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Shifting Operations and Graded Betti Numbers

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Abstract. The behaviour of graded Betti numbers under exterior and symmetric algebraic shifting is studied. It is shown that the extremal Betti numbers are stable under these operations. Moreover, the possible sequences of super extremal Betti numbers for a graded ideal with given Hilbert function are characterized. Finally it is shown that over a field of characteristic 0, the graded Betti numbers of a squarefree monomial ideal are bounded by those of the corresponding squarefree lexsegment ideal.

Keywords: algebraic shifting, shifted complexes, generic initial ideals, extremal Betti numbers

Introduction

The purpose of this paper is to discuss the relationship between symmetric algebraic shifting and exterior algebraic shifting. Both concepts are introduced by Kalai (see [15] and [16]). We also will consider the so-called combinatorial shifting.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring. An ideal in S is called a squarefree monomial ideal if it is generated by squarefree monomials. A map which assigns to each squarefree monomial ideal I in S a squarefree monomial ideal $\mathrm{Shift}(I)$ in S is called a shifting operation, if it satisfies the following conditions: (S_1) the ideal $\mathrm{Shift}(I)$ is squarefree strongly stable, (S_2) $\mathrm{Shift}(I) = I$ if I is squarefree strongly stable, (S_3) the rings S/I and $S/\mathrm{Shift}(I)$ have the same Hilbert function, (S_4) if $J \subset I$, then $\mathrm{Shift}(J) \subset \mathrm{Shift}(I)$. The exterior (resp. symmetric) algebraic shift of an ideal I will be denoted by I^e (resp. I^s), while a combinatorial shift of I will be denoted by I^c. A precise definition of these shifting operations will be given in Section 1.

It is clear that shifting operations may as well be defined in terms of simplicial complexes. In this paper however we prefer the algebraic interpretation of shifting operators since we want to relate them to generic initial ideals and want to study the graded Betti numbers of the free resolutions of the shifted ideals.

In combinatorial contexts shifting operations were first introduced by Erdös, Ko, and Rado (see [1]). Combinatorial shifting only depends on the simplicial complex associated with the ideal I, but not on the field K. On the other hand, symmetric algebraic shifting is only defined in characteric 0, while exterior algebraic shifting is defined for any base field, but may depend on its characteristic.

In Section 1 we recall the definitions of the various shifting operations and their basic properties. The properties (S_1)–(S_4) have been shown for the algebraic shiftings by Kalai in [15] and [16], and are easy to prove for combinatorial shifting. Since the proof of property (S_2) for the symmetric algebraic shifting is not explicitly given in [16], and since the proof is not obvious we include it in Section 1. Condition (S_2) is indeed equivalent to the fact that $\text{Gin}^S(I^s) = \text{Gin}^S(I)$ for any squarefee monomial ideal in I in S. Here $\text{Gin}^S(I)$ denotes the generic initial ideal of I with respect to the reverse lexicographical order induced by $x_1 > x_2 > \cdots > x_n$. As a consequence we obtain that $I^e = I^s$ if and only if $\text{Gin}^S(I^e) = \text{Gin}^S(I^s)$. A more combinatorial condition for the equality of exterior and algebraic shifting would be preferable.

One of the main results of Section 2 is the inequality $\beta_{ij}(I) \leq \beta_{ij}(I^s)$ which is valid for all i and j. We do not know whether a similar inequality holds for the exterior shifting, but we conjecture that $\beta_{ij}(I^s) \leq \beta_{ij}(I^e) \leq \beta_{ij}(I^c)$ for all i and j. However we show that the extremal Betti numbers (as defined in [6]) of I, I^s, and I^e coincide.

With techniques developed in Section 2 we prove a theorem on super extremal Betti numbers. This theorem can be derived from the Björner Kalai theorem [8] which extends the classical Euler Poincaré theorem. Let $I \subset S$ be a graded ideal and m the maximal shift in the minimal graded free resolution of S/I. We call the Betti numbers $\beta_{im}(S/I)$ super extremal. Note that the non-zero super extremal Betti numbers are extremal in the sense of [6]. In Theorem 2.8 we characterize all possible sequences of numbers which are the sequence of super extremal Betti numbers of a homogeneous K-algebra with a given Hilbert function.

Finally, as a consequence of the inequality $\beta_{ij}(I) \leq \beta_{ij}(I^s)$, we are able to show that if K is a field of characteristic 0, Δ is a simplicial complex and Δ^{lex} is the unique lexsegment simplicial complex with the same f-vector as Δ, then $\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^{lex}})$ for all i and j.

1. Shifting operations

Fix a field K, and let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over K with each $\deg x_i = 1$. The support of a monomial u of S is $\text{supp}(u) = \{i : x_i \text{ divides } u\}$. Let $m(u)$ denote the maximal integer belonging to $\text{supp}(u)$. If I is a monomial ideal of S, we write $G(I)$ for the (unique) minimal system of monomial generators of I, and $G(I)_j$ for the set of monomials of degree j belonging to $G(I)$.

Recall that a monomial ideal I of S is strongly stable if, for all $u \in G(I)$, one has $(x_j u)/x_i \in I$ for all $i \in \text{supp}(u)$ and all $j < i$. Similarly, a squarefree monomial ideal I of S is called squarefree strongly stable [3] if, for all $u \in G(I)$, one has $(x_j u)/x_i \in I$ for all $i \in \text{supp}(u)$ and all $j < i$ with $j \notin \text{supp}(u)$. Note that if I is (squarefree) strongly stable, then this exchange property holds for all (squarefree) monomials u of I.

Let E be the exterior algebra of the K-vector space V with basis $e_1, \ldots, e_n$. The canonical basis elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $i_1 < \cdots < i_k$, of E are called monomials. A monomial ideal in E is an ideal generated by monomials.

In order to explain exterior algebraic shifting of a monomial ideal $I \subset S$, we consider the corresponding monomial ideal $J \subset E$, and let $<$ be the reverse lexicographical order on the monomials induced by $e_1 > e_2 > \cdots > e_n$. In other words, if $u = e_{i_1} \wedge \cdots \wedge e_{i_k}$

and $v = e_{j_1} \wedge \cdots \wedge e_{j_l}$, then $u > v$ if either $k > l$, or else $k = l$ and there exists an r such that $i_s = j_s$ for $s > r$, and $i_r < j_r$. There exists a Zariski open set U of linear automorphisms $\varphi : E \to E$ such that the initial ideal $\mathrm{in}_<(\varphi(J))$ does not depend on the specific choice of $\varphi \in U$. This initial ideal, denoted $\mathrm{Gin}^E(J)$, is called the generic initial ideal of J: cf. [5]. Now the *exterior algebraic shifting* I^e *of* I is the squarefree monomial ideal in S corresponding to $\mathrm{Gin}^E(J)$. It is well known that I^e is squarefree strongly stable.

The generic ideal $\mathrm{Gin}^S(I)$ of a graded ideal $I \subset S$, is defined similarly as the generic initial ideal in the exterior algebra. For more detailed information we refer to [9] and [11]. Symmetric algebraic shifting is defined via $\mathrm{Gin}^S(I)$. Here we assume that K is a field of characteristic 0. It is known that in this case $\mathrm{Gin}^S(I)$ is a strongly stable ideal.

We will transform $\mathrm{Gin}^S(I)$ into a squarefree monomial ideal by applying a certain operator: for a monomial $u \in S$, $u = x_{i_1}x_{i_2}\cdots x_{i_j}\cdots x_{i_d}$ with $i_1 \le i_2 \le \cdots \le i_j \le \cdots \le i_d$, we set

$$u^\sigma = x_{i_1}x_{i_2+1}\cdots x_{i_j+(j-1)}\cdots x_{i_d+(d-1)}.$$

It then follows immediately

$$m(u^\sigma) - \deg u^\sigma = m(u) - 1. \tag{1}$$

If L is a monomial ideal with $G(L) = \{u_1, \ldots, u_s\}$, then we write L^σ for the squarefree monomial ideal generated by $u_1^\sigma, \ldots, u_s^\sigma$ in $K[x_1, \ldots, x_m]$, where $m = \max\{m(u) + \deg u - 1 : u \in G(L)\}$.

The *symmetric algebraic shifting of* I is defined to be the squarefree monomial ideal $I^s = (\mathrm{Gin}^S(I))^\sigma$. The definition of symmetric algebraic shifting presented here is formally different from that of Kalai [16]. However it is an easy exercise to see that both notions coincide.

A priori it is not clear from the definition of symmetric algebraic shifting that for a squarefree monomial ideal $I \subset S$, we also have $I^s \subset S$. The next lemma shows that this indeed is the case.

Lemma 1.1 *If I is a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$, then $m(u) + \deg u \le n + 1$ for all $u \in G(\mathrm{Gin}^S(I))$.*

Proof: The graded Betti numbers of a strongly stable ideal I are given by Eliahou-Kervaire [10]:

$$\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{m(u)-1}{i} \tag{2}$$

for all i and j.

Since $\mathrm{Gin}^S(I)$ is strongly stable, formula (2) implies that $\max\{m(u) + \deg u - 1 : u \in G(\mathrm{Gin}^S(I))\}$ is the highest shift in the resolution of $\mathrm{Gin}^S(I)$. The monomial ideal I being squarefree, Hochster's formula, e.g. [7, Theorem 5.5.1], guarantees that the highest shift in the resolution of I is less than or equal to n. Since the highest shift in the resolution of I and that of $\mathrm{Gin}^S(I)$ coincide [13], we have the desired inequalities. □

Note that condition (S_1) is satisfied for I^s since we have

Lemma 1.2 *Let I be a strongly stable ideal with $G(I) = \{u_1, \ldots, u_s\}$. Then the squarefree monomial ideal I^σ is squarefree strongly stable with $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.*

Proof: Suppose that, for some $u \in G(I)$, we have $u^\sigma \notin G(I^\sigma)$. Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$. Then, for some proper subset N of $\{1, 2, \ldots, d\}$ and for some $1 \leq q \leq s$, we have $u_q^\sigma = \prod_{j \in N} x_{i_j + (j-1)}$. Hence $u_q = \prod_{j \in N} x_{i_j + h_j}$, where h_j is the number of integers $1 \leq k < j$ with $k \notin N$. Since I is strongly stable, $\prod_{j \in N} x_{i_j}$ must belong to I. This contradicts $u \in G(I)$. Thus we have $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.

Next, to see why I^σ is squarefree strongly stable, let $u = x_{i_1} \cdots x_{i_d} \in G(I)$ and consider the monomial $(x_b u^\sigma)/x_{i_a + (a-1)}$ with $b \notin \mathrm{supp}(u^\sigma)$ and $b < i_a + (a-1)$. Let $i_p + (p-1) < b < i_{p+1} + p$ for some $p < a$ and set

$$v = \left(\prod_{j=1}^{p} x_{i_j} \right) x_{b-p} \left(\prod_{j=p+1}^{a-1} x_{i_j - 1} \right) \left(\prod_{j=a+1}^{d} x_{i_j} \right).$$

Then, since $b - p < i_{p+1} \leq i_a$ and since I is strongly stable, the monomial v belongs to I. Note that $v^\sigma = (x_b u^\sigma)/x_{i_a + (a-1)}$. Say, $v = x_{\ell_1} \cdots x_{\ell_d}$ with $\ell_1 \leq \cdots \leq \ell_d$. Again, since I is strongly stable, it follows that $w = x_{\ell_1} \cdots x_{\ell_c} \in G(I)$ for some $c \leq d$. Since w^σ divides v^σ, we have $(x_b u^\sigma)/x_{i_a + (a-1)} \in I^\sigma$, as desired. □

Next we give the proof of condition (S_2) for symmetric algebraic shifting.

Theorem 1.3 *Let $I \subset S$ be a squarefree strongly stable ideal of S. Then $I^s = I$.*

For the proof we introduce the operation τ which is inverse to σ: For a squarefree monomial $u = x_{i_1} x_{i_2} \cdots x_{i_j} \cdots x_{i_d}$ with $i_1 < i_2 < \cdots < i_j < \cdots < i_d$, we set

$$u^\tau = x_{i_1} x_{i_2 - 1} \cdots x_{i_j - (j-1)} \cdots x_{i_d - (d-1)}.$$

If $I \subset S$ is a squarefree monomial ideal with $G(I) = \{u_1, \ldots, u_s\}$, then we write I^τ for the monomial ideal generated by $u_1^\tau, \ldots, u_s^\tau$ in S.

Similarly to Lemma 1.2, we show:

Lemma 1.4 *Let I be a squarefree strongly stable ideal with $G(I) = \{u_1, \ldots, u_s\}$. Then the ideal I^τ is strongly stable with $G(I^\tau) = \{u_1^\tau, \ldots, u_s^\tau\}$.*

Proof: Assume that for some $u \in G(I)$, we have $u^\tau \notin G(I^\tau)$. Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 < \cdots < i_d$. Then for some proper subset $\{j_1, \ldots, j_t\}$ of $\{1, 2, \ldots, d\}$, where $j_1 < \cdots < j_t$, and for some $1 \leq q \leq s$, we have $u_q^\tau = \prod_{k=1}^{t} x_{i_{j_k} - (j_k - 1)}$. Hence $u_q = \prod_{k=1}^{t} x_{i_{j_k} - (j_k - k)}$.

Since $i_k \leq i_{j_k} - (j_k - k)$ for $1 \leq k \leq t$ and I is squarefree strongly stable, we get $x_{i_1} \cdots x_{i_t} \in I$ which contradicts $u \in G(I)$.

Now, we show that I^τ is strongly stable. Let $u = x_{i_1} \cdots x_{i_d} \in G(I)$ with $i_1 < \cdots < i_d$, and consider the monomial $v = (x_b u^\tau)/x_{i_k-(k-1)}$ with $b < i_k - (k-1)$. Let $i_p - (p-1) \leq b < i_{p+1} - p$ for some $p < k$. Then

$$v^\sigma = \left(\prod_{j=1}^{p} x_{i_j}\right) x_{b+p} \left(\prod_{j=p+1}^{k-1} x_{i_j+1}\right)\left(\prod_{j=k+1}^{d} x_{i_j}\right).$$

Since $b + p < i_{p+1}$ and $i_j + 1 \leq i_{j+1}$ for $p + 1 \leq j \leq k - 1$, and since I is squarefree strongly stable, we obtain that $v^\sigma \in I$. Say, $v^\sigma = x_{\ell_1} \cdots x_{\ell_d}$ with $\ell_1 < \cdots < \ell_d$. Again, since I is squarefree strongly stable, it follows that $w = x_{\ell_1} \cdots x_{\ell_c} \in G(I)$ for some $c \leq d$. Since w^τ divides $(v^\sigma)^\tau = v$, we have $v \in I^\tau$. □

If u is a monomial, denote by $B(u)$ the smallest strongly stable ideal in S containing u, and call it Borel principal. Similarly, for a squarefree monomial u, denote by $SqB(u)$ the smallest squarefree strongly stable ideal in S containing u, and call it squarefree Borel principal.

Lemma 1.5 *Let $I \subset S$ be a squarefree strongly stable ideal generated in degree d. Let $G(I) = \{u_1, \ldots, u_s\}$ where $u_1 > u_2 > \cdots > u_s$. Let $g = (a_{ij})_{1 \leq i,j \leq n}$ be a generic upper triangular matrix acting on S by $g(x_i) = \sum_{j=1}^{i} a_{ji} x_j$ for $1 \leq i \leq n$. Let c_{kj} denote the coefficient of u_j^τ in the polynomial $g(u_k)$ for $1 \leq k, j \leq s$. Then the determinant of the matrix $(c_{kj})_{1 \leq k,j \leq s}$ is different from zero.*

Proof: We may consider the generic coefficients a_{ij}, $1 \leq i \leq j \leq n$, as indeterminates over K. Let $\succ$ denote the degree lexicographic order on $K[a_{ij}, 1 \leq i \leq j \leq n]$ induced by:

$$a_{ij} \succ a_{kl} \quad \text{if } j > l \quad \text{or} \quad j = l \quad \text{and} \quad i > k.$$

Set $b_{kj} = \text{in}(c_{kj})$ for $1 \leq k, j \leq s$ where in (c) denotes the initial term of $c \in K[a_{ij}, 1 \leq i \leq j \leq n]$ with respect to $\succ$. We will show that $\Delta = \det(b_{kj}) \neq 0$ which will imply the claim of the lemma.

First note that from $a_{ij} = 0$ for $i > j$, it follows for $1 \leq k \leq s$:

$$g(u_k) = \sum_{w \in B(u_k)} c_k^w w \quad \text{where } c_k^w \in K[a_{ij}, 1 \leq i \leq j \leq n].$$

Since $u_k^\tau \in B(u_k)$, one has $b_{kk} \neq 0$ for $1 \leq k \leq s$. We will prove that in $(\Delta) = b_{11} b_{22} \cdots b_{ss}$.

We fix the following notation for the generators of I:

$$u_k = x_{k_1} \cdots x_{k_d} \quad \text{with} \quad k_1 < k_2 < \cdots < k_d \quad \text{for } 1 \leq k \leq s,$$

and we set $D = b_{11} b_{22} \cdots b_{ss}$.

Now, we will compute the initial terms b_{kj}, $1 \le k, j \le s$. Let $w = x_{j_1} \cdots x_{j_d} \in B(u_k)$ where $j_1 \le \cdots \le j_d$. Then $c_k^w = \sum a_{q_1,k_1} \cdots a_{q_d,k_d}$ where the summation is over all $(q_1, \ldots, q_d)$ such that $x_{q_1} \cdots x_{q_d} = x_{j_1} \cdots x_{j_d}$. Therefore, we obtain that in $(c_k^w) = a_{j_1,k_1} \cdots a_{j_d,k_d}$. In particular, for $w = u_j^\tau$ one has

$$b_{kj} = a_{j_1,k_1} a_{j_2-1,k_2} \cdots a_{j_d-(d-1),k_d}.$$

Let $P = b_{1\rho(1)} b_{2\rho(2)} \cdots b_{s\rho(s)}$ where $\rho \neq id$ is a permutation in the symmetric group S_s. We will show that $P \prec D$. We may assume that the claim is true for squarefree strongly stable ideals in S with number of generators less than s.

If $\rho(s) = s$, then the ideal $(u_1, \ldots, u_{s-1})$ being squarefree strongly stable, by induction hypothesis, one has $b_{1\rho(1)} b_{2\rho(2)} \cdots b_{s-1\rho(s-1)} \prec b_{11} b_{22} \cdots b_{s-1s-1}$, thus $P \prec D$.

So, we may suppose that $\rho(s) = t < s$. Let $\rho(\ell) = s$. Then $\ell < s$.

First, consider the case $\ell = t$. Then $P = \prod_{k \neq t,s} b_{k\rho(k)} b_{ts} b_{st}$. We will show that $b_{ts} b_{st} \prec b_{tt} b_{ss}$. Since $u_t > u_s$, there exists a p such that $t_p < s_p$ and $t_j = s_j$ for $p+1 \le j \le d$. We have:

$$b_{ts} = \prod_{j=1}^{p-1} a_{s_j-(j-1),t_j} \cdot a_{s_p-(p-1),t_p} \cdot \prod_{j=p+1}^{d} a_{s_j-(j-1),s_j};$$

$$b_{st} = \prod_{j=1}^{p-1} a_{t_j-(j-1),s_j} \cdot a_{t_p-(p-1),s_p} \cdot \prod_{j=p+1}^{d} a_{s_j-(j-1),s_j};$$

$$b_{ss} = \prod_{j=1}^{p-1} a_{s_j-(j-1),s_j} \cdot a_{s_p-(p-1),s_p} \cdot \prod_{j=p+1}^{d} a_{s_j-(j-1),s_j};$$

$$b_{tt} = \prod_{j=1}^{p-1} a_{t_j-(j-1),t_j} \cdot a_{t_p-(p-1),t_p} \cdot \prod_{j=p+1}^{d} a_{s_j-(j-1),s_j}.$$

Therefore $b = \prod_{j=p+1}^{d} a^2_{s_j-(j-1),s_j}$ divides both $b_{ts} b_{st}$ and $b_{tt} b_{ss}$. Then $a_{s_p-(p-1),s_p}$ is the biggest generic coefficient dividing $(b_{tt} b_{ss})/b$, and $a_{t_p-(p-1),s_p}$ is the biggest one dividing $(b_{ts} b_{st})/b$. Since $a_{s_p-(p-1),s_p} \succ a_{t_p-(p-1),s_p}$, we obtain $b_{ts} b_{st} \prec b_{tt} b_{ss}$. Thus $P \prec (\prod_{k \neq t,s} b_{k\rho(k)} b_{tt}) b_{ss}$. Again by induction hypothesis, we have $\prod_{k \neq t,s} b_{k\rho(k)} b_{tt} \preceq b_{11} \cdots b_{s-1s-1}$, and this completes the proof in this case.

Let now $\ell \neq t$. Set $m = m(u_s)$. Then there exists a q such that $m(u_k) < m$ for $1 \le k \le q$ and $m(u_k) = m$ for $q+1 \le k \le s$. First note that $a_{m-(d-1),m}$ is the biggest a_{ij}, $1 \le i \le j \le n$, which appears in Δ. We have:

$$D = \prod_{k=1}^{q} b_{kk} \prod_{k=q+1}^{s} a_{k_1,k_1} a_{k_2-1,k_2} \cdots a_{m-(d-1),m};$$

$$P = \prod_{k \le q, k \neq \ell} b_{k\rho(k)} \prod_{q+1 \le k \le s-1, k \neq \ell} \left(\prod_{j=1}^{d} a_{\rho(k)_j-(j-1),k_j} \right) b_{\ell s} b_{st}.$$

We see that $a^{s-q}_{m-(d-1),m}$ divides D and denoting by r the maximal power of $a_{m-(d-1),m}$ dividing P, one has $r \le s-q$. If $r < s-q$, we are done. Suppose $r = s-q$. Then

$\ell_d = t_d = m$, so that $q+1 \le t, \ell < s$. Moreover, one obtains that for all $q+1 \le k \le s$, $\rho(k)_d = m$ which implies that $q+1 \le \rho(k) \le s$ for $q+1 \le k \le s$. Therefore $1 \le \rho(k) \le q$ for $1 \le k \le q$, so that $\rho = \rho_1\rho_2$ where $\rho_1 \in S_q$ and $\rho_2 \in S_{s-q}$. Then

$$P = \prod_{k=1}^{q} b_{k\rho_1(k)} \prod_{k=q+1}^{s} b_{k\rho_2(k)}.$$

From our induction hypothesis it follows $P \preceq \prod_{k=1}^{q} b_{kk} \prod_{k=q+1}^{s} b_{k\rho_2(k)}$.

Now, the ideal $J = (u'_{q+1}, u'_{q+2}, \dots, u'_s) \subset S$, where $u'_k = u_k/x_m$ for $1+q \le k \le s$, is squarefree strongly stable with number of generators $< s$. Consider the same generic transformation g on S and let $(b'_{kj})_{q+1\le k,j\le s}$ denote the corresponding matrix of initial terms for J, i.e. b'_{kj} is the initial term of the coefficient of $(u'_j)^\tau$ in $g(u'_k)$. Then one obtains that $b'_{kj} = b_{kj}/a_{m-(d-1),m}$ for $q+1 \le k, j \le s$. By induction hypothesis applied to the ideal J, we have $\prod_{k=q+1}^{s} b'_{k\rho_2(k)} \prec \prod_{k=q+1}^{s} b'_{kk}$. Hence $\prod_{k=q+1}^{s} b_{k\rho_2(k)} \prec \prod_{k=q+1}^{s} b_{kk}$, and this completes the proof. □

Proof of Theorem (1.3): Since the ideal I is squarefree strongly stable, I is componentwise linear [3]. Therefore by [4, Theorem 1.1], for the graded Betti numbers of I and $\mathrm{Gin}^S(I)$ it holds: $\beta_{ii+j}(I) = \beta_{ii+j}(\mathrm{Gin}^S(I))$ for all i and j. On the other hand, the ideal $\mathrm{Gin}(I)$ being strongly stable, it follows from Lemma 2.2 below that $\beta_{ii+j}(\mathrm{Gin}^S(I)) = \beta_{ii+j}((\mathrm{Gin}^S(I))^\sigma)$ Thus, we obtain the equalities

$$\beta_{i,i+j}(I) = \beta_{i,i+j}((\mathrm{Gin}^S(I))^\sigma) \quad \text{for all} \quad i, j, \tag{3}$$

which imply that I and $(\mathrm{Gin}^S(I))^\sigma$ have the same Hilbert function. Hence it is enough to prove that $I \subseteq (\mathrm{Gin}^S(I))^\sigma$. By Lemma 1.2 and Lemma 1.4 this inclusion is equivalent to $I^\tau \subseteq \mathrm{Gin}^S(I)$. So, we will show that $u^\tau \in \mathrm{Gin}^S(I)$ for every $u \in G(I)$.

Since $I = \sum_{u\in G(I)} SqB(u)$, and $\mathrm{Gin}^S(SqB(u)) \subseteq \mathrm{Gin}^S(I)$ for every $u \in G(I)$, it is enough to show that the claim is true for squarefree Borel principal ideals. So, we may assume that $I = SqB(u)$. Set $d = \deg u$.

Let $G(I) = \{u_1, \dots, u_s\}$ where $u_1 > u_2 > \cdots > u_s$. Then $u_s = u$. We may assume that the claim is true for all u_k, $1 \le k \le s-1$. Then $(u_1^\tau, u_2^\tau, \dots, u_{s-1}^\tau) \subset \mathrm{Gin}^S(I)$, and since I^τ and $\mathrm{Gin}^S(I)$ have the same number of minimal monomial generators, one has $G(\mathrm{Gin}^S(I)) = \{u_1^\tau, u_2^\tau, \dots, u_{s-1}^\tau, v\}$, where v is a monomial of degree d. We have to prove that $v = u^\tau$.

Assume $v > u^\tau$. We will see that this is impossible. First, we show that $m(v) = m(u^\tau)$. It follows from formula (4) of Section (2) that

$$\beta_{ii+d}((\mathrm{Gin}^S(I))^\sigma) = \sum_{j=1}^{s-1} \binom{m(u_j)-d}{i} + \binom{m(v^\sigma)-d}{i};$$

$$\beta_{ii+d}(I) = \sum_{j=1}^{s-1} \binom{m(u_j)-d}{i} + \binom{m(u)-d}{i}.$$

Therefore, according to (3) we obtain $\binom{m(v^\sigma)-d}{i} = \binom{m(u)-d}{i}$ which implies $m(v^\sigma) = m(u)$, so that $m(v) = m(u^\tau)$.

We fix the following notation: $u = x_{s_1} \cdots x_{s_d}$ where $s_1 < \cdots < s_d$, and $v = x_{j_1} \cdots x_{j_d}$ where $j_1 \le \cdots \le j_d$. Since $v > u^\tau$, there exits a k such that $j_i = s_i - (i-1)$ for $k+1 \le i \le d$ and $j_k < s_k - (k-1)$. As $j_d = m(v) = m(u^\tau) = s_d - (d-1)$, one has $k < d$. If $j_i + (i-1) \le s_i$ for $1 \le i \le k$, then $I = SqB(u)$ being squarefree strongly stable, one obtains that $v^\sigma \in I$ which implies $v^\sigma = u_t$ for some $1 \le t \le s-1$ and the contradiction $v = u_t^\tau$. Thus, there exits an ℓ, $1 \le \ell < k$, such that $j_\ell + (\ell - 1) > s_\ell$. Then $j_\ell \le j_k < s_k - (k-1) \le s_d - (d-1) = m(v)$, therefore $x_{j_\ell} v / x_{m(v)}$ in $\mathrm{Gin}^S(I)$, because $\mathrm{Gin}^S(I)$ is strongly stable. Since $x_{j_\ell} v / x_{m(v)} > v$, we get $x_{j_\ell} v / x_{m(v)} = u_t^\tau$ for some $1 \le t \le s-1$. Say $u_t = x_{t_1} \cdots x_{t_d}$ where $t_1 < \cdots < t_d$. As $I = SqB(u)$ is a squarefree Borel principal ideal, we have $t_i \le s_i$ for $1 \le i \le d$, therefore $t_i - (i-1) \le s_i - (i-1)$ for $1 \le i \le d$. This contradicts $j_\ell > s_\ell - (\ell - 1)$.

Hence, $v \le u^\tau$. Now, we apply Lemma 1.5 using same notation. We have $\mathrm{Gin}^S(I) =$ in $(g(I))$ and $u_j^\tau \in \mathrm{Gin}^S(I)$ for $1 \le j \le s-1$. Since the rank of the matrix $(c_{kj})_{1 \le k, j \le s}$ is maximal, it follows that $v \ge u^\tau$, and so $v = u^\tau$. □

Corollary 1.6 *Let I be a strongly stable ideal of S. Then $Gin^S(I^\sigma) = I$.*

Proof: By 1.2 the ideal I^σ is squarefree strongly stable. Therefore, according to Theorem 1.3, we get $(\mathrm{Gin}^S(I^\sigma))^\sigma = I^\sigma$. Applying the operator τ, we obtain the desired equality. □

Corollary 1.7 *Let I be a squarefree monomial ideal of S. Then*

$$I^e = I^s \Longleftrightarrow \mathrm{Gin}^S(I^e) = \mathrm{Gin}^S(I^s).$$

Proof: Assume that $\mathrm{Gin}^S(I^e) = \mathrm{Gin}^S(I^s)$. Since I^e and I^s are squarefree strongly stable, our main result 1.3 implies $I^e = \mathrm{Gin}^S(I^e)^\sigma = \mathrm{Gin}^S(I^s)^\sigma = I^s$. □

The operator σ which establishes a bijection between strongly stable ideals and squarefree strongly stable ideals, restricts to a bijection between the lexsegment ideals and the squarefree lexsegment ideals, as shown in the following result. Recall that a (squarefree) monomial ideal is called a (squarefree) lexsegment ideal if for all (squarefree) monomials $u \in I$ and all (squarefree) monomials v of same degree with $u <_{lex} v$ it follows that $v \in I$.

Lemma 1.8 *Let L be a lexsegment ideal in S. Then L^σ is a squarefree lexsegment ideal in $K[x_1, \ldots, x_m]$ where $m = \max\{m(u) + \deg u - 1 : u \in G(L)\}$.*

Conversely, if L is a squarefree lexsegment ideal in S. Then L^τ is a lexsegment ideal in $K[x_1, \ldots, x_m]$ where $m = \max\{m(u) - \deg u + 1 : u \in G(L)\}$.

Proof: Suppose that L is a lexsegment ideal in S with $G(L) = \{u_1, \ldots, u_s\}$. Then by 1.2, the ideal L^σ is squarefree strongly stable with $G(L^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$. Note that this claim is true for any characteristic.

Let $w \in L^\sigma$ be a squarefree monomial, and let $w <_{lex} v$ be a squarefree monomial with $\deg v = \deg w$. We will show that $v \in L^\sigma$. Since L^σ is a squarefree strongly stable ideal, one can decompose $w = u^\sigma y$ where $u \in G(L)$ and y is a squarefree monomial with

$m(u^\sigma) < \min(y)$. Then $w^\tau = u\tilde{y} \in L$. Since L is lexsegment, and since $v^\tau > w^\tau$, one obtains that $v^\tau \in L$. Hence $v^\tau = u'y'$ where $u' \in G(L)$ and $m(u') \le \min(y')$. Therefore u'^σ divides $(v^\tau)^\sigma = v$, and we are done.

The converse direction with the operator τ is proved similarly. □

Corollary 1.9 *Let I be a squarefree strongly stable ideal in S. Then $\mathrm{Gin}^S(I)$ is lexsegment if and only if I is squarefree lexsegment.*

We conclude this section with a few remarks concerning combinatorial shifting. It is known that for the computation of the generic initial ideal in the exterior or symmetric algebra one may choose a linear automorphism φ which is represented by a generic upper triangular matrix with diagonal entries all 1; see for example [9] for the symmetric case. Any such automorphism is the product of elementary automorphisms φ_{ij}^a, $1 \le i < j \le n$, $a \in K$, which are defined as follows: $\varphi_{ij}^a(e_k) = e_k$ if $k \ne j$, and $\varphi_{ij}^a(e_j) = ae_i + e_j$.

In the combinatorics of finite sets one considers the following operation (cf. [1]): Let $\mathcal{A}$ be a collection of subsets of $[n]$. For given integers $1 \le i < j \le n$, and for all $A \in \mathcal{A}$ one defines:

$$S_{ij}(A) = \begin{cases} (A\backslash\{j\}) \cup \{i\}, & \text{if} \quad j \in A, \quad i \notin A, \quad (A\backslash\{j\}) \cup \{i\} \notin \mathcal{A}, \\ A, & \text{otherwise.} \end{cases}$$

For a set $A \in [n]$, $A = \{a_1 < a_2 < \cdots < a_i\}$, we set $e_A = e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge a_{a_i}$. The following fact is easily checked:

Lemma 1.10 *Let $I \subset E$ be a squarefree monomial ideal, and let $a \in K$, $a \ne 0$. Then in $(\varphi_{ij}^a(I))$ has the K-basis $\{e_{S_{ij}(A)} : e_A \in I\}$, where $\mathcal{A} = \{A \in [n] : e_A \in I\}$.*

It follows in particular that the ideal in $(\varphi_{ij}^a(I))$ does not depend on the choice of a. We may therefore denote it by $\mathrm{Shift}_{ij}(I)$. Notice that Shift_{ij} satisfies conditions (S_2)–(S_4), while condition (S_1) is usually not satisfied. However, it can be easily seen, that a finite number of iterated applications of this operation (with various $i < j$) leads to a strongly stable ideal; see for example [3]. We denote the resulting ideal by I^c. It can be shown by examples that the ideal I^c may depend on the choice on the sequence of operators Shift_{ij}. However we do not know whether for a suitable such choice one always has $I^c = I^e$.

2. Graded Betti numbers

In this section we study the relations between the graded Betti numbers of a squarefree monomial ideal $I \subset S$ and its shifted ideals.

As a main result we obtain:

Theorem 2.1 *Let $I \subset S$ be a squarefree monomial ideal. Then*

$$\beta_{ii+j}(I) \le \beta_{ii+j}(I^s) \quad \textit{for all} \quad i \quad \textit{and} \quad j.$$

This theorem is an immediate consequence of:

Lemma 2.2 *If I is a strongly stable monomial ideal, then $\beta_{ii+j}(I) = \beta_{ii+j}(I^\sigma)$ for all i and j.*

Proof: The graded Betti numbers of a squarefree strongly stable ideal I are given in [3] by:

$$\beta_{ii+j}(I) = \sum_{u \in G(I)_j} \binom{m(u) - j}{i} \tag{4}$$

for all i and j. Therefore, the assertion follows from (1), (2) and (4). □

Proof of Theorem (2.1): A simple semicontinuity argument (cf. [11]) shows that for any graded ideal $I \subset S$ one has $\beta_{ii+j}(I) \leq \beta_{ii+j}(\text{in}(I))$ for all i and j. In particular it follows that $\beta_{ii+j}(I) \leq \beta_{ii+j}(\text{Gin}^S(I))$ for all i and j. Thus the theorem follows from 2.2. □

Theorem 2.1 leads us to conjecture the following inequalities:

Conjecture 2.3 Let $I \subset S$ be a squarefree monomial ideal. Then for all i and j one has

$$\beta_{ii+j}(I^s) \leq \beta_{ii+j}(I^e) \leq \beta_{ii+j}(I^c).$$

In virtue of 2.1 the conjecture implies the inequalities

$$\beta_{ii+j}(I) \leq \beta_{ii+j}(I^e)$$

for all i and j. One should expect that there is direct proof of this inequality, avoiding a comparison with the symmmetric shifted ideal. Unfortunately no such proof is known so far. On the other hand the next result shows that the extremal Betti numbers of the algebraic shifted ideals behave as expected.

According to [6] a Betti number β_{kk+m} is called *extremal*, if $\beta_{ii+j} = 0$ for $(i, j) \neq (k, m)$, $i \geq k$ and $j \geq m$.

Theorem 2.4 *Let $I \subset S$ be a squarefree monomial ideal. Then for all i and j*
(a) *the following conditions are equivalent*:
  (i) *the ijth Betti number of I is extremal,*
  (ii) *the ijth Betti number of I^e is extremal,*
  (iii) *the ijth Betti number of I^s is extremal.*
(b) *the corresponding extremal Betti numbers of I, I^e and I^s are equal.*

Proof: The statements for I and I^e are proved in [2], and the corresponding statements for I and $\text{Gin}^S(I)$ are proved by Bayer-Charalambous-Popescu in [6]. Hence since $\beta_{ij}(\text{Gin}^S(I)) = \beta_{ij}(I^s)$ by 2.2, we obtain the assertions for I and I^s, too. □

The invariance of the extremal Betti numbers for combinatorial shifting is unknown. To prove it, it would suffice to show that I and $\mathrm{Shift}_{ij}(I)$ have the same extremal Betti numbers.

Now, let Δ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. Then $I_\Delta \subset S = K[x_1, \ldots, x_n]$ denote the squarefree monomial ideal, called the Stanley-Reisner ideal (cf. [7], [12] or [17]) arising from Δ. It follows from Lemma 1.1 that if $u = x_{i_1}x_{i_2}\cdots x_{m(u)} \in G(\mathrm{Gin}^S(I_\Delta))$ with $i_1 \le i_2 \le \cdots \le m(u)$, then $u^\sigma = x_{i_1}x_{i_2+1}\cdots x_{m(u)+\deg u-1}$ is a monomial belonging to S. Hence we can find a simplicial complex Δ^s on $[n]$ with $I_{\Delta^s} = (I_\Delta)^s$.

Similarly, Δ^e is defined to be the unique simplicial complex such that $J_{\Delta^e} = \mathrm{Gin}^E(J_\Delta)$ where $J_\Delta \subset E$ is the monomial ideal arising from Δ. Kalai [15] has shown that Δ and Δ^e have the same reduced simplicial homology. This is also true for symmetric algebraic shifting:

Corollary 2.5 *Let K be a field of characteristic* 0. *Then*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta^s; K) \quad \textit{for all} \quad i.$$

Proof: For any simplicial complex Γ on the vertex set $[n]$ one has

$$\beta_{in}(I_\Gamma) = \dim_K \tilde{H}_{n-i-2}(\Gamma; K) \quad \text{for all} \quad i. \tag{5}$$

This follows from Hochster's formulas ([14] and [7]). Hochster's formulas also imply that $\beta_{ij}(I_\Gamma) = 0$ for all i and all $j > n$. In particular, if $\beta_{in}(I_\Gamma) \neq 0$, then $\beta_{in}(I_\Gamma)$ is an extremal Betti number of I_Γ. Therefore 2.4 implies that $\beta_{in}(I_\Delta) = \beta_{in}(I_{\Delta^s})$ for all i, and the assertion follows from (5). □

The usefulness of 2.5 is partially explained by the fact that $\tilde{H}_\bullet(\Delta^s; K)$ can be computed combinatorially in a simple way. In fact, as noted by Kalai [16] (in a different terminology), one has

Lemma 2.6 *Let Δ be a simplicial complex on the vertex set $[n]$ such that I_Δ is squarefree strongly stable. Then*

$$\begin{aligned}\dim_K \tilde{H}_i(\Delta; K) &= \#\{u \in G(I_\Delta)_{i+2} : m(u) = n\}\\ &= \#\{F \in \Delta : \dim F = i, \quad F \cup \{n\} \notin \Delta\}\end{aligned}$$

Proof: The first equation follows from (5) and (4), while the second equation follows trivially from the definitions. □

As a further application of 2.4 we prove a non-squarefree version of a theorem of Björner and Kalai [8]. Since their result will be used in the sequel, we first give a more algebraic proof of their theorem, which applies to any graded ideal in the exterior algebra, and not just to monomial ideals, but nevertheless follows closely the arguments of the original proof of Björner and Kalai.

So let $I \subset E$ be a graded ideal. We set $f_{i-1} = \dim_K(E/I)_i$ for all $i \geq 0$, and call $f = (f_0, f_1, \ldots)$ the f-vector of E/I. Let $e = e_1 + e_2 + \cdots + e_n$. Since $e^2 = 0$, one gets a cocomplex

$$0 \to (E/I)_0 \xrightarrow{e} (E/I)_1 \xrightarrow{e} (E/I)_2 \xrightarrow{e} \cdots$$

We let $\beta_{i-1} = \dim_K H^i((E/I)_\bullet)$, and call $\beta = (\beta_{-1}, \beta_0, \beta_1 \ldots)$ the Betti sequence of E/I. In case $I = I_\Delta$ for some simplicial complex Δ, the β_i are the ordinary Betti numbers of Δ.

A pair of sequences $(f, \beta) \in \mathbb{N}_0^\infty$ is called *compatible* if there exists a graded K-algebra E/I such that f is the f-sequence and β the Betti sequence of E/I.

Let a and i be two integer. Then a has a unique binomial expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$; see [7] or [12].

We define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

We also set $0^{(i)} = 0$ for all $i \geq 1$.

Theorem 2.7 (*Björner and Kalai*) *Let K be a field. The following conditions are equivalent*:

(a) *The pair of sequences (f, β) is compatible.*

(b) *Set $\chi_i = (-1)^i \sum_{j=-1}^{i} (-1)^j (f_j - \beta_j)$ for all i. Then*

  (i) $\chi_{-1} = 1$ *and* $\chi_i \geq 0$ *for all* i,

  (ii) $\beta_i \leq \chi_{i-1}^{(i)} - \chi_i$ *for all* i.

Proof (a) $\Rightarrow$ (b)**:** It is well known and easy to see that E/I and E/I^e have the same Hilbert function, that is, the f-vectors of E/I and E/I^e coincide. In [2, Corollary 5.2] it is shown that $H^i((E/I)_\bullet) \cong H^i((E/I^e)_\bullet)$ for all i. Hence also the Betti sequences of E/I and E/I^e coincide. Thus we may replace I by I^e, and hence may as well assume that I is strongly stable.

Let I' be the ideal generated by all $u \in G(I)$ with $m(u) < n$ and all monomials $u \in E$ such that $u \wedge e_n \in G(I)$. Then I' is again strongly stable and $E_1 I' \subset I$. By 2.6, the last property implies that

$$\dim_K(I'/I)_i = \#\{u \in G(I)_{i+1}: m(u) = n\} = \beta_{i-1}(E/I).$$

It follows that $\dim_K(E/I')_i = f_{i-1} - \beta_{i-1}$ for all i. Now we notice that e_n is regular on E/I', in the sense that the complex

$$E/I' \xrightarrow{e_n} E/I' \xrightarrow{e_n} E/I'$$

is exact. Therefore, for each i we obtain an exact sequence of K-vector spaces

$$\to (E/I')_{i-1} \to (E/I')_i \to (E/I')_{i+1} \to (E/(I' + e_n E))_{i+1} \to 0, \tag{6}$$

and hence $\chi_i = \dim_K (E/(I' + e_n E))_{i+1}$.

Next we observe that $I'/I \cong (I' + e_n E)/(I + e_n E)$ and $E_1(I' + e_n E) \subset I + e_n E$, so that together with the Kruskal-Katona theorem (see for example [3]) we obtain

$$\begin{aligned} \chi_i + \beta_i &= \dim_K E_{i+1} - \dim_K (I + e_n E)_{i+1} \\ &\leq \dim_K E_{i+1} - \dim_K E_1(I' + e_n E)_i \leq \chi_{i-1}^{(i)}, \end{aligned}$$

as required.

(b) $\Rightarrow$ (a): The hypotheses imply that $\chi_i \leq \chi_{i-1}^{(i)}$ and $\chi_i + \beta_i \leq (\chi_{i-1} + \beta_{i-1})^{(i)}$. Thus the Kruskal-Katona theorem yields an integer m, and lexsegment ideals $L \subset N$ in the exterior algebra $E' = K\langle e_1, \ldots, e_{m-1}\rangle$ such that $\dim_K (E/N)_{i+1} = \chi_i$ and $\dim_K (E/L)_{i+1} = \chi_i + \beta_i$ that for all i.

Now let $I \subset E = K\langle e_1, \ldots, e_m\rangle$ be the ideal generated by the elements in $G(L)$ and all elements $u \wedge e_m$ with $u \in G(N)$. Moreover we set $I' = NE$. Then $I'/I \cong N/L$, and so

$$\begin{aligned} \dim_K (E/I)_{i+1} &= \dim_K (N/L)_{i+1} + \dim_K (E/I')_{i+1} \\ &= \beta_i + \dim_K (E/I')_{i+1}. \end{aligned} \tag{7}$$

On the other hand, e_m is regular on E/I', and so (6) yields

$$\dim_K (E/(I' + e_m E))_{i+1} = (-1)^{i+1} \sum_{j=0}^{i+1} (-1)^j \dim_K (E/I')_j \tag{8}$$

for all i. Thus, since $E/(I' + e_m E) \cong E'/N$, it follows from (8) that

$$\dim_K (E/I')_{i+1} = \dim_K (E'/N)_{i+1} + \dim_K (E'/N)_i = \chi_i + \chi_{i-1} = f_i - \beta_i.$$

This together with (7) implies that $\dim_K (E/I)_{i+1} = f_i$.

Finally it is clear from the construction of I that $\#\{u \in G(I)_{i+2} : m(u) = m\}$ equals $\dim_K (N/L)_{i+1}$ which is β_i. Thus, by 2.6, the assertion follows. □

The Björner-Kalai Theorem can be translated into a theorem on super extremal Betti numbers. Let $I \subset S$ be a graded ideal. We let m be the maximal integer j such that $\beta_{ij}(S/I) \neq 0$ for some i. In other words, m is the largest shift in the graded minimal free S-resolution of S/I. It is clear that $\beta_{im}(S/I)$ is an extremal Betti number for all i with $\beta_{im}(S/I) \neq 0$, and that there is at least one such i. These Betti numbers are distinguished by the fact that they are positioned on the diagonal $\{(i, m - i) : i = 0, \ldots, m\}$ on the Betti diagram, and that all Betti numbers on the right lower side of the diagonal are zero. The ring S/I may of course have other extremal Betti numbers, not sitting on this diagonal. We call the Betti numbers β_{im}, $i = 0, \ldots, m$, *super extremal*, regardless whether they are zero or not, and ask the question which sequences of numbers $(b_0, b_1, \ldots, b_m)$ appear as sequences of super extremal Betti numbers for graded rings with given Hilbert function.

Before answering the question we have to encode the Hilbert function $H_{S/I}(t)$ of S/I in a suitable way. Using the additivity of the Hilbert function, the graded minimal free resolution of S/I yields the following formula:

$$H_{S/I}(t) = \frac{a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m}{(1-t)^n}$$

with $a_i \in \mathbb{Z}$; see for example [7]. It follows that

$$(1-t)^{n-m} H_{S/I}(t) = \frac{a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m}{(1-t)^m}$$

Notice that $n-m$ may take positive or negative values. At any rate, the rational function $(1-t)^{n-m} H_{S/I}(t)$ has degree ≤ 0. One easily verifies that there is a unique expansion

$$(1-t)^{n-m} H_{S/I}(t) = \sum_{i=0}^{m} f_{i-1} \frac{t^i}{(1-t)^i}$$

with $f_i \in \mathbb{Z}$. It is clear that $f_{-1} = 1$, and we shall see later that all $f_i \geq 0$. We call $f = (f_{-1}, f_0, f_1, \ldots, f_{m-1})$ the *f-vector of S/I*. Given the highest shift in the resolution, the f-vector of S/I determines the Hilbert function of S/I, and vice versa.

We set $b_i = \beta_{m-i-1,m}$ and call $b = (b_{-1}, \ldots, b_{m-1})$ the *super extremal sequence of S/I*. Finally we set $\chi_i = (-1)^i \sum_{j=-1}^{i} (-1)^j (f_j - b_j)$ for $i = -1, 0. \ldots, m-1$. The Björner-Kalai theorem has the following counterpart.

Theorem 2.8 *Let K be a field of characteristic* 0. *Let $f = (f_{-1}, f_0, \ldots, f_{m-1})$ and $b = (b_{-1}, b_0, \ldots, b_{m-1})$ be sequences of non-negative integers. The following conditions are equivalent*:

(a) *there exists a homogeneous K-algebra S/I such that f is the f-vector, and b the super extremal sequence of S/I*;

(b) (i) $\chi_{-1} = 1$ *and* $\chi_i \geq 0$ *for all i*,
(ii) $b_i \leq \chi_{i-1}^{(i)} - \chi_i$ *for all i*.

Proof (a) $\Rightarrow$ (b)**:** Since the extremal Betti numbers are preserved when we pass from I to $\mathrm{Gin}^S(I)$, it follows that I and $\mathrm{Gin}^S(I)$ have the same highest shift m, and hence the same b-vector. Since S/I and $S/\mathrm{Gin}^S(I)$ have the same Hilbert function, it also follows that the f-vectors of S/I and $S/\mathrm{Gin}^S(I)$ coincide. Thus, since $\mathrm{char}(K) = 0$, we may assume that I is a strongly stable monomial ideal.

The ideal I^σ is defined in $S' = K[x_1, \ldots, x_m]$ and $\beta_{ii+j}(I) = \beta_{ii+j}(I^\sigma)$ by 2.2. This implies that

$$H_{S'/I^\sigma}(t) = (1-t)^{n-m} H_{S/I}(t).$$

Hence, if we let Δ be the simplicial complex with $I_\Delta = I^\sigma$, then Δ and S/I have the same f-vector, and one has $b_i = \dim_K \tilde{H}_i(\Delta; K)$; see (5). Therefore, the conclusion follows from Björner-Kalai Theorem.

(b) $\Rightarrow$ (a): Given an f-and b-sequence satisfying conditions (b), there exists by 2.7 an integer m and a simplicial complex Δ on the vertex set $[m]$ whose f-vector is f and whose β-sequence is b. Then $K[x_1, \ldots, x_m]/I_\Delta$ is a homogeneous K-algebra satisfying (a). □

As a last result we give an upper bound for the graded Betti numbers of a squarefree monomial ideal with a given f-vector defined over a field of characteristic 0. Let Δ be a simplicial complex on the vertex set $[n]$. Since I_Δ and $\mathrm{Gin}^S(I_\Delta)$ have the same Hilbert function, (3) guarantees that $f(\Delta) = f(\Delta^s)$. Moreover, by 2.1 we have

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^s}) \tag{9}$$

for all i and j.

Let Δ^{lex} denote the (unique) lexsegment simplicial complex on $[n]$ with the same f-vector as Δ, see [3]. Since $f(\Delta^s) = f(\Delta^{lex})$ and since I_{Δ^s} is squarefree strongly stable by Lemma 1.2, it follows from [3, Theorem 4.4] that

$$\beta_{ii+j}(I_{\Delta^s}) \leq \beta_{ii+j}(I_{\Delta^{lex}}) \tag{10}$$

for all i and j.

Combining inequalities (9) and (10), we obtain:

Theorem 2.9 *Let K be a field of characteristic 0, let Δ be a simplicial complex, and Δ^{lex} the (unique) lexsegment simplicial complex with the same f-vector as Δ. Then for the Betti numbers of the stanley Reisner ideals I_Δ and $I_{\Delta^{les}} \subset K[x_1, \ldots, x_n]$, we have*

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^{lex}})$$

for all i and j.

We expect that these inequalities are also valid when K is a field of positive characteristic.

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