



# Joint monotone and boolean numerical and spectral radii of $d$ -tuples of operators

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## Abstract

We study joint numerical and spectral radii defined for  $d$ -tuples of bounded operators on a Hilbert space and related to noncommutative notions of independence. The definitions are in analogy with the ones of Popescu, where his formulations turned out to be related with free creation operators, and in this way related to the free independence of Voiculescu. In our study the definitions are related with either weakly monotone creation operators, and thus associated with the monotone independence of Muraki, or with boolean creation operators, and hence related with the boolean independence.

**Keywords** Numerical radius · Spectral radius · Fock space · Monotone independence · Boolean independence

**Mathematics Subject Classification** 46L53 · 47A12 · 47A13

## 1 Introduction

The notion of numerical radius as well as the related notion of numerical range is an object of intensive studies since the work by Toeplitz [10] in 1918 until today. Numerical radius provides a norm on the space of bounded operators, which is equivalent to the operator norm. Its special features include unitarity invariance, the

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Dedicated to Franciszek Hugon Szafraniec on the occasion of his 80th birthday, with appreciation of his influence on our multidimensional collaboration.

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power inequality and the relation with the spectral radius, see e.g. [6]. The numerical range and radius link the properties of operators with geometry of a complex plane, allowing many interesting applications, from the approximate localization of spectrum via the stability results for differential equations (e.g. [5]) to the von Neumann type inequalities (e.g. [1]).

In 2009, Gelu Popescu [9], in his search for a free analogue of the Sz.-Nagy-Foias theory for row contractions, defined *free* analogues of numerical and spectral radii, namely the *joint numerical radius* and *joint spectral radius* for  $d$ -tuples of operators  $(T_1, \dots, T_d)$  acting on a Hilbert space  $\mathcal{H}$ . The definitions are as follows.

**Definition 1** The (free) joint numerical radius is

$$w_F(T_1, \dots, T_d) := \sup \left\{ \left| \sum_{\alpha \in \mathbb{F}_d^+} \sum_{j=1}^d \langle T_j h_{g_j \alpha} | h_\alpha \rangle \right| : \sum_{\alpha \in \mathbb{F}_d^+} \|h_\alpha\|^2 < \infty \right\}$$

where  $\mathbb{F}_d^+$  is the free semigroup with free generators  $g_1, \dots, g_d$  and each  $\alpha \in \mathbb{F}_d^+$  is a word in these generators.

**Definition 2** The (free) joint spectral radius is

$$r_F(T_1, \dots, T_d) := \lim_{k \rightarrow \infty} \left\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right\|^{1/2k}$$

where for  $\alpha \in \mathbb{F}_d^+$  one puts  $|\alpha| = k$  if it is a word in  $k$  generators  $\alpha = g_{i_1} \dots g_{i_k}$ .

These two notions are related not only to the free semigroup  $\mathbb{F}_d^+$ , but also to the model of *freeness* of Voiculescu [11], and more precisely to the creation operators on the full Fock space by the following result.

**Theorem 3** ([9], Corollary 1.2) *The joint free numerical and spectral radii can be computed as the ordinary numerical radius  $w$  and the spectral radius  $r$  of single operators:*

$$w_F(T_1, \dots, T_d) = w(S_1 \otimes T_1^* + \dots + S_d \otimes T_d^*) \tag{1}$$

$$r_F(T_1, \dots, T_d) = r(S_1 \otimes T_1^* + \dots + S_d \otimes T_d^*), \tag{2}$$

where  $S_1 := S(e_1), \dots, S_d := S(e_d)$  are the free creation operators on the full Fock space  $\mathcal{F}(H_d)$  on  $d$ -dimensional Hilbert space  $H_d$  with an orthonormal basis  $\{e_1, \dots, e_d\}$  and  $S(h)$  is the creation operator by the vector  $h \in H_d$ :  $S(h)v = h \otimes v$  for  $v \in \mathcal{F}(H_d)$ .

The main idea of this paper is to study analogues of these definitions in the case where we replace the (free) full Fock space by a Fock space associated to other noncommutative independences, and the free creation operators by the creation operators on the appropriate Fock space. We show that the joint (noncommutative) numerical radii, defined in analogy to Theorem 1, satisfy many basic properties similar to the free case. In particular, we show the unitarity invariance and the relation with the

appropriately defined spectral radius. We also compute some examples. In the paper we treat the *monotone* and *boolean* case, but the framework is more general. Our idea establishes yet another bridge between classical operator theory and the noncommutative probability and we believe this is a starting point for further investigations. However, in this paper there is no need to define the monotone and boolean independences, it is sufficient to consider the models of both of them, built on either weakly monotone Fock space or on the Boolean Fock space, respectively.

## 2 General scheme

Recall that the (classical) numerical radius of a linear operator  $T$ , bounded on a Hilbert space  $\mathcal{H}$ , is defined by

$$w_{\text{cl}}(T) := \sup\{|\langle Th, h \rangle| : h \in \mathcal{H}, \|h\| = 1\}$$

and the (classical) spectral radius of  $T$  is defined by

$$r_{\text{cl}}(T) = \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}}.$$

Let us consider one of the noncommutative independence, e.g. boolean, free, monotone and let us consider a noncommutative Fock space  $F_{\star}(H)$  associated to this independence. By this we mean the *full Fock space* ([11]) for the *free independence* of Voiculescu, the *weakly monotone Fock space* ([12]) for the *monotone independence* of Muraki and the *boolean Fock space* ([3]) for the *boolean independence* (c.f. [2]). On each of these Fock spaces we are given creation and annihilation operators, which we shall use to define relative joint (numerical and spectral) radii, following the work by Popescu [9] for the free case.

Let  $d \in \mathbb{N}$  and let  $H$  be the  $d$ -dimensional Hilbert space with the orthonormal basis  $\{e_j; 1 \leq j \leq d\}$ . Denote by  $A_1^{\star}, \dots, A_d^{\star}$  the creation operators associated to the basic vectors  $A_j^{\star} = A^{\star}(e_j)$  on the related Fock space  $F_{\star}(H)$  and define the  $\star$ -joint numerical radius of the  $d$ -tuple  $(T_1, \dots, T_d)$  of operators in  $B(\mathcal{H})$  by

$$w_{\star}(T_1, \dots, T_d) := w_{\text{cl}}(A_1^{\star} \otimes T_1^* + \dots + A_d^{\star} \otimes T_d^*).$$

Similarly, we define the  $\star$ -joint spectral radius of  $(T_1, \dots, T_d)$  by

$$r_{\star}(T_1, \dots, T_d) := r_{\text{cl}}(A_1^{\star} \otimes T_1^* + \dots + A_d^{\star} \otimes T_d^*).$$

The following properties of the  $\star$ -joint numerical and spectral radii are immediate consequences of the construction and of the properties of the classical numerical radius (compare with Theorem 1.1 in [9]).

**Proposition 4** *The  $\star$ -joint numerical radius and joint spectral radius satisfy:*

- (i)  $w_{\star}(\lambda T_1, \dots, \lambda T_d) = |\lambda| w_{\star}(T_1, \dots, T_d)$  for any  $\lambda \in \mathbb{C}$ ;
- (ii)  $w_{\star}(T_1 + T'_1, \dots, T_d + T'_d) \leq w_{\star}(T_1, \dots, T_d) + w_{\star}(T'_1, \dots, T'_d)$ ;
- (iii)  $w_{\star}(U^* T_1 U, \dots, U^* T_d U) = w_{\star}(T_1, \dots, T_d)$  for any unitary  $U : \mathcal{K} \rightarrow \mathcal{H}$ ;

- (iv)  $w_\star(I_{\mathcal{K}} \otimes T_1, \dots, I_{\mathcal{K}} \otimes T_d) = w_\star(T_1, \dots, T_d)$  for any separable Hilbert space  $\mathcal{K}$ ;
- (v)  $r_\star(T_1, \dots, T_d) \leq w_\star(T_1, \dots, T_d)$ .

**Proof** The properties (i) and (ii) follows from  $w_{\text{cl}}(\lambda T) = |\lambda|w_{\text{cl}}(T)$  and  $w_{\text{cl}}(T + T') \leq w_{\text{cl}}(T) + w_{\text{cl}}(T')$ . As for (iii), for a unitary  $U : \mathcal{K} \rightarrow \mathcal{H}$  we set  $V = I \otimes U$  and  $A := A_1^\star \otimes T_1^\star + \dots A_d^\star \otimes T_d^\star$  and observe that

$$\begin{aligned} w_\star(U^\star T_1 U, \dots, U^\star T_d U) &= w_{\text{cl}}(A_1^\star \otimes U^\star T_1^\star U + \dots A_d^\star \otimes U^\star T_d^\star U) \\ &= w_{\text{cl}}((I \otimes U)^\star (A_1^\star \otimes T_1^\star + \dots A_d^\star \otimes T_d^\star) (I \otimes U)) = w_{\text{cl}}(V^\star A V) \\ &= \sup\{|\langle AVh, Vh \rangle| : h \in \mathcal{H}, \|h\| = 1\} \\ &= \sup\{|\langle Ah', h' \rangle| : h' = Vh \in \mathcal{F}_*(H) \otimes \mathcal{K}, \|h'\| = \|Vh\| = 1\} \\ &= w_{\text{cl}}(A) = w_\star(T_1, \dots, T_d). \end{aligned}$$

The property (iv) goes exactly as in the proof of [9, Theorem 1.1], using  $w_{\text{cl}}(I_{\mathcal{K}} \otimes T) = w_{\text{cl}}(T)$ :

$$\begin{aligned} w_\star(I_{\mathcal{K}} \otimes T_1, \dots, I_{\mathcal{K}} \otimes T_d) &= w_{\text{cl}}\left(\sum_{j=1}^d A_j^\star \otimes (I_{\mathcal{K}} \otimes T_j)\right) \\ &= w_{\text{cl}}\left(I_{\mathcal{K}} \otimes \left(\sum_{j=1}^d A_j^\star \otimes T_j\right)\right) = w_{\text{cl}}\left(\sum_{j=1}^d A_j^\star \otimes T_j\right) \\ &= w_\star(T_1, \dots, T_d). \end{aligned}$$

Finally, to show (v) we just use the classical result  $r_{\text{cl}}(T) \leq w_{\text{cl}}(T)$ .  $\square$

### 3 Joint boolean numerical and spectral radii

Recall after [3] that the *boolean Fock space* (over the  $d$ -dimensional space  $H$ ) is defined as the direct sum

$$\mathcal{F}_B(H) = \mathbb{C}\Omega \oplus H, \tag{3}$$

where  $\Omega$  is a unit vector, called the *vacuum*. The boolean creation and annihilation operators are given by

$$B^*(f)h = \begin{cases} f & \text{if } h = \Omega \\ 0 & \text{if } h \in H \end{cases}, \quad B(f)h = \begin{cases} 0 & \text{if } h = \Omega \\ \langle f, h \rangle \Omega & \text{if } h \in H \end{cases}. \tag{4}$$

For a fixed orthonormal basis  $\{e_j : 1 \leq j \leq d\}$  in  $H$ , we shall use the notation  $B_j^* := B^*(e_j)$ ,  $B_j := B(e_j)$ ,  $j = 1, \dots, d$ , for the creation and annihilation operators (respectively) by the basic vectors, and  $e_0 := \Omega$ . It is easy to see that  $B_j B_k^* = \delta_{jk} P_\Omega$ , where  $P_\Omega$  is the projection onto the vacuum vector  $e_0 = \Omega$ .

Let now  $(T_1, \dots, T_d)$  be the  $d$ -tuple of bounded operators on a Hilbert space  $\mathcal{H}$ . We define the *joint boolean numerical radius* of  $(T_1, \dots, T_d)$  as

$$w_B(T_1, \dots, T_d) := w_{cl}(B_1^* \otimes T_1^* + \dots B_d^* \otimes T_d^*)$$

and the spectral *joint boolean spectral radius* of  $(T_1, \dots, T_d)$  as

$$r_B(T_1, \dots, T_d) := r_{cl}(B_1^* \otimes T_1^* + \dots B_d^* \otimes T_d^*).$$

We first provide the explicit formula for computing the joint boolean numerical radius, which is the analogue of Popescu’s definition in the free case (see Definition 1).

**Proposition 5** *The joint boolean numerical radius can be expressed as*

$$w_B(T_1, \dots, T_d) = \sup \left\{ \left| \langle g_0, \sum_{j=1}^d T_j g_j \rangle \right| : g_0, g_1, \dots, g_d \in \mathcal{H}, \sum_{j=0}^d \|g_j\|^2 = 1 \right\}.$$

**Proof** By the definition of the classical numerical radius, we have

$$w_B(T_1, \dots, T_d) := \sup \{ |\langle \sum_{j=1}^d B_j^* \otimes T_j^* h, h \rangle| : h \in \mathcal{F}_B(H) \otimes \mathcal{H}, \|h\| = 1 \}.$$

Expressing  $h \in \mathcal{F}_B(H) \otimes \mathcal{H}$  as  $h = \sum_{k=0}^d e_k \otimes g_k$  with  $g_0, g_1, \dots, g_d \in \mathcal{H}$  (recall that  $e_0 := \Omega$ ), we observe that

$$\|h\|^2 = \langle \sum_{k=0}^d e_k \otimes g_k, \sum_{m=0}^d e_m \otimes g_m \rangle = \sum_{k,m=0}^d \langle e_k, e_m \rangle \langle g_k, g_m \rangle = \sum_{k=0}^d \|g_k\|^2,$$

whereas, using the relation  $B_j e_m = \delta_{jm} e_0$ , for  $1 \leq j, m \leq d$ , and  $B_j e_0 = 0$  for  $1 \leq j \leq d$ , we get

$$\begin{aligned} \langle \sum_{j=1}^d B_j^* \otimes T_j^* h, h \rangle &= \sum_{j=1}^d \sum_{k,m=0}^d \langle (B_j^* \otimes T_j^*)(e_k \otimes g_k), (e_m \otimes g_m) \rangle \\ &= \sum_{j=1}^d \sum_{k,m=0}^d \langle B_j^* e_k, e_m \rangle \langle T_j^* g_k, g_m \rangle \\ &= \sum_{j=1}^d \sum_{k,m=0}^d \langle e_k, B_j e_m \rangle \langle T_j^* g_k, g_m \rangle \\ &= \sum_{m=1}^d \sum_{k=0}^d \langle e_k, e_0 \rangle \langle g_k, T_m g_m \rangle \\ &= \sum_{m=1}^d \langle g_0, T_m g_m \rangle = \langle g_0, \sum_{m=1}^d T_m g_m \rangle. \end{aligned}$$

□

It turns out that all of the properties that were shown to hold for the (free) joint numerical radius (see [9, Theorem 1.1]), remains true in the boolean case. Some of them were already observed in Proposition 4; here we prove the remaining ones.

**Proposition 6** *We have*

- (i)<sub>B</sub>  $w_B(T_1, \dots, T_d) = 0$  if and only if  $T_1 = \dots = T_d = 0$ ,
- (ii)<sub>B</sub>  $\frac{1}{2} \left\| \sum_{j=1}^d T_j T_j^* \right\|^{\frac{1}{2}} \leq w_B(T_1, \dots, T_d) \leq \left\| \sum_{j=1}^d T_j T_j^* \right\|^{\frac{1}{2}}$ ,
- (iii)<sub>B</sub>  $w_B(X^* T_1 X, \dots, X^* T_d X) \leq \|X\|^2 w_B(T_1, \dots, T_d)$  for any bounded operator  $X : \mathcal{H} \rightarrow \mathcal{K}$ .

**Proof** Ad (i)<sub>B</sub>. By the properties of the classical numerical radius,  $w_B(T_1, \dots, T_d) = 0$  implies  $\sum_{j=1}^d B_j^* \otimes T_j^* = 0$ . Hence for any  $h = \sum_{k=0}^d e_k \otimes g_k \in \mathcal{F}_B(H) \otimes \mathcal{H}$ , one gets

$$0 = \sum_{j=1}^d B_j^* \otimes T_j^* h = \sum_{j=1}^d \sum_{k=0}^d B_j^* e_k \otimes T_j^* g_k = \sum_{j=1}^d B_j^* e_0 \otimes T_j^* g_0 = \sum_{j=1}^d e_j \otimes T_j^* g_0.$$

Thus, for any  $j = 1, \dots, d$  and  $g_0 \in \mathcal{H}$ , one gets  $T_j^* g_0 = 0$ . Consequently,  $T_j^* = 0$  and, equivalently,  $T_j = 0$  for any  $j = 1, \dots, d$ .

Ad (ii)<sub>B</sub>. Thanks to  $\frac{1}{2} \|T\| \leq w_{cl}(T) \leq \|T\|$ , we have

$$\begin{aligned} w_B(T_1, \dots, T_d) &= w_{cl}\left(\sum_{j=1}^d B_j^* \otimes T_j^*\right) \leq \left\| \sum_{j=1}^d B_j^* \otimes T_j^* \right\| \\ &= \left\| \left(\sum_{j=1}^d B_j^* \otimes T_j^*\right)^* \left(\sum_{k=1}^d B_k^* \otimes T_k^*\right) \right\|^{\frac{1}{2}} = \left\| \sum_{j,k=1}^d \underbrace{B_j^* B_k}_{=\delta_{jk} P_\Omega} \otimes T_j^* T_k \right\|^{\frac{1}{2}} \\ &= \left\| P_\Omega \otimes \sum_{j=1}^d T_j^* T_j \right\|^{\frac{1}{2}} = \underbrace{\|P_\Omega\|}_{=1}^{\frac{1}{2}} \cdot \left\| \sum_{j=1}^d T_j^* T_j \right\|^{\frac{1}{2}}. \end{aligned}$$

Similarly, we prove that  $w_B(T_1, \dots, T_d) \geq \frac{1}{2} \left\| \sum_{j=1}^d T_j^* T_j \right\|^{\frac{1}{2}}$ .

Ad (iii)<sub>B</sub>. Let  $X : \mathcal{H} \rightarrow \mathcal{K}$  be a bounded operator and let  $g_0, \dots, g_d \in \mathcal{H}$  satisfy  $\sum_{k=0}^d \|g_k\|^2 = 1$ . Define  $C := \sqrt{\sum_{k=0}^d \|X g_k\|^2}$  and  $h_k := \frac{1}{C} X g_k$ ,  $k = 0, 1, \dots, d$ . Then  $\sum_{k=0}^d \|h_k\|^2 = 1$  and  $C \leq \|X\|$ . Consequently,

$$\begin{aligned}
 w_B(X^*T_1X, \dots, X^*T_dX) &= \sup \left\{ \left| \langle Xg_0, \sum_{j=1}^d T_jXg_j \rangle \right| : \sum_{j=0}^d \|g_j\|^2 = 1 \right\} \\
 &\leq \sup \left\{ \left| \langle Ch_0, \sum_{j=1}^d T_jCh_j \rangle \right| : \sum_{j=0}^d \|h_j\|^2 = 1 \right\} \\
 &= C^2 w_B(T_1, \dots, T_d) \leq \|X\|^2 w_B(T_1, \dots, T_d).
 \end{aligned}$$

□

**Remark 7** The properties (i), (ii) and  $(i)_B$  show that the joint boolean numerical radius is a norm on  $B(\mathcal{H})^d$ , which, by  $(ii)_B$ , is actually equivalent to the operator norm  $\left\| \sum_{j=1}^d T_j^*T_j \right\|^{\frac{1}{2}}$  of the operator row matrix  $[T_1, \dots, T_d]$ , hence  $w_B$  is a continuous map in the norm topology.

We now compute some examples and, in particular, show that  $w_B \neq w_F$ .

**Example 8** For  $T_j = I_{\mathcal{H}}, j = 1, \dots, d$  we have

$$w_B(I_1, \dots, I_d) = \frac{\sqrt{d}}{2}.$$

Using the relation between the  $\ell_1$ - and  $\ell_2$ -norm on  $\mathcal{H}^d$ :

$$\sum_{k=1}^d \|g_k\| \leq \sqrt{d} \sqrt{\sum_{k=1}^d \|g_k\|^2} \quad \text{for } g_1, \dots, g_d \in \mathcal{H},$$

and Proposition 5, we observe that

$$\left| \langle g_0, \sum_{k=1}^d g_k \rangle \right| \leq \|g_0\| \cdot \left\| \sum_{k=1}^d g_k \right\| \leq \|g_0\| \cdot \sum_{k=1}^d \|g_k\| \leq \sqrt{d} \|g_0\| \cdot \sqrt{\sum_{k=1}^d \|g_k\|^2}.$$

Hence for any  $g_0, \dots, g_d \in \mathcal{H}$  such that  $\sum_{k=0}^d \|g_k\|^2 = 1$ , denoting  $t := \|g_0\| \in [0, 1]$ , we get

$$\begin{aligned}
 w_B(I_1, \dots, I_d) &\leq \sup \left\{ \sqrt{d} \|g_0\| \cdot \sqrt{\sum_{k=1}^d \|g_k\|^2} : g_0, \dots, g_d \in \mathcal{H}, \sum_{k=0}^d \|g_k\|^2 = 1 \right\} \\
 &= \sup \{ t \cdot \sqrt{d} \cdot \sqrt{1-t^2} : t \in [0, 1] \} = \frac{\sqrt{d}}{2},
 \end{aligned}$$

and the supremum is achieved when  $t = \frac{1}{\sqrt{2}}$ . This show that  $w_B(I_1, \dots, I_d) \leq \frac{\sqrt{d}}{2}$ .

To see that the equality holds, take  $g_0 \in \mathcal{H}$  with  $\|g_0\|^2 = \frac{1}{2}$  and  $g_k = \frac{g_0}{\sqrt{d}}, k = 1, \dots, d$ . Then  $\sum_{k=1}^d \|g_k\|^2 = 1$  whereas  $|\langle g_0, \sum_{k=1}^d g_k \rangle| = |\langle g_0, \frac{d}{\sqrt{d}} g_0 \rangle| = \frac{\sqrt{d}}{2}$ .

**Example 9** For  $T_j = B_j, j = 1, \dots, d$ , the boolean annihilators, we have

$$w_B(B_1, \dots, B_d) = \frac{\sqrt{d}}{2}.$$

Since  $B_j h = \langle h, e_j \rangle \Omega$  for  $h \in \mathcal{H}$ , we compute

$$\begin{aligned} |\langle g_0, \sum_{j=1}^d B_j g_j \rangle| &= |\langle g_0, \sum_{j=1}^d \langle g_j, e_j \rangle \Omega \rangle| \leq \|g_0\| \left\| \sum_{j=1}^d \langle g_j, e_j \rangle \right\| \\ &\leq \|g_0\| \sum_{j=1}^d \|g_j\| \leq \sqrt{d} \|g_0\| \sqrt{\sum_{j=1}^d \|g_j\|^2} \leq \frac{\sqrt{d}}{2} \end{aligned}$$

as shown in the previous Example.

Taking  $g_0 = \frac{1}{\sqrt{2}} \Omega$  and  $g_k = \frac{1}{\sqrt{2d}} e_j$  for  $j = 1, \dots, d$ , we find that  $\sum_{k=1}^d \|g_k\|^2 = 1$  while

$$|\langle g_0, \sum_{j=1}^d B_j g_j \rangle| = \frac{1}{2\sqrt{d}} |\langle \Omega, \sum_{j=1}^d \langle e_j, e_j \rangle \Omega \rangle| = \frac{\sqrt{d}}{2}.$$

**Example 10** For  $d = 1$  we find out that

$$\frac{1}{2} w_F(T) = \frac{1}{2} w_{cl}(T) \leq w_B(T) \leq w_{cl}(T) = w_F(T) \tag{5}$$

and  $w_B(T) = \frac{1}{2} w_F(T)$  when  $\mathcal{H} = \mathbb{C}, Tz := az, a \in \mathbb{C}^*$ . This shows that  $w_F(T_1, \dots, T_d) \neq w_B(T_1, \dots, T_d)$  in general.

It was shown in [9, Sect. 1] that  $w_F(T) = w_{cl}(T)$ . Now, observe that

$$\begin{aligned} w_B(T) &= \sup \left\{ |\langle g_0, Tg_1 \rangle| : g_0, g_1 \in \mathcal{H}, \|g_0\|^2 + \|g_1\|^2 = 1 \right\} \\ &\geq \sup \left\{ |\langle g_0, Tg_0 \rangle| : g_0 \in \mathcal{H}, \|g_0\|^2 = \frac{1}{2} \right\} \\ &= \sup \left\{ \frac{1}{2} |\langle h, Th \rangle| : h := \sqrt{2} g_0 \in \mathcal{H}, \|h\|^2 = \frac{1}{2} \right\} = \frac{1}{2} w_{cl}(T). \end{aligned}$$

On the other hand, given  $g_0, g_1 \in \mathcal{H}$ , with  $\|g_0\|^2 + \|g_1\|^2 = 1$ , we can repeat the idea of Popescu, setting  $f_\theta := g_0 + e^{i\theta} g_1$  for  $\theta \in [0, 2\pi]$ . Then



$$\begin{aligned} \int_0^{2\pi} \|f_\theta\|^2 d\theta &= \int_0^{2\pi} \langle g_0 + e^{i\theta} g_1, g_0 + e^{i\theta} g_1 \rangle d\theta \\ &= \|g_0\|^2 \int_0^{2\pi} d\theta + \langle g_0, g_1 \rangle \int_0^{2\pi} e^{-i\theta} d\theta + \langle g_1, g_0 \rangle \int_0^{2\pi} e^{i\theta} d\theta \\ &\quad + \|g_1\|^2 \int_0^{2\pi} d\theta \\ &= 2\pi(\|g_0\|^2 + \|g_1\|^2) = 2\pi, \end{aligned}$$

and

$$\begin{aligned} \int_0^{2\pi} \langle f_\theta, Tf_\theta \rangle e^{i\theta} d\theta &= \int_0^{2\pi} \langle g_0 + e^{i\theta} g_1, T(g_0 + e^{i\theta} g_1) \rangle e^{i\theta} d\theta \\ &= \int_0^{2\pi} \langle g_0, Tg_0 \rangle e^{i\theta} d\theta + \int_0^{2\pi} \langle g_0, Tg_1 \rangle d\theta + \int_0^{2\pi} \langle g_1, Tg_0 \rangle e^{2i\theta} d\theta + \int_0^{2\pi} \langle g_1, Tg_1 \rangle e^{i\theta} d\theta \\ &= 2\pi \langle g_0, Tg_1 \rangle. \end{aligned}$$

So, using the fact that  $|\langle h, Th \rangle| \leq w_{cl}(T) \|h\|^2$ , we get

$$|\langle g_0, Tg_1 \rangle| \leq \frac{1}{2\pi} \int_0^{2\pi} |\langle f_\theta, Tf_\theta \rangle e^{i\theta}| d\theta \leq \frac{1}{2\pi} w_{cl}(T) \int_0^{2\pi} \|f_\theta\|^2 d\theta = w_{cl}(T).$$

For the special choice  $\mathcal{H} = \mathbb{C}$  and  $T_a z := az$  for some fixed  $a \in \mathbb{C}^*$ , we get  $w_B(T_a) = \frac{1}{2} w_{cl}(T_a)$ . Indeed,

$$\begin{aligned} w_B(T_a) &= \sup \left\{ |\langle z, aw \rangle| : \|z\|^2 + \|w\|^2 = 1 \right\} = \left| \begin{matrix} z = e^{i\phi} \sin t, w = e^{i\psi} \cos t, \\ \phi, \psi, t \in [0, 2\pi] \end{matrix} \right| \\ &= \frac{|a|}{2} \sup \{ |\sin(2t)|, t \in [0, 2\pi] \} = \frac{|a|}{2} = \frac{w_{cl}(T_a)}{2}. \end{aligned}$$

It is an open problem to check if the equality  $w_B(T) = w_F(T)$  can hold.

We end this Section with an observation that the joint boolean spectral radius degenerates.

**Proposition 11** *The joint boolean spectral radius is always 0.*

**Proof** Since the boolean creation operators satisfy  $B_j^* B_k^* = 0$  for any  $j, k$ , we have

$$\begin{aligned} r_B(T_1, \dots, T_d) &= r(B_1^* \otimes T_1^* + \dots + B_d^* \otimes T_d^*) = \lim_{m \rightarrow \infty} \left\| \left( \sum_{j=1}^d B_j^* \otimes T_j^* \right)^m \right\|^{1/m} \\ &= \lim_{m \rightarrow \infty} \left\| \sum_{j_1, \dots, j_m=1}^d B_{j_1}^* \dots B_{j_m}^* \otimes T_{j_1}^* \dots T_{j_m}^* \right\|^{1/m} = 0. \end{aligned}$$

□

### 4 Joint monotone numerical and spectral radii

We consider the model of the *monotone independence* on the *discrete weakly monotone* Fock space  $\mathcal{F}_{WM}(H)$ , defined in [13]. This space is built upon a  $d$ -dimensional Hilbert space  $H$  with a given orthonormal basis  $\{e_j : 1 \leq j \leq d\}$ , as a closed subspace of the full Fock space  $\mathcal{F}(H)$ , spanned by the vacuum vector  $\Omega := e_0$  and the simple tensors  $e_{j_k} \otimes \dots \otimes e_{j_1}$ , where the indices are in weakly monotonic order:  $1 \leq j_1 \leq \dots \leq j_k$ . By the standard convention we identify  $e_0 \otimes h = h \otimes e_0 = h$  for any  $h \in \mathcal{F}_{WM}(H)$ .

The creation operator  $M_j^*$  by the vector  $e_j$  is defined as follows:

$$M_j^*(e_{j_k} \otimes \dots \otimes e_{j_1}) = e_j \otimes e_{j_k} \otimes \dots \otimes e_{j_1} \quad \text{if } j \geq j_k \geq \dots \geq j_1 \geq 1,$$

$$M_j^*(e_{j_k} \otimes \dots \otimes e_{j_1}) = 0 \quad \text{if } j < j_k.$$

The annihilation operator is defined as

$$M_j(e_j) = \Omega, \quad M_j(e_{j_k} \otimes \dots \otimes e_{j_1}) = \delta_{jk} e_{j_{k-1}} \otimes \dots \otimes e_{j_1},$$

and they are mutually adjoint:  $(M_k)^\star = M_k^*$ .

It is useful to introduce the following orthogonal projections. By definition  $P_0$  is the orthogonal projection onto  $e_0 = \Omega$  and for  $m \geq 1$ :

$$P_m(e_{j_k} \otimes \dots \otimes e_{j_1}) = e_{j_k} \otimes \dots \otimes e_{j_1} \quad \text{if } m = j_k \geq \dots \geq j_1 \geq 1,$$

$$P_m(e_{j_k} \otimes \dots \otimes e_{j_1}) = 0 \quad \text{if } m \neq j_k \text{ and } m \geq 1,$$

$$Q_m := P_0 + P_1 + \dots + P_m.$$

Then  $Q_m$  is the orthogonal projection onto the span of  $e_0 = \Omega$  and vectors of the form  $e_{j_k} \otimes \dots \otimes e_{j_1}$  with  $j_k \leq m$ , and  $P_m$  is the orthogonal projection onto the span of vectors of the form  $e_{j_k} \otimes \dots \otimes e_{j_1}$  with  $j_k = m, k \geq 1$ .

The weakly monotone creation and annihilation operators satisfy the following commutation relations

$$M_j^* M_k^* = M_k M_j = 0 \quad \text{if } j < k \tag{6}$$

$$M_j M_k^* = 0 \quad \text{if } j \neq k \tag{7}$$

$$M_k M_k^* = Q_k \quad \text{for } 1 \leq k \leq d \tag{8}$$

$$M_k^* M_k = P_k \quad \text{for } 1 \leq k \leq d. \tag{9}$$

In particular, for  $j < k$  we have

$$M_j M_k M_k^* M_j^* = M_j Q_k M_j^* = Q_j. \tag{10}$$

**Remark 12** The monotone creation operators are bounded and generate  $*$ -subalgebras which are monotone independent in the sense of Muraki [7, 8] (for the proof see [13] and [4]).

In this setting the *joint monotone numerical radius* is defined as

$$w_M(T_1, \dots, T_d) := w_{cl}(M_1^* \otimes T_1^* + \dots M_d^* \otimes T_d^*)$$

and the *joint monotone spectral radius* of  $(T_1, \dots, T_d)$  is defined as

$$r_M(T_1, \dots, T_d) := r_{cl}(M_1^* \otimes T_1^* + \dots M_d^* \otimes T_d^*).$$

To provide explicit formulas for the joint monotone numerical and spectral radii we introduce some operations on weakly monotone sequences. For each  $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  we define

$$\begin{aligned} \mathcal{M}_k &:= \{(i_k, \dots, i_1) \in \mathbb{N}^k : d \geq i_k \geq \dots \geq i_1 \geq 1\}, k \geq 1 \\ \mathcal{M}_0 &:= \{0\} \\ \mathcal{M}_{-1} &:= \{\emptyset\} \end{aligned}$$

Note that each  $\mathcal{M}_k$  with  $k \geq 1$  is finite with cardinality  $\binom{k+d-1}{k}$ . We also set  $\mathcal{M} := \bigcup_{k \geq 0} \mathcal{M}_k$  to be the union of all  $\mathcal{M}_k$  for  $k \geq 0$  and  $\widetilde{\mathcal{M}} := \mathcal{M} \cup \mathcal{M}_{-1}$ .

There is a natural comparison relation  $R$  of the sequences from  $\mathcal{M}$ , namely for  $\alpha = (i_k, \dots, i_1) \in \mathcal{M}_k$  and  $\beta = (j_m, \dots, j_1) \in \mathcal{M}_m$  with  $k, m \geq 1$  we put

$$\begin{aligned} (\alpha, \beta) \in R &\quad \text{if } i_1 \geq j_m \\ (\alpha, \beta) \notin R &\quad \text{if } i_1 < j_m \\ (\alpha, 0) \in R &\quad \text{if } \alpha \in \mathcal{M}_k, k \geq 1. \end{aligned}$$

Observe that this relation is neither symmetric nor antisymmetric.

For convenience we shall write  $\alpha \geq \beta$  iff  $(\alpha, \beta) \in R$ .

**Definition 13** The *weakly monotone concatenation* of such sequences is defined on  $\widetilde{\mathcal{M}}$  as follows: Let  $\alpha = (i_k, \dots, i_1) \in \mathcal{M}_k$  and  $\beta = (j_m, \dots, j_1) \in \mathcal{M}_m$  with  $k, m \geq 1$ , then

$$\begin{aligned} \alpha\beta &:= (i_k, \dots, i_1, j_m, \dots, j_1) \quad \text{if } \alpha \geq \beta \\ \alpha\beta &:= \emptyset \quad \text{if } (\alpha, \beta) \notin R, \\ 0\alpha = \alpha 0 &:= \alpha \\ \emptyset\alpha = \alpha\emptyset &:= \emptyset. \end{aligned}$$

If  $\alpha \in \mathcal{M}_k, \beta \in \mathcal{M}_m$  with  $k, m \geq 0$  and  $\alpha \geq \beta$ , then  $\alpha\beta \in \mathcal{M}_{k+m}$ .

In particular, for  $j \in \mathcal{M}_1$  and  $\alpha = (i_k, \dots, i_1) \in \mathcal{M}_k$  we shall have  $j\alpha = (j, i_k, \dots, i_1)$  if  $j \geq \alpha$ . As one can see for this concatenation  $0$  plays the role of neutral element and the empty set  $\emptyset$  behaves like  $0$  in multiplication of numbers.

Then, with the notation

$$e_\alpha = \begin{cases} 0 & \text{if } \alpha = \emptyset, \\ \Omega & \text{if } \alpha \in \mathcal{M}_0, \\ e_{i_k} \otimes \dots \otimes e_{i_1} & \text{if } \alpha = (i_k, \dots, i_1) \in \mathcal{M}_k, k \geq 1, \end{cases}$$

the set  $\{e_\alpha : \alpha \in \mathcal{M}\}$  is an orthonormal basis of  $\mathcal{F}_{WM}(H)$ . We shall extend this notation to the operators: if  $T_1, \dots, T_d \in B(\mathcal{H})$  are given and  $\alpha \in \mathcal{M}$ , then

$$T_\alpha = \begin{cases} I & \text{if } \alpha \in \mathcal{M}_0, \\ T_{i_k} \otimes \dots \otimes T_{i_1} & \text{if } \alpha = (i_k, \dots, i_1) \in \mathcal{M}_k, k \geq 1. \end{cases}$$

Now we are ready to describe the explicit formula for the joint monotone numerical radius of  $d$ -tuple of operators.

**Proposition 14** *Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of bounded operators on a Hilbert space  $\mathcal{H}$ . Their joint monotone numerical radius can be computed as*

$$w_M(T_1, \dots, T_d) = \sup \left\{ \left| \sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} \langle g_\alpha | T_j g_{j\alpha} \rangle \right| : g_\alpha \in \mathcal{H}, \sum_{\alpha \in \mathcal{M}} \|g_\alpha\|^2 = 1 \right\}. \tag{11}$$

**Proof** We express  $h \in \mathcal{F}_M(H) \otimes \mathcal{H}$  as  $h = \sum_{\alpha \in \mathcal{M}} e_\alpha \otimes g_\alpha$  with  $g_\alpha \in \mathcal{H}$ . Then

$$\begin{aligned} w_M(T_1, \dots, T_d) &= \sup \{ |\langle \sum_{j=1}^d M_j^* \otimes T_j^* h, h \rangle| : h \in \mathcal{F}_M(H) \otimes \mathcal{H}, \|h\| = 1 \} \\ &= \sup \left\{ \left| \langle \sum_{j=1}^d M_j^* \otimes T_j^* (\sum_{\alpha \in \mathcal{M}} e_\alpha \otimes g_\alpha), (\sum_{\beta \in \mathcal{M}} e_\beta \otimes g_\beta) \rangle \right| : g_\alpha \in \mathcal{H}, \|\sum_{\alpha \in \mathcal{M}} e_\alpha \otimes g_\alpha\| = 1 \right\}. \end{aligned}$$

Since

$$\begin{aligned} \|\sum_{\alpha \in \mathcal{M}} e_\alpha \otimes g_\alpha\|^2 &= \langle \sum_{\alpha \in \mathcal{M}} e_\alpha \otimes g_\alpha, \sum_{\beta \in \mathcal{M}} e_\beta \otimes g_\beta \rangle \\ &= \sum_{\alpha, \beta \in \mathcal{M}} \langle e_\alpha, e_\beta \rangle \langle g_\alpha, g_\beta \rangle = \sum_{\alpha \in \mathcal{M}} \|g_\alpha\|^2, \end{aligned}$$

and

$$\begin{aligned} &\left\langle \sum_{j=1}^d M_j^* \otimes T_j^* (\sum_{\alpha \in \mathcal{M}} e_\alpha \otimes g_\alpha), (\sum_{\beta \in \mathcal{M}} e_\beta \otimes g_\beta) \right\rangle = \\ &= \sum_{j=1}^d \sum_{\alpha, \beta \in \mathcal{M}} \langle M_j^* e_\alpha, e_\beta \rangle \langle T_j^* g_\alpha, g_\beta \rangle \\ &= \sum_{j=1}^d \sum_{\alpha, \beta \in \mathcal{M}} \langle e_{j\alpha}, e_\beta \rangle \langle T_j^* g_\alpha, g_\beta \rangle = \sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} \langle g_\alpha, T_j g_{j\alpha} \rangle, \end{aligned}$$

the formula (11) follows. □

Note that in the summation over  $\alpha \in \mathcal{M}$  the nonzero terms might be only for those  $\alpha$  for which  $j\alpha \neq \emptyset$ , i.e. when  $j \geq i_k$  if  $\alpha = (i_k, \dots, i_1)$ . For example, if  $j = 1$  then it must be  $\alpha = 0$  or  $\alpha = (1, \dots, 1)$ , so in this case the summation over  $\alpha \in \mathcal{M}$  reduces to the summation over  $k = 0, 1, \dots$ , the length of the sequence of 1's.

To study the properties of the joint monotone numerical radius we need the following lemma.

**Lemma 1** For any  $T_1, \dots, T_d \in B(\mathcal{H})$  we have

$$\left\| \sum_{j=1}^d M_j^* \otimes T_j^* \right\| = \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d T_j T_j^* \right\|^{\frac{1}{2}}. \tag{12}$$

**Proof** Using (7) and (8) and the definition of  $Q_j$ , we observe that

$$\begin{aligned} \left\| \sum_{j=1}^d M_j^* \otimes T_j^* \right\| &= \left\| \left( \sum_{j=1}^d M_j \otimes T_j \right) \left( \sum_{k=1}^d M_k^* \otimes T_k^* \right) \right\|^{\frac{1}{2}} = \left\| \sum_{j,k=1}^d \underbrace{M_j M_k^*}_{=\delta_{jk} Q_j} \otimes T_j T_k^* \right\|^{\frac{1}{2}} \\ &= \left\| \sum_{j=1}^d Q_j \otimes T_j T_j^* \right\|^{\frac{1}{2}} = \left\| \sum_{j=1}^d \left( \sum_{m=0}^j P_m \right) \otimes T_j T_j^* \right\|^{\frac{1}{2}} \\ &= \left\| P_0 \otimes \sum_{j=1}^d T_j T_j^* + \sum_{m=1}^d P_m \otimes \sum_{j=m}^d T_j T_j^* \right\|^{\frac{1}{2}} \end{aligned}$$

Now, we use the mutual orthogonality of the orthogonal projections  $P_0, P_1, \dots, P_d$  to get

$$\left\| \sum_{j=1}^d M_j^* \otimes T_j^* \right\| = \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d T_j T_j^* \right\|^{\frac{1}{2}}.$$

□

In analogy to Proposition 6 we have the following properties of the joint monotone numerical radius.

**Proposition 15** We have

- (i)<sub>M</sub>  $w_M(T_1, \dots, T_d) = 0$  if and only if  $T_1 = \dots = T_d = 0$ ,
- (ii)<sub>M</sub>  $\frac{1}{2} \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d T_j T_j^* \right\|^{\frac{1}{2}} \leq w_M(T_1, \dots, T_d) \leq \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d T_j T_j^* \right\|^{\frac{1}{2}}$ ,
- (iii)<sub>M</sub>  $w_M(X^* T_1 X, \dots, X^* T_d X) \leq \|X\|^2 w_M(T_1, \dots, T_d)$  for any bounded operator  $X : \mathcal{H} \rightarrow \mathcal{K}$ .

**Proof** Ad (i)<sub>M</sub>. As in the proof of Proposition 6,  $w_M(T_1, \dots, T_d) = 0$  implies  $\sum_{j=1}^d M_j^* \otimes T_j^* = 0$ . Hence for any  $h = \sum_{k=0}^d e_k \otimes g_k \in \mathcal{F}_B(H) \otimes \mathcal{H}$ , one gets

$$0 = \sum_{j=1}^d M_j^* \otimes T_j^* h = \sum_{j=1}^d \sum_{k=0}^d M_j^* e_k \otimes T_j^* g_k = \sum_{k=0}^d \sum_{j>k} e_j \otimes e_k \otimes T_j^* g_0.$$

Hence  $T_j = 0$  for any  $j = 1, \dots, d$ .

Ad (ii)<sub>M</sub>. Due to  $w_{cl}(T) \leq \|T\|$  and Lemma 1 we have

$$w_M(T_1, \dots, T_d) = w_{cl}\left(\sum_{j=1}^d M_j^* \otimes T_j^*\right) \leq \left\| \sum_{j=1}^d M_j^* \otimes T_j^* \right\| = \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d T_j T_j^* \right\|^{\frac{1}{2}}.$$

The other part follows from  $\frac{1}{2}\|T\| \leq w_{cl}(T)$ .

Ad (iii)<sub>M</sub>. Let  $X : \mathcal{H} \rightarrow \mathcal{K}$  be a bounded operator and let  $g_0, \dots, g_d \in \mathcal{H}$  satisfy  $\sum_{k=0}^d \|g_k\|^2 = 1$ . Define  $C := \sqrt{\sum_{k=0}^d \|Xg_k\|^2}$  and  $h_k := \frac{1}{C}Xg_k$ ,  $k = 0, 1, \dots, d$ . Then  $\sum_{k=0}^d \|h_k\|^2 = 1$  and  $C \leq \|X\|$ . Consequently,

$$\begin{aligned} w_M(X^*T_1X, \dots, X^*T_dX) &= \sup \left\{ \left| \langle Xg_0, \sum_{j=1}^d T_j Xg_j \rangle \right| : \sum_{j=0}^d \|g_j\|^2 = 1 \right\} \\ &\leq \sup \left\{ \left| \langle Ch_0, \sum_{j=1}^d T_j Ch_j \rangle \right| : \sum_{j=0}^d \|h_j\|^2 = 1 \right\} \\ &= C^2 w_M(T_1, \dots, T_d) \leq \|X\|^2 w_M(T_1, \dots, T_d). \end{aligned}$$

□

**Remark 16** The properties (i)<sub>M</sub>, (ii)<sub>M</sub> and (i)<sub>M</sub> show that the joint monotone numerical radius is a norm on  $B(\mathcal{H})^d$ , which, by (ii)<sub>M</sub>, is actually equivalent to the maximum, over  $1 \leq m \leq d$ , of the operator norms of the operator row matrices  $[T_m, \dots, T_d]$ .

**Example 17** For  $d = 1$  and  $T \in B(\mathcal{H})$  we get  $w_M(T) = w_F(T) = w_{cl}(T)$ .

This follows immediately from the fact that for  $d = 1$  the set  $\mathcal{M}_k$  consists of the unique term  $(1, \dots, 1)$  ( $k$ -times). Hence,

$$w_M(T) = \sup \left\{ \left| \sum_{k=0}^{\infty} \langle g_k, Tg_{k+1} \rangle \right| : \sum_{k=0}^{\infty} \|g_k\|^2 = 1 \right\},$$

which is exactly the same as the joint free numerical radius for a single operator, and which in turn was shown to be equal to the classical numerical radius (see [9, Sect. 1]).

The joint monotone spectral radius can be computed similarly to the (free) joint spectral radius of Popescu.

**Proposition 18** *The joint monotone spectral radius satisfies*

$$r_M(T_1, \dots, T_d) = \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d \sum_{\alpha \in \mathcal{M}_k^j} T_\alpha T_\alpha^* \right\|^{1/2k},$$

where

$$\mathcal{M}_k^j := \{ \alpha = (i_k, \dots, i_1) \in \mathcal{M}_k : i_k \geq \dots \geq i_1 = j \} \tag{13}$$

is a set of monotone sequences of length  $k$ , starting from  $j$ .

**Proof** First observe that

$$\begin{aligned} \left( \sum_{j=1}^d M_j^* \otimes T_j^* \right)^k &= \sum_{j_1, \dots, j_k=1}^d M_{j_k}^* \dots M_{j_1}^* \otimes T_{j_k}^* \dots T_{j_1}^* \\ &= \sum_{j_k \geq \dots \geq j_1 \geq 1} M_{j_k}^* \dots M_{j_1}^* \otimes T_{j_k}^* \dots T_{j_1}^* = \sum_{\alpha \in \mathcal{M}_k} M_\alpha^* \otimes T_\alpha^*, \end{aligned}$$

due to (6). This also implies

$$\begin{aligned} &\left( \sum_{m=1}^d M_m \otimes T_m \right)^k \left( \sum_{j=1}^d M_j^* \otimes T_j^* \right)^k \\ &= \sum_{m_1, \dots, m_k=1}^d \sum_{j_1, \dots, j_k=1}^d M_{m_1} \dots M_{m_k} M_{j_k}^* \dots M_{j_1}^* \otimes T_{m_1} \dots T_{m_k} T_{j_k}^* \dots T_{j_1}^* \\ &= \sum_{1 \leq m_1 \leq \dots \leq m_k} \sum_{j_k \geq \dots \geq j_1 \geq 1} M_{m_1} \dots M_{m_k} M_{j_k}^* \dots M_{j_1}^* \otimes T_{m_1} \dots T_{m_k} T_{j_k}^* \dots T_{j_1}^* \\ &= \sum_{j_k \geq \dots \geq j_1 \geq 1} M_{j_1} \dots M_{j_k} M_{j_k}^* \dots M_{j_1}^* \otimes T_{j_1} \dots T_{j_k} T_{j_k}^* \dots T_{j_1}^* \\ &= \sum_{j_k \geq \dots \geq j_1 \geq 1} Q_{j_1} \otimes T_{j_1} \dots T_{j_k} T_{j_k}^* \dots T_{j_1}^* \\ &= \sum_{m=1}^d Q_m \otimes T_m \left( \sum_{m \leq j_2 \leq \dots \leq j_k \leq d} T_{j_2} \dots T_{j_k} T_{j_k}^* \dots T_{j_2}^* \right) T_m^* \\ &= \sum_{m=1}^d (Q_1 + P_2 + \dots + P_m) \otimes T_m \left( \sum_{m \leq j_2 \leq \dots \leq j_k \leq d} T_{j_2} \dots T_{j_k} T_{j_k}^* \dots T_{j_2}^* \right) T_m^* \\ &= Q_1 \otimes T_1 \left( \sum_{1 \leq j_2 \leq \dots \leq j_k \leq d} T_{j_2} \dots T_{j_k} T_{j_k}^* \dots T_{j_2}^* \right) T_1^* + \\ &+ \sum_{m=2}^d P_m \otimes \sum_{i=m}^d T_i \left( \sum_{i \leq j_2 \leq \dots \leq j_k \leq d} T_{j_2} \dots T_{j_k} T_{j_k}^* \dots T_{j_2}^* \right) T_i^* \end{aligned}$$

Since the projections  $Q_1, P_2, \dots, P_d$  are mutually orthogonal, we obtain

$$\begin{aligned} & \left\| \left( \sum_{m=1}^d M_m \otimes T_m \right)^k \left( \sum_{j=1}^d M_j^* \otimes T_j^* \right)^k \right\| \\ &= \max_{1 \leq m \leq d} \left\| \sum_{j_1=m}^d \sum_{j_1 \leq j_2 \leq \dots \leq j_k \leq d} T_{j_1} T_{j_2} \dots T_{j_k} T_{j_k}^* \dots T_{j_2}^* T_{j_1}^* \right\|. \end{aligned}$$

Therefore we can write

$$\begin{aligned} r_M(T_1, \dots, T_d) &= r_{cl} \left( \sum_{j=1}^d M_j^* \otimes T_j^* \right) = \lim_{k \rightarrow \infty} \left\| \left( \sum_{j=1}^d M_j^* \otimes T_j^* \right)^k \right\|^{\frac{1}{k}} \\ &= \lim_{k \rightarrow \infty} \left\| \left( \sum_{m=1}^d M_m \otimes T_m \right)^k \left( \sum_{j=1}^d M_j^* \otimes T_j^* \right)^k \right\|^{1/2k} \\ &= \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\| \sum_{j_1=m}^d \sum_{j_1 \leq j_2 \leq \dots \leq j_k \leq d} T_{j_1} T_{j_2} \dots T_{j_k} T_{j_k}^* \dots T_{j_2}^* T_{j_1}^* \right\|^{1/2k} \\ &= \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d \sum_{\alpha \in \mathcal{M}'_k} T_\alpha T_\alpha^* \right\|^{1/2k}. \end{aligned}$$

□

### 4.1 Example: joint monotone numerical radius for weakly monotone annihilation operators

We shall compute the joint monotone numerical radius for  $d$ -tuple of weakly monotone annihilators  $M_1, \dots, M_d$ . An upper bound for  $w_M(M_1, \dots, M_d)$  is easily obtained.

**Proposition 19** For any  $d \geq 2$  we have

$$w_M(M_1, \dots, M_d) \leq d.$$

**Proof** Using the fact that the norm of any creation operator  $M_j$  is 1 and the Cauchy-Schwarz inequality, we have the following

$$\begin{aligned} \left| \sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} \langle g_\alpha | M_j g_{j\alpha} \rangle \right| &\leq \sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} |\langle M_j^* g_\alpha | g_{j\alpha} \rangle| \leq \sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} \|g_\alpha\| \|g_{j\alpha}\| \\ &\leq \sum_{j=1}^d \left( \sum_{\alpha \in \mathcal{M}} \|g_\alpha\|^2 \right)^{\frac{1}{2}} \left( \sum_{\alpha \in \mathcal{M}} \|g_{j\alpha}\|^2 \right)^{\frac{1}{2}} \leq \sum_{j=1}^d 1 = d \end{aligned}$$



for any family  $(g_\alpha)_{\alpha \in \mathcal{M}}$  satisfying  $\sum_{\alpha \in \mathcal{M}} \|g_\alpha\|^2 = 1$ . □

This is a rough estimate and the following calculations show that perhaps the optimal one would be related to  $\sqrt{d}$ . To support this claim, we provide lower estimates of that sort.

It will be instructive to treat first the case  $d = 2$ .

**Proposition 20** *For two weakly monotone annihilation operators  $T_1 = M_1$  and  $T_2 = M_2$ , we have  $w_M(M_1, M_2) \geq \frac{4}{3\sqrt{3}}\sqrt{2} \approx 0,77 \cdot \sqrt{2}$ .*

**Proof** For the proof we shall analyze in details the expression

$$\sum_{j=1}^2 \sum_{\alpha} \langle g_\alpha | M_j g_{j\alpha} \rangle.$$

For this purpose we shall use the notation  $\alpha = [k, m]$  if  $\alpha = (2, \dots, 2, 1, \dots, 1)$  is a sequence of  $m$  times the digit 2 followed by  $k$  times the digit 1. Then we have explicit formulas for the composition  $j\alpha$  with  $j = 1, 2$ , namely  $2[k, m] = [k, m + 1]$  if  $k, m \geq 0$ ,  $1[k, 0] = [k + 1, 0]$  if  $k \geq 1$  and  $1[k, m] = 0$  if  $m \geq 1$ .

We shall also denote the orthonormal basis  $e_\alpha = e_{m,k}$  if  $\alpha = [k, m]$ ; in particular  $\langle e_{s,r} | e_{u,t} \rangle = 1$  if and only if  $s = u, r = t$ . Each vector  $g_{[k,m]}$  can be written in the orthonormal basis as  $g_{[k,m]} = \sum_{r,s \geq 0} g_{[k,m]}^{r,s} e_{s,r}$  and then we get

$$\sum_{\alpha} \|g_\alpha\|^2 = \sum_{k,m,r,s \geq 0} |g_{[k,m]}^{r,s}|^2 \tag{14}$$

and

$$\begin{aligned} \sum_{j=1}^2 \sum_{\alpha} \langle g_\alpha | M_j g_{j\alpha} \rangle &= \sum_{j=1}^2 \sum_{k,m \geq 0} \sum_{r,s \geq 0} \sum_{t,u \geq 0} g_{[k,m]}^{r,s} g_{j[k,m]}^{t,u} \langle e_{s,r} | M_j e_{u,t} \rangle \\ &= \sum_{j=1}^2 \sum_{k,m \geq 0} \sum_{r,s \geq 0} \sum_{t,u \geq 0} g_{[k,m]}^{r,s} g_{j[k,m]}^{t,u} \langle M_j^* e_{s,r} | e_{u,t} \rangle. \end{aligned} \tag{15}$$

We split this summation into two parts.

- For  $j = 1$  we have necessarily  $m = 0$  and  $\langle M_1^* e_{s,r} | e_{u,t} \rangle = \langle e_{0,r+1} | e_{u,t} \rangle$  if  $s = 0$  and  $\langle M_j^* e_{s,r} | e_{u,t} \rangle = 0$  if  $s \geq 1$ . Hence for  $j = 1$  we get only the following nonzero term

$$\sum_{k \geq 0} \sum_{r \geq 0} \sum_{t,u \geq 0} g_{[k,0]}^{r,s} g_{[k+1,0]}^{t,u} \langle e_{0,r+1} | e_{u,t} \rangle = \sum_{k,r \geq 0} g_{[k,0]}^{r,0} g_{[k+1,0]}^{r+1,0}.$$

- For  $j = 2$  and  $m \geq 0$  in a similar manner we get

$$\langle M_2^* e_{s,r} | e_{u,t} \rangle = \langle e_{s+1,r} | e_{u,t} \rangle = 1 \text{ if and only if } u = s + 1, t = r.$$

Hence the nonzero term in (15) is

$$\sum_{k \geq 0} \sum_{m \geq 1} \sum_{r \geq 0} \sum_{t, u \geq 0} g_{[k,m]}^{r,s} g_{[k,m+1]}^{t,u} \langle e_{s+1,r} | e_{u,t} \rangle = \sum_{k \geq 0} \sum_{m \geq 1} \sum_{r,s \geq 0} g_{[k,m]}^{r,s} g_{[k,m+1]}^{r,s+1}.$$

So we get

$$\begin{aligned} \sum_{j=1}^2 \sum_{\alpha} \langle g_{\alpha} | M_j g_{j\alpha} \rangle &= \sum_{j=1}^2 \sum_{k,m \geq 0} \sum_{r,s \geq 0} \sum_{t,u \geq 0} g_{[k,m]}^{r,s} g_{j[k,m]}^{t,u} \langle M_j^* e_{s,r} | e_{u,t} \rangle \\ &= \sum_{k,r \geq 0} g_{[k,0]}^{r,0} g_{[k+1,0]}^{r+1,0} + \sum_{k,m \geq 0} \sum_{r,s \geq 0} g_{[k,m]}^{r,s} g_{[k,m+1]}^{r,s+1}. \end{aligned}$$

Now we consider special vectors, defined for a constant  $x \in (0, 1)$  and  $k, m, r, s \geq 0$ :

$$g_{[k,m]}^{rs} = \begin{cases} x^{m+k}, & \text{if } r = k, s = m \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

We have that in this case the quantity in (14) equals

$$\sum_{\alpha \in \mathcal{M}} \|g_{\alpha}\|^2 = \sum_{k,m,r,s=0}^{\infty} |g_{[k,m]}^{rs}|^2 = \sum_{k,m=0}^{\infty} x^{2k} x^{2m} = \frac{1}{(1-x^2)^2}. \tag{17}$$

On the other hand, we have

$$\begin{aligned} &\sum_{j=1}^2 \sum_{\alpha} \langle g_{\alpha} | M_j g_{j\alpha} \rangle \\ &= \sum_{k \geq 0} x^{2k+1} + \sum_{k \geq 0} \sum_{m \geq 0} x^{m+k} x^{m+k+1} \\ &= \frac{x}{1-x^2} + \frac{x}{(1-x^2)^2} = \frac{2x-x^3}{(1-x^2)^2}. \end{aligned} \tag{18}$$

Hence maximizing the quotient of (18) by (17) we get the function  $f(x) := 2x - x^3$ , which in  $(0, 1)$  has the local maximum  $\frac{4}{3} \sqrt{\frac{2}{3}}$  at  $x = \sqrt{\frac{2}{3}}$ .

To sum up, we have

$$w_M(M_1, M_2) \geq \sup_{x \in (0,1)} f(x) = \frac{4}{3} \sqrt{\frac{2}{3}}.$$

□

We are now ready to treat the general case.

**Proposition 21** For  $d \geq 3$  and the monotone annihilators  $T_j = M_j, j = 1, \dots, d$ , we have

$$w_M(M_1, \dots, M_j) \geq \frac{5}{9} \sqrt{d}.$$

**Proof** We generalize the idea from the proof of the case  $d = 2$ . For a fixed  $d \geq 3$ , any  $\alpha \in \mathcal{M}$  is of the form

$$\alpha = (\underbrace{d, \dots, d}_{k_d\text{-times}}, \underbrace{d-1, \dots, d-1}_{k_{d-1}\text{-times}}, \dots, \underbrace{2, \dots, 2}_{k_2\text{-times}}, \underbrace{1, \dots, 1}_{k_1\text{-times}}),$$

where  $k_d, \dots, k_1 \geq 0$ . In the sequel, such an element will be simply denoted by  $\alpha = [k_d, \dots, k_1]$ . For arbitrary (but fixed)  $x \in (0, 1)$  we shall consider the vectors

$$g_\alpha = g_{[k_d, \dots, k_1]} := x^{k_d + \dots + k_1} e_\alpha,$$

for which we have

$$\sum_{\alpha \in \mathcal{M}} \|g_\alpha\|^2 = \sum_{k_d, \dots, k_1 \geq 0} x^{2(k_d + \dots + k_1)} = \frac{1}{(1-x^2)^d}. \tag{19}$$

Now, we would like to compute

$$\sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} \langle g_\alpha | M_j g_{j\alpha} \rangle. \tag{20}$$

For this purpose observe that, for a fixed  $1 \leq j \leq d$ , if  $\alpha = [k_d, \dots, k_1]$ , then  $j\alpha \neq \emptyset$  if and only if there are no indices bigger than  $j$  in  $\alpha$ , that is if  $k_{j+1} = 0, \dots, k_d = 0$ . In such case  $j\alpha = [0, \dots, 0, k_j + 1, k_{j-1}, \dots, k_1]$  and

$$\begin{aligned} \langle g_\alpha | M_j g_{j\alpha} \rangle &= \langle g_{[0, \dots, 0, k_j, \dots, k_1]} | M_j g_{[0, \dots, 0, k_j + 1, \dots, k_1]} \rangle \\ &= \langle x^{k_j + \dots + k_1} e_{[0, \dots, 0, k_j, \dots, k_1]} | x^{k_j + \dots + k_1 + 1} M_j e_{[0, \dots, 0, k_j + 1, \dots, k_1]} \rangle \\ &= x^{2(k_j + \dots + k_1) + 1} \langle e_{[0, \dots, 0, k_j, \dots, k_1]} | e_{[0, \dots, 0, k_j + 1, \dots, k_1]} \rangle = x^{2(k_j + \dots + k_1) + 1}. \end{aligned}$$

Otherwise  $j\alpha = \emptyset$  and  $\langle g_\alpha | M_j g_{j\alpha} \rangle = 0$ .

This simplifies the expression (20) considerably:

$$\sum_{j=1}^d \sum_{\alpha \in \mathcal{M}} \langle g_\alpha | M_j g_{j\alpha} \rangle = \sum_{j=1}^d \sum_{k_j, \dots, k_1 \geq 0} x^{2(k_j + \dots + k_1) + 1} = \sum_{j=1}^d \frac{x}{(1-x^2)^j}. \tag{21}$$

Taking into account the normalization (19) of the family  $(g_\alpha)_\alpha$ , we arrive to the problem of maximizing the function:

$$f_d(x) := \frac{\sum_{j=1}^d \frac{x}{(1-x^2)^j}}{\frac{1}{(1-x^2)^d}} = \sum_{j=1}^d x(1-x^2)^{d-j} = x \sum_{k=0}^{d-1} (1-x^2)^k \tag{22}$$

over  $(0, 1)$ . Setting  $y = 1 - x^2$  this reduces to the maximization of the function

$$g_d(y) := \sqrt{1-y} \sum_{k=0}^{d-1} y^k = \frac{1-y^d}{\sqrt{1-y}}$$

over  $(0, 1)$ .

Trying to resolve the optimization problem explicitly, one arrives to the equation

$$(2d - 1)y^d - 2dy^{d-1} + 1 = 0, \tag{23}$$

which describes the critical points of  $g_d$ . We shall not go this way. We just observe that the value of  $g_d$  at the points  $y_d = 1 - \frac{1}{d}$  provides the required estimates. Namely,

$$\sup_{y \in (0,1)} g(y) \geq g(1 - \frac{1}{d}) = \sqrt{d}(1 - (1 - \frac{1}{d})^d) \geq \frac{5}{9}\sqrt{d} \quad \text{for } d \geq 3.$$

This shows that the numerical radius satisfies

$$w_M(M_1, \dots, M_d) \geq \sup_{x \in (0,1)} f(x) = \sup_{y \in (0,1)} g(y) \geq \frac{5}{9}\sqrt{d}.$$

□

Let us remark that numerical calculations reveal that the point  $y_d = 1 - \frac{1}{d}$  seems to be a good approximation of  $y_{d,max}$ , at which the maximum of  $g_d$  is achieved. Moreover, for  $3 \leq d \leq 100$ , the relative error appearing when  $g_d(y_{d,max})$  is replaced by  $g_d(y_d)$  is smaller than 2%.

All these calculations and observations have led us to the formulation of the following statement.

**Conjecture 1** *There exists an absolute constant  $c > 0$  such that the joint monotone numerical radius of the  $d$ -tuple of weakly monotone annihilators equals*

$$w_M(M_1, \dots, M_d) = c\sqrt{d}.$$

### 4.2 Example: joint monotone spectral radius for weakly monotone creation operators

Using Proposition 18 we compute the joint monotone spectral radius of  $d$  weakly monotone annihilation operators  $M_1, \dots, M_d$ . In this case we have

$$r_M(M_1, \dots, M_d) = \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d \sum_{\alpha \in \mathcal{M}_k^j} M_\alpha M_\alpha^* \right\|^{1/2k} = 1.$$

Indeed, let us denote by  $c_k^j$  the cardinality of the set  $\mathcal{M}_k^j$ , see (13). Then  $c_{jk} = |\mathcal{M}_k^j| = \binom{k+d-j}{k-1}$ . Since for  $\alpha \in \mathcal{M}_k^j$ , we have  $M_\alpha M_\alpha^* = Q_j$ , the right-hand side above can be written as

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d c_k^j \cdot Q_j \right\|^{1/2k} &= \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\| \sum_{j=m}^d c_k^j \cdot (P_0 + \sum_{i=1}^j P_i) \right\|^{1/2k} \\
 &= \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\{ \left\| Q_m \sum_{j=0}^{d-m} c_k^{m+j} \right\|^{1/2k}, \left\| \sum_{s=1}^{d-m} P_{m+s} \sum_{j=s}^{d-m} c_k^{m+j} \right\|^{1/2k} \right\} \\
 &= \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\{ \left\| Q_m \sum_{j=0}^{d-m} c_k^{m+j} \right\|^{1/2k}, \max_{1 \leq s \leq d-m} \left\| P_{m+s} \sum_{j=s}^{d-m} c_k^{m+j} \right\|^{1/2k} \right\} \\
 &= \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \left\{ \left( \sum_{j=0}^{d-m} c_k^{m+j} \right)^{1/2k}, \max_{1 \leq s \leq d-m} \left( \sum_{j=s}^{d-m} c_k^{m+j} \right)^{1/2k} \right\}
 \end{aligned}$$

We see that the maximum is achieved for  $m = 1$  and the last expression equals

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \max_{1 \leq m \leq d} \max_{0 \leq s \leq d-m} \left\{ \left( \sum_{j=s}^{d-m} c_k^{m+j} \right)^{1/2k} \right\} &= \lim_{k \rightarrow \infty} \left( \sum_{j=1}^d c_k^j \right)^{1/2k} \\
 &= \lim_{k \rightarrow \infty} \left( \sum_{j=1}^d \binom{k+d-j}{k-1} \right)^{1/2k} = \lim_{k \rightarrow \infty} \binom{k+d}{k-1}^{1/2k} = 1.
 \end{aligned}$$

### 5 Concluding remarks and open problems

This paper is a beginning of the studies of joint numerical and spectral radii related to creation operators independent in noncommutative sense: the monotone and boolean ones. We have showed that some properties are analogous as in the work of Popescu for free creation operators. However, especially for the monotone case, which we base on the weakly monotone creation operators, our results differ from the free case.

There is still a lot about the noncommutative joint numerical and spectral radii to be understood. Some open problems appeared in our study, in particular to find the exact formulas for the joint monotone numerical radius of  $d \geq 2$  weakly monotone annihilation operators (at first glance it seems to be related to  $\sqrt{d}$  rather than to  $d$ ). But to see the real power of these objects one should search for analogues of the classical power inequality or von Neumann type inequalities. Finally, it would be interesting to know what are the relations between the three types (free, boolean and monotone) of numerical radii.

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## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

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