LETTER TO THE EDITOR

General Non-Rotating Perfect-fluid Solution with an Abelian Spacelike C_3 Including Only One Isometry

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The general solution for non-rotating perfect-fluid spacetimes admitting one Killing vector and two conformal (non-isometric) Killing vectors spanning an abelian three-dimensional conformal algebra (C_3) acting on spacelike hypersurfaces is presented. It is of Petrov type D; some properties of the family such as matter contents are given. This family turns out to be an extension of a solution recently given in [9] using completely different methods. The family contains Friedman–Lemaître–Robertson–Walker particular cases and could be useful as a test for the different FLRW perturbation schemes. There are two very interesting limiting cases, one with a non-abelian G_2 and another with an abelian G_2 acting non-orthogonally transitively on spacelike surfaces and with the fluid velocity non-orthogonal to the group orbits. No examples are known to the authors in these classes.

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Very few exact perfect fluid solutions of Einstein field equations with a low degree of symmetry are known even though they may prove important for the study of inhomogeneities in the Universe or in parts of the Universe. The high complexity of the inhomogeneities of the real Universe makes the modeling of such structures using exact solutions intractable. Thus, perturbation schemes (see e.g. Refs. 1,2 and references therein) are usually used in order to understand the evolution of such inhomogeneities (in a Friedmann–Lemaître–Robertson–Walker background). However, these perturbation schemes smooth out any unexpected behaviours due to the high non-linearity of the theory of general relativity. These possible new behaviours can only be fully understood by finding and analyzing exact solutions of Einstein field equations. Moreover, they can be significant in testing the ranges in which the linear approximation of the perturbation theory is valid. In order to accomplish these objectives it is necessary and convenient to find an increasing number of exact solution with less possible symmetries. In the recent past, the use of conformal Killing vectors has proved useful in finding new families of exact solutions with a two-dimensional isometry group (the so-called G_2 cosmologies). For instance, the general perfect-fluid solution with a two-dimensional isometry group acting orthogonally transitively [3] on spacelike orbits and admitting one conformal Killing vector is known [4,5]. The next natural step is trying to determine the perfect-fluid solutions when the spacetime admits *one* isometry and two conformal Killing vectors (all of them being spacelike). The simplest case is when this three-dimensional conformal algebra is abelian. Since we are interested in cosmological models we will also impose that the fluid velocity is non-rotating (although rotating cosmological models are also of great interest, their study is significantly more difficult than the non-rotating one) and that $\rho + p$ is positive at least in an open region (ρ and p being the energy-density and pressure of the fluid respectively). In this letter we present the general solution of Einstein field equations under these assumptions.[5]

The spacetimes admitting an abelian C_3 algebra of conformal Killing vectors, with one Killing vector and two conformal Killing vectors (CKVs) acting on spacelike hypersurfaces, admit coordinates (using Defrise–Carter theorem, Ref. 8) in which the line-element reads

$$ds^2 = \Omega^2(y, z, t)\left[-\frac{dt^2}{M(t)} + M(t)dx^2 + P(t)dy^2 + S(t)dz^2\right], \qquad (1)$$

where M, P and S are arbitrary functions of t and the conformal factor Ω

[5] We exclude the conformally flat families since all of them are known [7].

is an arbitrary function of t, y and z. The metric (1) admits one Killing vector ∂_x and two CKVs ∂_y and ∂_z. The main result of this letter is the following.

The general non-rotating non-conformally flat perfect-fluid solution of Einstein's field equations (having $\rho + p > 0$ somewhere) with one Killing and two conformal Killing vectors spanning an abelian Lie algebra acting on spacelike hypersurfaces is

$$ds^2 = \frac{N^{(1-\alpha)/\alpha}(y)}{Q^{(1+\alpha)/\alpha}(z)}\left[-\frac{dt^2}{A(t)} + A(t)dx^2 + t^{1+\alpha}dy^2 + t^{1-\alpha}dz^2\right], \tag{2}$$

where α is a non-vanishing constant, the function $A(t)$ reads

$$A(t) = r_0 + r_1 t^{1-\alpha} + r_2 t^{1+\alpha} \tag{3}$$

(r_0, r_1 and r_2 are arbitrary constants), and the functions $N(y)$, $Q(z)$ satisfy the following trivial differential equations:

$$\left(\frac{dN}{dy}\right)^2 = \alpha^2(r_1 N^2 - v_1), \quad \left(\frac{dQ}{dz}\right)^2 = \alpha^2(r_2 Q^2 - v_2), \quad v_1, v_2 \text{ const.}$$

Thus, the metric is completely explicit and very simple in form. The velocity one-form of the fluid is

$$\mathbf{u} = -\frac{N^{(1-\alpha)/2\alpha}}{Q^{(1+\alpha)/2\alpha}}\frac{1}{\sqrt{R}}\left(\mathbf{dt} + \frac{tN_{,y}}{\alpha N}\mathbf{dy} - \frac{tQ_{,z}}{\alpha Q}\mathbf{dz}\right), \tag{4}$$

where R stands for

$$R \equiv r_0 + v_1\frac{t^{1-\alpha}}{N^2} + v_2\frac{t^{1+\alpha}}{Q^2}. \tag{5}$$

This expression must be strictly positive in order to have a perfect-fluid spacetime. When $R < 0$ the matter contents is a tachyon fluid and $R = 0$ represents a null fluid. For certain values of the parameters the spacetime has a region where the matter contents is a perfect fluid, there exists a transition hypersurface where the fluid becomes null (with pressure, in general) and there is also a non-empty open region where the perfect fluid is tachyonic. This kind of behaviour is very common when solving perfect-fluid Einstein field equations in non-comoving coordinates (see Ref. 5 for other explicit examples of this fact). From (4) we learn that the fluid is highly tilted with respect to the orbits of the conformal group. It is orthogonal to the Killing vector ∂_x but it is not orthogonal to either of the two conformal Killings (and consequently it is not orthogonal to the three-dimensional conformal orbits). It is convenient to define a new function τ by

$$\tau \equiv t \frac{N^{1/\alpha}}{Q^{1/\alpha}},$$

so that the fluid velocity one-form can be rewritten in a compact form as

$$\mathbf{u} = -\frac{\mathbf{d}\boldsymbol{\tau}}{\sqrt{r_0 N^{(1+\alpha)/\alpha} Q^{(\alpha-1)/\alpha} + v_1 \tau^{1-\alpha} + v_2 \tau^{1+\alpha}}}.$$

This expression shows that the fluid velocity is hypersurface orthogonal and therefore non-rotating, which is one of our main assumptions. The hypersurfaces orthogonal to the fluid are given by τ = const and therefore τ is a cosmic time for the spacetime. The energy density and pressure are (using this new time τ)

$$\begin{aligned} \rho = {} & \frac{3}{4}(1+\alpha)^2 \frac{v_2}{\tau^{1-\alpha}} + \frac{3}{4}(1-\alpha)^2 \frac{v_1}{\tau^{1+\alpha}} \\ & + \frac{r_0}{4}(1-\alpha^2)\frac{N^{(1+\alpha)/\alpha}}{\tau^2 Q^{(1-\alpha)/\alpha}}, \\ p = {} & -\frac{(1+\alpha)(1+5\alpha)v_2}{4\tau^{1-\alpha}} + \frac{(1-\alpha)(5\alpha-1)v_1}{4\tau^{1+\alpha}} \\ & + \frac{r_0}{4}(1-\alpha^2)\frac{N^{(1+\alpha)/\alpha}}{\tau^2 Q^{(1-\alpha)/\alpha}}, \end{aligned} \tag{6}$$

which imply

$$\rho + p = \frac{1}{2}(1-\alpha^2)\frac{RN^{(1+\alpha)/\alpha}}{\tau^2 Q^{(1-\alpha)/\alpha}},$$

and therefore the energy condition $\rho + p > 0$ is fulfilled everywhere provided the constant α is restricted to $\alpha^2 < 1$. The family of solutions is invariant under the simultaneous changes $\alpha \leftrightarrow -\alpha$, $y \leftrightarrow z$, $N \leftrightarrow Q$. Thus, we can assume without loss of generality that α is strictly positive and then the energy condition imposes

$$0 < \alpha < 1.$$

Expression (6) shows that the spacetime is singular at $\tau = 0$ where a big bang singularity occurs. Assuming $r_0 > 0$ there always exist a non-empty open region near the big bang singularity where both the density and pressure are positive. For v_1 and v_2 non-negative we have $R > 0$ everywhere (so that the matter contents is perfect fluid in the whole spacetime) and the energy-density is positive everywhere.

The Petrov type of the spacetime is D and in a null tetrad adapted to the two repeated null principal directions

$$\mathbf{l} = \frac{1}{\sqrt{2}} \frac{N^{(1-\alpha)/2\alpha}}{Q^{(1+\alpha)/2\alpha}} \left(\frac{\mathbf{d}t}{\sqrt{A}} + \sqrt{A}\, \mathbf{d}x \right),$$

$$\mathbf{k} = \frac{1}{\sqrt{2}} \frac{N^{(1-\alpha)/2\alpha}}{Q^{(1+\alpha)/2\alpha}} \left(\frac{\mathbf{d}t}{\sqrt{A}} - \sqrt{A}\, \mathbf{d}x \right),$$

the only non-vanishing Weyl spinor component reads

$$\Psi_2 = \frac{r_0(\alpha^2 - 1) Q^{(1+\alpha)/\alpha}}{12 t^2 N^{(1-\alpha)/\alpha}}. \tag{7}$$

These expressions show that the fluid velocity does not lie in the two-plane generated at each point by the repeated null directions. Both null directions are geodesic and non-rotating and they are shearing and expanding. Furthermore, the acceleration of the fluid $\dot{\mathbf{u}}$ does not lie in the plane generated by $\mathbf{l}$ and $\mathbf{k}$ (see Ref. 7).

From (7) we have that the conformally flat subcases of the solution are obtained when either $\alpha = 1$ or $r_0 = 0$. The metric with $\alpha = 1$ is de Sitter ($v_2 > 0$), anti-de Sitter ($v_2 < 0$) or Minkowski ($v_2 = 0$). When $r_0 = 0$ (arbitrary α) the condition $R > 0$ (5) implies that at least one of the v_1, v_2 must be positive. The fluid satisfies a barotropic equation of state (6) and therefore the spacetime must be a Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology (see e.g. Ref. 6).

It is remarkable that the general family we present in this letter is an extension of a solution recently found in [9] using completely different methods. The two families coincide when either v_1 or v_2 are positive and, therefore, the solutions with $v_1 \leq 0$ and $v_2 \leq 0$ presented here are new. In [9], the authors use the Kerr–Schild transformation to find perfect-fluid solutions starting from a FLRW seed metric. This seed metric is exactly the subcase $r_0 = 0$, $v_1 > 0$ in (2) (written in different coordinates). After performing the Kerr–Schild transformation, they find a family of solutions which is equivalent[6] to the subfamily $v_1 > 0$ (or $v_2 > 0$) in (2). The solutions with $v_1 \leq 0$, $v_2 \leq 0$ were not found using the Kerr–Schild method in [9] because their conformally flat limit is not FLRW. Instead, one gets

[6] The two coordinate systems are different and therefore it is not obvious that the two families coincide.

a tachyon fluid (admitting a G_6) when $v_1 < 0$ and $v_2 \leq 0$ (or equivalently $v_1 \leq 0$ and $v_2 < 0$) and a radiation solution admitting G_7 when $v_1 = v_2 = 0$. It can be seen, however, that the whole family presented here can be generated using the Kerr–Schild ansatz starting from the seed metric (2) with $r_0 = 0$. This fact is most remarkable since the Kerr–Schild transformation and the existence of conformal Killing vectors have no apparent relationship. Thus, the family of perfect-fluid solutions (2) has two completely different and apparently unrelated characterizations as the most general solution with an abelian spacelike C_3 including one isometry and the most general solution which can be found from the metric with $r_0 = 0$ using the Kerr–Schild transformation. The possible reason for such an unexpected connection should be a matter for further investigation.

The analysis of the Killing equations for the metric (2) shows that there is only one Killing vector $\vec{k}_1 = \partial_x$ except for the three following subcases (apart from $\alpha^2 = 1$ and $r_0 = 0$ which have been discussed above).

A) When $r_1 = r_2 = 0$, $v_1 < 0$ and $v_2 < 0$ (therefore, this is a new solution not included in Ref. 9) the metric admits a non-abelian G_2 and the line-element can be written in the form

$$ds^2 = D^2 \frac{y^{(1-\alpha)/\alpha}}{z^{(1+\alpha)/\alpha}} [-dt^2 + dx^2 + t^{1+\alpha}dy^2 + t^{1-\alpha}dz^2]$$

(where D cannot be reabsorbed into the coordinates). This metric has only two Killing vectors,

$$\vec{k}_1 = \frac{\partial}{\partial x}, \qquad \vec{k}_2 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{1-\alpha}{2} y\frac{\partial}{\partial y} + \frac{1+\alpha}{2} z\frac{\partial}{\partial z},$$

with commutator $[\vec{k}_1, \vec{k}_2] = \vec{k}_1$. $\vec{k}_1$ is spacelike everywhere, but $\vec{k}_2$ changes its spacelike, null and timelike character through the spacetime. The fluid velocity also changes its character. It is timelike (and thus a perfect fluid) in a region near the big-bang and it becomes spacelike for big enough values of τ. It can be proved that in the region where the matter contents is perfect fluid (the physical region) the Killing vector $\vec{k}_2$ is spacelike everywhere. After the fluid has become tachyonic, $\vec{k}_2$ becomes timelike and thus the spacetime is stationary. As far as we know, this solution is the first explicit exact solution for a perfect fluid with a non-abelian two-dimensional maximal isometry group (see Ref. 10 for a discussion of some properties of non-abelian G_2 perfect-fluid spacetimes).

B) When $r_1 = v_1 = 0$ (i.e $N_{,y} = 0$) and $Q_{,z} \neq 0$ the metric has an abelian G_2 acting on spacelike orbits. The two Killings are ∂_x and ∂_y

(there is a similar case when $Q_{,z} = 0$ and $N_{,y} \neq 0$, then the Killings are obviously ∂_x and ∂_z).

C) When $v_2 = v_1 = 0$ (then necessarily r_1 and r_2 must be non-negative and we can write them as $r_1 = c_1^2$, $r_2 = c_2^2$). The metric takes the form

$$\mathrm{d}s^2 = e^{(1-\alpha)c_1 y - (1+\alpha)c_2 z}\Bigg[- \frac{\mathrm{d}t^2}{r_0 + c_2^2 t^{1+\alpha} + c_1^2 t^{1-\alpha}} + (r_0 + c_2^2 t^{1+\alpha} + c_1^2 t^{1-\alpha})\mathrm{d}x^2 + t^{1+\alpha}\mathrm{d}y^2 + t^{1-\alpha}\mathrm{d}z^2\Bigg], \tag{8}$$

and the perfect-fluid satisfies a barotropic equation of state for a stiff fluid,

$$p = \rho = \frac{r_0(1-\alpha^2)}{4t^2}\, e^{(\alpha-1)c_1 y + (1+\alpha)c_2 z} > 0\,.$$

Since this metric has $v_1 = v_2 = 0$ it is not contained in the family previously found in [9]. This metric has two Killing vectors, $\vec{k}_1$ and $\vec{k}_2$, and one homothety $\vec{h}$ (assuming c_1 or c_2 non-zero, otherwise the metric is a Bianchi I cosmology) given by

$$\vec{k}_1 = \frac{\partial}{\partial x},$$
$$\vec{k}_2 = (1+\alpha)c_2\frac{\partial}{\partial y} + (1-\alpha)c_1\frac{\partial}{\partial z},$$
$$\vec{h} = (1+\alpha)c_1\frac{\partial}{\partial y} - (1-\alpha)c_2\frac{\partial}{\partial z},$$

which are commuting and spacelike everywhere. The one-form associated with $\vec{k}_1$ is clearly integrable while the one-form associated with $\vec{k}_2$ is not only non-integrable (assuming c_1 and c_2 non-zero, otherwise the metric is a so-called diagonal cosmology, Ref. 3) but also satisfies

$$\mathbf{k_1} \wedge \mathbf{k_2} \wedge \mathbf{d}\,\mathbf{k_2} \neq 0,$$

which means that the orbits of the two-dimensional group are not surface orthogonal and thus the isometry group is not orthogonally transitive. In addition the fluid velocity is not orthogonal to the isometry group orbits. This solution belongs to A(ii) in Wainwright's classification of G_2 cosmologies [3]. Very few exact solutions in this class are known and to the best of our knowledge all of them have the fluid velocity orthogonal to the isometry group orbits [11,12]. This is a general property for B(i) and B(ii) classes, but it is an added assumption for A(i) and A(ii) classes. In our case the fluid velocity is *not* orthogonal to the isometry group orbits and as far as we know this is the first example with this property.

As a final comment, let us emphasize that the family of solutions (2) is given explicitly in terms of elementary functions and is of very simple form, despite the low isometry group. Thus, it may be suitable for testing the range of validity of the different FLRW perturbation schemes. In addition, the r_0 parameter controls the deviation from FLRW in a very neat way and hence any possible disagreement between the exact model and the predictions of the perturbation schemes can easily be detected and interpreted.

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