reduced to Lü, Mei, and Pope (LMP)'s solution [8] as well as Kehagias and Sfetsos (KS)'s solution [10].

Now let us consider (14) in detail. For $r\gg[\beta/|\omega(\omega-2\Lambda_W)|]^{1/3}$, we can arrive at two solutions. First one is the asymptotically AdS case with $\Lambda_W<0$ and $\omega>0$,

$$f=1+\frac{|\Lambda_W|}{2}\left|\frac{\Lambda_W}{\omega}\right|r^2-\frac{2M}{\sqrt{1+2|\Lambda_W/\omega|}}\frac{1}{r}$$
$$+\mathcal{O}(r^{-4}), \tag{15}$$

and the second one is the asymptotically dS case with $\Lambda_W>0$ and $\omega<0$,

$$f=1-\frac{\Lambda_W}{2}\left|\frac{\Lambda_W}{\omega}\right|r^2-\frac{2M}{\sqrt{1+2|\Lambda_W/\omega|}}\frac{1}{r}+\mathcal{O}(r^{-4}). \tag{16}$$

The thermodynamics of the Park black hole has been studied in [21,49]. Now we will further investigate the different behaviors of these potentials. In general, the Park black hole solution has two horizons, one cosmological horizon and the other black hole horizon. By considering the black hole horizon r_+, the mass of the Park black hole can be written as

$$M=\frac{1+2(\omega-\Lambda_W)r_+^2+\Lambda_W^2r_+^4}{4\omega r_+}. \tag{17}$$

Using the Bekenstein–Hawking area law,

$$S=\frac{A}{4}=\pi r_+^2, \tag{18}$$

and the relation

$$\Lambda=\frac{(N-1)(N-2)}{2l^2}, \tag{19}$$

which connects the radius of the curvature l of dS or AdS space with Λ, the cosmological constant (where N is the dimension), one can arrive at the mass–entropy relation,

$$M=\frac{4S^2-4l^2\pi S+l^4\pi(\pi+2S\omega)}{4l^4\pi^{\frac{3}{2}}\omega\sqrt{S}}. \tag{20}$$

Particularly in the dS case, also there exists an upper mass bound given by

$$M_{\mathrm{bound}}=\frac{\left(\frac{4}{l^2}-\omega\right)}{4}\left(\frac{S}{\pi}\right)^{3/2}. \tag{21}$$

The thermodynamics regarding this upper mass bound is studied in [49].

Now other thermodynamic quantities like temperature, heat capacity and free energy can be obtained from the usual definitions of them,

$$T=\left(\frac{\partial M}{\partial S}\right),$$
$$C=T\left(\frac{\partial S}{\partial T}\right),$$
$$F=M-T\,S. \tag{22}$$

Here the temperature of the black hole is obtained as

$$T=\frac{12S^2-4l^2\pi S+l^4\pi(\pi-2S\omega)}{8l^2\pi^{\frac{3}{2}}\sqrt{S}(l^2(\pi+S\omega)-2S)}, \tag{23}$$

the heat capacity as

$$C=\frac{2S(12S^2-4l^2\pi S+l^4\pi(\pi-2S\omega)(l^2(\pi+S\omega)-2S)}{-24S^3+4l^2S^2(7\pi+3S\omega)+2l^4\pi S(-5\pi+4S\omega)+l^6\pi(\pi^2+5\pi S\omega-2S^2\omega^2)}, \tag{24}$$

and the free energy as

$$F=\frac{-16S^3-4l^2S^2(-6\pi+S\omega)-12l^4\pi S(\pi+S\omega)+l^6\pi(2\pi^2+7\pi S\omega+2S^2\omega^2)}{8l^4\pi^{\frac{3}{2}}\sqrt{S}\omega(l^2(\pi+S\omega)-2S)}. \tag{25}$$

We have plotted the variation of the mass against the entropy in Figs. 1 and 2 for dS and AdS case respectively. Similarly, in Figs. 3 and 4 the temperature variations are plotted. For the dS case (Fig. 3), we can see that there is an infinite discontinuity in temperature and for a certain range of S values temperature becomes negative also, which indicates the existence of some unphysical regions. These two anomalous behaviors are due to the existence of the mass bound given by (21). For the AdS case (Fig. 4) also there exist some unphysical regions. The

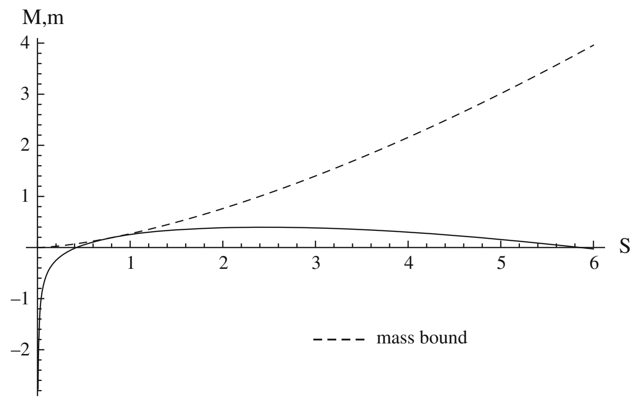

Fig. 1 Plots of mass M vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$

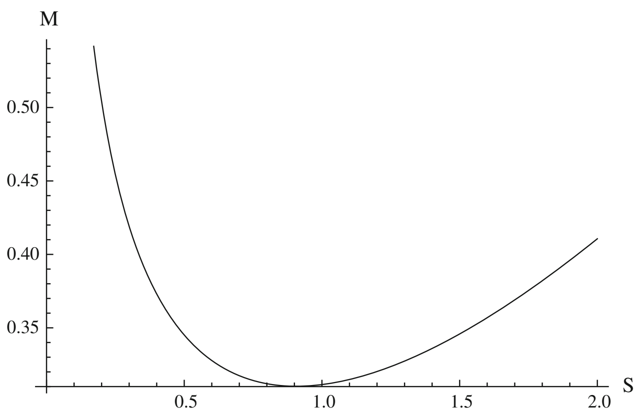

Fig. 2 Plots of mass M vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$

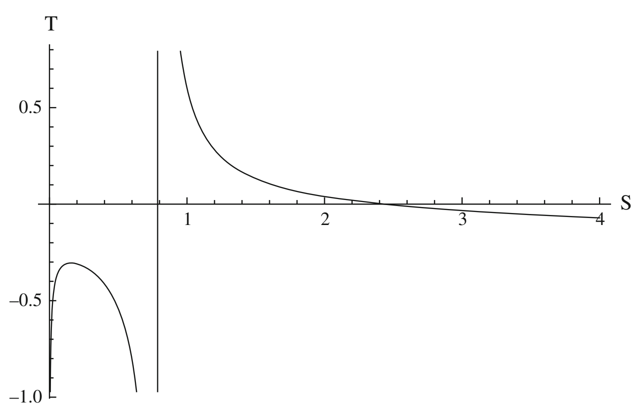

Fig. 3 Plots of temperature T vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$

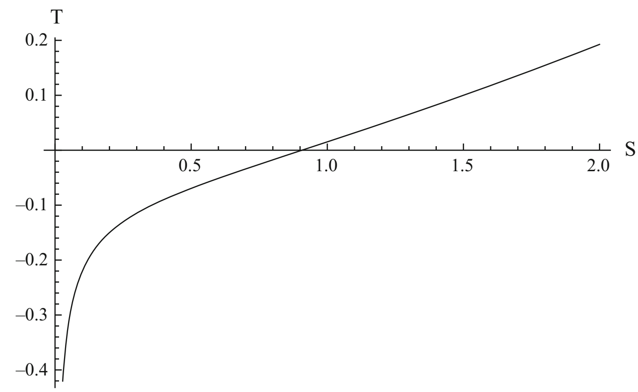

Fig. 4 Plots of temperature T vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$

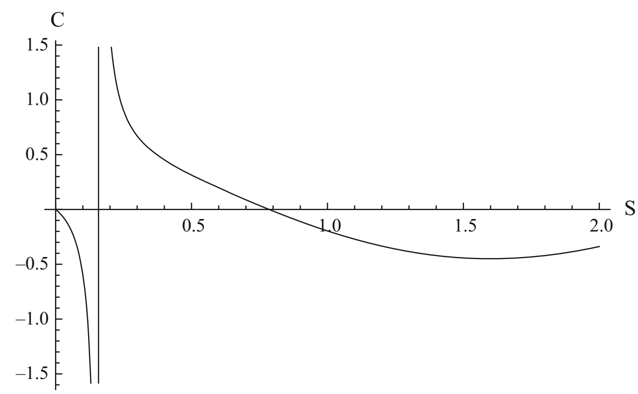

Fig. 5 Plots of the specific heat C vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$

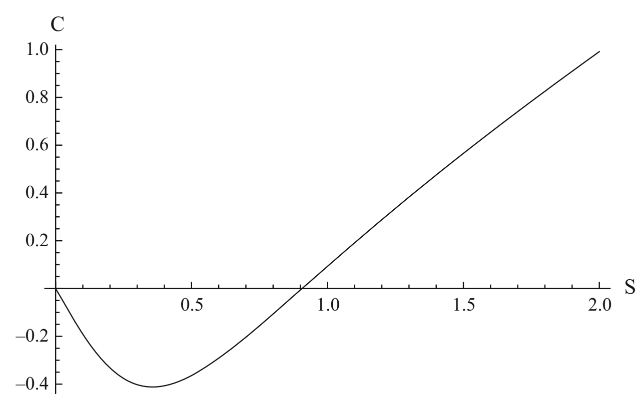

Fig. 6 Plots of the specific heat C vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$

temperature changes continuously in this case without any discontinuities.

In Figs. 5 and 6 we have plotted the specific heat of the Park black hole with entropy, while in Figs. 7 and 8, the variation of the free energy against entropy is plotted. From Fig. 5 we can see that the Park–dS black hole undergoes a phase transition from thermodynamically unstable state to a thermodynamically stable state. In Fig. 7, free energy changes from positive to negative, supportingly the black hole changes from unstable to stable state via phase transition. But for Park–AdS black hole, from Figs. 6 and 8, we can see that black hole undergoes a continuous transition from initial thermodynamically unstable phase to a stable phase and no phase transition takes place.

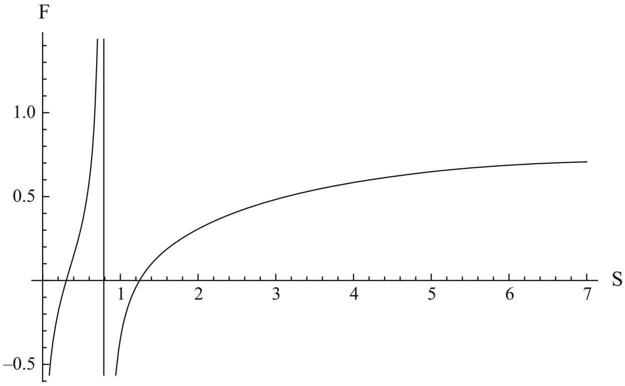

Fig. 7 Plots of free energy F vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$

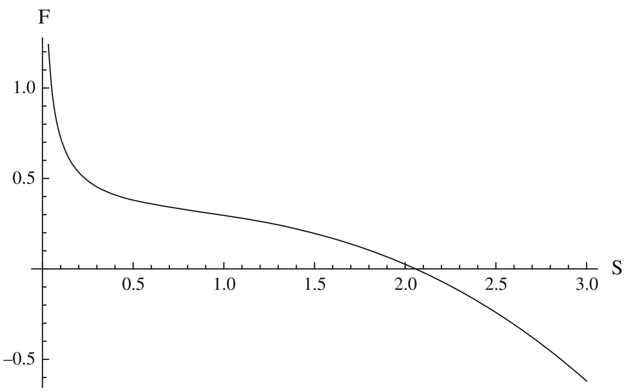

Fig. 8 Plots of free energy F vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$

$$g^W = \begin{bmatrix} M_{SS} & M_{Sl} & M_{S\omega} \\ M_{lS} & M_{ll} & M_{l\omega} \\ M_{\omega S} & M_{\omega l} & M_{\omega\omega} \end{bmatrix}$$

where $M_S = \partial M/\partial S$, etc. On calculating the curvature scalar of this metric, we arrive at

$$R^W = \frac{A(S, l, \omega)}{3[l^2\pi - 4S]^3[8l^2\pi S - 36S^2 + l^4\pi(5\pi - 4\omega S)]^2}. \tag{26}$$

where $A(S, l, \omega)$ is a complicated expression with no physical interest. From the above expression, R^W diverges at the points $S = 0.785$, $S = 2.06$ for the dS case and at $S = 1.171$ for AdS case. (From now on, throughout this paper we are choosing $l = 1$ and $\omega = -2$ for dS case and $l = -1$ and $\omega = 2$ for the AdS case. Also we are not considering imaginary as well as negative roots.) The point $S = 0.785$ or $r_+ = 0.5$ corresponds to the infinite discontinuity of temperature and free energy, and one of the points at which the specific heat becomes zero. Moreover, the mass bound is saturated at this point. But Weinhold's metric fails to explain any physical singularities in the AdS case.

Now we will consider the Ruppeiner geometry. The Ruppeiner metric can be written from (2) as

$$g^R = \frac{1}{T} \begin{bmatrix} M_{SS} & M_{Sl} & M_{S\omega} \\ M_{lS} & M_{ll} & M_{l\omega} \\ M_{\omega S} & M_{\omega l} & M_{\omega\omega} \end{bmatrix}.$$

The curvature of this metric is given by

$$R^R = \frac{B(S, l, \omega)}{[l^2\pi - 4S]^3[8l^2\pi S - 36S^2 + l^4\pi(5\pi - 4\omega S)]^2[4l^2\pi S - 12S^2 + l^4\pi(\pi - 2\omega S)][-2S + l^2(\pi + \omega S)]^3}. \tag{27}$$

So among Park–dS and Park–AdS black hole, only the dS case shows a phase transition. Also there are many regions of these plots, like the negative temperature regions, an upper mass bound, infinite discontinuity etc., whose physical meaning are still unrevealed. In the next section we will investigate further regarding this abnormalities shown by the black hole. We are aiming at a good explanation of these observations in terms of different thermodynamic geometric methods.

3 Thermodynamic geometry of the Park black hole

We now turn to the thermodynamic geometry of the Park black hole. In order to incorporate the differential geometry in to the thermodynamic case we will consider l and ω as the other extensive variables of the present thermodynamic system. Therefore the Weinhold metric can be written from (1) as

where $B(S, l, \omega)$ is also a long complicated expression with less physical interest. For dS and AdS cases, R^R possesses singularities at points $S = 0.785, 2.43$ and $S = 0.906$, respectively. The point, $S = 0.785$ is well explained by Weinhold's metric. But the point, $S = 2.43$ or $r_+ = 0.879$ is the new one that corresponds to a zero value of temperature and specific heat. For the AdS case, the point $S = 0.906$ or $r_+ = 0.537$ corresponds to the zeros in mass, temperature and specific heat.

As we mentioned in the introduction, the main problem with Weinhold's and Ruppeiner's metrics is that they are not Legendre invariant. Hence we will consider the geometrothermodynamics to explain the thermodynamics, since Legendre invariance is preserved in GTD.

For geometrothermodynamic calculations, we will consider 7-dimensional thermodynamic phase space $\mathcal{T}$. This phase space is constituted by the coordinates $Z^A = \{M, S, l, \omega, T, \iota, \vartheta\}$, where S, l, ω are extensive variables and T, ι, ϑ are their dual intensive variables. Then the fundamental Gibbs 1-form defined on $\mathcal{T}$ can be written as

$$\Theta = dM - TdS - \iota dl - \vartheta d\omega \tag{28}$$

The equilibrium phase space $\mathcal{E}$ can be defined as a simple mapping $\varphi : \{S, l, \omega\} \to \{M(S, l, \omega), S, l, \omega, T(S, l, \omega), \iota(S, l, \omega), \vartheta(S, l, \omega)$. The Quevedo metric is given from (6),

$$g^{\mathrm{GTD}} = (SM_S + lM_l + \omega M_\omega) \begin{bmatrix} -M_{SS} & 0 & 0 \\ 0 & M_{ll} & M_{l\omega} \\ 0 & M_{\omega l} & M_{\omega\omega} \end{bmatrix}.$$

The curvature scalar corresponding to the above metric is found to be

$$R^{\mathrm{GTD}} = \frac{D(S, l, \omega)}{[l^2\pi - 2s]^3[4l^2\pi s + 12s^2 + l^4\pi(3\pi - 2\omega s)]^2[-20l^2\pi s + 28s^2 + l^4\pi(3\pi - 2\omega s)]^3}, \tag{29}$$

in which $D(S, l, \omega)$ is a complicated expression of less physical interest. At points $S = 0.785$ and at $S = 0.477$ and 2.1 for dS and AdS, respectively, the Legendre invariant scalar curvature becomes zero or shows infinite discontinuities. The point $S = 0.785$ or $r_+ = 0.5$ is the same point where the phase transition takes place. To get an exact idea regarding this, we will consider the Fig. 9, which shows the correspondence between the divergence of scalar curvature R^{GTD} and the specific heat C. It is very interesting to note that the point $S = 0.477$ or $r_+ = 0.386$ in AdS case corresponds to the point of inflection in the curves of temperature, specific heat, and free energy, where the convex nature of the curve changes to a concave nature or vice versa. Similarly the point $S = 2.1$ or $r_+ = 0.817$ coincides with the point of the free energy curve where it becomes zero. So using geometrothermodynamics and hence by constructing the Legendre invariant metric, we are able to reproduce the behavior of thermodynamic potentials and their interactions. The correspondence of divergence and zeros of the thermodynamic potentials with the divergence of the Legendre invariant scalar curvature leads to the complete understanding of the Park black hole thermodynamics.

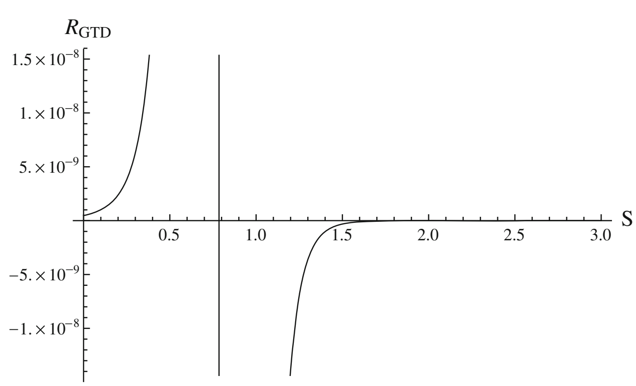

Fig. 9 Plots of scalar curvature vs. entropy for the dS black hole with $l = 1, \omega = -2$

4 Conclusion and discussion

In this paper, we have investigated the thermodynamics as well as thermodynamic geometry of the Park black hole. We have considered both the dS and the AdS cases. We have analyzed the usual thermodynamics of both these cases and found that there exist many abnormal behaviors like the existence of an upper mass bound, negative temperature, infinite discontinuity in temperature, heat capacity, and free energy, etc. We have incorporated the geometric ideas into the usual thermodynamics by means of different thermodynamic geometric methods.

We have analyzed first the thermodynamic geometry based on Weinhold's metric and Ruppeiner's metric and the GTD. We have found that the corresponding thermodynamic scalar curvature possesses many singularities, and these singularities are in accordance with the behaviors of mass, temperature, specific heat, and free energy. As we have mentioned in this work, these two methods depend entirely on the choice of the thermodynamic potential to describe the system. Even though this particular choice gives almost good results, but the lack of Legendre invariance leads us to consider a much more general geometrothermodynamic method. The potential independence of the results or in other words the Legendre invariance is ensured in this metric.

When we use GTD to explain the thermodynamics, we find that it possesses a true curvature singularity. The singularity corresponds to the points where the mass bound gets saturated, temperature shows infinite discontinuity and specific heat also shows infinite discontinuity. Park–dS black holes undergo a second-order phase transition from a thermodynamically unstable state to thermodynamically stable state while in the AdS case, there exist no such behaviors. So GTD reproduces the thermodynamics of the Park black hole, irrespective of the potential choice to explain the system. When we consider the GTD metric, it is found to be finite and smooth at the regions where the black hole is stable. But when the black hole becomes unstable, this metric possesses true singularities, and as mentioned above, this corresponds to the second-order phase transition shown by the black hole. So by incorporating the Legendre invariance as well as differential geometry, GTD is an important method to well explain the thermodynamics of black holes. Here GTD explains the second-order phase transition, the existence of a negative temperature, the point of inflection, and the upper mass bound of the Park black hole.

Acknowledgments The authors wish to thank UGC, New Delhi for financial support through a major research project sanctioned to VCK. VCK also wishes to acknowledge Associateship of IUCAA, Pune, India.

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