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Risk Aversion, Downside Risk Aversion and Paying for Stochastic Improvements

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This paper considers the relationship between risk preferences and the willingness to pay for stochastic improvements. We show that if the stochastic improvement satisfies a double-crossing condition, then a decision maker with utility v is willing to pay more than a decision maker with utility u, if v is both more risk averse and less downside risk averse than u. As the condition always holds in the case of self-protection, the result implies novel characterizations of individuals' willingness to pay to reduce the probability of loss. By establishing a general result on the correspondence between an individual's willingness to pay, and his optimal purchase of stochastic improvement when there is a given relationship between stochastic improvements and the amount paid for them, we further show that all results on the willingness to pay can be applied directly to characterize the conditions under which a more risk averse individual will optimally choose to buy more stochastic improvement. Generalizations of existing results on optimal choice of self-protection can be obtained as corollaries.
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Introduction

Most decisions under risk can be thought of as decisions on paying for stochastic improvements.[1] An insurance policy, for example, pays a sum to offset part or all of a loss depending on the realized size of the loss and thus effects a stochastic improvement for the purchaser. Whether an individual will buy an insurance policy depends on whether his willingness to pay for the improvement exceeds the premium demanded by the insurer. The same can be said for decisions ranging from buying a burglar alarm to investing in a financial asset to get a college degree. How do an individual's risk

[1] A stochastic improvement here is meant mainly to be a first-degree stochastic dominant improvement in established terminology, but it can also be one in higher degrees.

preferences relate to his willingness to pay for a stochastic improvement? Is a more risk averse individual (in the Arrow-Pratt sense) always willing to pay more for, say, an insurance policy?[2] It has long been established that if the improvement in distribution satisfied a single-crossing condition, then a more risk averse individual is indeed always willing to pay more for the improvement (Hammond, 1974; Diamond and Stiglitz, 1974). Diamond and Stiglitz (1974) and Jewitt (1989) also provide necessary and sufficient conditions on the distributions for a more risk averse individual to be always willing to pay more for the change in distribution. And these conditions do describe well many real-world stochastic improvements that individuals pay for. For example, if the risks faced by an individual are all insurable, then it can be easily shown that the stochastic improvements effected by practically all real-world insurance policies satisfy the single-crossing condition and hence are worth more to a more risk averse individual. Nevertheless, it has also been demonstrated in the simplest setting where individuals either suffer a loss of a given size or they do not, a more risk averse individual in the Arrow-Pratt sense may not always be willing to pay more to reduce the probability of loss.[3] In this simple setting, subsequent authors further obtain interesting characterizations and interpretations of the relationship between risk aversion and the willingness to pay for (or optimal purchase of) such an improvement. But studies on the subject matter have so far focused exclusively on the simple case with just two possible outcomes mainly for reasons of tractability. The magnitude of the financial loss to a home owner in the event of a burglary, for example, is rarely, if ever, certain and the purchase of a burglar alarm or similar equipments may reduce but not eliminate the likelihood of the largest possible losses. In cases like this, it can be easily shown (as will be in the text) that the necessary and sufficient conditions of Diamond and Stiglitz (1974) and Jewitt (1989) are never satisfied and hence a more risk averse individual's willingess to pay for such equipments may not be greater.

This paper first addresses the question of a more risk averse individual's willingness to pay for a general stochastic improvement that does not satisfy

[2] In its original definition, an individual's risk premium is his willingness to pay for complete insurance for a zero-mean risk. A fundamental result in Pratt (1964) establishes its equivalence with the degree of risk aversion.

[3] This is, of course, the self-protection problem first studied by Ehrlich and Becker (1972) who have given hints on its peculiar relationship with risk aversion by showing that self-protection can be attractive to both risk lover and risk averter, and market insurance and self-protection can be complements. Dionne and Eeckhoudt (1985) show explicitly that a more risk-averse individual does not always purchase more self-protection. Briys and Schlesinger (1990) first relate the choice of self-protection to downside risk aversion.

the necessary and sufficient condition of Diamond and Stiglitz (1974). We find that an individual's willingness to pay for such an improvement may be determined not only by his risk aversion, but also by his downside risk aversion, a concept first introduced by Menezes *et al.* (1980) and extended by Keenan and Snow (2002, 2009).[4] Specifically, we show that if the stochastic improvement satisfies a double-crossing condition, then a decision maker with utility v is willing to pay more than a decision maker with utility u, if v is both more risk averse and less downside risk averse than u. As the condition always holds in the case of self-protection, the result implies novel characterizations of individuals' willingness to pay to reduce the probability of loss. Another corollary of the result gives conditions for an individual who is both risk averse and downside risk averse to be willing to pay more than the fair price (i.e., the increase in expected value) for a stochastic improvement.

When there is a given relationship between stochastic improvements and the amount paid for them, the correspondence between an individual's willingness to pay for a stochastic improvement and his optimal purchase of stochastic improvement is intuitive but has never been established formally.[5] By establishing a general result on such a correspondence under regularity conditions usually assumed, we show that all results on the willingness to pay can be applied directly to characterize the conditions under which a more risk averse individual will optimally choose to buy more stochastic improvement. In particular, generalizations of existing results on optimal choice of self-protection can be obtained as corollaries.

The rest of the paper is organized as follows: Section 2 sets out the basic notation and reviews established concepts and results on increases in risk and downside risk. Section 3 analyses the relationship between risk aversion and individuals' willingness to pay for stochastic improvements. Section 4 characterizes the conditions under which a more risk averse individual will optimally choose to buy more stochastic improvement. Section 5 concludes.

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[4] A downside risk averse individual dislikes an increase in riskiness in the lower possible values of his prospective wealth distribution combined with a decrease in riskiness in the higher possible values such that the variance remains the same and is characterized by a positive third derivative of his von Neumann-Morgenstern utility function, if he is an Expected Utility maximizer. Keenan and Snow (2002, 2009) define an individual to be more downside risk averse than another, if the former dislikes a compensated increase in downside risk for the latter.

[5] Some authors (e.g., Ross, 1981) consider regularities of an individual's willingness to pay (i.e., risk premium) and his optimal choice of stochastic improvement as distinct results to be derived separately, while others (e.g., Jewitt, 1989; Jullien *et al.*, 1999) appear to take the equivalence of the two for granted.

Preliminaries and changes in risk and downside risk

Throughout the paper, $F(x)$ and $G(x)$ etc. denote (cumulative) distribution functions and their variances are denoted by $\sigma^2(F(x))$, $\sigma^2(G(x))$, etc., and their density functions, if they exist, are denoted by $f(x)$ and $g(x)$ etc. The supports of all distributions considered are contained in the interval $[a, b]$. Individuals preferences over distributions are represented by von Neumann-Morgenstern (VNM) utility $u(\ )$, $v(\ )$, etc., which are assumed to be strictly increasing and concave and at least three times differentiable. $[(y, p)(z, 1-p)]$ denotes a Bernoulli distribution that gives y with probability p and z with probability $(1-p)$.

G is said to be a first-degree stochastic dominant (FSD) improvement of F, if $G(x) \leqslant F(x)$ for all $x \in [a, b]$ and G is said to be a mean-preserving spread (MPS) of F, and F a mean-preserving contraction (MPC) of G if

1. $\int_a^b [G(y) - F(y)]dy = 0$;
2. $\int_a^x [G(y) - F(y)]dy \geqslant 0$ for all $x[a, b]$.

It is well-known that an FSD improvement is desirable to any Expected Utility (EU) maximizer whose VNM utility function is increasing and any EU maximizer whose VNM utility function is concave is averse to MPSs. Following Menezes *et al.* (1980), G is said to be a downside risk increase of F if and only if

1. $\int_a^b [G(y) - F(y)]dy = 0$;
2. $\int_a^b \int_a^y [G(s) - F(s)]ds dy = 0$;
3. $\int_a^x \int_a^y [G(s) - F(s)]ds dy \geqslant 0$ for all $x \in [a, b]$.

Menezes *et al.* (1980) show that conditions (i) and (ii) are equivalent to F and G having the same mean and variance, and condition (iii), together with (i) and (ii), means that the shift from F to G can be decomposed into an MPS and an MPC with the MPS "coming before" the MPC and thus corresponds to a dispersion transfer from higher to lower wealth levels. They further show that any EU maximizer whose VNM utility function has a positive third derivative dislikes any increase in downside risk, and that an increase in downside risk implies an increase in skewness, as measured by the third central moment.

Extending the definitions and analysis of Menezes *et al.* (1980) and Diamond and Stiglitz (1974), Keenan and Snow (2002, 2009) define the concept of "compensated increase in downside risk" for an individual u and show that another individual v dislikes a compensated increase in downside risk for u and thus more downside risk averse than u, if and only if $v(x) = T(u(x))$ and $T'''(\ ) \geqslant 0$. In our notation, $u^{-1}(\ )$ being well-defined given $u(\ )$ being strictly increasing, the shift from $F(x)$ to $G(x)$ is a compensated increase in downside risk for u if

1. $\int_{u(a)}^{u(b)}[G(u^{-1}(s))-F(u^{-1}(s))]ds=0$;
2. $\int_{u(a)}^{u(b)}\int_{u(a)}^{t}[G(u^{-1}(s))-F(u^{-1}(s))]dsdt=0$;
3. $\int_{u(a)}^{u}\int_{u(a)}^{t}[G(u^{-1}(s))-F(u^{-1}(s))]dsdt\geqslant 0$ for all $u\in[u(a),u(b)]$,

where $G(u^{-1}(u))$ and $F(u^{-1}(u))$ are u's utility distributions induced by $G(x)$ and $F(x)$, respectively. Thus, the conditions (i)–(iii) mean that the shift from $F(x)$ to $G(x)$ induces a downside risk increase in u's utility distribution, which can be interpreted as a compensated downside risk increase from u's viewpoint since condition (i) means u is indifferent to the shift.

As will be seen in the sequel, paying for stochastic improvements may not always result in just a change in risk or just a change in downside risk but a combination of them. We next characterize a mean-preserving change in distribution, which can be decomposed into a change in risk and a change in downside risk.[6]

Proposition 1 Let F, G be distribution functions with the same mean.

1. There exists a distribution function $H(x)$ such that $G(x)$ is a downside risk increase of $H(x)$ and $H(x)$ an MPC of $F(x)$, if and only if $\int_x^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$ for all $x\in[a,b]$.
2. There exists a distribution function $H(x)$ such that $G(x)$ is a downside risk increase of $H(x)$ and $H(x)$ an MPS of $F(x)$, if and only if $\int_a^x\int_a^y[G(s)-F(s)]dsdy\geqslant 0$ for all $x\in[a,b]$.[7]

We next show that if F and G have the same mean and $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ only once, then the shift from F to G can always be decomposed into a change in risk and a change in downside risk.

Proposition 2 Suppose F and G have the same mean and $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ once from above. Then

1. If $\sigma^2(F(x))\leqslant\sigma^2(G(x))$, then $\int_a^x\int_a^y[G(s)-F(s)]dsdy\geqslant 0$ for all $x\in[a,b]$.
2. If $\sigma^2(F(x))\geqslant\sigma^2(G(x))$, then $\int_x^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$ for all $x\in[a,b]$.

Clearly if G and F cross twice and have the same mean, then $\int_a^x G(y)dy$ and $\int_a^x F(y)dy$ cross once.

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[6] Proofs of all new formal results not proved in the text can be found in the Appendix.

[7] The proof of the Proposition in the Appendix contains an example illustrating the construction of the distribution $H(x)$, which should be helpful for understanding the conditions (i) and (ii).

Willingness to pay for stochastic improvements

Suppose an individual u with initial prospective wealth distribution $F(x)$ can pay to secure an FSD improvement in $F(x)$ and the improved distribution is denoted by $G(x)$. Then the maximum u is willing to pay for the improvement, π^u, is such that

$$\int_a^b u(y)dF(y) = \int_a^b u(y-\pi^u)dG(y) = \int_a^b u(y)dG(y+\pi^u). \tag{1}$$

Will a more risk averse individual in the Arrow-Pratt sense v necessarily be willing to pay more (or less) for the same improvement? That is, letting $v(x)=T(u(x))$ and $T''()\leqslant 0$[8] and π^v be such that

$$\int_a^b v(y)dF(y) = \int_a^b v(y-\pi^v)dG(x). \tag{2}$$

Will we necessarily have $\pi^v \geqslant \pi^u$ (or $\pi^v \leqslant \pi^u$)? It has long been established that if $G(x+\pi^u)$ crosses $F(x)$ exactly once, then the answer is definite (Hammond (1974) and Diamond and Stiglitz (1974)). Diamond and Stiglitz (1974) and Jewitt (1989) further provide necessary and sufficient conditions for $\pi^v \geqslant \pi^u$ when the distributions do not necessarily cross only once. The existing results are summarized as follows.

Proposition 3 For any distributions F and G and premia π^u and π^v satisfying (1) and (2),

1. $\pi^u \leqslant (\geqslant) \pi^v$ for any $v(x)=T(u(x))$ such that $T'()>0$ and $T''()\leqslant 0$, if $G(x+\pi^u)$ crosses $F(x)$ once from below (above);
2. $\pi^u \leqslant (\geqslant) \pi^v$ for any $v(x)=T(u(x))$ such that $T'()>0$ and $T''()\leqslant 0$, if and only if $\int_{u(a)}^{u}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds \leqslant (\geqslant) 0$ for all $u\in[u(a),u(b)]$.[9]

In the simple self-insurance problem considered by Ehrlich and Becker (1972), where $F(x)$ is the distribution function for $[(w-l,p)(w,1-p)]$ and $G(x)$ for $[(w-l+\theta,p)(w,1-p)]$, $G(x+\pi)$ clearly crosses $F(x)$ once from below for

[8] It is well-known that $v(x)=T(u(x))$ and $T''()\leqslant 0$, if and only if $(-v''(x)/v'(x)) \geqslant (-u''(x)/u'(x))$ for all x.
[9] Since (1) holds, the condition $\int_{u(a)}^{u}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds \leqslant (\geqslant) 0$ for all $u\in[u(a),u(b)]$ indicates that $G(x+\pi^u)$ is a mean-utility-preserving contraction (spread) of $F(x)$.

$\pi \in (0, l)$.[10] We thus have the following:

Corollary 1 (Self-insurance) Let $F(x)$ and $G(x)$ be the distribution functions for $[(w-l, p)(w, 1-p)]$ and $[(w-l+\theta, p)(w, 1-p)]$, respectively, where $\theta \in (0, l)$ and premia π^u and π^v satisfy (1) and (2). Then, $\pi^u \leqslant \pi^v$ for $v(x) = T(u(x))$ such that $T'(\,) \geqslant 0$ and $T''(\,) \leqslant 0$.

It is instructive to note that a necessary condition for a more risk averse v to be always willing to pay more (less) than u for the FSD improvement from $F(x)$ to $G(x)$ is that $G(x+\pi^u)$ crosses $F(x)$ for an odd number of times first from below (above).

Proposition 4 For any distributions F and G and premia π^u and π^v satisfying (1) and (2), if $\int_a^b u(y)dF(y) = \int_a^b u(y-\pi^u)dG(y)$ and $\int_{u(a)}^{u}[G(u^{-1}(s)+\pi^u) - F(u^{-1}(s))]\,ds \leqslant (\geqslant)\, 0$ for all $u \in [u(a), u(b)]$, then $G(x+\pi^u)$ crosses $F(x)$ an odd number of times first from below (above).

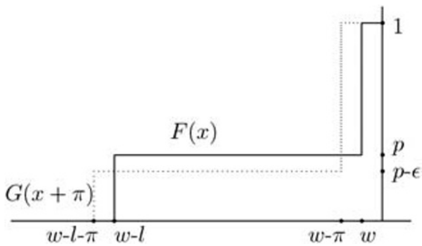
If $G(x+\pi^u)$ and $F(x)$ cross each other an odd number of times first from below, then their densities, $g(x+\pi^u)$ and $f(x)$ if they exist, cross each other an even number of times first from below and hence the support of $g(x+\pi^u)$ is contained in that of $f(x)$. Thus, for a more risk averse v to be always willing to pay more for the improvement, $G(x+\pi^u)$ needs to have a "smaller spread" than $F(x)$ (in the sense of the support of $g(x+\pi^u)$ being contained in that of $f(x)$). The results in Proposition 3 can thus *potentially* apply to stochastic improvements resulting in $G(x+\pi^u)$ having a smaller or larger spread than $F(x)$.

What Proposition 4 also indicates, however, is that the results in Proposition 3 never apply to one of the simplest risk management problems. Specifically, in the basic self-protection problem where $F(x)$ is the distribution function for $[(w-l, p)(w, 1-p)]$ and $G(x)$ for $[(w-l, p-\varepsilon)(w, 1-p+\varepsilon)]$, $G(x+\pi)$ crosses $F(x)$ twice (first from above) for $\pi \in (0, l)$ as illustrated below.

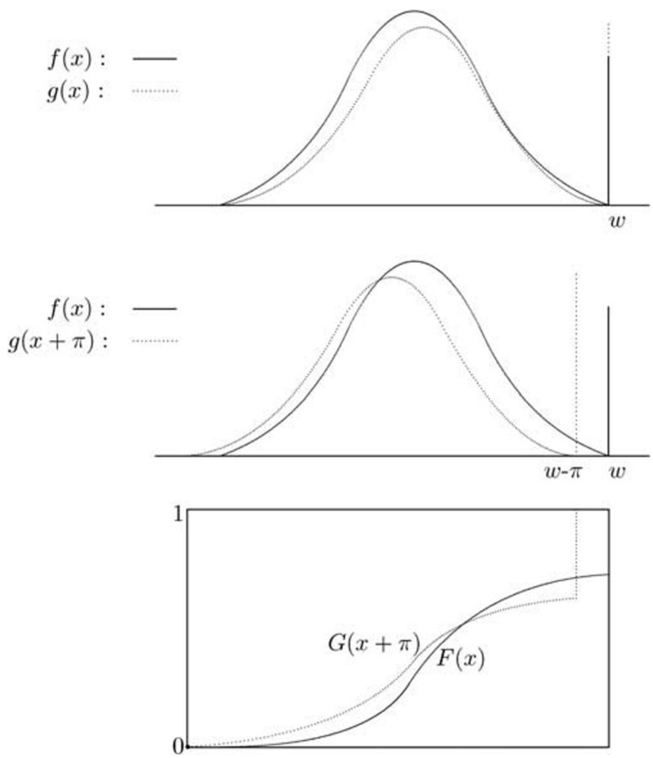
That is, in this problem the necessary condition for a more risk averse individual to be *always* willing to pay more for an improvement is violated.[11] In the case where there are many different possible outcomes, paying for a stochastic

[10] More generally, if $F(x)$ is the distribution function for $\tilde{x} = w - \tilde{l}$ and the improvement takes the form of an insurance policy that pays an indemnity $I(\tilde{l})$ such that both $I(\tilde{l})$ and $[\tilde{l} - I(\tilde{l})]$ are non-decreasing, a condition satisfied by practically all real-world insurance policies, then a more risk averse individual will always be willing to pay more for the policy because it can be shown that $G(x+\pi^u)$ always crosses $F(x)$ once from below.

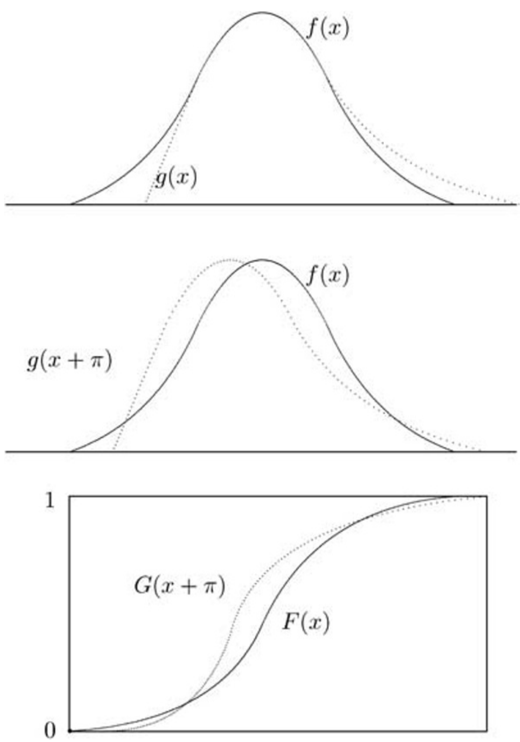
[11] It is thus not surprising that a more risk averse individual has been shown not to always choose more self-protection.



improvement can also easily result in a new distribution that crosses the old twice. For example, if the magnitude of the financial loss to a home owner in the event of a burglary is uncertain and a burglar alarm *reduces, but does not eliminate*, the likelihood of the largest possible losses as alluded to in the Introduction, then if $f(x)$ and $g(x)$, the density functions of the wealth distributions before and after installing a burglar alarm (which have a mass point at w, the wealth level if no burglary occurs), are as illustrated below, paying a price π for the improvement will result in $G(x+\pi)$ crossing $F(x)$ twice first from above.



Since the usual self-protection problem corresponds to the special case where the magnitude of the loss is certain, the problem illustrated above where the new distribution crosses the old twice first from above can be thought of as a "generalized self-protection problem". On the other hand, the stochastic improvement, say the effect of a marketing campaign on a firm owner's profit, can take the form of eliminating the possibility of the worst outcomes and significantly extending the range of possible good outcomes as illustrated below, also resulting in $G(x+\pi^u)$ crossing $F(x)$ twice, albeit first from below.



We next show that if $G(x+\pi^u)$ crosses $F(x)$ twice, then whether a more risk averse individual is always willing to pay more (or less) for the improvement depends also on whether he is also more (or less) downside risk averse.

Given $u'(\,)>0$, we clearly have $u^{-1'}(\,)>0$. Hence for π^u as defined in (1), u's utility distributions induced by $G(x+\pi^u)$ and $F(x)$, $G(u^{-1}(u)+\pi^u)$ and $F(u^{-1}(u))$, cross each other the same number of times as $G(x+\pi^u)$ and $F(x)$.

Since (1) is equivalent to

$$\int_{u(a)}^{u(b)} u dG(u^{-1}(u) + \pi^u) = \int_{u(a)}^{u(b)} u dF(u^{-1}(u))$$

or $G(u^{-1}(u) + \pi^u)$ and $F(u^{-1}(u))$ have the same mean, if $G(x + \pi^u)$ crosses $F(x)$ twice first from above and hence $G(u^{-1}(u) + \pi^u)$ crosses $F(u^{-1}(u))$ twice first from above, then $\int_{u(a)}^{u} G(u^{-1}(s) + \pi^u) ds$ crosses $\int_{u(a)}^{u} F(u^{-1}(s)) ds$ once from above. If $\sigma^2(F(u^{-1}(u))) \geqslant \sigma^2(G(u^{-1}(u) + \pi^u))$ in addition, then, by Propositions 1 and 2, there exists a distribution function $\hat{H}(u)$ such that $G(u^{-1}(u) + \pi^u)$ is a downside risk increase of $\hat{H}(u)$ and $\hat{H}(u)$ is an MPC of $F(u^{-1}(u))$. We thus have

$$\int_a^b v(y - \pi^u) dG(y) - \int_a^b v(y) dF(y) = \int_a^b v(y) dG(y + \pi^u) - \int_0^{\bar{x}} v(y) dF(y)$$

$$= \int_a^b T(u(y)) d[G(y + \pi^u) - F(y)] = \int_{u(a)}^{u(b)} T(u) d[G(u^{-1}(u) + \pi^u) - F(u^{-1}(u))]$$

$$= \int_{u(a)}^{u(b)} T(u) d[G(u^{-1}(u) + \pi^u) - \hat{H}(u)] + \int_{u(a)}^{u(b)} T(u) d[\hat{H}(u) - F(u^{-1}(u))] \geqslant 0$$

if $T''(\,) \leqslant 0$ and $T'''(\,) \leqslant 0$, which implies $\pi^v \geqslant \pi^u$.

Proposition 5 For any distributions F and G and premia π^u and π^v satisfying (1) and (2), suppose $G(x + \pi^u)$ crosses $F(x)$ twice first from above (below). Then

1. $\pi^u \leqslant (\geqslant) \pi^v$ for $v(x) = T(u(x))$ such that $T'(\,) \geqslant 0, T''(\,) \leqslant 0$, and $T'''(\,) \leqslant 0$ if $\sigma^2(F(u^{-1}(u))) \geqslant (\leqslant) \sigma^2(G(u^{-1}(u) + \pi^u))$;
2. $\pi^u \geqslant (\leqslant) \pi^v$ for $v(x) = T(u(x))$ such that $T'(\,) \geqslant 0$, $T''(\,) \leqslant 0$, and $T'''(\,) \geqslant 0$ if $\sigma^2(F(u^{-1}(u))) \leqslant (\geqslant) \sigma^2(G(u^{-1}(u) + \pi^u))$.

Note first that for $F(x)$ and $G(x)$ being any two distributions, the change in overall riskiness from u's viewpoint (as measured by the variance of u's utility) brought about by the change in distribution from $F(x)$ to $G(x + \pi^u)$ is either positive or negative, that is, either $\sigma^2(F(u^{-1}(u))) \geqslant \sigma^2(G(u^{-1}(u) + \pi^u))$ or $\sigma^2(F(u^{-1}(u))) \leqslant \sigma^2(G(u^{-1}(u) + \pi^u))$ must be true. The result says that in the case where $G(x + \pi^u)$ crosses $F(x)$ twice first from above and the overall riskiness from u's viewpoint is reduced, a more risk averse individual v will pay more for the improvement if v is also less downside risk averse than u as is defined by Keenan and Snow (2002, 2009), that is, $T'''(\,)\leqslant 0$. On the other hand, in the case where $G(x+\pi^u)$ crosses $F(x)$ twice first from above and the overall riskiness from u's viewpoint is raised, a more risk averse individual v will actually pay *less* for the improvement if he is also more downside risk averse. The intuition for the result can be gleaned from its derivation: If $G(x+\pi^u)$ crosses $F(x)$ twice first from above and the variance of u's utility is reduced, the induced change in u's utility distribution is a combination of an MPC and a downside risk increase (by Propositions 1 and 2). Thus, a more risk averse and less downside risk averse v will be willing to pay more for such a change.

Proposition 1 and the reasoning for Proposition 5 clearly indicate that we can have a more general result which gives the necessary and sufficient condition for a more risk averse but less downside risk averse individual to be willing to pay more for a stochastic improvement. The result can be seen as a generalization of Keenan and Snow (2002, 2009) and is given in the Appendix.

Considering the case where u is linear, Proposition 5 immediately implies the following result, which provides a sufficient condition on a stochastic improvement for a more risk averse but less (more) downside risk averse individual to be willing to pay more (less) than the fair price (i.e., the increase in expected value) for the improvement. In stating the formal results that follow, we shall denote by π^{GF} the fair price for the improvement from F to G, $\int_a^b y\,d[G(y)-F(y)]$.

Corollary 2 Suppose $G(x+\pi^{GF})$ crosses $F(x)$ twice first from above (below). Then

1. If $\sigma^2(F(x))\geqslant(\leqslant)\sigma^2(G(x))$ and $v'''(\,)\leqslant 0$, then $\pi^v\geqslant(\leqslant)\pi^{GF}$;
2. If $\sigma^2(F(x))\leqslant(\geqslant)\sigma^2(G(x))$ and $v'''(\,)\geqslant 0$, then $\pi^v\leqslant(\geqslant)\pi^{GF}$.

That is, in the case where $G(x+\pi^{GF})$ crosses $F(x)$ twice first from above, a risk averse v will pay more than the fair price for the improvement from $F(x)$ to $G(x)$ if the improvement reduces the variance and he is also downside risk loving. On the other hand, if the improvement increases the variance and he is downside risk averse, he will be willing to pay *less* than the fair price for the improvement.

These results clearly apply to the basic self-protection problem where $F(x)$ is the distribution function for $[(w-l,p)(w,1-p)]$ and $G(x)$ for $[(w-l,p-\varepsilon)(w,1-p+\varepsilon)]$ and $G(x+\pi)$ crosses $F(x)$ twice first from above for $\pi\in(0,l)$ as shown earlier. Thus, if the variance of u's utility is reduced by the improvement, then a more risk averse but less downside risk averse v will be willing to pay more for the improvement.

Corollary 3 Let $F(x)$ and $G(x)$ be the distribution functions for $[(w-l, p)(w, 1-p)]$ and $[(w-l, p-\varepsilon)(w, 1-p+\varepsilon)]$, respectively, and premia π^u and π^v satisfy (1) and (2). Then

1. $\pi^u \leqslant \pi^v$ for $v(x) = T(u(x))$ such that $T'() \geqslant 0$, $T''() \leqslant 0$, and $T'''() \leqslant 0$ if $\sigma^2(F(u^{-1}(u))) \geqslant \sigma^2(G(u^{-1}(u) + \pi^u))$;
2. $\pi^u \geqslant \pi^v$ for $v(x) = T(u(x))$ such that $T'() \geqslant 0$, $T'''() \leqslant 0$, and $T'''() \geqslant 0$ if $\sigma^2(F(u^{-1}(u))) \leqslant \sigma^2(G(u^{-1}(u) + \pi^u))$.

Moreover, a risk averse and downside risk loving individual will pay more than the fair price for the improvement from $F(x)$ to $G(x)$ if the improvement reduces the variance.

Corollary 4 Let $F(x)$ and $G(x)$ be the distribution functions for $[(w-l, p)(w, 1-p)]$ and $[(w-l, p-\varepsilon)(w, 1-p\ +\varepsilon)]$, respectively. Then,

1. If $\sigma^2(F(x)) \geqslant \sigma^2(G(x))$ and $v'''(\ ) \leqslant 0$, then $\pi^v \geqslant \pi^{GF}$;
2. If $\sigma^2(F(x)) \leqslant \sigma^2(G(x))$ and $v'''(\ ) \geqslant 0$, then $\pi^v \leqslant \pi^{GF}$.

Corollaries 3 and 4 offer novel characterizations of the relationship between risk aversion, downside risk aversion, and the willingness to pay for self-protection that illuminate existing results on the matter. It was first pointed out by Briys and Schlesinger (1990) that purchasing self-protection does not result in a reduction of risk in an individual's wealth distribution in the sense of an MPC. Instead, if the price paid for reducing the loss probability is fair and hence the mean of the wealth distribution is kept the same, the effect of self-protection is in fact a downside risk increase provided the overall variance is also kept the same. The overall variance, however, as they also point out, does not in general stay the same. Chiu (2000) explicitly examines the relationship between risk aversion and the willingness to pay for self-protection and finds, among other things, that if the initial loss probability p is below a threshold level, then a risk averse individual will be willing to pay more than the fair price for self-protection and the threshold is determined by both risk aversion and downside risk aversion. This result can be better understood in the context of Corollaries 3 and 4 as the effects of purchasing self-protection on the variances of an individual's wealth distribution and his induced utility distribution are determined in part by the initial loss probability p. In particular, since it can be easily shown that $\sigma^2(F(x)) \geqslant \sigma^2(G(x))$ for all $\varepsilon \in (0, p)$ if and only if $p \leqslant 1/2$, Corollary 4 implies the following that complements Chiu's (2000) result.

Corollary 5 Let $F(x)$ and $G(x)$ be the distribution functions for $[(w-l, p)(w, 1-p)]$ and $[(w-l, p-\varepsilon)(w, 1-p+\varepsilon)]$, respectively. If $p \leqslant 1/2$ and $v'''(\ ) \leqslant 0$, then $\pi^v \geqslant \pi^{GF}$.

The effect of self-protection on the variance of an individual's induced utility distribution as a function of the initial loss probability is more complex. But, as will be shown in the next section, if the changes in the loss probability are small, then the variance of an individual's utility is an increasing function of the initial loss probability, which explains the important role played by the initial loss probability in determining the relationship between risk aversion and the *optimal choice* of self-protection expenditure uncovered in previous studies (to be discussed in the next section).

Optimal purchase of stochastic improvements

There have so far been few attempts to characterize the relationship between risk aversion and optimal purchase of stochastic improvements in a general setting with more than two possible outcomes and the results obtained by Lee (1998) and Jullien *et al.* (1999), for example, do not go beyond applications of known properties of single-crossing distributions. In this section, we consider a more general and yet less complex model[12] and show all our earlier results on the willingness to pay for stochastic improvements can be applied directly to obtain more general characterizations on optimal purchase of stochastic improvements. In particular, by first establishing a general result on the correspondence between an individual's willingness to pay and his optimal purchase of stochastic improvement when there is a given relationship between stochastic improvements and the amount paid for them, we show that a more risk averse but less downside risk averse individual will optimally choose to buy more stochastic improvement if a marginal change in the amount of stochastic improvement purchased entails a new distribution crossing the old twice and reduces the overall riskiness as measured by the variance of the less risk averse individual's utility. Given that the double-crossing condition is satisfied globally in the self-protection problem, a corollary of this result significantly generalizes an existing result on the relationship between downside risk aversion (or equivalently prudence) and the optimal choice of self-protection.

Denote an individual prospective wealth distribution by $F(x, \theta)$ and for $\theta_2 > \theta_1$, $F(x, \theta_2)$ is an FSD improvement of $F(x, \theta_1)$. Individuals can pay a sum e

[12] The general model encompasses paying for stochastic improvements, which reduce both the probability of loss and the severity of loss considered by Lee (1998).

to secure an FSD improvement. So $\theta = \theta(e)$ and $\theta'(e) > 0$. Individual u will thus choose e to maximize

$$EU(e) = \int_a^b u(x - e)dF(x, \theta(e)).$$

It is assumed that the functions $u(x)$, $\theta(e)$, and $F(x, \theta)$ are such that the second-order condition $EU''(e) \leqslant 0$ holds and the solution is internal and hence u's optimal choice e_u is given by the first-order condition

$$EU'(e_u) = -\int_a^b u'(x - e_u)dF(x, \theta(e_u)) + \theta'(e_u)\int_a^b u(x - e_u)dF_\theta(x, \theta(e_u)) = 0.$$

For notational simplicity, we define $\theta_u \equiv \theta(e_u)$.

We now show that an individual's optimal purchase of stochastic improvement will be larger than another's if and only if, given the latter's optimal purchase, the former's marginal willingness to pay for a small additional improvement, which may be positive or negative, is greater than the latter's (in absolute value). To state the formal results that follow, for $\Delta\theta \in R$, the functions $\Delta e_u = \Delta e_u(\Delta\theta)$ and $\Delta e_v = \Delta e_v(\Delta\theta)$, which represent, respectively, u's and v's marginal willingness to pay for an additional improvement $\Delta\theta$ are defined respectively by the following equations:

$$\int_a^b u(x - e_u - \Delta e_u)dF(x, \theta_u + \Delta\theta) = \int_a^b u(x - e_u)dF(x, \theta_u);$$

$$\int_a^b v(x - e_u - \Delta e_u)dF(x, \theta_u + \Delta\theta) = \int_a^b v(x - e_u)dF(x, \theta_u).$$

Lemma 1 $e_v \geqslant e_u$, if and only if there exists $\delta > 0$ such that for $\Delta\theta \in (-\delta, \delta)$, $|\Delta e_u(\Delta\theta)| \leqslant |\Delta e_v(\Delta\theta)|$.

The result allows for direct application of the results on the willingness to pay to characterize the conditions under which a more risk averse individual optimally chooses to purchase more stochastic improvement. The result of Diamond and Stiglitz (1974) is first used to give a necessary and sufficient condition for a more risk averse individual to optimally choose to purchase more stochastic improvement.

Proposition 6 $e_v \geqslant e_u$ for all $v(x) = T(u(x))$ such that $T'() > 0$ and $T''() \leqslant 0$ if and only if there exists $\delta > 0$ such that, for $\Delta\theta \in (-\delta,\delta)$, $\Delta\theta \int_{u(a)}^{u} [F(u^{-1}(s) + e_u + \Delta e_u, \theta_u + \Delta\theta) - F(u^{-1}(s) + e_u, \theta_u)]ds \leqslant 0$ for all $u \in [u(a), u(b)]$.[13]

The result generalizes those of Lee (1998) and Jullien *et al.* (1999), which can be seen as applications of the following corollary to Proposition 6.

Corollary 6 $e_v \geqslant e_u$ for all $v(x) = T(u(x))$ such that $T'(\ ) > 0$ and $T''(\ ) \leqslant 0$ if there exists $\delta > 0$ such that for $\Delta\theta \in (-\delta, \delta)$, $\Delta\theta \cdot F(x + e_u + \Delta e_u, \theta_u + \Delta\theta)$ crosses $\Delta\theta \cdot F(x + e_u, \theta_u)$ once from below.

Clearly if the single-crossing condition holds "globally", as in the case of self-insurance (where $F(x,\theta)$ is the distribution function for $[(w-l+\theta, p)(w, 1-p)]$ and $w > l > 0$), then a more risk averse individual always chooses to buy more stochastic improvement.

Corollary 7 (Self-insurance) Let $F(x, \theta)$ be the distribution function for $[(w-l+\theta, p)(w, 1-p)]$ where $w > l > 0$. Then, $e_u \leqslant e_v$ for $v(x) = T(u(x))$ such that $T'(\ ) \geqslant 0$ and $T''(\ ) \leqslant 0$.

Proposition 5, together with Lemma 1, implies that a more risk averse but less downside risk averse individual will optimally choose to buy more stochastic improvement if a marginal change in the amount of stochastic improvement purchased entails a new distribution crossing the old twice and reduces the overall riskiness, as measured by the variance of the less risk averse individual's utility.

Proposition 7

1. $e_v \geqslant e_u$ for $v(x) = T(u(x))$ such that $T'(\ ) > 0$ and $T''(\ ) \leqslant 0$ and $T'''(\ ) \leqslant 0$ if there exists $\delta > 0$ such that for $\Delta\theta \in (-\delta, \delta)$, $\Delta\theta \cdot F(x + e_u + \Delta e_u, \theta_u + \Delta\theta)$ crosses $\Delta\theta \cdot F(x + e_u, \theta_u)$ twice first from above and $\Delta\theta \cdot \sigma^2(F(u^{-1}(u) + e_u + \Delta e_u, \theta_u + \Delta\theta)) \leqslant \Delta\theta\ \sigma^2(F(u^{-1}(u) + e_u, \theta_u))$;
2. $e_v \leqslant e_u$ for $v(x) = T(u(x))$ such that $T'\ (\ ) > 0$ and $T''(\ ) \leqslant 0$ and $T'''(\ ) \geqslant 0$ if there exists $\delta > 0$ such that for $\Delta\theta \in (-\delta, \delta)$, $\Delta\theta \cdot F(x + e_u + \Delta e_u, \theta_u + \Delta\theta)dy$

[13] The condition $\Delta\theta \int_{u(a)}^{u} [F(u^{-1}(s) + e_u + \Delta e_u, \theta_u + \Delta\theta) - F(u^{-1}(s) + e_u, \theta_u)]ds \leqslant 0$ for all $u \in [u(a), u(b)]$ simply means that for $\Delta > 0$, $\int_{u(a)}^{u} [F(u^{-1}(s) + e_u + \Delta e_u, \theta_u + \Delta\theta) - F(u^{-1}(s) + e_u, \theta_u)]ds \leqslant 0$ for all $u \in [u(a), u(b)]$, and for $\Delta\theta < 0$, $\int_{u(a)}^{u} [F(u^{-1}(s) + e_u + \Delta e_u, \theta_u + \Delta\theta) - F(u^{-1}(s) + e_u, \theta_u)]ds \leqslant 0$ for all $u \in [u(a), u(b)]$. The same device is used in the statements of other results that follow to make the descriptions less tedious.

crosses $\Delta\theta \cdot F(x+e_u, \theta_u)$ twice first from above and $\Delta\theta \cdot \sigma^2(F(u^{-1}(u)+e_u+\Delta e_u, \theta_u+\Delta\theta)) \geqslant \Delta\theta \cdot \sigma^2(F(u^{-1}(u)+e_u, \theta_u))$.

Again, the condition on the number of crossings between the distributions is required only for small deviations from u's optimal choice. The same results of course obtain in cases where purchasing any amount of stochastic improvement within the range of feasible choice entails a new distribution crossing the old twice. This is clearly the case in the basic self-protection problem where $F(x, \theta)$ is the distribution function for $[(w-l, p-\theta)(w, 1-p+\theta)]$ and $F(x+e, \theta_2)$ crosses $F(x, \theta_1)$ twice first from above for all $\theta_2, \theta_1 \in (0, p)$ such that $\theta_2 > \theta_1$ and $e \in (0, l)$. Furthermore, there is a monotonic relationship between the change in the variance of u's utility caused by a marginal change in the amount of self-protection purchased and the probability of loss given u's purchase of self-protection, as is shown in the following lemma.

Lemma 2 Let $F(x, \theta)$ be the distribution function for $[(w-l, p-\theta)(w, 1-p+\theta)]$ where $w > l > 0$. Then, there exists $\bar{p}_u \in (0,1)$ such that $\sigma^2_{\Delta\theta}(F(u^{-1}(u)+e_u+\Delta e_u(\Delta\theta), \theta_u\ \Delta\theta))|_{\Delta\theta=0} < (=, >)\, 0$ if $p-\theta_u < (=, >) \bar{p}_u$.

We are thus able to characterize the condition under which a more risk averse but less (or more) downside risk averse individual chooses to purchase more self-protection solely in terms of the loss probability.

Corollary 8 Let $F(x, \theta)$ be the distribution function for $[(w-l, p-\theta)(w, 1-p+\theta)]$ where $w > l > 0$ and $v(x) = T(u(x))$ and $T''(\ ) < 0$. Then there exists $\bar{p}_u \in (0, 1)$ such that

1. if $p-\theta_u \leqslant \bar{p}_u$ and $T'''(\ ) \leqslant 0$, then $e_u \leqslant e_v$;
2. if $p-\theta_u \geqslant \bar{p}_u$ and $T'''(\ ) \geqslant 0$, then $e_u \geqslant e_v$.

The result is a characterization of the relationship between risk aversion and the choice of self-protection that highlights the role of downside risk aversion in determining the relationship. Its derivation from more general results also serves to make plain the intuition for the importance of downside risk aversion and the probability of loss and thus illuminates existing results involving the probability of loss. Specifically, McGuire *et al.* (1991) and Jullien *et al.* (1999) show that an individual v more risk averse than u will choose more self-protection, if and only if the probability of loss given u's optimal choice is less than a critical value. This critical value, however, depends on both v and u, as well as e_u. Lemma 2 and Corollary 8, on the other hand, show that the crucial role played by the probability of loss is due to its effect on the variance of u and whether a more risk averse individual is also more (or less) downside risk averse is also important in determining whether he will purchase more self-protection.[14]

Noting that $\bar{p}_u = 1/2$ when u is linear,[15] Corollary 8 is also a generalization of the main result obtained in Eeckhoudt and Gollier (2005), which states that a prudent, or equivalently downside risk averse, individual optimally chooses to purchase less self-protection than a risk neutral individual if the initial loss probability is less than one-half. The result corresponds to the special case of Corollary 8 where u is risk neutral as stated in what follows.

Corollary 9 Let $F(x,\theta)$ be the distribution function for $[(w-l, p-\theta)(w, 1-p+\theta)]$ where $w > l > 0$, $v(x)$ be concave and $u(x)$ linear. Then,

1. if $p-\theta_u \leqslant 1/2$ and $v'''(\ ) \leqslant 0$, then $e_u \leqslant e_v$;
2. if $p-\theta_u \geqslant 1/2$ and $v'''(\ ) \geqslant 0$, then $e_u \geqslant e_v$.

Conclusions

This paper addresses the question of whether a more risk averse individual is always willing to pay more for a stochastic improvement. We show that if the stochastic improvement satisfies a double-crossing condition, then whether a more risk averse individual is willing to pay more (or less) depends also on whether he is less downside risk averse. As the condition always holds in the case of self-protection, the result implies novel characterizations of individuals' willingness to pay to reduce the probability of loss. Corollaries of these results give conditions for a risk averse individual to be willing to pay more than the fair price (i.e., the increase in the expected value) for a stochastic improvement. By establishing a general result on the correspondence between an individual's willingness to pay and his optimal purchase of stochastic improvement when there is a given relationship between stochastic improvements and the amount paid for them, we further show that all results on the willingness to pay can be applied directly to characterize the conditions under which a more risk averse individual will optimally choose to buy more stochastic improvement. Corollaries of the results generalize, and provide intuitive explanation for,

[14] A crucial difference between our result and McGuire *et al.* (1991) is that their threshold depends on u and a specific v. Ours, on the other hand, applies to *any* v who is more risk averse and less downside risk averse.

[15] As can be seen in the proof of Lemma 2, in the case where $u(w) = w$,

$$\frac{\partial \sigma^2}{\partial \Delta\theta}\Big|_{\Delta\theta=0} = w_2^2 - w_1^2 - 2[pw_1 + (1-p)w_2](w_2 - w_1)$$
$$= (w_2 - w_1)^2(2p-1) > (=, <) 0 \text{ if } p > (=, <) \frac{1}{2}.$$

existing results on how risk aversion, downside risk aversion and the initial loss probability determine an individual's optimal choice of self-protection.

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About the Author

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Appendix

Proofs

Proof of Proposition 1

(i) ($\Rightarrow$) Since $\int_a^b\int_0^y[G(s)-H(s)]dsdy=0$ and $\int_a^x\int_a^y[G(s)-H(s)]dsdy\geqslant 0$ for all x, we have $\int_x^b\int_a^y[G(s)-H(s)]dsdy\leqslant 0$ for all x, which together with $\int_a^x[H(y)-F(y)]dy\leqslant 0$ for all x, implies $\int_x^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$ for all x.
($\Leftarrow$) Suppose $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ n times and $x_1,x_2,\dots,x_n$ are the crossings. $\int_x^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$ for all $x\in[a,b]$ clearly implies that $\int_a^x G(y)dy\leqslant\int_a^x F(y)dy$ for $x\in[x_n,b]$ and $\int_a^x G(y)dy\geqslant\int_a^x F(y)dy$ for $x\in[x_{n-1},x_n]$. Consider first the case where $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ first from above. $H(x)$ is constructed as follows: (a) $H(x)=F(x)$ for $x\leqslant x_1$; (b) if $\int_a^{x_1}\int_0^y[G(s)-H(s)]dsdy+\int_{x_1}^{x_2}\int_a^y[G(s)-F(s)]dsdy<0$, then for $x\in[x_1,x_2]$, let $H(x)$ be such that $\min\{F(x),G(x)\}\leqslant H(x)\leqslant\max\{F(x),G(x)\}$, $\int_a^x G(y)dy\leqslant\int_a^x H(y)dy\leqslant\int_a^x F(y)dy$, and $\int_{x_1}^{x_2}\int_a^y[H(s)-G(s)]dsdy=\int_a^{x_1}\int_a^y[G(s)-H(s)]dsdy$; otherwise, $H(x)=F(x)$; (c) for $x\in[x_2,x_3]$, $H(x)=F(x)$; (d) if $\int_a^{x_3}\int_a^y[G(s)-H(s)]dsdy+\int_{x_3}^{x_4}\int_a^y[G(s)-F(s)]dsdy<0$, then for $x\in[x_3,x_4]$, let $H(x)$ be such that $\min\{F(x), G(x)\leqslant H(x)\leqslant\max\{F(x), G(x)\}$, $\int_a^x G(y)dy\leqslant\int_a^x H(y)dy\leqslant\int_a^x F(y)dy$, and $\int_{x_3}^{x_4}\int_a^y[H(s)-G(s)]dsdy=\int_a^{x_3}\int_a^y[G(s)-H(s)]dsdy$; otherwise, $H(x)=F(x)\dots$ for $x\in[x_{n-1},x_n]$, $H(x)=F(x)$; (such construction and $\int_x^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$ for all x imply that $\int_0^{x_n}\int_a^y[G(s)-H(s)]dsdy+\int_{x_n}^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$ since, by the construction, there exists a crossing $x_m\leqslant x_{n-1}$ of $\int_a^x G(y)dy$ and $\int_a^x F(y)dy$ such that $\int_a^{x_m}\int_a^y[G(s)-H(s)]dsdy=0$ and $H(x)=F(x)$ for $x\in[x_m,x_n]$, which, together with $\int_{x_m}^b\int_a^y[G(s)-F(s)]dsdy\leqslant 0$, gives $\int_a^{x_n}\int_a^y[G(s)-H(s)]dsdy+\int_{x_n}^b\int_0^y[G(s)-F(s)]dsdy=\int_a^{x_m}\int_a^y[G(s)-H(s)]dsdy+\int_{x_m}{}^b\int_0^y[G(s)-F(s)]dsdy\leqslant 0$) for $x\in[x_n,b]$, let $H(x)$ be such that $\min\{F(x), G(x)\}\leqslant H(x)\leqslant\max\{F(x), G(x)\}$, $\int_a^x G(y)dy\leqslant\int_a^x H(y)dy\leqslant\int_a^x F(y)dy$, and $\int_{x_n}^b\int_a^y[H(s)-G(s)]dsdy=\int_a^{xn}\int_a^y[G(s)-H(s)]dsdy$. We thus have $\int_a^x\int_a^y[G(s)-H(s)]dsdy\geqslant 0$ for all x, $\int_a^b\int_a^y[G(s)-H(s)]dsdy=0$, and $\int_a^b[G(y)-H(y)]dy=0$ (by our construction $\int_a^b H(y)dy=\int_a^b G(y)dy=\int_a^b F(y)dy$ given that $G(x)$ and $F(x)$ have the same mean), $\int_a^x[H(y)-F(y)]dy\leqslant 0$ for all x and $\int_a^b[H(y)-F(y)]dy=0$. That is, $G(x)$ is a downside risk increase of $H(x)$ and $H(x)$ is an MPC of $F(x)$.
The construction of $H(x)$ in the case where $F(x)$ is an uniform distribution and $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ three times first from above is illustrated graphically in Figure 1. $H(x)$ coincides with $F(x)$ for $x\in[a,x_1]$. Since $\int_a^{x_1}\int_a^y[G(s)-H(s)]dsdy+\int_{x_1}^{x_2}\int_a^y[G(s)-F(s)]dsdy<0$ as depicted (in the lower panel), $H(x)$ for $x\in[x_1,x_2]$ is constructed so that $\min\{F(x),G(x)\}\leqslant H(x)\leqslant\max\{F(x),G(x)\}$, $\int_a^x G(y)dy\leqslant\int_a^x H(y)dy\leqslant\int_a^x F(y)dy$, and $\int_{x_1}^{x_2}\int_a^y[H(s)-G(s)]dsdy=\int_a^{x_1}\int_a^y[G(s)-H(s)]dsdy$ and its graph is illustrated by the dotted curve (in the upper panel). $H(x)$ then coincides with $F(x)$ for $x\in[x_2,x_3]$ and for $x\in[x_3,b]$ is again represented by the dotted curve.

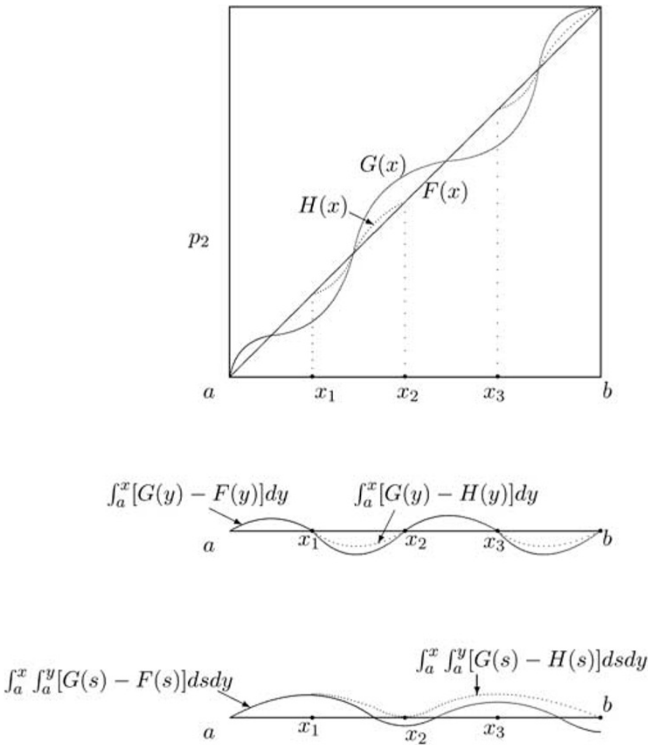

Figure 1. The construction of $H(x)$.

If $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ first from below, let $\hat{x}_1$ and $\hat{x}_2$ be two consecutive crossings of $G(x)$ and $F(x)$ such that $\hat{x}_1 < x_1 < \hat{x}_2$ and $G(x)-F(x) \geqslant 0$ for $x \in [\hat{x}_1, \hat{x}_2]$. $H(x)$ is constructed as follows: (a) for $x \leqslant \hat{x}_1$, $H(x) = G(x)$; (b) for $x \in [\hat{x}_1, \hat{x}_2]$, let $H(x)$ be such that $F(x) \leqslant H(x) \leqslant G(x)$, and $\int_{\hat{x}_1}^{\hat{x}_2}[H(y)-F(y)]dy = -\int_a^{\hat{x}_1}[G(y)-F(y)]dy$; (c) for $x \in [\hat{x}_1, \hat{x}_2]$, $H(x) = F(x)$; (d) if $\int_a^{x_2}\int_a^y[G(s)-H(s)]dsdy + \int_{x2}^{x3}\int_a^y[G(s)-F(s)]dsdy < 0$, then for $x \in [x_2, x_3]$, let $H(x)$ be such that $\min\{F(x), G(x)\} \leqslant H(x) \leqslant \max\{F(x), G(x)\}$, $\int_a^x G(y) dy \leqslant \int_a^x H(y)dy \leqslant \int_a^x F(y)dy$, and $\int_{x_2}^{x_3}\int_a^y[H(s)-G(s)]dsdy = \int_a^{x_2}\int_a^y[G(s)-H(s)] dsdy$, otherwise, $H(x) = F(x)$. The rest of the proof is the same as the case where $\int_a^x G(y)dy$ crosses $\int_a^x F(y)dy$ first from above.

(ii) The proof is completely analogous. $\square$

Proof of Proposition 2

(i) Let μ be the common mean of F and G. It can be shown by integration by parts that F and G have the same mean, if and only if $\int_a^b yd[G(y)-F(y)] = -\int_a^b[G(y)-F(y)]dy = 0$.

$$\sigma^2(G(x)) - \sigma^2(F(x)) = \int_a^b (y-\mu)^2 d[G(y) - F(y)]$$
$$= \int_a^b y^2 d[G(y)] - F(y) - \int_a^b 2y\mu d[G(y) - F(y)]$$
$$+ \int_a^b \mu^2 d[G(y) - F(y)]$$
$$= \int_a^b y^2 d[G(y) - F(y)]$$
$$= y^2[G(y)] - F(y)]|_a^b - \int_a^b 2y[G(y) - F(y)]dy$$
$$= -2\int_a^b yd\int_a^y [G(s) - F(s)]ds$$
$$= -2y\int_a^y [G(s) - F(s)]|_a^b + 2\int_a^b\int_a^y [G(s) - F(s)]$$
$$\times dsdy = 2\int_a^b\int_a^y [G(s) - F(s)]dsdy.$$

Thus, $\sigma^2(F(x))\sigma^2(G(x))$, if and only if $\int_a^b\int_a^y[G(s)-F(s)]dsdy \geqslant 0$. $\int_a^x G(y)dy$ crossing $\int_a^x F(y)dy$ once from above means there exist $\hat{x}$ such that $\int_a^x\int_a^y[G(s)-F(s)]dsdy$ is non-negative and non-decreasing for $x \in [a, \hat{x}]$ and is non-increasing for $x \in [\hat{x}, b]$. $\sigma^2(F(x)) \leqslant \sigma^2(G(x))$ therefore implies $\int_a^x\int_a^y[G(s)-F(s)]dsdy \geqslant 0$ for all $x \in [a, b]$.

(ii) The proof is completely analogous. □

Proof of Proposition 4

Note first that $\int_a^b u(y)dF(y) = \int_a^b u(y-\pi^u)dG(x)$ is equivalent to $\int_{u(a)}^{u(b)}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds = 0$. Second, if $\int_{u(a)}^{u}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds \leqslant 0$ for all $u \in [u(a), u(b)]$, then $G(x+\pi^u)$ clearly must cross $F(x)$ first from below. Now suppose $G(x+\pi^u)$ crosses $F(x)$ n times first from below and n is even. Then, letting x_n be the nth crossing, $G(x+\pi^u)-F(x)\leqslant 0$ for $x\in(x_n,b)$ with the inequality strict for some subinterval(s) of (x_n,b). Equivalently, $[G(u^{-1}(u)+\pi^u)-F(u^{-1}(u))]\leqslant 0$ for $u\in(u(x_n),u(b))$ with the inequality strict for some interval(s) of $(u(x_n),u(b))$, which, given $\int_{u(a)}^{u}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds\leqslant 0$ for all $u\in[u(a),u(b)]$ and hence $\int_{u(a)}^{u(x_n)}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds\leqslant 0$, implies $\int_{u(a)}^{u(b)}[G(u^{-1}(s)+\pi^u)-F(u^{-1}(s))]ds<0$. □

Proof of Lemma 1

Since the first-order condition can be written as

$$\frac{1}{\theta'(e_u)}=\frac{\int_a^b u(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b u'(x-e_u)dF(x,\theta(e_u))}$$

and given the second-order condition, $e_v\geqslant e_u$ if and only if

$$-\int_a^b v'(x-e_u)dF(x,\theta(e_u)+\theta'(e_u)\int_a^b v(x-e_u)dF_\theta(x,\theta(e_u))\geqslant 0,$$

we have $e_v\geqslant e_u$ if and only if

$$\frac{\int_a^b u(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b u'(x-e_u)dF(x,\theta(e_u))}\leqslant\frac{\int_a^b v(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b v'(x-e_u)dF(x,\theta(e_u))},$$

which we will show to be equivalent to the existence of δ with the prescribed properties. From the definition of Δe_u,

$$\begin{aligned}
0&=\int_a^b u(x-e_u-\Delta e_u)dF(x,\theta_u+\Delta\theta)-\int_a^b u(x-e_u)dF(x,\theta_u)\\
&=\int_a^b[u(x-e_u-\Delta e_u)-u(x-e)]dF(x,\theta_u+\Delta\theta)\\
&\quad+\int_a^b u(x-e_u)d[F(x,\theta_u+\Delta\theta)-F(x,\theta_u)]\\
&=\frac{\int_a^b[u(x-e_u-\Delta e_u)-u(x-e_u)]dF(x,\theta_u+\Delta\theta)}{\Delta e_u}\Delta e_u\\
&\quad+\frac{\int_a^b u(x-e_u)d[F(x,\theta_u+\Delta\theta)-F(x,\theta_u)]}{\Delta\theta}\Delta\theta
\end{aligned}$$

$$\frac{\Delta e_u}{\Delta\theta} = -\left[\frac{\int u(x-e_u)d[F(x,\theta_u+\Delta\theta)-F(x,\theta_u)]]/\Delta\theta}{\left[\int_a^b[u(x-e_u-\Delta e_u)-u(x-e_u)]dF(x,\theta_u+\Delta\theta)\right]/\Delta e_u}\right.$$

For $\Delta\theta\in(0,\delta)$, $|\Delta e_u(\Delta\theta)|\leqslant|\Delta e_v(\Delta\theta)|$ means

$$\int_a^b v(x-e_u-\Delta e_u)dF(x,\theta_u+\Delta\theta) - \int_a^b v(x-e_u)dF(x,\theta_u)\geqslant 0$$

$$\frac{\int_a^b[v(x-e_u-\Delta e_u)-v(x-e_u)]dF(x,\theta_u+\Delta\theta)}{\Delta e_u}\Delta e_u + \frac{\int_a^b v(x-e_u)d[F(x,\theta_u+\Delta\theta)-F(x,\theta_u)]}{\Delta\theta}\Delta\theta\geqslant 0$$

$$\frac{\Delta e_u}{\Delta\theta}\leqslant -\frac{\left[\int_a^b v(x-e_u)d[F(x,\theta_u+\Delta\theta)-F(x,\theta_u)]\right]/\Delta\theta}{\left[\int_a^b[v(x-e_u-\Delta e_u)-v(x-e_u)]dF(x,\theta_u+\Delta\theta)\right]/\Delta e_u}$$

For $\Delta\theta\in(-\delta,0)$, $|\Delta e_u(\Delta\theta)|\leqslant|\Delta e_v(\Delta\theta)|$ means $\int_a^b v(x-e_u-\Delta e_u)dF(x,\theta_u+\Delta\theta)\leqslant\int_a^b v(x-e_u)dF(x,\theta_u)$, which can be similarly shown to be equivalent to

$$\frac{\Delta e_u}{\Delta\theta}\leqslant -\frac{\left[\int_a^b v(x-e_u)d[F(x,\theta_u+\Delta\theta)-F(x,\theta_u)]\right]/\Delta\theta}{\left[\int_a^b[v(x-e_u-\Delta e_u)-v(x-e_u)]dF(x,\theta_u+\Delta\theta)\right]/\Delta e_u}$$

We thus have

$$-\frac{\left[\int_a^b u(x-e_u)d[F(x,\ \theta_u+\Delta\theta)-F(x,\ \theta_u)]\right]/\Delta\theta}{\int_a^b[u(x-e_u-\Delta e_u)-u(x-e_u)]dF(x,\ \theta_u+\Delta\theta)/\Delta e_u} = \frac{\Delta e_u}{\Delta\theta}\leqslant\frac{\left[\int_a^b v(x-e_u)d[F(x,\ \theta_u+\Delta\theta)-F(x,\ \theta_u)]/\Delta\theta\right]}{\int_a^b[v(x-e_u-\Delta e_u)-v(x-e_u)]dF(x,\ \theta_u+\Delta\theta)/\Delta e_u}$$

As $\Delta\theta \to 0$ and $\Delta e_u \to 0$, we have

$$\frac{\int_a^b u(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b u'(x-e_u)dF(x,\theta(e_u))} \leqslant \frac{\int_a^b v(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b v'(x-e_u)dF(x,\theta(e_u))}$$ [16]

Conversely, if there exists no δ with the prescribed properties, that is, for all $\delta > 0$, there exists $\Delta\theta \in (0,\delta)$ such that $\int_a^b v(x-e_u-\Delta e_u)dF(x,\theta_u+\Delta\theta) < \int_a^b v(x-e_u)dF(x,\theta_u)$ or there exists $\Delta\theta \in (-\delta, 0)$ such that $\int_a^b v(x-e_u-\Delta e_u)$ $dF(x,\theta_u+\Delta\theta) > \int_a^b v(x-e_u)dF(x,\theta_u)$, then clearly we cannot have

$$\frac{\int_a^b u(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b u'(x-e_u)dF(x,\theta(e_u))} \leqslant \frac{\int_a^b v(x-e_u)dF_\theta(x,\theta(e_u))}{\int_a^b v'(x-e_u)dF(x,\theta(e_u))} \quad \square$$

Proof of Lemma 2

It is sufficient to show that $\sigma^2_{\Delta\theta}(F(u^{-1}(u)+e_u+\Delta e_u(\Delta\theta),\theta_u+\Delta\theta))|_{\Delta\theta=0}$ is increasing in $p-\theta_u$ and is negative when $p-\theta_u=0$ and positive when $p-\theta_u=1$. For $\Delta\theta \in R$, Δe_u is defined by $(p-\theta_u-\Delta\theta)u(w-1-e_u-\Delta e_u)+(1-p+\theta_u+\Delta\theta)$ $u(w-e_u-\Delta e_u)=(p-\theta_u)u(w-l-e_u)+(1-p+\theta_u)u(w-e_u)\equiv\mu$ Defining $p_u=p-\theta_u$, $w_1=w-l-e_u$, and $w_2=w-e_u$, we have $(p_u-\Delta\theta)u(w_1-\Delta e_u)+(1-p_u+\Delta\theta)$ $u(w_2-\Delta e_u)=(p_u)u(w_1)+(1-p_u)u(w_2)\equiv\mu$ from which we obtain

$$\frac{d\Delta e_u}{d\Delta\theta} = \frac{u(w_2-\Delta e_u)-u(w_1-\Delta e_u)}{(p_u-\Delta\theta)u'(w_1-\Delta e_u)+(1-p_u+\Delta\theta)u'(w_2-\Delta e_u)}$$

$$\begin{aligned}&\sigma^2(F(u^{-1}(u)+e_u+\Delta e_u(\Delta\theta),\theta_u+\Delta\theta))\\&=(p_u-\Delta\theta)[u(w_1-\Delta e_u)-\mu]^2+(1-p_u+\Delta\theta)\\&\quad\times[u(w_2-\Delta e_u)-\mu]^2=(p_u-\Delta\theta)u(w_1-\Delta e_u)^2\\&\quad+(1-p_u+\Delta\theta)u(w_2-\Delta e_u)^2-\mu^2\end{aligned}$$

[16] Given the differentiability of

$$\int_a^b u(x(e)dF(x,\theta(e))$$

$\int_a^b u(x-e)dF(x,\theta(e))$with respect to e, as $\Delta\theta \to 0$, $\Delta e_v \to 0$, and $\Delta e_u \to 0$,

$$\frac{\int_a^b [v(x-e-\Delta e_u)v(x-e)]dF(x,\theta+\Delta\theta)}{\Delta e_u} = \frac{\int_a^b [v(xe-\Delta e_v)-v(x-e)]dF(x,\theta+\Delta\theta)}{\Delta e_v}$$

.

$$\frac{\partial\sigma^2}{\partial\Delta\theta}|_{\Delta\theta=0} = [u(w_2)^2 - u(w_1)^2] - 2[p_u u'(w_1)u(w_1) + (1-p_u)u'(w_2)u(w_2)]\frac{u(w_2)-u(w_1)}{p_u u'(w_1)+(1-p_u)u'(w_2)}$$
$$\times\frac{\partial}{\partial p_u}[\frac{\partial\sigma^2}{\partial\Delta\theta}|_{\Delta\theta=0}] = 2[u'(w_1)u(w_1) - u'(w_2)u(w_2)] \times\frac{u(w_2)-u(w_1)}{p_u u'(w_1)+(1-p_u)u'(w_2)} + 2[p_u u'(w_1)u(w_1) + (1-p_u)u'(w_2)u(w_2)]\frac{[u(w_2)-u(w_1)][u'(w_1)-u'(w_2)]}{[p_u u'(w_1)+(1-p_u)u'(w_2)]^2}$$

With $[p_u u'(w_1)+(1-p_u)u'(w_2)]^2$ being the denominator, the numerator of the expression is $-2[u'(w_1)u(w_1)-u'(w_2)u(w_2)][u(w_2)-u(w_1)][p_u u'(w_1)+(1-p_u)u'(w_2)]+2[p_u u'(w_1)u(w_1)+(1-p_u)u'(w_2)u(w_2)][u(w_2)-u(w_1)][u'(w_1)-u'(w_2)] = -2p_u[u'(w_1)]^2u(w_1)u(w_2)+2p_u[u'(w_1)]^2[u(w_1)]^2-2(1-p_u)u'(w_1)u(w_1)u(w_2)u'(w_2)+2(1-p_u)u'(w_1)[u(w_1)]^2u'(w_2)+2p_u u'(w_2)[u(w_2)]^2u'(w_1)-2p_u u'(w_2)u(w_2)u(w_1)u'(w_1)-2(1-p_u)[u'(w_2)]^2u(w_2)u(w_1)+2(1-p_u)[u'(w_2)]^2[u(w_2)]^2+2p_u[u'(w_1)]^2u(w_1)u(w_2)-2p_u u'(w_2)u(w_2)u(w_1)u'(w_1)+2u'p_u(w_1)[u(w_1)]^2u'(w_2)-2p_u[u'(w_1)]^2[u(w_1)]^2+2(1-p_u)u'(w_2)[u(w_2)]^2u'(w_1)-2(1-p_u)u'(w_1)u(w_1)u(w_2)u'(w_2)+2(1-p_u)[u'(w_2)]^2u(w_2)u(w_1)-2(1-p_u)[u'(w_2)]^2[u(w_2)]^2 = -4u'(w_1)u(w_1)u(w_2)u'(w_2)+2u'(w_2)[u(w_2)]^2u'(w_1)+2u'(w_1)[u(w_1)]^2u'(w_2) = 2u'(w_1)u'(w_2)[u(w_2)^2+u(w_1)^2-2u(w_1)u(w_2)] = 2u'(w_1)u'(w_2)[u(w_2)-u'(w_1)]^2>0$.

$$p_u = 0 \Rightarrow \frac{\partial\sigma^2}{\partial\Delta\theta}|_{\Delta\theta=0} = [u(w_2)^2 - u(w_1)^2] - 2u(w_2)^2 + 2u(w_2)u(w_1) = -[u(w_2)-u(w_1)]^2 < 0$$

$$p_u = 1 \Rightarrow \frac{\partial\sigma^2}{\partial\Delta\theta}|_{\Delta\theta=0} = [u(w_2)^2 - u(w_1)^2] - 2u(w_1)u(w_2) + 2u(w_1)^2 = [u(w_2)-u(w_1)]^2 > 0 \quad \square$$

A necessary and sufficient condition for a higher willingness to pay for a stochastic improvement

Proposition 8

(i) $\pi^u \leqslant (\geqslant) \pi^v$ for any $v(x) = T(u(x))$ such that $T'(\,) > 0$, $T''(\,) \leqslant 0$, and $T'''(\,) \leqslant 0$, if and only if $\int_u^{u(b)}\int_{u(a)}^t [G(u^{-1}(s)+\pi^u) - F(u^{-1}(s))]dsdt \leqslant (\geqslant) 0$ for all $u \in [u(a), u(b)]$;

(ii) $\pi^u \geqslant (\leqslant) \pi^v$ for any $v(x) = T(u(x))$ such that $T'(\ ) > 0$, $T''(\ ) \leqslant 0$, and $T'''(\ ) \geqslant 0$, if and only if $\int_{u(a)}^{u} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt \geqslant (\leqslant) 0$ for all $u \in [u(a), u(b)]$.

Proof

(i) Given $\int_a^b u(y - \pi^u) dG(y) = \int_a^b u(y) dG(y + \pi^u) = \int_a^b u(y) dF(y)$, which is equivalent to $\int_{u(a)}^{u(b)} u dG(u^{-1}(u) + \pi^u) = \int_{u(a)}^{u(b)} u dF(u^{-1}(u))$, and $\int_u^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt \leqslant 0$ for all $u \in [u(a), u(b)]$, there exists $\hat{H}(x)$ such that $G(u^{-1}(u) + \pi^u)$ is a downside risk increase of $\hat{H}(u)$ and $\hat{H}(u)$ is an MPC of $F(u^{-1}(u))$. Thus, given $T''(\ ) \leqslant 0$ and $T'''(\ ) \leqslant 0$, $\int_a^b v(y) d[G(y + \pi^u) - F(y)] = \int_a^b T(u(y)) d[G(y + \pi^u) - F(y)] = \int_{u(a)}^{u(b)} T(u) d[G(u^{-1}(u) + \pi^u) - F(u^{-1}(u))] = \int_{u(a)}^{u(b)} T(u) d[G(u^{-1}(u) + \pi^u) - \hat{H}(u)] + \int_{u(a)}^{u(b)} T(u) d[\hat{H}(u) - F(u^{-1}(u))] \geqslant 0$. To prove the converse, by repeated integration by parts and using $\int_{u(a)}^{u(b)} [G(u^{-1}(u) + \pi^u) - F(u^{-1}(u))] du = 0$, we have $\int_a^b v(y) d[G(y + \pi^u) - F(y)] = \int_{u(a)}^{u(b)} T(u) d[G(u^{-1}(u) + \pi^u) - F(u^{-1}(u))] T'' = (u(a)) \int_{u(a)}^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt + \int_{u(a)}^{u(b)} T'''(u) \int_u^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt$. If it is not true that $\int_u^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt \leqslant 0$ for all $u \in [u(a), u(b)]$, that is, there exists an interval $J \subset [u(a), u(b)]$ such that $\int_u^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt > 0$ for $u \in J$, then we can always choose a $T()$ such that $T''() \leqslant 0$ and $T'''() \leqslant 0$ but $|T'''(u)|$ is sufficiently large for $u \in J$ relative to $|T''(u(a))|$ and $|T'''(u)|$ elsewhere that $T''(u(a)) \int_{u(a)}^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt + \int_{u(a)}^{u(b)} (u) \int_u^{u(b)} \int_{u(a)}^{t} [G(u^{-1}(s) + \pi^u) - F(u^{-1}(s))] ds dt < 0$ and thus $\pi^u > \pi^v$.

(ii) The proof is completely analogous. □