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# Entanglement in joint $\Lambda\bar{\Lambda}$ decay; cont.

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**Abstract** We have previously investigated joint $\Lambda\bar{\Lambda}$ decay in the reaction $e^+e^- \to \gamma\Lambda(\to p\pi^-)\bar{\Lambda}(\to \bar{p}\pi^+)$. The cross-section-distribution functions encountered were relativistically covariant and expressed in terms of scalar products of the four-momentum vectors of the particles involved. In the present, sequel investigation, we work instead in the $\Lambda\bar{\Lambda}$ rest system and with three-momentum scalars. In this configuration our results become directly comparable to those of others, including experiment.

## 1 Introduction

The *BABAR* Collaboration [1] has measured initial-state-radiation in the annihilation reaction $e^+e^- \to \gamma\Lambda(\to p\pi^-)\bar{\Lambda}(\to \bar{p}\pi^+)$. Such measurements are interesting since they offer opportunities to determine electromagnetic form factors of the Lambda hyperons in the time-like region.

Theoretical analyses of this reaction are presented in Ref. [2], for the $\Lambda\bar{\Lambda}\gamma$ final state with single hyperon polarization, and in Ref. [3], for the $\Lambda\bar{\Lambda}\gamma$ final state with double hyperon polarizations.

A Lorentz-covariant description of the cross-section-distribution functions, including those of the hyperon decays but neglecting polarizations, is presented in Ref. [4]. The arguments of the covariant functions encountered in this analysis are various scalar products of the four-momentum vectors of the particles involved. Working in the covariant formalism is cumbersome, and therefore we take advantage of the covariance and pick a particular reference frame for our considerations, the $\Lambda\bar{\Lambda}$ rest frame, which also is the choice of Ref. [2].

Replacing four-dimensional arguments by three-dimensional ones also requires considerable work, but this work is worth-while, as we shall see.

## 2 Cross-section distribution

Our notation follows Pilkuhn [5]. The cross-section distribution for the reaction $e^+e^- \to \gamma\Lambda(\to p\pi^-)\bar{\Lambda}(\to \bar{p}\pi^+)$ is written as

$$d\sigma = \frac{1}{2\sqrt{\lambda(s, m_e^2, m_e^2)}}\,\overline{|\mathcal{M}|^2}\,\mathrm{dLips}(k_1+k_2; q, l_1, l_2, q_1, q_2), \tag{1}$$

where the average over the squared matrix element indicates summation over final proton and anti-proton spins and average over initial electron and positron spins. The definitions of the particle momenta are explained in Fig. 1.

We remove some trivial factors from the squared matrix element, collected in a factor denoted $\mathcal{K}$,

$$\overline{|\mathcal{M}|^2} = \mathcal{K}\overline{|\mathcal{M}_{red}|^2}. \tag{2}$$

## 3 Previous analysis

We start where our previous analysis ended, Ref. [4], but before we can do so it is necessary to repeat some of the important definitions and results.

The form factors of the hyperon-electromagnetic couplings are denoted $G_1$ and $G_2$, a standard choice. The designations of particle four-momenta can be seen in the Feynman diagrams of Fig. 1.

The cross-section-distribution function, or rather the covariant square of the annihilation matrix element $\overline{|\mathcal{M}_{red}|^2}$,

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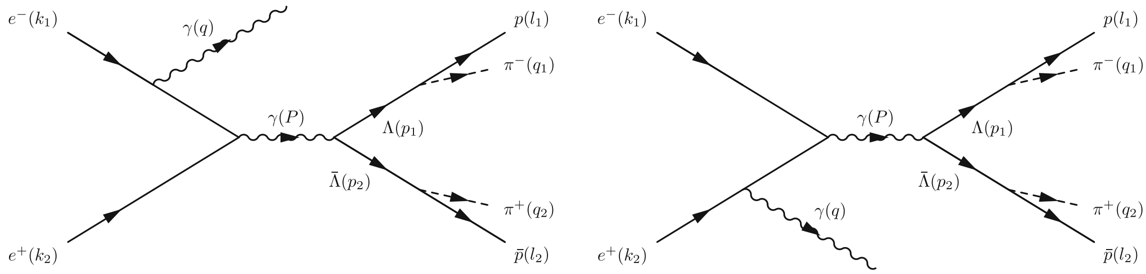


**Fig. 1** Graphs included in our calculation of the reaction $e^+e^- \to \gamma\Lambda(\to p\pi^-)\bar{\Lambda}(\to \bar{p}\pi^+)$

is obtained by contracting hadronic $H_{\mu\nu}$ and leptonic $L^{\mu\nu}$ tensors, so that

$$\overline{|\mathcal{M}_{red}|^2} = L^{\mu\nu}H_{\mu\nu}. \tag{3}$$

Now, the right-hand-side of this equation can be rewritten as a sum of four terms,

$$\overline{|\mathcal{M}_{red}|^2} = \bar{R}_\Lambda R_\Lambda M^{RR} + \bar{R}_\Lambda S_\Lambda M^{RS} + \bar{S}_\Lambda R_\Lambda M^{SR} + \bar{S}_\Lambda S_\Lambda M^{SS}, \tag{4}$$

with coefficients $R_\Lambda$, $S_\Lambda$ and $R_{\bar{\Lambda}}$, $S_{\bar{\Lambda}}$ that refer to the $\Lambda$ and $\bar{\Lambda}$ decay constants of Ref. [4], and with $R$ the spin-independent and $S$ the spin-dependent ones.

From the structure of the lepton tensor, Eq. (24) of Ref. [4], it follows that each of the $M^{XY}$ functions of Eq. (4) has two parts,

$$M^{XY} = -a_y A^{XY}(G_1, G_2) - b_y B^{XY}(G_1, G_2), \tag{5}$$

where the $A^{XY}$ factor is obtained by contracting the hadron tensor with the symmetric tensor $k_{1\mu}k_{1\nu} + k_{2\mu}k_{2\nu}$, and the $B^{XY}$ factor by contraction with the tensor $g_{\mu\nu}$. For details see Ref. [4]. The weight factors $a_y$ and $b_y$ are defined in appendix A.

The functions $A^{XY}$ and $B^{XY}$ are bilinear forms of $G_1$ and $G_2$, and we expand them accordingly, for $A^{XY}$,

$$A^{XY}(G_1, G_2) = |G_1|^2\mathcal{K}_1^{AXY} + |G_2|^2\mathcal{K}_2^{AXY} + 2\Re(G_1G_2^\star)\mathcal{K}_3^{AXY} + 2\Im(G_1G_2^\star)\mathcal{K}_4^{AXY}, \tag{6}$$

and similarly for $B^{XY}$. We refer to the functions $\{\mathcal{K}\}$ as co-factors. They are Lorentz covariant functions of the particle four-momenta and the functions of our attention.

## 4 Previous results

The leading term of Eq. (4) is $M^{RR}$ as it is independent of variables that relate to spin dependences in the hyperon decay distributions. We have

$$\begin{aligned} A^{RR} &= 2|G_1|^2\Big[(k_1\cdot P)^2 + (k_2\cdot P)^2 - (k_1\cdot Q)^2 \\ &\quad -(k_2\cdot Q)^2\Big] \\ &\quad +4\Re(G_1G_2^\star)\Big[(k_1\cdot Q)^2 + (k_2\cdot Q)^2\Big] \\ &\quad -|G_2|^2\frac{Q^2}{2M^2}\Big[(k_1\cdot Q)^2 + (k_2\cdot Q)^2\Big], \end{aligned} \tag{7}$$

with $Q = p_1 - p_2$. Furthermore,

$$\begin{aligned} B^{RR} &= -4|G_1|^2(P^2 + 2M^2) + 4\Re(G_1G_2^\star)Q^2 \\ &\quad -|G_2|^2\frac{(Q^2)^2}{2M^2}. \end{aligned} \tag{8}$$

Thus, the distribution function $M^{RR}$ does not depend on the decay momenta $l$ and $q$ of the Lambda hyperons.

Next in order are terms linear in the spin variables,

$$\begin{aligned} A^{RS} &= -4\Im(G_1G_2^\star)\Big[k_1\cdot Q\,\det(p_2p_1l_1k_1) \\ &\quad + k_2\cdot Q\,\det(p_2p_1l_1k_2)\Big], \end{aligned} \tag{9}$$

$$\begin{aligned} A^{SR} &= -4\Im(G_1G_2^\star)\Big[k_1\cdot Q\,\det(p_2p_1l_2k_1) \\ &\quad + k_2\cdot Q\,\det(p_2p_1l_2k_2)\Big], \end{aligned} \tag{10}$$

with $\det(abcd) = \epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta$ and

$$B^{RS} = 0, \tag{11}$$
$$B^{SR} = 0. \tag{12}$$

The expressions for the spin-spin contributions are more complicated. We have for the $A^{SS}$ contribution

$$\begin{aligned} A^{SS} &= -2|G_1|^2\Big[\Big((k_1\cdot P)^2 \\ &\quad +(k_2\cdot P)^2 - (k_1\cdot Q)^2 - (k_2\cdot Q)^2\Big) \\ &\quad \Big(p_1\cdot l_1\, p_2\cdot l_2 + M^2 l_1\cdot l_2\Big) \end{aligned}$$

$$+2M^2\Big(P^2(k_1\cdot l_1 k_1\cdot l_2+k_2\cdot l_1 k_2\cdot l_2)$$
$$-2P\cdot l_2(k_1\cdot l_1 k_1\cdot p_2+k_2\cdot l_1 k_2\cdot p_2)$$
$$-2P\cdot l_1(k_1\cdot l_2 k_1\cdot p_1+k_2\cdot l_2 k_2\cdot p_1)\Big)\Big]$$
$$-4\Re(G_1G_2^\star)\Big[M^2 l_1\cdot l_2\Big((k_1\cdot Q)^2+(k_2\cdot Q)^2\Big)$$
$$-M^2\Big(P\cdot l_2(k_1\cdot Qk_1\cdot l_1+k_2\cdot Qk_2\cdot l_1)$$
$$-P\cdot l_1(k_1\cdot Qk_1\cdot l_2+k_2\cdot Qk_2\cdot l_2)\Big)$$
$$-k_1\cdot Q\Big(k_1\cdot p_1 p_2\cdot l_1 p_2\cdot l_2-k_1\cdot p_2 p_1\cdot l_1 p_1\cdot l_2$$
$$-\tfrac{1}{2}P^2k_1\cdot l_1 p_2\cdot l_2+\tfrac{1}{2}P^2k_1\cdot l_2 p_1\cdot l_1\Big)$$
$$-k_2\cdot Q\Big(k_2\cdot p_1 p_2\cdot l_1 p_2\cdot l_2-k_2\cdot p_2 p_1\cdot l_1 p_1\cdot l_2$$
$$-\tfrac{1}{2}P^2k_2\cdot l_1 p_2\cdot l_2+\tfrac{1}{2}P^2k_2\cdot l_2 p_1\cdot l_1\Big)\Big]$$
$$-|G_2|^2\frac{1}{2M^2}\Big((k_1\cdot Q)^2$$
$$+(k_2\cdot Q)^2\Big)\Big[Q^2\Big(p_1\cdot l_1 p_2\cdot l_2-M^2 l_1\cdot l_2\Big)$$
$$+2M^2Q\cdot l_1 Q\cdot l_2\Big], \tag{13}$$

and for the $B^{SS}$ contribution

$$B^{SS}=+4|G_1|^2\Big[(P^2+2M^2)(p_1\cdot l_1 p_2\cdot l_2+M^2 l_1\cdot l_2)$$
$$-M^2\Big(P^2 l_1\cdot l_2+2P\cdot l_2 l_1\cdot p_1+2P\cdot l_1 l_2\cdot p_2\Big)\Big]$$
$$-4\Re(G_1G_2^\star)\Big[Q^2M^2 l_1\cdot l_2$$
$$-M^2\Big(Q\cdot l_1 P\cdot l_2-Q\cdot l_2 P\cdot l_1\Big)$$
$$-\Big(p_1\cdot Qp_2\cdot l_1 p_2\cdot l_2-p_2\cdot Qp_1\cdot l_1 p_1\cdot l_2$$
$$-\tfrac{1}{2}P^2Q\cdot l_1 p_2\cdot l_2+\tfrac{1}{2}P^2Q\cdot l_2 p_1\cdot l_1\Big)\Big]$$
$$-|G_2|^2\frac{Q^2}{2M^2}\Big[Q^2\Big(p_1\cdot l_1 p_2\cdot l_2-M^2 l_1\cdot l_2\Big)$$
$$+2M^2Q\cdot l_1 Q\cdot l_2\Big]. \tag{14}$$

The functions $A^{SS}$ and $B^{SS}$ describe the joint-decay distributions of the Lambda and anti-Lambda hyperons. The distributions are correlated, *i.e.*, they cannot be written as a product of Lambda and anti-Lambda distribution functions. Our distribution functions are explicitly covariant, as they are expressed in terms of the four-momentum vectors of the participating particles. It is not necessary to work in several coordinate systems, as in Refs. [1] and [3]. Another important point is that our calculation correctly counts the number of intermediate hyperon states.

## 5 Reference frames

The cross-section distribution function of sect. 4 involves expressions that are functions of scalar products of particle four-momenta. To determine the scalar product of two four-vectors requires knowledge of those vectors in one and the same reference frame. Our task in this section is to demonstrate how this is achieved.

Designations of the particle four-momenta follow from the energy-momentum-balance condition in the reaction $e^+e^-\to\bar\Lambda(\to\bar p\pi^+)\Lambda(\to p\pi)\gamma$,

$$k_1+k_2=p_1+p_2+q. \tag{15}$$

Additional information is contained in Fig. 1.

The gamma three-momentum $\mathbf{q}$, and electron three-momentum $\mathbf{k}$, are momenta defined in the $e^+e^-$ centre-of-momentum (c.m.) reference frame, in which $\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}=\cos\theta$. We refer to this frame as $S_0$. In $S_0$ electron and positron four-momenta are $k_1=\epsilon(1,\hat{\mathbf{k}})$ and $k_2=\epsilon(1,-\hat{\mathbf{k}})$, with $\epsilon$ the common lepton energy. With $\omega$ the gamma energy, the gamma four-momentum is denoted $q=\omega(1,\hat{\mathbf{q}})$. Furthermore, the four-momenta of Lambda and anti-Lambda are $p_1=(E_1,\mathbf{p}_1)$ and $p_2=(E_2,\mathbf{p}_2)$.

Now, we shall not perform our calculations in $S_0$ but in $S_1$ which is the c.m. frame of the $\Lambda\bar\Lambda$ pair. We indicate variables in this frame by a prime, so that

$$p'_{1,2}=(E_\Lambda,\pm p_\Lambda\mathbf{f}) \tag{16}$$

with $E_\Lambda=\sqrt{p_\Lambda^2+M^2}$ and $\mathbf{f}$ a unit vector. The $\Lambda\bar\Lambda$ c.m. energy $W=2E_\Lambda$ may be obtained from the identity

$$W^2=4\epsilon(\epsilon-\omega). \tag{17}$$

The next question concerns the relation between frames $S_1$ and $S_0$. Since $\mathbf{p}_1+\mathbf{p}_2=-\mathbf{q}$ in $S_0$, we argue that $S_1$ can be reached from $S_0$ through a boost along the direction of motion of the gamma, and of magnitude,

$$v=\frac{-(\mathbf{p}_1+\mathbf{p}_2)\cdot\hat{\mathbf{q}}}{E_1+E_2}=\frac{\omega}{\sqrt{\omega^2+W^2}}, \tag{18}$$

and with Lorentz-transformation (LT) coefficient

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}} = \frac{\sqrt{\omega^2+W^2}}{W}. \tag{19}$$

Also, note that $v$ is the relative velocity between two reference frames, it is not a particle velocity.

A Lorentz boost from $S_0$ to $S_1$ leads to new four-momentum vectors for the initial state leptons, namely

$$k'_{1,2} = \epsilon\gamma\left[(1 \pm v\mathbf{n}\cdot\hat{\mathbf{k}});\ v(\mathbf{n}\pm\mathbf{N})\right], \tag{20}$$

and with $\mathbf{n}$ and $\mathbf{N}$ by definition

$$\mathbf{n} = \hat{\mathbf{q}}, \tag{21}$$

$$\mathbf{N} = \frac{1}{v\gamma}\left[\hat{\mathbf{k}} + (\gamma-1)(\mathbf{n}\cdot\hat{\mathbf{k}})\mathbf{n}\right]. \tag{22}$$

Relations (21) and (22) are identical to those introduced by the BaBar collaboration [1].

The photon radiated in our annihilation process carries energy $\omega$ and three-momentum $\mathbf{q} = \omega\mathbf{n}$ , when observed in $S_0$. A boost from $S_0$ to $S_1$, sends this vector into $\mathbf{q}' = \omega'\mathbf{n}$, with

$$\omega' = \omega\sqrt{\frac{1+v}{1-v}}. \tag{23}$$

However, we should not forget the decay products of the hyperons, the antiproton and the proton. In the rest system $S_2$ of the Lambda the proton is represented by the four-vector

$$l_1 = (E_g, p_g\,\mathbf{g}), \tag{24}$$

with $\mathbf{g}$ a unit vector, and with decay parameters $p_g$ and $E_g$.

Similarly, in the rest system $S_3$ of the anti-Lambda the anti-proton is represented by the four-vector

$$l_2 = (E_h, p_h\mathbf{h}), \tag{25}$$

with $\mathbf{h}$ a unit vector, and decay parameters $p_h = p_g$ and $E_h = E_g$. A passage from $S_3$ to $S_1$ is achieved by a Lorentz boost with velocity $v_h$ and direction $\mathbf{f}$, whereas a passage from $S_2$ to $S_1$ is achieved by a Lorentz boost with velocity $v_g$ and direction $-\mathbf{f}$.

The boost equations for the massive hyperons are well known. Vectors orthogonal to the boost velocity, $\mathbf{v} = v\mathbf{n}$, are unchanged, those parallel are changed according to the Lorentz-transformation prescription

$$\begin{aligned}\mathbf{p}'_{1,2} = {} & \left[\mathbf{p}_{1,2} - \mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{1,2})\right] \\ & + \gamma(v)\mathbf{n}\left[\mathbf{n}\cdot\mathbf{p}_{1,2} + vE_{1,2}\right],\end{aligned} \tag{26}$$

$$E'_{1,2} = \gamma(v)\left[E_{1,2} - \mathbf{v}\cdot\mathbf{p}_{1,2}\right]. \tag{27}$$

The inverse-transformation equations, going from $S_1$ to $S_0$, are obtained by changing the sign of the velocity, from $\mathbf{n}$ to $-\mathbf{n}$.

After this elementary discussion we are ready for the proton and antiproton four-momentum vectors in $S_1$; for the proton

$$\mathbf{p}'_g = \gamma_\Lambda E_g\mathbf{f}\left[\mathbf{f}\cdot(\mathbf{v}_g + \mathbf{v}_\Lambda)\right] + \mathbf{p}_{g\perp}, \tag{28}$$

$$E'_g = \gamma_\Lambda E_g\left[1 + \mathbf{v}_\Lambda\cdot\mathbf{v}_g\right], \tag{29}$$

the transverse-vector component $\mathbf{p}_{g\perp}$ being

$$\mathbf{p}_{g\perp} = \mathbf{p}_g - (\mathbf{p}_g\cdot\mathbf{f})\,\mathbf{f}, \tag{30}$$

with $\mathbf{p}_{g\perp}\cdot\mathbf{f} = 0$; and for the antiproton

$$\mathbf{p}'_h = \gamma_\Lambda E_h\mathbf{f}\left[\mathbf{f}\cdot(\mathbf{v}_h - \mathbf{v}_\Lambda)\right] + \mathbf{p}_{h\perp}, \tag{31}$$

$$E'_h = \gamma_\Lambda E_h\left[1 - \mathbf{v}_\Lambda\cdot\mathbf{v}_h\right]. \tag{32}$$

Our calculations make use of the shorthand notations,

$$G_g = (\mathbf{v}_g + \mathbf{v}_\Lambda)\cdot\mathbf{f},\quad H_g = 1 + \mathbf{v}_\Lambda\cdot\mathbf{v}_g, \tag{33}$$

$$G_h = (\mathbf{v}_h - \mathbf{v}_\Lambda)\cdot\mathbf{f},\quad H_h = 1 - \mathbf{v}_\Lambda\cdot\mathbf{v}_h. \tag{34}$$

## 6 Calculating co-factors

Co-factors can be identified in the $A^{XY}$ and the $B^{XY}$ functional distributions of Sect. 4. The results are co-factors expressed in terms of scalar products of four-vector momenta. However, our goal was to find simpler expressons, and this by evaluating all scalars in one and the same reference frame, the c.m. reference frame of the $\Lambda\bar{\Lambda}$ pair.

We have evaluated cross-section distributions for two sets of form-factor parameters. The two sets have attached form-factor sets, that we indicate by different letters, such that $(G_1, G_2) \Rightarrow \{\mathcal{K}\}$ and $(G_M, G_E) \Rightarrow \{\mathcal{L}\}$ . We start with the $\mathcal{K}$ set and return to the $\mathcal{L}$ set in sect.7.

The much needed $\Omega$ functions are defined in appendix B. The spin-correlation functions are equally important. Their definitions are,

$$X_a(\mathbf{g},\mathbf{h}) = 2\mathbf{g}\cdot\mathbf{f}\,\mathbf{h}\cdot\mathbf{f} - \mathbf{g}\cdot\mathbf{h}, \tag{35}$$

$$X_b(\mathbf{g},\mathbf{h}) = \mathbf{g}\cdot\mathbf{f}\,\mathbf{h}\cdot\mathbf{f}. \tag{36}$$

A parameter that appears in practically every formula is the $Z$ parameter,

$$Z = \frac{4M_\Lambda^2}{Q^2} = \frac{-1}{\gamma_\Lambda^2 v_\Lambda^2} = 1 - \frac{1}{v_\Lambda^2}. \tag{37}$$

Other important parameters are $v_\Lambda = p_\Lambda/E_\Lambda$ and $\gamma_\Lambda = E_\Lambda/M_\Lambda$, with $M_\Lambda = M$.

### 6.1 Spin-independent co-factors

We start with the $\mathcal{K}$ base and the $\mathcal{K}^{ARR}$ co-factors, which can be extracted from the $A^{RR}$ contribution to the $M^{RR}$ functional distribution of Eq. (7). Since he calculation is straightforward we are satisfied with the result,

$$\mathcal{K}_1^{ARR} = (2\epsilon\omega)^2\Omega_f v_\Lambda^2\Big[-Z+\Omega_\perp/(\Omega_f v_\Lambda^2)\Big], \tag{38}$$

$$\mathcal{K}_2^{ARR} = (2\epsilon\omega)^2\Omega_f v_\Lambda^2\Big[\frac{-1}{Z}\Big], \tag{39}$$

$$\mathcal{K}_3^{ARR} = (2\epsilon\omega)^2\Omega_f v_\Lambda^2\Big[1\Big]. \tag{40}$$

The $\Omega$ functions are described in appendix B.

Next, we extract the $\mathcal{K}^{BRR}$ co-factors from the $B^{RR}$ contribution (8) to the $M^{RR}$ functional distribution of Eq. (5);

$$\mathcal{K}_1^{BRR} = -2Q^2\Big[Z-2(1-Z)\Big], \tag{41}$$

$$\mathcal{K}_2^{BRR} = -2Q^2\Big[\frac{1}{Z}\Big], \tag{42}$$

$$\mathcal{K}_3^{BRR} = -2Q^2\Big[-1\Big]. \tag{43}$$

Knowledge of these co-factors leads immediately to the $A^{RR}$ and $B^{RR}$ functions of Eq. (6),

$$A^{RR} = (2\epsilon\omega)^2\frac{1}{Z(1-Z)}\Big[-|ZG_1-G_2|^2\Omega_f + Z(1-Z)|G_1|^2\Omega_\perp\Big], \tag{44}$$

$$B^{RR} = \frac{2Q^2}{Z}\Big[|ZG_1-G_2|^2 + 2Z(1-Z)|G_1|^2\Big], \tag{45}$$

expressions which are well-known and also displayed in Ref. [2] and [3].

### 6.2 Linearly spin-dependent co-factors

Next in order are terms linear in the spin variables, represented by the functions of Eqs. (9) and (10), and their only non-vanishing co-factors $\mathcal{K}_4^{ARS}$ and $\mathcal{K}_4^{ASR}$. The expressions for the prefactors of the above-mentioned equations are quite easily obtained, and equals

$$Q\cdot k_{1,2} = -\frac{2p_\Lambda\epsilon\omega}{W}\mathbf{f}\cdot(\mathbf{n}\pm\mathbf{N}). \tag{46}$$

In the $S_1$ frame the determinant boils down to

$$\det(p_2'p_1'l_1'k_2') = 2E_\Lambda\mathbf{p}_1'\cdot(\mathbf{p}_g'\times\mathbf{k}_2') = 2E_\Lambda p_\Lambda\mathbf{f}\cdot(\mathbf{p}_g'\times\mathbf{k}_2'). \tag{47}$$

Hence, components of the vectors $\mathbf{p}_g'$ or $\mathbf{k}_2'$ along $\mathbf{f}$ will not contribute to the value of the determinant, so that by Eq. (28) we may replace $\mathbf{p}_g'$ by $\mathbf{p}_g$. For $\mathbf{k}_2'$ we make recourse to Eq. (20). All this yields,

$$\mathcal{K}_4^{ARS} = 2p_\Lambda p_g(2\epsilon\omega)^2 v_\Lambda \Big[\mathbf{f}\cdot\mathbf{nf}\cdot(\mathbf{g}\times\mathbf{n})+\mathbf{f}\cdot\mathbf{Nf}\cdot(\mathbf{g}\times\mathbf{N})\Big], \tag{48}$$

$$\mathcal{K}_4^{ASR} = 2p_\Lambda p_g(2\epsilon\omega)^2 v_\Lambda \Big[\mathbf{f}\cdot\mathbf{nf}\cdot(\mathbf{h}\times\mathbf{n})+\mathbf{f}\cdot\mathbf{Nf}\cdot(\mathbf{h}\times\mathbf{N})\Big]. \tag{49}$$

The co-factors that relate to $B^{XY}$ of Eq. (5) vanish,

$$\mathcal{K}_4^{BRS} = 0, \tag{50}$$

$$\mathcal{K}_4^{BSR} = 0. \tag{51}$$

With the co-factors of Eqs. (48) and (49) in hand we can determine $A^{RS}$ and $A^{SR}$ from Eq. (6). The related functions $B^{RS}$ and $B^{SR}$ vanish identically. In agreement with Ref. [2],

$$A^{RS} = 2p_\Lambda v_\Lambda p_g(2\epsilon\omega)^2 2\Im(G_1G_2^\star) \Big[\mathbf{f}\cdot\mathbf{nf}\cdot(\mathbf{g}\times\mathbf{n})+\mathbf{f}\cdot\mathbf{Nf}\cdot(\mathbf{g}\times\mathbf{N})\Big], \tag{52}$$

$$A^{SR} = 2p_\Lambda v_\Lambda p_g(2\epsilon\omega)^2 2\Im(G_1G_2^\star) \Big[\mathbf{f}\cdot\mathbf{nf}\cdot(\mathbf{h}\times\mathbf{n})+\mathbf{f}\cdot\mathbf{Nf}\cdot(\mathbf{h}\times\mathbf{N})\Big]. \tag{53}$$

### 6.3 Doubly spin-dependent co-factors

Now, the co-factors are suffixed $ASS$ and $BSS$. Those suffixed $ASS$ are obtained by analysing $A^{SS}$ of Eq. (13);

$$\mathcal{K}_1^{ASS} = (2p_\Lambda p_g\epsilon\omega)^2\Omega_f v_\Lambda^2 \Big[-Z^2X_a+\frac{Z\Omega_\perp}{v_\Lambda^2\Omega_f}(X_a-2X_b)+\frac{Z}{v_\Lambda^2\Omega_f}B_1\Big], \tag{54}$$

$$\mathcal{K}_2^{ASS} = (2p_\Lambda p_g\epsilon\omega)^2\Omega_f v_\Lambda^2\Big[-X_a\Big], \tag{55}$$

$$\mathcal{K}_3^{ASS} = (2p_\Lambda p_g\epsilon\omega)^2\Omega_f v_\Lambda^2\Big[ZX_a+\frac{Z}{v_\Lambda^2\Omega_f}B_3\Big], \tag{56}$$

with functions $B_1$ and $B_3$ as defined in appendix B. The co-factors suffixed $BSS$ are dug out from $B^{SS}$ of Eq. (14);

$$\mathcal{K}_1^{BSS} = 2(2p_\Lambda^2 p_g)^2\Big[Z^2X_a+2Z(1-Z)X_b\Big], \tag{57}$$

$$\mathcal{K}_2^{BSS} = 2(2p_\Lambda^2 p_g)^2\Big[X_a\Big], \tag{58}$$

$$\mathcal{K}_3^{BSS} = 2(2p_\Lambda^2 p_g)^2\Big[-ZX_a\Big], \tag{59}$$

where $Z$ is defined in Eq. (37).

Starting from the $\mathcal{K}^{ASS}$ and $\mathcal{K}^{BSS}$ co-factors we easily derive functions $A^{SS}$ and $B^{SS}$,

$$A^{SS} = 4(p_\Lambda p_g \epsilon\omega)^2 \Big[ -|ZG_1 - G_2|^2 \frac{\Omega_f}{1-Z} X_a + Z\Omega_\perp |G_1|^2 (X_a - 2X_b) + 2Z\Re(G_1 G_2^\star) B_3 + Z|G_1|^2 B_1 \Big], \quad (60)$$

$$B^{SS} = 8(p_\Lambda^2 p_g)^2 \Big[ |ZG_1 - G_2|^2 X_a + 2Z(1-Z)|G_1|^2 X_b \Big], \quad (61)$$

two functions which have not been investigated before. Now, we have all the ingrediants needed to calculate the cross-section-distribution function from Eqs. (4–6).

## 7 The $G_E/G_M$ set

Until now, we have only considered an expansion of the cross-section-distribution functions $A^{XY}$ and $B^{XY}$ in terms of the form factors $G_1$ and $G_2$ and their co-factors. Other choices of form factors are possible and we shall in particular consider the pair $G_E$ and $G_M$. The two sets are related by

$$G_M = G_1, \quad (62)$$
$$G_E = (ZG_1 - G_2)/Z. \quad (63)$$

The arguments of the form factors are all equal to $P^2$. In particular, when $P^2 = 4M^2$ then $G_M = G_E$.

The functions $A^{XY}$ and $B^{XY}$ are bilinear forms of $G_1$ and $G_2$, and expanded according to Eq. (6), but they can also be expanded in terms of $G_M$ and $G_E$ in which case

$$A^{XY}(G_M, G_E) = |G_M|^2 \mathcal{L}_1^{AXY} + |G_E|^2 \mathcal{L}_2^{AXY} \quad (64)$$
$$+ 2\Re(G_M G_E^\star)\mathcal{L}_3^{AXY} + 2\Im(G_M G_E^\star)\mathcal{L}_4^{AXY}, \quad (65)$$

and similarly for $B^{XY}$. The relation between the two sets of functions, $\{\mathcal{K}_i^{XYZ}\}$ and $\{\mathcal{L}_i^{XYZ}\}$ becomes

$$\mathcal{L}_1 = \mathcal{K}_1 + Z^2\mathcal{K}_2 + 2Z\mathcal{K}_3, \quad (66)$$
$$\mathcal{L}_2 = Z^2\mathcal{K}_2, \quad (67)$$
$$\mathcal{L}_3 = -Z^2\mathcal{K}_2 - Z\mathcal{K}_3 \quad (68)$$
$$\mathcal{L}_4 = -Z\mathcal{K}_4, \quad (69)$$

and with the parameter $Z$ defined in Eq. (37). The most notible fact about the new set is that several co-factors vanish;

$$\mathcal{L}_3^{ARR} = \mathcal{L}_3^{BRR} = 0. \quad (70)$$

Also, as we shall see, $\mathcal{L}_3^{BSS} = 0$ but $\mathcal{L}_3^{ASS} \neq 0$. Our results are the following.

Co-factors suffixed $ARR$;

$$\mathcal{L}_1^{ARR} = (2\epsilon\omega)^2 \Big[ (\mathbf{n}\times\mathbf{f})^2 + (\mathbf{N}\times\mathbf{f})^2 \Big], \quad (71)$$
$$\mathcal{L}_2^{ARR} = (2\epsilon\omega)^2 \frac{1}{\gamma_\Lambda^2} \Big[ (\mathbf{n}\cdot\mathbf{f})^2 + (\mathbf{N}\cdot\mathbf{f})^2 \Big], \quad (72)$$
$$\mathcal{L}_3^{ARR} = 0. \quad (73)$$

Co-factors suffixed $BRR$;

$$\mathcal{L}_1^{BRR} = -4P^2, \quad (74)$$
$$\mathcal{L}_2^{BRR} = 8M^2, \quad (75)$$
$$\mathcal{L}_3^{BRR} = 0. \quad (76)$$

Co-factors suffixed $ARS$ and $ASR$;

$$\mathcal{L}_4^{ARS} = -2Zp_\Lambda p_g (2\epsilon\omega)^2 v_\Lambda \Big[ \mathbf{f}\cdot\mathbf{nf}\cdot(\mathbf{g}\times\mathbf{n}) + \mathbf{f}\cdot\mathbf{Nf}\cdot(\mathbf{g}\times\mathbf{N}) \Big], \quad (77)$$
$$\mathcal{L}_4^{ASR} = -2Zp_\Lambda p_g (2\epsilon\omega)^2 v_\Lambda \Big[ \mathbf{f}\cdot\mathbf{nf}\cdot(\mathbf{h}\times\mathbf{n}) + \mathbf{f}\cdot\mathbf{Nf}\cdot(\mathbf{h}\times\mathbf{N}) \Big]. \quad (78)$$

The co-factors that relate to $B^{XY}$ of Eq. (5) vanish,

$$\mathcal{L}_4^{BRS} = 0, \quad (79)$$
$$\mathcal{L}_4^{BSR} = 0. \quad (80)$$

With these co-factors in hand we can determine $A^{RS}$ and $A^{SR}$ from Eq.(6) whereas the related functions $B^{RS}$ and $B^{SR}$ vanish.

Co-factors suffixed $ASS$ ;

$$\mathcal{L}_1^{ASS} = (2p_\Lambda p_g \epsilon\omega)^2 Z \Big[ \Omega_\perp (X_a - 2X_b) + 2(L_0 + ZL_M/\gamma_M) \Big], \quad (81)$$
$$\mathcal{L}_2^{ASS} = (2p_\Lambda p_g \epsilon\omega)^2 \Omega_f v_\Lambda^2 \Big[ -Z^2 X_a \Big], \quad (82)$$
$$\mathcal{L}_3^{ASS} = (2p_\Lambda p_g \epsilon\omega)^2 \Big[ -Z^2 \gamma_\Lambda L_M \Big]. \quad (83)$$

The functions $L_0$ and $L_M$ appearing in Eqs. (81) and (83) are defined in appendix B.

Co-factors suffixed $BSS$;

$$\mathcal{L}_1^{BSS} = 2(2p_\Lambda^2 p_g)^2 \Big[ 2Z(1-Z)X_b \Big], \quad (84)$$
$$\mathcal{L}_2^{BSS} = 2(2p_\Lambda^2 p_g)^2 \Big[ Z^2 X_a \Big], \quad (85)$$
$$\mathcal{L}_3^{BSS} = 0. \quad (86)$$

## 8 Discussion

In a previous report, Ref. [4], we derived a set of cross-section-distribution functions for the reaction $e^+e^- \to \gamma\Lambda(\to p\pi^-)\bar{\Lambda}(\to \bar{p}\pi^+)$. It involved functions whose arguments were Lorentzian scalar products of four-momentum vectors of the particles participating in the reaction. Since some of the functons are quite intricate we have here tried another approach, replacing the Lorentz scalars by Euclidean scalars of three-momentum vectors. In doing so it should be remembered that one can form a scalar product of two vectors only if they are defined in the same reference system. The functions which multiply the coupling constants are called co-factors, or weight factors, and can be retrieved from Eqs. (7–14). For a complete determination of the cross-section-distribution function fourteen co-factors are needed.

The cross-section-distribution function describing our $e^+e^-$ annihilation reaction is detailed in Sect. 6. However, we are also interested in knowing the final-state-distribution function after integrating over one or both hyperon decays. Let us start with the integral over the $\Omega_g$ hyperon decay, while keeping the anti-hyperon decay angle $\Omega_h$ fixed. Then, terms which are linear in the three-vector $\mathbf{g}$ vanish. Thus,

$$\int \frac{\mathrm{d}\Omega_g}{4\pi} A^{SS}, B^{SS}, A^{RS} = 0. \tag{87}$$

Also, since $B^{RS}$ is non-excisting we can add $B^{RS} = 0$ leaving the three-vector $\mathbf{h}$ described by the co-vector of Eq. (49).

The next step is integration over the hyperon decay angles $\Omega_h$,

$$\int \frac{\mathrm{d}\Omega_h}{4\pi} A^{SR} = 0. \tag{88}$$

Since by definition, $B^{SR} = 0$, we end up with the cross-section-distribution function for the reaction $e^+e^- \to \gamma\Lambda\bar{\Lambda}$, as expected. This distribution function is proportional to the function $M^{RR}$ of Eq. (4).

The angular integrations just described can also be performed in the four-dimensional formulation of the co-factors, as in Sect. 4, by exploiting Eq. (7.48) of Ref. [4] for the integration.

There is an alternative approach to the angular integration [6], which employs the Euler angles. In this case the angular measure is written as

$$\mathrm{d}\Omega_h \mathrm{d}\Omega_g = \mathrm{d}(\cos\theta_{gh})\mathrm{d}\alpha\mathrm{d}(\cos\beta)\mathrm{d}\gamma. \tag{89}$$

Since we know that terms linear in the vectors $\mathbf{g}$ or $\mathbf{h}$ vanish upon angular integration, we need only concern ourselves with the co-factors of $A^{SS}$ and $B^{SS}$ of Sect. 6.3. We notice that $\cos\theta_{gh}$ only appears in the function $X_a(\mathbf{g}, \mathbf{h})$ of Eq. (35),

$$X_a(\mathbf{g}, \mathbf{h}) = 2\mathbf{g}\cdot\mathbf{f}\,\mathbf{h}\cdot\mathbf{f} - \mathbf{g}\cdot\mathbf{h}.$$

The first term in this expression vanishes on the $\alpha\beta\gamma$ integrations, and $\mathbf{g}\cdot\mathbf{h} = \cos\theta_{gh}$. Hence, with a little help from the co-factors of sect. [6] we get for the $\alpha\beta\gamma$ averages,

$$\begin{aligned}\bar{B}^{SS} &= \left\langle B^{SS}\right\rangle_{\alpha\beta\gamma} \\ &= 8\cos\theta_{gh}(p_\Lambda^2 p_g)^2\Big(-|ZG_1 - G_2|^2\Big),\end{aligned} \tag{90}$$

$$\begin{aligned}\bar{A}^{SS} &= \left\langle A^{SS}\right\rangle_{\alpha\beta\gamma} \\ &= 4\cos\theta_{gh}(p_\Lambda p_g \epsilon\omega)^2 \\ &\quad\times \frac{1}{1-Z}\Big(|ZG_1 - G_2|^2\Omega_f + Z(1-Z)|G_1|^2\Omega_\perp\Big).\end{aligned} \tag{91}$$

Both functions, $\bar{B}^{SS}$ and $\bar{A}^{SS}$, vanish on integration over the $\cos\theta_{gh}$ variable, but a finite result is obtained by weighting the integration with the factor $\cos\theta_{gh}$. Moreover, $ZG_1 - G_2 = ZG_E$ and $G_1 = G_M$, and the functions $\Omega_f$ and $\Omega_\perp$, are defined in appendix B. As a consequence, we may write our result on a more compact form,

$$\bar{B}^{SS} = 8\cos\theta_{gh}(p_\Lambda^2 p_g)^2\Big(-Z^2|G_E|^2\Big), \tag{92}$$

$$\bar{A}^{SS} = 4\cos\theta_{gh}(p_\Lambda p_g \epsilon\omega)^2 Z\left(\frac{-1}{\gamma_\Lambda^2}|G_E|^2\Omega_f + |G_M|^2\Omega_\perp\right). \tag{93}$$

Thus ends our exposé.

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## Appendix A: Kinematics explained

The parameters describing the decay of Lambda into proton and pion are $p_g$ and $E_g$, with

$$p_g = \frac{1}{2M_\Lambda}\Big[\big((M_\Lambda + m_p)^2 - \mu^2\big)\big((M_\Lambda - m_p)^2 - \mu^2\big)\Big]^{1/2}, \tag{A1}$$

$$E_g = \frac{1}{2M_\Lambda}\big(M_\Lambda^2 + m_p^2 - \mu^2\big), \tag{A2}$$

representing the proton in the Lambda rest system. The hyperon mass is sometimes denoted $M$, sometimes $M_\Lambda$.

The kinematic variables $P^2$, $y_1$, and $y_2$ of Eq. (5) are defined by,

$$P^2 = (p_1 + p_2)^2, \tag{A3}$$
$$y_1 = 2k_1 \cdot q = 2\epsilon\omega(1 - \cos\theta), \tag{A4}$$
$$y_2 = 2k_2 \cdot q = 2\epsilon\omega(1 + \cos\theta), \tag{A5}$$

and the normalisation factors $a_y$ and $b_y$ of the same equation by

$$a_y = 4P^2/(y_1 y_2), \tag{A6}$$
$$b_y = (2sP^2 + y_1^2 + y_2^2)/(y_1 y_2). \tag{A7}$$

## Appendix B: Notations explained

The $\Omega$ fuctions are by definition, and $\mathbf{n}^2 = 1$,

$$\Omega(\mathbf{n}, \mathbf{N}) = \mathbf{n}^2 + \mathbf{N}^2 = \frac{1}{v^2} + (\mathbf{n}\cdot\hat{\mathbf{k}})^2, \tag{B1}$$
$$= \Omega_f(\mathbf{n}, \mathbf{N}) + \Omega_\perp(\mathbf{n}, \mathbf{N}), \tag{B2}$$
$$\Omega_f(\mathbf{n}, \mathbf{N}) = (\mathbf{n}\cdot\mathbf{f})^2 + (\mathbf{N}\cdot\mathbf{f})^2, \tag{B3}$$
$$\Omega_\perp(\mathbf{n}, \mathbf{N}) = (\mathbf{n}\times\mathbf{f})^2 + (\mathbf{N}\times\mathbf{f})^2. \tag{B4}$$

The $\mathbf{N}$ vector is defined in Eq. (22), and fulfils the relations,

$$\mathbf{N}^2 = \frac{W^2}{\omega^2} + (\mathbf{n}\cdot\hat{\mathbf{k}})^2, \tag{B5}$$
$$\mathbf{n}\cdot\mathbf{N} = \frac{2\epsilon - \omega}{\omega}\mathbf{n}\cdot\hat{\mathbf{k}}. \tag{B6}$$

Also, introduced are co-factor functions $B_1$ and $B_3$

$$L_0 = \mathbf{n}\cdot\mathbf{g}_\perp\,\mathbf{n}\cdot\mathbf{h}_\perp + \mathbf{N}\cdot\mathbf{g}_\perp\,\mathbf{N}\cdot\mathbf{h}_\perp, \tag{B7}$$
$$L_M = (\mathbf{f}\cdot\mathbf{gh}_\perp + \mathbf{f}\cdot\mathbf{hg}_\perp)\cdot(\mathbf{nf}\cdot\mathbf{n} + \mathbf{Nf}\cdot\mathbf{N}), \tag{B8}$$
$$B_1 = 2\Big[L_0 + \frac{1}{\gamma_\Lambda}L_M\Big], \tag{B9}$$
$$B_3 = \gamma_\Lambda L_M, \tag{B10}$$

with $\gamma_\Lambda = E_\Lambda/M_\Lambda$.

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