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# A generalized entropy optimization and Maxwell–Boltzmann distribution*

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**Abstract.** Based on the results of the diffusion entropy analysis of Super-Kamiokande solar neutrino data, a generalized entropy, introduced earlier by the first author is optimized under various conditions and it is shown that Maxwell–Boltzmann distribution, Raleigh distribution and other distributions can be obtained through such optimization procedures. Some properties of the entropy measure are examined and then Maxwell–Boltzmann and Raleigh densities are extended to multivariate cases. Connections to geometrical probability problems, isotropic random points, and spherically symmetric and elliptically contoured statistical distributions are pointed out.

## 1 Introduction

Historically, the notion of entropy emerged in conceptually distinct contexts. This paper deals with the connection between entropy, probability, and fractional dynamics as they appeared in solar neutrino astrophysics since the 1970s [1–4].

Boltzmann's derivation of the second law of thermodynamics was based on mechanics arguments. In his paper of 1872, Boltzmann considered the dynamics of binary collisions and stated that "One has therefore rigorously proved that, whatever the distribution of the kinetic energy at the initial time might have been, it will, after a very long time, always necessarily approach that found by Maxwell" [5]. Boltzmann's Stosszahlansatz, i.e. the assumption of molecular chaos used in his equation, was a statistical assumption which had no dynamical basis. His equally famous relation between entropy and probability, $S \sim \log W$, in his paper "On the relation between the second law of the mechanical theory of heat and probability theory with respect to the laws of thermal equilibrium" [6,7] was not based on dynamics. At that time Boltzmann's Stosszahlansatz was heavily criticized by Loschmidt's reversibility paradox [6,7] and Zermelo's recurrence paradox [8–10].

In the remarkable year 1900 for physics, Planck elaborated on the connection between entropy and probability based on the universality of the second law of thermodynamics and the established laws of probability and put in writing the final form of the relation between entropy S and permutability $P \sim W$ in its definitive form $S = k \log W$. He called $k$ Boltzmann's constant and came to the conclusion that in every finite region of phase space the thermodynamic probability has a finite magnitude limited by $h$, representing Planck's constant. At this point Planck introduced his quantum hypothesis [11]. Concerning Planck's hypothesis of light quanta he strictly preserved Maxwell's theory in vacuum and applied the quantum hypothesis only to matter that interacts with radiation [12].

In 1911 at the first Solvay Conference, Einstein literally put it as a requirement that one needs a fundamental theory of dynamics to make sense of Boltzmann's connection between entropy and probability, even in the case of Planck's use of Boltzmann's formula in the process of discovery of the quantum of action. Einstein's immediate reaction to Planck's extensive report at the first Solvay Congress was [13]:

"What I find strange about the way Mr. Planck applies Boltzmann's equation is that he introduces a state probability $W$ without giving this quantity a physical definition. If one proceeds in such a way, then, to begin with, Boltzmann's equation does not have a physical meaning. The circumstance that $W$ is equated to the number of complexions belonging to a state does not change anything here; for there is no indication of what is supposed to be meant by the statement that two complexions are equally probable. Even if it were possible to define the complexions in such a manner that the $S$ obtained from Boltzmann's equation agrees with experience, it seems to me that with this conception of Boltzmann's principle it is not possible to draw any conclusions about the admissibility of any fundamental theory whatsoever on the basis of the empirically known thermodynamic properties of a system."

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Recently, Brush and Segal [14] commented on the above Boltzmann–Planck–Einstein dispute from a historical point of view on how the interaction of theory and experiment in physics with available applicable mathematics and statistics lead to established theories and subsequently to predictions and explanations of natural phenomena. He perceives Planck's derivation of an equation for black-body radiation that this equation, when explored with Boltzmann's formula for entropy, implied that radiation is composed of particles. Planck, as a strong supporter of the wave theory of electromagnetic radiation, could not believe what the mathematics was telling him. Similarly, Kuhn [15] pointed out that Planck did not propose a physical quantum theory but he used quantization only as a convenient method of approximation.

Following the above reasoning of Boltzmann, Planck, and Einstein, in this paper we utilize the statistical methodology developed by Scafetta [16] by evaluating the scaling exponent of the probability density function through Boltzmann's entropy of a kind of diffusion process generated by complex fluctuations in the measurements of the solar neutrino flux in the Super-Kamiokande experiment [17–20]. This method does focus on the scaling properties of the Super-Kamiokande time series (see Fig. 1) generated by a supposedly unknown complex dynamical phenomenon. By summing the terms of such a time series one gets a trajectory and this trajectory can be used to generate a diffusion process. The method is thus based upon the evaluation of the Boltzmann entropy of the probability density function of a diffusion process. The numerical result of diffusion entropy analysis of the solar neutrino data from Super-Kamiokande is shown in Figure 2.

In principle, one can perceive the graphical result in Figure 2 of the diffusion entropy analysis of solar neutrino radiation similar to Planck's analysis of black body radiation. What physical meaning this carries remains to be seen. Assuming that the solar neutrino signal is governed by a probability density function (pdf) with scaling given by the asymptotic time evolution of a pdf, obeying the property:

$$p(x,t) = \frac{1}{t^{\delta}} F\left(\frac{x}{t^{\delta}}\right),$$

where $\delta$ denotes the scaling exponent of the pdf.

The quantum mechanics of neutrino flavour oscillations can be analyzed in a variety of ways in physics. There are treatments of this oscillation phenomenon based on plane waves, on wave packets, and on quantum field theory. These treatments have yielded the standard expression for the probability of oscillations. Neutrinos have been detected in three distinct flavours which interact in particular ways with electrons, muons, and tau leptons, respectively. Flavour oscillations occur because the flavour states are distinct from the neutrino mass states. In particular, a given flavour state may be represented as a coherent superposition of different mass states. In a recent MINOS experiment it was discovered that the phenomenon of neutrino oscillations violates the Leggett–Garg inequality, an analogue of Bell's inequality, involving correlations of

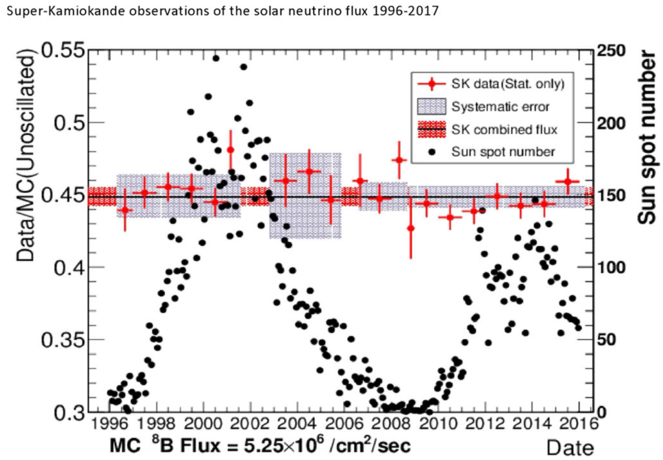


**Fig. 1.** Super-Kamiokande I, II, III, and IV solar neutrino data, http://vietnam.in2p3.fr/2017/neutrinos/program.php.

measurements on neutrino oscillations at different times [21]. The MINOS experiment analysis did show a violation of the classical limits imposed by the Leggett–Garg inequality. This provided evidence for the existence of the quantum effect of entanglement between the mass eigenstates which make up a flavour state. The entropy of entanglement [22] is an entanglement measure for a many-body quantum state and the question arises if the results shown in Figure 2 may find an interpretation in terms of the evolution of an entanglement entropy over time.

Back to Figure 2, it shows a phenomenon that follows certain scaling laws. This Diffusion Entropy Analysis (DEA) measures the correlated variations in the Super-Kamiokande solar neutrino time series. The analysis is based on the diffusion process generated by the time series and measures the time evolution of the Boltzmann entropy of the probability density function of this diffusion process. As in Brownian motion trajectories the value of a time series is interpreted as the steps of a diffusion process. The trajectories of this process are defined by the cumulative sum of these steps and obtain a different trajectory for each value of the time series over the full period of time of measurements. Subsequently the probability density function $p(x,t)$ is evaluated that describes the probability that a given trajectory has a displacement of $x$ after $t$ steps. For every particular $t$ the temporal Boltzmann entropy of the probability density function $p(x,t)$ at time $t$ is evaluated by $S(t) = \delta \log t$, where $\delta$ is the diffusion exponent. For a random uncorrelated diffusion process with finite variance, the $p(x,t)$ will converge according to the Central Limit Theorem to a Gaussian pdf which exhibits $\delta = 1/2$. Figure 2 shows that all $\delta$'s are different from the value $\delta = 1/2$. These diffusion exponents are non-Gaussian and exhibit diffusive fluctuations that cannot be modeled by random Gaussian diffusion processes.

To evaluate the Boltzmann entropy of the diffusion process at time $t$, [16] defined $S(t)$ as:

$$S(t) = -\int_{-\infty}^{+\infty} dx\, p(x,t) \ln\, p(x,t),$$

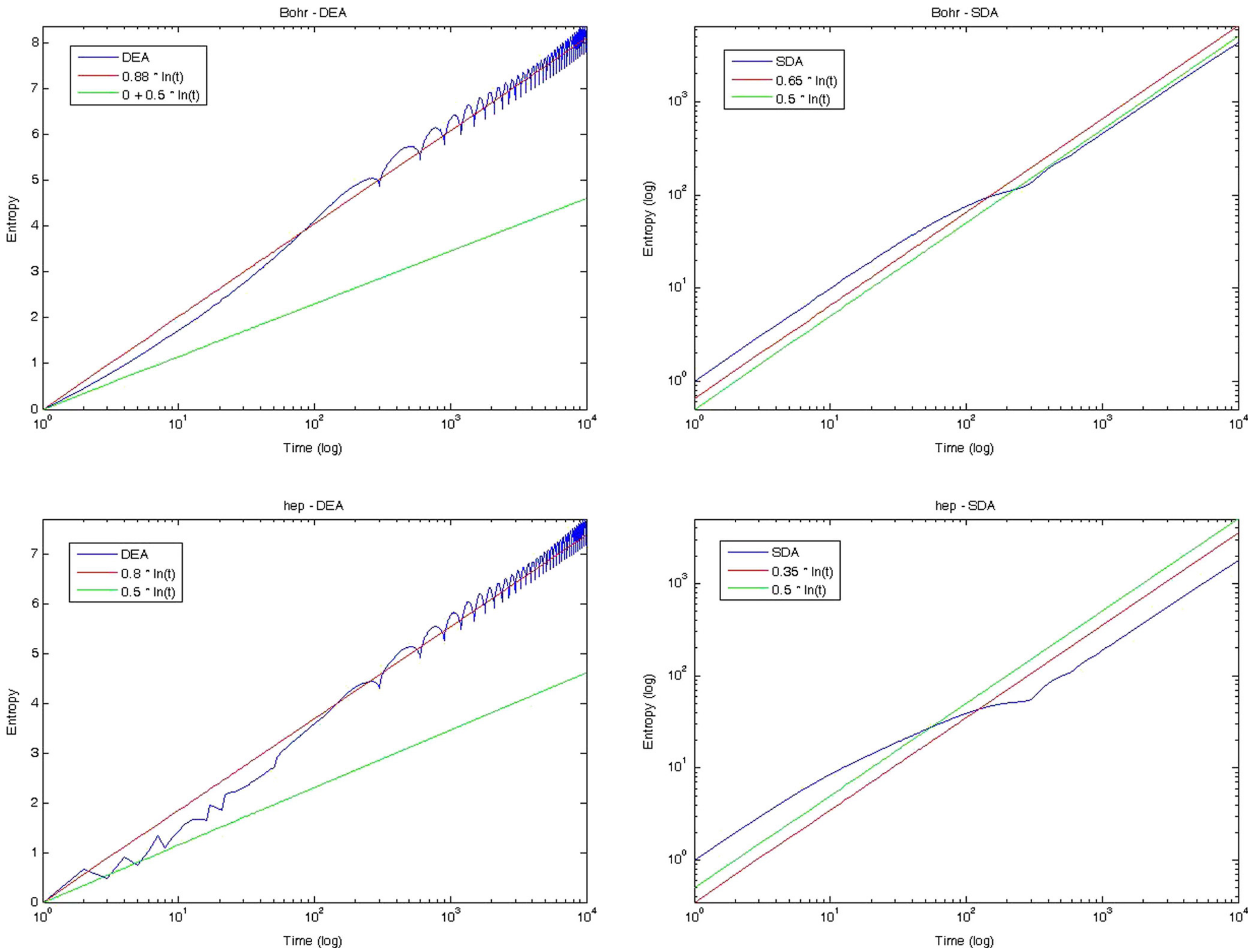


**Fig. 2.** Diffusion Entropy Analysis (DEA) and Standard Deviation Analysis (SDA) of the Super-Kamiokande I and II solar neutrino data [20].

and with the previous $p(x,t)$, one has:

$$S(t) = A + \delta \ln(t), \quad A = -\int_{-\infty}^{+\infty} dy F(y) \ln F(y).$$

The scaling exponent, $\delta$, is the slope of the entropy against the logarithmic time scale. The slope is visible in Figure 2 for the Super-Kamiokande data measured for $^8B$ and *hep*. The Hurst exponents of the Standard Deviation Analysis (SDA) of the same time series are $H = 0.66$ and $H = 0.36$ for $^8B$ and *hep*, respectively, shown in Figure 2. The pdf scaling exponents for DEA are $\delta = 0.88$ and $\delta = 0.80$ for $^8B$ and *hep*, respectively. The values for both SDA and DEA indicate a deviation from Gaussian behavior, which would require that $H = \delta = 1/2$.

Based on the discussion above in the following we consider the entropy measure

$$M_\alpha(f) = \frac{\int_x [f(x)]^{1+\frac{\delta-\alpha}{\eta}} \mathrm{d}x - 1}{\alpha - \delta}, \alpha \neq \delta, \eta > 0, \alpha < \eta + \delta, \tag{1}$$

where $\delta$ is a real number or anchoring point, $\alpha$ is a varying parameter, $\eta > 0$ is the measuring unit of the distance $\delta - \alpha$, $x$ is a real scalar or vector or matrix variable. We will use small $x$ to denote a scalar variable and capital $X$ to denote a vector or matrix variable. The $\mathrm{d}X$ stands for the wedge product of differentials. The $f(x)$ is a real-valued scalar function of $x$ such that $f(x) \geq 0$ for all $x$ and $\int_x f(x)\mathrm{d}x = 1$ or $f(x)$ is a statistical density. If $X$ is a $p \times 1$ vector with $X' = (x_1, \ldots, x_p)$ denoting its transpose, then $\mathrm{d}X = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_p$. If $X = (x_{ij})$ is a $m \times n$ matrix with distinct real scalar variable elements $x_{ij}$'s then $\mathrm{d}X = \wedge_{i=1}^m \wedge_{j=1}^n \mathrm{d}x_{ij}$. Here $\delta$ is an anchoring point, any real number including zero. When $\alpha \to \delta$ then $1 + \frac{\delta-\alpha}{\eta} \to 1$ and $\int_x [f(x)]^{1+\frac{\delta-\alpha}{\eta}} \mathrm{d}x - 1 \to 0$. Also $\alpha - \delta \to 0$. Hence

$$\begin{aligned} \lim_{\alpha \to \delta} M_\alpha(f) &= \lim_{\alpha \to \delta} \left\{ \frac{\int_x [f(x)]^{1+\frac{\delta-\alpha}{\eta}} \mathrm{d}x - 1}{\alpha - \delta} \right\} \\ &= -\frac{1}{\eta} \int_x f(x) \ln f(x) \mathrm{d}x, \end{aligned}$$

which is Boltzmann's measure of entropy when $x$ is a real scalar variable. Hence (1) is a generalization of Shannon's entropy measure in the real scalar variable case as well as extended to vector and matrix-variate cases. Observe that

$$\int_x [f(x)]^{1+\frac{\delta-\alpha}{\eta}}\mathrm{d}x = \int_x [f(x)]^{\frac{\delta-\alpha}{\eta}} f(x)\mathrm{d}x = E[f(x)]^{\frac{\delta-\alpha}{\eta}},$$

where $E$ denotes statistical expectation. When $\alpha=\delta$ this expected value is $E(1)=1$. Here $\delta$ is a fixed point on the real line and the entropy is anchored at the point $x=\delta$ when $x$ is a scalar variable. When $\alpha=\delta$, $M_\alpha(f)$ goes to Boltzmann's entropy and hence $M_\alpha(f)$ can be taken as a measure of departure from Boltzmann's entropy, departure measured in terms of $\frac{\delta-\alpha}{\eta}$ units. If $\alpha=\delta$ gives a stable stage in a physical situation then when $\alpha$ moves away from $\delta$ then $M_\alpha(f)$ will measure the entropy in the neighborhood of the stable stage. Later it will be shown that $\alpha$ can describe a pathway for the movement from a stable situation to the unstable neighborhoods.

We will start with our discussion when $x$ is a real scalar variable. First we will optimize (1) and obtain the Maxwell–Boltzmann density.

### 1.1 Entropy optimization for the Maxwell–Boltzmann density

One form of the Maxwell–Boltzmann velocity density is the following:

$$f(v) = \begin{cases} \frac{4}{\sqrt{\pi}}\beta^{\frac{3}{2}}v^2\mathrm{e}^{-\beta v^2}, 0\le v<\infty, \beta=\frac{m}{2kT} \\ 0,\ \text{elsewhere.}\end{cases} \quad (2)$$

Note that $f(v)\ge 0$ for all $v$ and $\int_0^\infty f(v)\mathrm{d}v = 1$. It is a statistical density. Consider an arbitrary density for a real scalar positive variable $x$, denoted by $f(x)$, and consider the following moments for the real scalar positive variable $x$:

$$\mu_1 = E[x^{2(\frac{\delta-\alpha}{\eta})}] = \int_0^\infty x^{2(\frac{\delta-\alpha}{\eta})} f(x)\mathrm{d}x, \eta>0, \quad (3)$$

and

$$\mu_2 = E[x^{2(\frac{\delta-\alpha}{\eta})+2}] = \int_0^\infty x^{2(\frac{\delta-\alpha}{\eta})+2} f(x)\mathrm{d}x, \eta>0. \quad (4)$$

Observe that when $\alpha=\delta$, (3) says $E(1)=1$ with respect to any density $f(x)$, and (4) says about the second moment of the arbitrary density $f(x)$. Let us assume that $\mu_1$ and $\mu_2$ are fixed or given in the class of all densities $f(x)$ or in the set of all real-valued scalar functions $f(x)$ such that $f(x)\ge 0$ for all $x$ and $\int_x f(x)\mathrm{d}x=1$. Let us optimize the entropy in (1) under the constraints that $\mu_1$ and $\mu_2$ are given for fixed $\delta,\alpha,\eta$ with $\eta>0$. If we use calculus of variation to optimize (1) then the Euler equation is the following:

$$(i)\quad \frac{\partial}{\partial f}\{f^{1+\frac{\delta-\alpha}{\eta}} - \lambda_1 x^{2(\frac{\delta-\alpha}{\eta})} f + \lambda_2 x^{2(\frac{\delta-\alpha}{\eta})+2} f\} = 0, \quad (5)$$

where $\lambda_1$ and $\lambda_2$ are Lagrangian multipliers. Then (5) gives

$$(ii)\quad \left(1+\frac{\delta-\alpha}{\eta}\right) f^{\frac{\delta-\alpha}{\eta}} - \lambda_1 x^{2(\frac{\delta-\alpha}{\eta})} + \lambda_2 x^{2(\frac{\delta-\alpha}{\eta})+2} = 0. \quad (6)$$

That is,

$$(iii)\quad f = \left[\frac{1}{1+\frac{\delta-\alpha}{\eta}}\right]^{\frac{\eta}{\delta-\alpha}} x^2 \left[1-\frac{\lambda_2}{\lambda_1}x^2\right]^{\frac{\eta}{\delta-\alpha}}. \quad (7)$$

By taking $\frac{\lambda_2}{\lambda_1} = a(\delta-\alpha), \alpha<\delta, a>0$ and $c_1 = \left[\frac{1}{1+\frac{\delta-\alpha}{\eta}}\right]^{\frac{\eta}{\delta-\alpha}}$ we have

$$f_1(x) = c_1 x^2[1-a(\delta-\alpha)x^2]^{\frac{\eta}{\delta-\alpha}}, a>0, \alpha<\delta, \quad (8)$$

where $1-a(\delta-\alpha)x^2>0$, $c_1$ can act as the normalizing constant when $f_1(x)$ is a statistical density, and $f_1(x)=0$ elsewhere. If $\alpha>\delta$ then write $\delta-\alpha = -(\alpha-\delta), \alpha>\delta$ and then (8) is transformed to

$$f_2(x) = c_2 x^2[1+a(\alpha-\delta)x^2]^{-\frac{\eta}{\alpha-\delta}}, \quad \alpha>\delta, \eta>0, a>0, x\ge 0, \quad (9)$$

and zero elsewhere, where $c_2$ can act as the normalizing constant. Observe that $c_2$ is different from $c_1$, and in fact that the two functions $f_1(x)$ and $f_2(x)$ are structurally different, one belonging to the generalized type-1 beta family of densities and the other belonging to the generalized type-2 beta family of densities, where the support of $f_1(x)$ is finite $0\le x\le [a(\delta-\alpha)]^{-\frac{1}{2}}$, and $f_2(x)$ has the support $0\le x<\infty$. When $\alpha\to\delta$ then both $f_1(x)$ and $f_2(x)$ go to

$$f_3(x) = c_3 x^2\mathrm{e}^{-a\eta x^2}, \eta>0, a>0, x\ge 0, \quad (10)$$

and zero elsewhere, where $c_3$ is the normalizing constant. Note that (10) is a form of the Maxwell–Boltzmann velocity density. Observe that from (8) one can go to (9) and (10). Also one can go from (9) to (8) and (10). Hence (8) or (9) is called a pathway version of the Maxwell–Boltzmann density, where $\delta$ is a fixed point such as $\delta=1$ and $\alpha$ is the pathway parameter and the departure from the point $x=\delta$ is measured in terms of $\eta$ units, $\eta>0$. If we want to incorporate the parameters in the Maxwell–Boltzmann density then we may take $a\eta = \frac{m}{2kT}$ or $a=\frac{m}{\eta 2kT}$. Then (8)–(10) reduce to the following:

$$f_1^*(x) = c_1^* x^2\left[1-(\delta-\alpha)\left(\frac{m}{\eta 2kT}\right)x^2\right]^{\frac{\eta}{\delta-\alpha}}, \alpha<\delta, \eta>0, \quad (11)$$

$$1-(\delta-\alpha)\left(\frac{m}{\eta 2kT}\right)x^2>0,$$

$$f_2^*(x) = c_2^* x^2 \left[1 + (\alpha - \delta)\left(\frac{m}{\eta 2kT}\right)x^2\right]^{-\frac{\eta}{\alpha-\delta}},$$
$$\alpha > \delta, x \geq 0, \tag{12}$$
$$f_3^*(x) = c_3^* x^2 \mathrm{e}^{-\frac{m}{2kT}x^2}, x \geq 0. \tag{13}$$

We can evaluate $c_1^*$ by using a type-1 beta integral, $c_2^*$ by using a type-2 beta integral and $c_3^*$ by using a gamma integral. The results are the following:

$$c_1^* = \frac{4\Gamma(\frac{\eta}{\delta-\alpha}+\frac{5}{2})(\delta-\alpha)^{\frac{3}{2}}(\frac{m}{\eta 2kT})^{\frac{3}{2}}}{\sqrt{\pi}\Gamma(\frac{\eta}{\delta-\alpha}+1)}, \alpha < \delta, \eta > 0, \tag{14}$$
$$c_2^* = \frac{4}{\sqrt{\pi}} \frac{\Gamma(\frac{\eta}{\alpha-\delta})(\alpha-\delta)^{\frac{3}{2}}(\frac{m}{\eta 2kT})^{\frac{3}{3}}}{\Gamma(\frac{\eta}{\alpha-\delta}-\frac{3}{2})},$$
$$\alpha > \delta, \frac{\eta}{\alpha-\delta} - \frac{3}{2} > 0, \tag{15}$$
$$c_3^* = \frac{4}{\sqrt{\pi}}\left(\frac{m}{\eta 2kT}\right)^{\frac{3}{2}}. \tag{16}$$

We will call (11) and (12) as the pathway generalized Maxwell–Boltzmann density.

## 2 Raleigh density and optimization of the generalized entropy

One form of Raleigh density is the following:

$$g(x) = \frac{1}{\gamma^2} x \mathrm{e}^{-\frac{x^2}{2\gamma^2}}, 0 \leq x < \infty, \gamma > 0, \tag{17}$$

and zero elsewhere. If (17) is to be obtained from the generalized entropy (1) then consider the following constraints:

$$\nu_1 = E[x^{\frac{\delta-\alpha}{\eta}}] = \int_0^\infty x^{\frac{\delta-\alpha}{\eta}} g(x)\mathrm{d}x, \eta > 0, \tag{18}$$

and

$$\nu_2 = E[x^{\frac{\delta-\alpha}{\eta}+2}] = \int_0^\infty x^{\frac{\delta-\alpha}{\eta}+2} g(x)\mathrm{d}x, \eta > 0. \tag{19}$$

Assuming that $\nu_1$ and $\nu_2$ are fixed for an arbitrary density $g(x)$, for fixed $\alpha, \delta, \eta$ and proceeding as in Section 1.1 we have the following densities corresponding to $f_1, f_2, f_3$ in Section 1.1:

$$g_1(x) = d_1\ x[1 - a(\delta-\alpha)x^2]^{\frac{\eta}{\delta-\alpha}}, a > 0, \alpha < \delta, \eta > 0, \tag{20}$$
$$1 - (\delta - \alpha)x^2 > 0,$$
$$g_2(x) = d_2\ x[1 + a(\alpha-\delta)x^2]^{-\frac{\eta}{\alpha-\delta}}, a > 0,$$
$$\alpha > \delta, \eta > 0, x \geq 0 \tag{21}$$
$$g_3(x) = d_3\ x\mathrm{e}^{-a\eta x^2}, a > 0, \eta > 0, x \geq 0, \tag{22}$$

where $g_1, g_2, g_3$ are zero outside the support indicated above, and $d_1, d_2, d_3$ are the respective normalizing constants. Comparing (22) with the Raleigh density in (17) we may take $a\eta = \frac{1}{2\gamma^2}$ or $a = \frac{1}{\eta 2\gamma^2}$. Then $g_1, g_2$ transform to the following:

$$g_1^*(x) = d_1^*\ x\left[1 - (\delta-\alpha)\left(\frac{1}{\eta 2\gamma^2}\right)x^2\right]^{\frac{\eta}{\delta-\alpha}},$$
$$\alpha < \delta, \eta > 0, \tag{23}$$
$$1 - (\delta-\alpha)\left(\frac{1}{\eta 2\gamma^2}\right)x^2 > 0,$$
$$g_2^*(x) = d_2^*\ x\left[1 + (\alpha-\delta)\left(\frac{1}{\eta 2\gamma^2}\right)x^2\right]^{-\frac{\eta}{\alpha-\delta}},$$
$$\alpha > \delta, \eta > 0, x \geq 0, \tag{24}$$

where the normalizing constants $d_1^*$ and $d_2^*$ can be evaluated by using a type-1 beta integral and a type-2 beta integral, respectively. Then the resulting densities are the following:

$$g_1^*(x) = \frac{\eta+\delta-\alpha}{\eta\gamma^2} x\left[1 - \frac{(\delta-\alpha)}{2\eta\gamma^2}x^2\right]^{\frac{\eta}{\delta-\alpha}}, \alpha < \delta, \eta > 0, \tag{25}$$
$$g_2^*(x) = \frac{\eta+\delta-\alpha}{\eta\gamma^2} x\left[1 + \frac{(\alpha-\delta)}{2\eta\gamma^2}x^2\right]^{-\frac{eta}{\alpha-\delta}},$$
$$\alpha > \delta, \eta > 0, x \geq 0, \tag{26}$$
$$\eta + \delta - \alpha > 0,$$
$$g_3^*(x) = \frac{1}{\gamma^2} x\mathrm{e}^{-\frac{x^2}{2\gamma^2}}, x \geq 0, \gamma > 0. \tag{27}$$

If we take the anchoring point $\delta = 1$ and the parameter $\eta = 1$ then the normalizing constants $d_1^* = d_2^* = \frac{2-\alpha}{\gamma^2}$. Then for $\alpha < 1$ one has type-1 beta case in (25), for $\alpha > 1$ one has the type-2 beta case in (26) and for $\alpha \to 1$ one has the gamma case in (27).

## 3 Some general observations

Consider the generalized entropy in (1). This can be written as

$$\begin{aligned} M_\alpha(f) &= \frac{\int_x [f(x)]^{1+\frac{\delta-\alpha}{\eta}}\mathrm{d}x - 1}{\alpha-\delta} \\ &= \frac{\int_x [f(x)]^{1+\frac{\delta-\alpha}{\eta}}\mathrm{d}x - \int_x f(x)\mathrm{d}x}{\alpha-\delta} \\ &= \int_x f(x)\frac{[f(x)^{\frac{\delta-\alpha}{\eta}} - 1]}{\alpha-\delta}\mathrm{d}x \\ &= E\left[\frac{f^{\frac{\delta-\alpha}{\eta}} - 1}{\alpha-\delta}\right], \end{aligned} \tag{28}$$

where $E$ denotes the expected value. When $\alpha = \delta$ we have $f^{\frac{\delta-\alpha}{\eta}} = 1$. Thus, depending upon the departure of $\alpha$ from the fixed anchoring point $x=\delta$ we have a very small or larger departure from 1 in $f^{\frac{\delta-\alpha}{\eta}}$. Also

$$\lim_{\alpha\to\delta}\frac{f^{\frac{\delta-\alpha}{\eta}}-1}{\alpha-\delta}=\lim_{\alpha\to\delta}\frac{\mathrm{e}^{\frac{\delta-\alpha}{\eta}\ln f}-1}{\alpha-\delta}$$
$$=\frac{\lim_{\alpha\to\delta}\frac{\partial}{\partial\alpha}[\mathrm{e}^{\frac{\delta-\alpha}{\eta}\ln f}-1]}{\lim_{\alpha\to\delta}\frac{\partial}{\partial\alpha}[\alpha-\delta]}$$
$$=-\frac{1}{\eta}\ln f. \quad (29)$$

That is,

$$\lim_{\alpha\to\delta}M_\alpha(f)=-\frac{1}{\eta}E[\ln f]=-\frac{1}{\eta}\int_x f(x)\ln f(x)\mathrm{d}x=S,$$

where $S$ denotes Boltzmann's entropy. Hence

$$S=-\frac{1}{\eta}E[\ln f]=E\left[-\frac{1}{\eta}\ln f\right]=\lim_{\alpha\to\delta}M_\alpha(f).$$

Hence, what is done in $M_\alpha(f)$ is to approximate $-\frac{1}{\eta}\ln f$ by $\frac{f^{\frac{\delta-\alpha}{\eta}}-1}{\alpha-\delta}$, where $\delta$ is a fixed anchoring point, $\eta$ is fixed and positive and $\alpha$ can vary, where $\alpha<\delta,\alpha>\delta,\alpha\to\delta$. Consider a simple example. Let $x=$ energy generated in a physical system. Then the physical law of conservation of energy can be stated as an expected value $E$ in statistical terms, that is, $E(x)$ is fixed in the density $f(x)$ of $x$. That is,

$$E(x)=\int_0^\infty xf(x)\mathrm{d}x=\text{ fixed}.$$

For example, if $f(x)$ is the exponential density, $f(x)=c\mathrm{e}^{-cx},c>0,x\geq 0$ then $E(x)=\frac{1}{c}$. Instead of $E(x)=$ fixed, let us consider a slight disturbance and consider the following constraints:

$$E[x^{\frac{\delta-\alpha}{\eta}}]=\text{ fixed and } E[x^{\frac{\delta-\alpha}{\eta}+1}]=\text{ fixed}. \quad (30)$$

When $\alpha\to\delta$ then the two restrictions above are $E(1)=1$ and $E(x)$ is fixed or the law of conservation of energy. Hence in (30) we consider only a slight disturbance to the law of conservation of energy. Let us consider an arbitrary density $f(x)$ and let us optimize $M_\alpha(f)$ of (1) under the constraints in (30). Then proceeding as in the derivation from (17) to (22) we see that the Euler equation, if we use calculus of variations for the optimization, as the following:

$$\frac{\partial}{\partial f}\{f^{1+\frac{\delta-\alpha}{\eta}}-\lambda_1x^{\frac{\delta-\alpha}{\eta}}f+\lambda_2x^{1+\frac{\delta-\alpha}{\eta}}f\}=0, \quad (31)$$

where $\lambda_1$ and $\lambda_2$ are the Lagrangian multipliers. Then (31) gives

$$f_1(x)=c_1\ x[1-a(\delta-\alpha)x]^{\frac{\eta}{\delta-\alpha}},\alpha<\delta,\eta>0, \quad (32)$$

for $1-a(\delta-\alpha)x>0,a>0$ and zero elsewhere, where $\frac{\lambda_2}{\lambda_1}$ is taken as $a(\delta-\alpha),a>0$ and $c_1$ can act as the normalizing constant. For $\alpha>\delta$, (32) changes to the following:

$$f_2(x)=c_2\ x[1+a(\alpha-\delta)x]^{-\frac{\eta}{\alpha-\delta}},\alpha>\delta,a>0,x\geq 0, \quad (33)$$

and zero elsewhere. When $\alpha\to\delta$, both (32) and (33) go to

$$f_3(x)=c_3\ x\mathrm{e}^{-a\eta x},a>0,\eta>0,x\geq 0, \quad (34)$$

and zero elsewhere. This is Maxwell–Boltzmann's energy density. Note that (32)–(34) give the pathway form of the Maxwell–Boltzmann density which is available by optimizing $E[\frac{f^{\frac{\delta-\alpha}{\eta}}-1}{\alpha-\delta}]$ which is an approximation to $E[-\frac{1}{\eta}\ln f]$, under the constraints $E[x^{\frac{\delta-\alpha}{\eta}}]$ is fixed and $E[x^{1+\frac{\delta-\alpha}{\eta}}]$ is fixed, which correspond to a slight disturbance from the law of conservation of energy. Note that if in (34), $a\eta=\frac{1}{kT}$ where $k$ is Boltzmann's constant and $T$ is the temperature, then the densities in (32)–(34) change to the following:

$$f_{11}(x)=c_1\ x\left[1-\frac{1}{kT}\left(\frac{\delta-\alpha}{\eta}\right)x\right]^{\frac{\eta}{\delta-\alpha}},\alpha<\delta,\eta>0, \quad (35)$$
$$0\leq x\leq\frac{\eta kT}{\delta-\alpha},$$
$$f_{21}(x)=c_2\ x\left[1+\frac{1}{kT}\left(\frac{\alpha-\delta}{\eta}\right)x\right]^{-\frac{\eta}{\alpha-\delta}},$$
$$\alpha>\delta,\eta>0,x\geq 0, \quad (36)$$
$$f_{31}(x)=c_3\ x\mathrm{e}^{-\frac{x}{kT}},x\geq 0. \quad (37)$$

Note that if $a$ in (32)–(34) is taken as 1 then $\eta=\frac{1}{kT}$. Then the densities in (35)–(37) change to the following:

$$f_{12}(x)=c_1\ x[1-(\delta-\alpha)x]^{\frac{1}{kT(\delta-\alpha)}},$$
$$\alpha<\delta,1-(\delta-\alpha)x>0, \quad (38)$$
$$f_{22}(x)=c_2\ x[1+(\alpha-\delta)x]^{-\frac{1}{kT(\alpha-\delta)}},\alpha>\delta,x\geq 0, \quad (39)$$
$$f_{32}(x)=c_3\ x\mathrm{e}^{-\frac{x}{kT}},x\geq 0. \quad (40)$$

In this case the restrictions can be stated as the following:

$$E[x^{kT(\delta-\alpha)}]=\text{ fixed and } E[x^{kT(\delta-\alpha)+1}]=\text{ fixed},\alpha<\delta.$$

These are only slight deviations from $E(1)=1$ and $E(x)=$ fixed. As before, the normalizing constants can be evaluated with the help of type-1 beta, type-2 beta, and gamma integrals and they are the following:

$$c_1=\frac{(\delta-\alpha)^2\Gamma(\frac{1}{kT(\delta-\alpha)}+3)}{\Gamma(\frac{1}{kT(\delta-\alpha)}+1)},\alpha<\delta, \quad (41)$$

$$c_2 = \frac{(\alpha-\delta)^2 \Gamma(\frac{1}{kT(\alpha-\delta)})}{\Gamma(\frac{1}{kT(\alpha-\delta)} - 2)}, \alpha > \delta, \frac{1}{kT(\alpha-\delta)} - 2 > 0, \tag{42}$$

$$c_3 = \frac{1}{(kT)^2}. \tag{43}$$

### 3.1 Differential pathway

Consider the differential equation for (38). Denoting $\frac{\mathrm{d}}{\mathrm{d}x} f_{12}(x) = f'_{12}(x)$ we have the following:

$$f'_{12}(x) = f_{12}(x)\left[\frac{1}{x} - \frac{1}{kT[1-(\delta-\alpha)x]}\right], \alpha < \delta. \tag{44}$$

Similar differential equations corresponding to (42) and (43) can be derived. This will provide a differential pathway.

If Boltzmann's entropy is optimized subject to the constraint that the first moment is fixed then we automatically arrive at (28) or (40).

### 3.2 Evaluation of $\delta - \alpha$ for $\alpha < \delta$

From (38) or (41) the first moment $E(x)$ is the following:

$$\begin{aligned} E(x) &= \frac{2}{\delta-\alpha}\left[\frac{1}{\frac{1}{kT(\delta-\alpha)} + 3}\right] \Rightarrow (\delta-\alpha) \\ &= \frac{2kT - E(x)}{3kTE(x)}, \alpha < \delta. \end{aligned} \tag{45}$$

But, through the constraint, $E(x)$ is fixed. Then $(\delta - \alpha)$ is evaluated in terms of $E(x)$. Note that in the stable situation $\alpha \to \delta$ we have $E(x) = kT$. For $\alpha > \delta$,

$$\alpha - \delta = \frac{E(x) - 2kT}{3kTE(x)}. \tag{46}$$

## 4 Generalized energy density

If $M_\alpha(f)$ of (1) is optimized under the conditions

$$E[x^{\gamma(\frac{\delta-\alpha}{\eta})}] = \text{ fixed and } E[x^{\gamma(\frac{\delta-\alpha}{\eta})+\rho}] = \text{ fixed} \tag{47}$$

for some $\gamma > 0, \rho > 0$, then proceeding as in (31)–(34) one gets the following models:

$$\begin{aligned} g_1(x) &= \hat{c}_1 x^\gamma \left[1 - a\frac{(\delta-\alpha)}{\eta} x^\rho\right]^{\frac{\eta}{\delta-\alpha}}, \\ &\qquad \alpha < \delta, a > 0, \rho > 0, \gamma > 0, \end{aligned} \tag{48}$$

$$0 \le x \le \left[\frac{\eta}{a(\delta-\alpha)}\right]^{\frac{1}{\rho}},$$

$$\begin{aligned} g_2(x) &= \hat{c}_2 x^\gamma \left[1 + a\frac{(\alpha-\delta)}{\eta} x^\rho\right]^{-\frac{\eta}{\alpha-\delta}}, \\ &\qquad \alpha > \delta, a > 0, \rho > 0, \gamma > 0, x \ge 0, \end{aligned} \tag{49}$$

$$g_3(x) = \hat{c}_3 x^\gamma \mathrm{e}^{-ax^\rho}, a > 0, \rho > 0, x \ge 0. \tag{50}$$

The models in (48) or (49) is the pathway model of [23] for the real scalar positive variable case. For $\gamma = \rho - 1$ one has the power transformed version of the energy densities in (32)–(34). If $\gamma = 1, \rho = 2$ then we have extended Raleigh density in (47) and (48) and when $\alpha \to \delta$ it is the Raleigh density. If $\gamma = 2, \rho = 2$ then we have extended Maxwell–Boltzmann density in (47) and (48), and when $\alpha \to \delta$ it is the Maxwell–Boltzmann density.

## 5 Multicomponent energy generation

Consider the matrix $X = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$. Let the total energy produced by $X$ be a norm of $X$, say $\|X\|$. Taking the Euclidean norm $\|X\| = (x_1^2 + \dots + x_p^2)^{\frac{1}{2}}$, where $x_1, \dots, x_p$ are the real components of $X$, we can look at the density of $u = \|X\|^2 = x_1^2 + \dots + x_p^2$. Let $\mathrm{d}X = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_p$ the wedge product of the differentials $\mathrm{d}x_j$'s. Then Mathai's entropy (1) in this case is the following:

$$\begin{aligned} M_\alpha(f) &= \frac{\int_X [f(X)]^{1+\frac{\delta-\alpha}{\eta}} \mathrm{d}X - 1}{\alpha - \delta}, \alpha \ne \delta, \eta > 0, \\ \mathrm{d}X &= \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_p, \end{aligned} \tag{51}$$

and $f$ is a density, that is $f(X) \ge 0$ for all $X$ and $\int_X f(X)\mathrm{d}X = 1$. Let us optimize (1) under the conditions

$$E[u^{\gamma\frac{(\delta-\alpha)}{\eta}}] = \text{ fixed and } E[u^{\gamma\frac{(\delta-\alpha)}{\eta}+\rho}] = \text{ fixed}. \tag{52}$$

Observe that when $\alpha = \delta$ the first condition is $E(1) = 1$ and the second condition is that the $\rho$th moment is fixed. Going through the steps as in earlier sections the density $f(X)$ is the following for the three cases $\alpha < \delta, \alpha > \delta, \alpha \to \delta$, denoted by $g_1(X), g_2(X), g_3(X)$, respectively:

$$\begin{aligned} g_1(X) &= C_1 u^\gamma \left[1 - a\left(\frac{\delta-\alpha}{\eta}\right) u^\rho\right]^{\frac{\eta}{\delta-\alpha}}, \\ &\qquad \alpha < \delta, a > 0, \eta > 0, \rho > 0, \gamma > 0 \\ &= C_1 (x_1^2 + \dots + x_p^2)^\gamma \left[1 - a\left(\frac{\delta-\alpha}{\eta}\right)\right. \\ &\qquad \left.\times (x_1^2 + \dots + x_p^2)^\rho\right]^{\frac{\eta}{\delta-\alpha}}, \alpha < \delta, \end{aligned} \tag{53}$$

$$0 \le \|X\| \le \left[\frac{\eta}{a(\delta-\alpha)}\right]^{\frac{1}{\rho}},$$

$$\begin{aligned} g_2(X) &= C_2 (x_1^2 + \dots + x_p^2)^\gamma \left[1 + a\left(\frac{\alpha-\delta}{\eta}\right)\right. \\ &\qquad \left.\times (x_1^2 + \dots + x_p^2)^\rho\right]^{-\frac{\eta}{\alpha-\delta}}, \\ &\qquad \alpha > \delta, \eta > 0, a > 0, \|X\| \ge 0, \end{aligned} \tag{54}$$

$$g_3(X) = C_3(x_1^2 + \cdots + x_p^2)^\gamma \mathrm{e}^{-a\eta(x_1^2+\cdots+x_p^2)^\rho},$$
$$a > 0, \eta > 0, \|X\| \geq 0, \quad (55)$$

where $C_1, C_2, C_3$ are the normalizing constants. The densities in (53)–(55) are also connected with isotropic random points, type-1 beta, type-2 beta and gamma distributed, in geometrical probability problems, see for example [24]. Observe that $\|X\|^2$ is invariant under orthonormal transformations on $X$. That is, if $Y = QX$ with $QQ' = I, Q'Q = I$ where a prime denotes the transpose and $I$ is the identity matrix, then

$$Y'Y = X'X = \|X\|^2 = \|Y\|^2 = y_1^2 + \cdots + y_p^2.$$

In statistical problems, the components of $X$, namely $x_j$'s, may be correlated and one may want to make the components noncorrelated. Then we consider the transformation $Y = V^{-\frac{1}{2}}X$ where the $p \times p$ matrix $V > O$ (positive definite) is the covariance matrix and $V^{\frac{1}{2}}$ denotes the positive definite square root to $V$. In this case the norm $\|V^{-\frac{1}{2}}X\| = X'V^{-1}X = \|Y\|^2$, or it is a positive definite quadratic form of the type $X'AX, A = A' > O$ where $A = V^{-1}$ in the correlated case. Observe that $X'AX = c > 0$ is an ellipsoid in the $p$-space. In this situation, if we consider an orthonormal transformation $X = Q'Y, QQ' = I, Q'Q = I$ then

$$X'AX = Y'QAQ'Y = Y'DY = \lambda_1 y_1^2 + \cdots + \lambda_p y_p^2, \quad (56)$$

where $\lambda_j > 0, j = 1, \ldots, p$ are the eigenvalues of $A > O$. Then, instead of the spherically symmetric densities in (53)–(55) we end up with the elliptically contoured densities. If Mathai's entropy $M_\alpha(f)$ is optimized under the conditions $E[v^{\gamma(\frac{\delta-\alpha}{\eta})}]$ is fixed and $E[v^{\gamma(\frac{\delta-\alpha}{\eta})+\rho}]$ is fixed, where $v = X'AX$ then we have the following densities:

$$g_4(X) = C_4(X'AX)^\gamma \left[1 - a\left(\frac{\delta-\alpha}{\eta}\right)(X'AX)^\rho\right]^{\frac{\eta}{\delta-\alpha}},$$
$$\alpha < \delta, \quad (57)$$
$$\eta > 0, a > 0, \gamma > 0, \rho > 0, 0 \leq X'AX \leq \left[\frac{\eta}{a(\delta-\alpha)}\right]^{\frac{1}{\rho}},$$
$$g_5(X) = C_5(X'AX)^\gamma \left[1 + a\left(\frac{\alpha-\delta}{\eta}\right)(X'AX)^\rho\right]^{-\frac{\eta}{\alpha-\delta}},$$
$$\alpha > \delta, \quad (58)$$
$$a > 0, \gamma > 0, \rho > 0, X'AX > 0,$$
$$g_6(X) = C_6(X'AX)^\gamma \mathrm{e}^{-a(X'AX)^\rho}, a > 0, X'AX \geq 0, \quad (59)$$

where $C_4, C_5, C_6$ are the normalizing constants. Let us evaluate the normalizing constant $C_1$ in (57). This procedure will also hold for the evaluation of $C_5$ and $C_6$. From (57) the total probability is 1. Hence $1 = \int_X g_4(X)\mathrm{d}X$. Consider the transformation $A^{\frac{1}{2}}X = Y$ then $\mathrm{d}X = |A|^{-\frac{p+1}{2}}\mathrm{d}Y$, see [25] where $|(\cdot)|$ denotes the determinant of $(\cdot)$. Then

$$1 = C_4|A|^{-\frac{p+1}{2}} \int_Y (Y'Y)^\gamma \left[1 - a\left(\frac{\delta-\alpha}{\eta}\right)(Y'Y)^\rho\right]^{\frac{\eta}{\delta-\alpha}} \mathrm{d}Y.$$

Put $Y'Y = s = r^2$. Then

$$\mathrm{d}Y = \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} s^{\frac{p}{2}-1}\mathrm{d}s = 2\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} r^{p-1}\mathrm{d}r,$$

see [25]. Then

$$1 = C_4|A|^{-\frac{p+1}{2}} \int_{r=0}^\infty r^{2\gamma} \left[1 - a\left(\frac{\delta-\alpha}{\eta}\right) r^{2\rho}\right]^{\frac{\eta}{\delta-\alpha}}$$
$$\times \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} r^{p-1}\mathrm{d}r, \alpha < \delta.$$

Put

$$z = a\left(\frac{\delta-\alpha}{\eta}\right) r^{2\rho} \Rightarrow r = z^{\frac{1}{2\rho}} \left[\frac{\eta}{a(\delta-\alpha)}\right]^{\frac{1}{2\rho}},$$

and

$$\mathrm{d}r = \frac{1}{2\rho}\left[\frac{\eta}{a(\delta-\alpha)}\right]^{\frac{1}{2\rho}} z^{\frac{1}{2\rho}-1}\mathrm{d}z,$$

$$r^{2\gamma+p-1}\mathrm{d}r = \frac{z^{\frac{\gamma}{\rho}+\frac{p}{2\rho}-1}}{2\rho}\left[\frac{\eta}{a(\delta-\alpha)}\right]^{\frac{\gamma}{\rho}+\frac{p}{2\rho}}.$$

Integration over $r$ gives

$$\frac{\Gamma(\frac{\gamma}{\rho}+\frac{p}{2\rho})\Gamma(\frac{\eta}{\delta-\alpha}+1)}{\Gamma(\frac{\eta}{\delta-\alpha}+1+\frac{\gamma}{\rho}+\frac{p}{2\rho})}.$$

Therefore

$$C_4 = \frac{|A|^{\frac{p+1}{2}}\Gamma(\frac{p}{2})\Gamma(\frac{\eta}{\delta-\alpha}+1+\frac{\gamma}{\rho}+\frac{p}{2\rho})}{\rho\pi^{\frac{p}{2}}\Gamma(\frac{\gamma}{\rho}+\frac{p}{2\rho})\Gamma(\frac{\eta}{\delta-\alpha}+1)}$$
$$\times \left[\frac{a(\delta-\alpha)}{\eta}\right]^{\frac{\gamma}{\rho}+\frac{p}{2\rho}}, \alpha < \delta. \quad (60)$$

For $\alpha > \delta$, follow through the same steps as above and then evaluate the integral by using a type-2 beta integral, then one has the following:

$$C_5 = \frac{|A|^{\frac{p+1}{2}}\Gamma(\frac{p}{2})\Gamma(\frac{\eta}{\alpha-\delta})}{\rho\pi^{\frac{p}{2}}\Gamma(\frac{\gamma}{\rho}+\frac{p}{2\rho})} \frac{[a\frac{(\alpha-\delta)}{\eta}]^{\frac{\gamma}{\rho}+\frac{p}{2\rho}}}{\Gamma(\frac{\eta}{\alpha-\delta}-\frac{\gamma}{\rho}-\frac{p}{2\rho})}, \quad (61)$$

for $\alpha > \delta, \frac{\eta}{\alpha-\delta} - \frac{\gamma}{\rho} - \frac{p}{2\rho} > 0$ and

$$C_6 = \frac{|A|^{\frac{p+1}{2}} \Gamma(\frac{p}{2})(a)^{\frac{\gamma}{\rho}+\frac{p}{2\rho}}}{\rho \pi^{\frac{p}{2}} \Gamma(\frac{\gamma}{\rho}+\frac{p}{2\rho})}, a > 0, \gamma > 0, \rho > 0. \quad (62)$$

In all the above cases, $A > O, \gamma > 0, \rho > 0, a > 0, \eta > 0$. Note that the energy density is the case for $\gamma = 1, p = 1$. Recently, [26] constructed matrix-variate analogues of the Maxwell–Boltzmann and Raleigh densities.

## 6 Concluding remarks

In this paper we are going back to the roots of the original solar neutrino problem. This problem was solved through the discovery of neutrino oscillations and was recently enriched by the experimental and theoretical consideration of neutrino entanglement. To reconsider possible new properties of solar neutrinos we performed diffusion entropy analysis (DEA), utilizing Boltzmann entropy, and standard deviation analysis (SDA) with Super-Kamiokande solar neutrino data. Surprisingly this analysis revealed a non-Gaussian signal with harmonic behavior. The Hurst exponent is different from the scaling exponent of the probability density function and both Hurst exponent and scaling exponent of the Super-Kamiokande data deviate considerably from the value of 1/2 which indicates that the statistics of the underlying phenomenon is anomalous. We recapitulate arguments from the so-called Boltzmann–Planck–Einstein discussions related to Planck's discovery of the black-body radiation law and emphasize from this discussion that a meaningful implementation of the complex entropy-probability-dynamics may offer two ways for explaining the results of DEA and SDA [20]. One way is to consider an anomalous diffusion process that needs to use the fractional space-time diffusion equation and the other way to generalize Boltzmann's entropy by assuming a power law probability density function. We consider in this paper the second way and postulate a generalized Boltzmann entropy called Mathai entropy. This entropy contains a varying parameter that is used to construct an entropic pathway covering generalized type-1 beta, type-2 beta, and gamma families of densities. Similarly pathways for respective distributions and differential equations can be developed. Mathai's entropy is optimized under various conditions reproducing the well-known Maxwell–Boltzmann distribution, Raleigh distribution, and other distributions used in physics. Properties of the entropy measure for the generalized entropy are examined and their extension to multivariate cases, including Maxwell–Boltzmannian and Raleighian, are obtained. To make visible the usefulness of this theory of generalized entropy, we also extend the formalism to include geometrical probability problems, isotropic random points, spherically symmetric and elliptically contoured statistical distributions. All results are given for real scalar, vector, as well as matrix variables. Future work will have to link the theoretical results in this paper to the above mentioned second way of reproducing the analysis results of the Super-Kamiokande solar neutrino data considering fractional space–time diffusion processes.

## Author contribution statement

All authors designed and performed the research, analyzed the data, and wrote the paper.



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