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Property ($UW\Pi$) under perturbations

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Abstract
Property ($UW\Pi$), introduced in Berkani and Kachad (Bull Korean Math Soc 49:1027–1040, 2015) and studied more recently in Aiena and Kachad (Acta Sci Math (Szeged) 84:555–571, 2018) may be thought as a variant of Browder's theorem, or Weyl's theorem, for bounded linear operators acting on Banach spaces. In this article we study the stability of this property under some commuting perturbations, as quasi-nilpotent perturbation and, more in general, under Riesz commuting perturbations. We also study the transmission of property ($UW\Pi$) from T to $f(T)$, where f is an analytic function defined on a neighborhood of the spectrum of T. Furthermore, it is shown that this property is transferred from a Drazin invertible operator T to its Drazin inverse S.

Keywords Property ($UW\Pi$) · SVEP

Mathematics Subject Classification 47A53 · 47A10; 47A11

1 Introduction and preliminaries

Let $T \in L(X)$ be a bounded linear operator on an infinite-dimensional complex Banach spaces X, and denote by $\alpha(T)$ and $\beta(T)$ the dimension of the kernel $N(T)$ and the codimension of the range $R(T) = T(X)$, respectively. Let $\Phi_+(X) := \{T \in L(X : \alpha(T) < \infty, T(X)$ is closed$\}$ be the class of all *upper semi-Fredholm operators* and $\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$ the class of all *lower semi-Fredholm*

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operators. If $T \in \Phi_{\pm}^{+}(X) := \Phi_{+}(X) \cup \Phi_{-}(X)$, the index of T is defined by $ind(T) := \alpha(T) - \beta(T)$. If $\Phi(X) := \Phi_{+}(X) \cap \Phi_{-}(X)$, denotes the set of all Fredholm operators, the class of *Weyl operators* is defined by

$$W(X) = \{T \in \Phi(X) : ind(T) = 0\},$$

the class of *upper semi-Weyl* operators is defined by

$$W_{+}(X) = \{T \in \Phi_{+}(X) : ind(T) \leqslant 0\},$$

while the class *lower semi-Weyl* operators is defined by

$$W_{-}(X) = \{T \in \Phi_{-}(X) : ind(T) \geqslant 0\}.$$

Evidently, $W(X) = W_{-}(X) \cap W_{+}(X)$. If T^* denotes the dual of $T \in L(X)$, it is well known that $T \in W_{+}(X)$ (respectively, $T \in W_{-}(X)$) if and only if $T^* \in W_{-}(X^*)$ (respectively, $T^* \in W_{+}(X^*)$). The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *upper semi-Weyl spectrum*, defined by

$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_{+}(X)\},$$

and the *lower semi-Weyl spectrum*, defined by

$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_{-}(X)\}.$$

Two classical quantities in operator theory are defined as follows. The *ascent* of an operator T, is the smallest non-negative integer $p := p(T)$ such that $N(T^p) = N(T^{p+1})$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* of T, is the smallest non-negative integer $q := q(T)$ such that $R(T^q) = R(T^{q+1})$, and if such integer q does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 1.20]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ exactly when λ is a pole of the resolvent of T, see [14, Proposition 50.2]. The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty, \},$$

while the class of all *upper semi-Browder operators* is defined

$$B_{+}(X) := \{T \in \Phi_{+}(X) : p(T) < \infty, \}.$$

Obviously, $B(X) \subseteq W(X)$ and $B_{+}(X) \subseteq W_{+}(X)$, see [1, Chapter 3]. If $\sigma_b(T)$ and $\sigma_{ub}(T)$ denote the *Browder spectrum* and the *upper semi-Browder spectrum*, respectively, then $\sigma_w(T) \subseteq \sigma_b(T)$ and $\sigma_{uw}(T) \subseteq \sigma_{ub}(T)$.

In [6,8,10] Berkani et al. generalize semi-Fredholm operators in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set T= $T_{[0]}$). $T \in L(X)$ is said to be *semi B-Fredholm*,(respectively, *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm*) if, for some integer n, the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). If $T_{[n]}$ is a semi-Fredholm operator, then $T_{[n]}$ is a semi-Fredholm operator for all $m \geqslant n$ ([10]), with the same index of $T_{[n]}$. This enables one to define the index of a semi-B-Fredholm as $ind(T) = ind(T_{[n]})$. Analogously, a bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, upper semi B-Weyl, lower semi B-Weyl) if for some integer $n \geqslant 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl). Analogous definitions are given for semi *B-Browder operators*. The B-Weyl spectrum is defined as

$$\sigma_{bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \; is \; not \; B - Weyl\},$$

and analogously, the upper semi B-Weyl spectrum of T is defined by

$$\sigma_{ubw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \; is \; not \; upper \; semi \; B - Weyl\}.$$

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case the Banach algebra $L(X)$, an operator $T \in L(X)$ is said to be *Drazin* if $p(T) = q(T) < \infty$. Evidently, if T is Drazin invertible then either $\lambda I - T$ is invertible or λ is a pole of the resolvent of T. An operator $T \in L(X)$ is said to be *left Drazin invertible* if $p = p(T) < \infty$ and $R(T^{p+1})$ is closed. The *Drazin spectrum* is then defined as

$$\sigma_d(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

while the *left Drazin spectrum* is defined as

$$\sigma_{ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\}.$$

In the sequel we denote by $\sigma_a(T)$ the approximate point spectrum, defined by $\sigma_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$, where an operator is said to be bounded below if it is injective and has closed range. The classical surjective spectrum of T is denoted by $\sigma_s(T)$.

The following property has a fundamental role in local spectral theory. An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has this property at every $\lambda \in \mathbb{C}$. (See [1] and [15] for more details about this concept). Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the isolated points of the spectrum. Note that

$$p(\lambda I - T) < \infty \Longrightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually,

$$q(\lambda I - T) < \infty \Longrightarrow T^* \text{ has SVEP at } \lambda, \tag{2}$$

see [1, Chapter 2]. Furthermore, if acc F denote the set of all cluster points of $F \subseteq \mathbb{C}$, we have:

$$\lambda \notin \operatorname{acc} \sigma_a(T) \Longrightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually,

$$\lambda \notin \operatorname{acc} \sigma_s(T) \Longrightarrow T^* \text{ has SVEP at } \lambda, \tag{4}$$

see [1, Chapter 2].

Remark 1.1 In [1, Chapter 2] it is shown that the implications above are equivalences if $\lambda I - T$ semi B-Fredholm, in particular semi-Fredholm.

2 Property ($UW\Pi$)

Denote by $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$ the set of all poles of T having finite rank, by $\Pi(T) = \sigma(T) \setminus \sigma_d(T)$ the set of all poles of T, and by $p_{00}^a(T) = \sigma_a(T) \setminus \sigma_{ub}(T)$ the set of all left poles of T having finite rank. It is easy to check that $p_{00}(T) \subseteq p_{00}^a(T)$ for all $T \in L(X)$, and obviously every point of $p_{00}(T)$ is an isolated point of $\sigma(T)$, and hence an isolated point of $\sigma_a(T)$ (since every isolated point of the spectrum belongs to $\sigma_a(T)$). Set

$$\Delta(T) := \sigma(T) \setminus \sigma_w(T) \quad \text{and} \quad \Delta_a(T) := \sigma_a(T) \setminus \sigma_{uw}(T).$$

Let $\Pi_a(T) := \sigma_a(T) \setminus \sigma_{ld}(T)$ be the set of all left poles of the resolvent of T. Obviously, $p_{00}(T) \subseteq \Delta_a(T)$, since each point of $p_{00}(T)$ is an eigenvalue of T and every Browder operator is upper semi-Weyl.

Hereafter, the symbol $\bigsqcup$ stands for disjoint union.

Definition 2.1 A bounded operator $T \in L(X)$ is said to satisfy:

(i) Browder's theorem if $\sigma_w(T) = \sigma_b(T)$ or equivalently $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$, ([13]).

(ii) a-Browder's theorem if $\sigma_{uw}(T) = \sigma_{ub}(T)$ or equivalently $\Delta_a(T) = p_{00}^a(T)$, ([13], [1, Chapter 5]).

(iii) Property ($UW\Pi$) if $\Delta_a(T) = \Pi(T)$, or equivalently $\sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi(T)$, ([2,9]).

Remark 2.2 a-Browder's theorem entails Browder's theorem. Furthermore, a-Browder's theorem is equivalent to generalized a-Browder's theorem, i.e., a-Browder's theorem is equivalent to saying that the equality $\sigma_{ubw}(T) = \sigma_{ld}(T)$ holds, where $\sigma_{ubw}(T)$ denotes the upper semi B-Weyl spectrum, see [5].

Property $(UW\Pi)$ may be characterized as follows ([2]):

Theorem 2.3 *Let $T \in L(X)$. Then the following assertions are equivalent:*

(i) *T satisfies property $(UW\Pi)$;*
(ii) *T satisfies a-Browder's theorem, and $p_{00}^a(T) = \Pi(T)$;*
(iii) *T^* has SVEP at every $\lambda \in \Delta_a(T)$ and $\sigma_b(T) = \sigma_d(T)$.*

Property $(UW\Pi)$ entails some relevant equalities between parts of the spectrum:

Theorem 2.4 *Suppose that T satisfies property $(UW\Pi)$. Then we have:*

(i) $\sigma_w(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_a(T)$.
(ii) $\sigma_{uw}(T) \setminus \sigma_{ubw} = \Pi_a(T) \setminus \Pi(T)$.
(iii) $\sigma(T) \setminus \sigma_w(T) = \Pi(T)$.

Proof If $\lambda \in \sigma_w(T)\setminus\sigma_{uw}(T)$ then $\alpha(\lambda I - T) < \infty$ and $(\lambda I - T)(X)$ is closed. Suppose that $0 < \alpha(\lambda I - T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{uw}(T) = \Pi(T)$. Hence, $p(\lambda I - T) = q(\lambda I - T) < \infty$ and consequently, by [1, Theorem 1.22], $\lambda I - T$ is Browder, in particular $\lambda \notin \sigma_w(T)$, a contradiction. Hence $\alpha(\lambda I - T) = 0$, so $\lambda \notin \sigma_a(T)$, and consequently $\lambda \in \sigma(T) \setminus \sigma_a(T)$.

Conversely, if $\lambda \in \sigma(T)\setminus\sigma_a(T)$ then $\lambda \notin \sigma_{uw}(T)$, since $\sigma_{uw}(T) \subseteq \sigma_a(T)$. Suppose that $\lambda \notin \sigma_w(T)$. Then $\alpha(\lambda I - T) = \beta(\lambda I - T)$, and since $\alpha(\lambda I - T) = 0$ we have $\lambda \notin \sigma(T)$, a contradiction. Hence $\lambda \in \sigma_w(T) \setminus \sigma_{uw}(T)$.

(ii) Suppose that T has property $(UW\Pi)$. Then a-Browder's theorem holds for T, or equivalently generalized a-Browder's theorem holds for T. If $\lambda \in \sigma_{uw}(T) \setminus \sigma_{ubw}$ then $\lambda \notin \sigma_{ubw}(T) = \sigma_{ld}(T)$, and $\lambda \in \sigma_a(T)$, since $\sigma_{uw}(T) \subseteq \sigma_a(T)$. Hence, $\lambda \in \Pi_a(T)$. Since a-Browder's theorem holds for T we then have $\lambda \in \sigma_{ub}(T)$, hence $\lambda \notin \sigma_a(T) \setminus \sigma_{ub}(T) = \pi_{00}^a$, and hence $\lambda \notin \Pi(T)$, by Theorem 2.3.

Conversely, suppose that $\lambda \in \Pi_a(T) \setminus \Pi(T)$. Then $\lambda \in \Pi_a(T)$ and hence $\lambda \notin \sigma_{ld}(T) = \sigma_{ubw}(T)$, since generalized a-Browder's theorem holds for T. On the other hand, $\lambda \notin \Pi(T) = p_{00}^a(T)$, by Theorem 2.3, and since $\lambda \in \sigma_a(T)$, then $\lambda \in \sigma_{ub}(T) = \sigma_{uw}(T)$.

(iii) Write $\sigma(T) = \sigma(T) \setminus \sigma_a(T) \bigsqcup \sigma_a(T)$. Since $\sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi(T)$, by part (i) we then have

$$\begin{aligned}\sigma(T) &= \sigma_w(T) \setminus \sigma_{uw}(T) \bigsqcup \sigma_a(T) \\ &= \sigma_w(T) \setminus \sigma_{uw}(T) \bigsqcup \sigma_{uw}(T) \bigsqcup \Pi(T) = \sigma_w(T) \bigsqcup \Pi(T),\end{aligned}$$

so $\sigma(T) \setminus \sigma_w(T) = \Pi(T)$. □

The following variant of Weyl's theorem and Browder's theorem has been introduced recently by Zariouh [19]. An operator $T \in L(X)$ is said to satisfies *property* (Z_{Π_a}) if $\Delta(T) := \sigma(T) \setminus \sigma_w(T)$ coincides with the set $\Pi_a(T)$ of left poles of T.

The next theorem improves a result of [2], which was proved by assuming that T^* has SVEP.

Theorem 2.5 *If $T \in L(X)$ satisfies property $(UW\Pi)$, then T satisfies property $(Z\Pi_a)$.*

Proof Suppose that T satisfies property $(UW\Pi)$, i.e. $\sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi(T)$. By part (i) and part (iii) of Theorem 2.4 we have

$$\begin{aligned}\sigma(T) = \sigma_w(T) \bigsqcup \Pi(T) &= (\sigma_w(T) \setminus \sigma_{uw}(T)) \bigsqcup \sigma_{uw}(T) \bigsqcup \Pi(T)\\ &= (\sigma(T) \setminus \sigma_a(T)) \bigsqcup \sigma_{uw}(T) \bigsqcup \Pi(T)\\ &= (\sigma(T) \setminus \sigma_a(T)) \bigsqcup [(\sigma_{uw}(T) \setminus \sigma_{ubw}(T)) \bigsqcup \sigma_{ubw}(T)] \bigsqcup \Pi(T).\end{aligned}$$

From part (ii) of Theorem 2.4 we then obtain

$$\begin{aligned}\sigma(T) &= (\sigma_w(T) \setminus \sigma_{uw}(T)) \bigsqcup (\Pi_a(T) \setminus \Pi(T)) \bigsqcup \sigma_{ubw}(T) \bigsqcup \Pi(T)\\ &= (\sigma_w(T) \setminus \sigma_{uw}(T)) \bigsqcup \Pi_a(T) \bigsqcup \sigma_{ubw}(T).\end{aligned}$$

Since $\sigma_{ubw}(T) \subseteq \sigma_{uw}(T)$ then

$$\sigma(T) \subseteq (\sigma_w(T) \setminus \sigma_{uw}(T)) \bigsqcup \Pi_a(T) \bigsqcup \sigma_{uw}(T) = \sigma_w(T) \bigsqcup \Pi_a(T).$$

Trivially, $\sigma_w(T) \bigsqcup \Pi_a(T) \subseteq \sigma(T)$ Consequently, $\sigma(T) = \sigma_w(T) \bigsqcup \Pi_a(T)$ and hence property $(Z\Pi_a)$ holds for T. □

The converse of Theorem 2.5 holds if T^* has SVEP, see [2]. The precise relationship between property $(UW\Pi)$ and property $(Z\Pi_a)$ is described by the following theorem.

Theorem 2.6 *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T satisfies property $(UW\Pi)$;*
(ii) *T satisfies property $(Z\Pi_a)$ and $\sigma_w(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_a(T)$;*
(iii) *T satisfies property $(Z\Pi_a)$ and $\Delta_a(T) \cap \sigma_w(T) = \emptyset$.*

Proof (i) $\Longleftrightarrow$ (ii) Suppose that T satisfies property $(UW\Pi)$. Then T has property $(Z\Pi_a)$ and, by Theorem 2.4, we have $\sigma_w(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_a(T)$. Conversely, if T has property $(Z\Pi_a)$ and $\sigma_w(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_a(T)$, we have $\sigma(T) = \sigma_w(T) \bigsqcup \Pi_a(T)$. Our assumption $\sigma_w(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_a(T)$ entails that $\sigma(T) = [\sigma(T) \setminus \sigma_a(T)] \bigsqcup [\sigma_{uw}(T) \bigsqcup \Pi_a(T)]$, hence $\sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi_a(T)$. From [19, Lemma 2.9] we have that $\Pi_a(T) = \Pi(T)$, so $\sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi(T)$, hence T satisfies property $(UW\Pi)$.

(i) $\Longleftrightarrow$ (iii) Suppose that T satisfies property $(UW\Pi)$. Then T has property $(Z\Pi_a)$. Let $\lambda \in \Delta_a(T) = \Pi(T)$. Then $\lambda I - T \in W_+(X)$ and since $p(\lambda I - T) = q(\lambda I - T) < \infty$ it follows, from [1, Theorem 1.22] that $\lambda I - T \in B(X)$ and in particular $\lambda I - T \in W(X)$, so $\lambda \notin \sigma_w(T)$.

Conversely, suppose that T satisfies property $(Z\Pi_a)$ and $\Delta_a(T) \cap \sigma_w(T) = \emptyset$. If $\lambda \in \Delta_a(T)$ then $\lambda \notin \sigma_w(T)$ and hence $\lambda \notin \sigma_{uw}(T)$. This implies that $\lambda \in \Delta_a(T) = \Pi_a(T) = \Pi(T)$, always by [19, Lemma 2.9]. Therefore T has property $(UW\Pi)$. □

3 Property ($UW\Pi$) and perturbations

An operator $R \in L(X)$ is called a *Riesz operator* if $\lambda I - R$ is Fredholm for all nonzero $\lambda \in \mathbb{C}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. Also every operator K for which, for some $n \in \mathbb{N}$, K^n is finite-dimensional is a Riesz operator. Indeed, K^n is a Riesz operator and hence K is a Riesz operator, see [14]. The spectrum of a Riesz operator is either a finite set or a sequence of eigenvalues which clusters at 0. It is known that if $T \in L(X)$ then $\sigma_b(T + R) = \sigma_b(T)$ for every Riesz operator R commuting with T, see [1, Chapter 2], and $\sigma_{uw}(T + R) = \sigma_{uw}(T)$ see [17].

- Every Riesz operator T having infinite spectrum has property $(UW\Pi)$. To see this, observe first that $\sigma_a(T) = \sigma(T)$, since T^* has SVEP. Further, $\sigma_{uw}(T) = \sigma_d(T) = \{0\}$, so $\Delta_a(T) = \Pi(T)$.
- An operator $T \in L(X)$ is said to be *algebraic* if there exists a nontrivial complex polynomial h such that $h(T)$=0. Examples of algebraic operators are idempotent operators, nilpotent operators and every operator K such that K^n is finite-dimensional for some $n \in \mathbb{N}$.If T is algebraic then $\sigma(T)$ is a finite set of poles [1, Chapter 3], say $\{\lambda_1, \dots, \lambda_n\}$. Therefore, $\sigma_d(T) = \emptyset$ and hence $\Pi(T) = \{\lambda_1, \dots, \lambda_n\}$. Moreover, $\sigma_a(T) = \sigma(T)$, since a pole is always an eigenvalue and $\sigma_{uw}(T) \neq \emptyset$ for every operator $T \in L(X)$. Therefore, for an algebraic operator we have $\Delta_a(T) \neq \Pi(T)$, i.e., for every algebraic operator property $(UW\Pi)$ fails.
- A Riesz operator which has finite spectrum may fail property $(UW\Pi)$. Indeed, every finite-dimensional operator K does not satisfy this property, since K is algebraic.
- If Q be a quasi-nilpotent operator, then

$$Q \text{ has property } (UW\Pi) \iff 0 \notin \sigma_d(Q).$$

  Indeed, if 0 is not a pole then $\{0\} = \sigma(Q) = \sigma_{uw}(Q) = \sigma_a(Q) = \sigma_d(Q)$, since both $\sigma_{uw}(Q)$ and $\sigma_a(Q)$ are non-empty. Therefore $\Delta_a(Q) = \Pi(Q) = \emptyset$, so Q has property $(UW\Pi)$. Conversely, if Q has property $(UW\Pi)$ then, we have $\emptyset = \sigma_a(Q) \setminus \sigma_{uw}(Q) = \Pi(T)$ and since $\sigma(Q) = \{0\}$ it then follows that $\sigma_d(Q) = \{0\}$.
- It should be noted that if K^n is finite-dimensional for some $n \in \mathbb{N}$ and $KT = TK$ then

$$\sigma_d(T) = \sigma_d(T + K),$$

  see [21], or [1, Chapter 3] for an alternative proof. Moreover, if acc F denotes the set of all cluster points of $F \subseteq \mathbb{C}$, then

$$\operatorname{acc} \sigma(T) = \operatorname{acc} \sigma(T + K) \quad \text{and} \quad \operatorname{acc} \sigma_a(T) = \operatorname{acc} \sigma_a(T + K)$$

  see [20].

Theorem 3.1 *Suppose that $T \in L(X)$ has property $(UW\Pi)$ and T^* has SVEP. If $K \in L(X)$ commutes with T and K^n is finite-dimensional for some $n \in \mathbb{N}$, then $T + K$ has property $(UW\Pi)$.*

Proof Since T has property $(UW\Pi)$, by Theorem 2.3 we have $\sigma_b(T) = \sigma_d(T)$. The spectrum $\sigma_b(T)$ is invariant under Riesz commuting perturbations, and being K a Riesz operator, we then have $\sigma_b(T + K) = \sigma_b(T)$, while the equality $\sigma_d(T + K) = \sigma_d(T)$, has been already observed. Therefore, $\sigma_b(T + K) = \sigma_d(T + K)$. Now, K is a Riesz operator and hence also K^* is a Riesz operator. Consequently, $T^* + K^*$ has SVEP, see [1, Chapter 2]. By Theorem 2.3 it then follows that $T + K$ has property $(UW\Pi)$. □

Corollary 3.2 *Suppose that $Q \in L(X)$ is a quasi-nilpotent such that $\alpha(Q) < \infty$. If $K \in L(X)$ commutes with Q and K^n is finite-dimensional for some $n \in \mathbb{N}$, then $Q + K$ satisfies property $(UW\Pi)$.*

Proof If $\alpha(Q) < \infty$ then $0 \in \sigma_d(Q)$, otherwise, by [1, Theorem 1.22], we would have $0 \notin \sigma_b(Q)$, hence $\sigma_b(Q) = \emptyset$, and this is impossible. Therefore, 0 is not a pole of the resolvent of T and, as noted before, Q has property $(UW\Pi)$. Theorem 3.1 then applies, since Q^* has SVEP. □

Property $(UW\Pi)$ is also transmitted to $T + K$ if we assume that the approximate-point spectrum $\sigma_a(T)$ has no isolated points:

Theorem 3.3 *Suppose that $T, K \in L(X)$ commute, and K^n is finite-dimensional for some $n \in \mathbb{N}$. If $\operatorname{iso}\sigma_a(T) = \emptyset$ and T has property $(UW\Pi)$, then $T + K$ has property $(UW\Pi)$.*

Proof Observe that if $\operatorname{iso}\sigma_a(T) = \emptyset$ then $\operatorname{iso}\sigma(T) = \emptyset$, since $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma_a(T)$. Therefore,

$$\sigma(T) = \operatorname{iso}\sigma(T) \cup \operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T),$$

and, similarly, $\sigma_a(T) = \operatorname{acc}\sigma_a(T)$. Since $\operatorname{acc}\sigma_a(T) = \operatorname{acc}\sigma_a(T+K)$, and $\operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T+K)$ see [20], we then have $\sigma_a(T+K) = \sigma_a(T)$ and $\sigma(T+K) = \sigma(T)$. Also the spectrum $\sigma_{uw}(T)$ is invariant under Riesz commuting perturbations, so $\sigma_{uw}(T + K) = \sigma_{uw}(T)$. Hence,

$$\Delta_a(T + K) = \sigma_a(T + K) \setminus \sigma_{uw}(T + K) = \sigma_a(T) \setminus \sigma_{uw}(T) = \Delta_a(T).$$

As observed before, $\sigma_d(T + K) = \sigma_d(T)$, so

$$\Pi(T + K) = \sigma(T + K) \setminus \sigma_d(T + K) = \sigma(T) \setminus \sigma_d(T) = \Pi(T).$$

Therefore, $\Delta_a(T + K) = \Pi(T + K) = \Pi(T) = \Delta_a(T)$. □

Theorem 3.3 applies to nilpotent commuting perturbations. However, in this case no assumption on the spectrum is required. It is well-known that $\sigma(T)$ and $\sigma_a(T)$ are invariant under commuting quasi-nilpotent perturbations.

Theorem 3.4 *Let $T \in L(X)$ and let N be a nilpotent operator which commutes with T. If T has property $(UW\Pi)$, then $T + N$ has property $(UW\Pi)$.*

Proof We have that $\sigma(T+N) = \sigma(T)$, $\sigma_a(T+N) = \sigma_a(T)$ and $\sigma_d(T+N) = \sigma_d(T)$, so $\Delta_a(T + N) = \Delta_a(T) = \Pi(T) = \Pi(T + N)$. □

The following example shows that the result of Theorem 3.4 cannot be extended to quasi-nilpotent commuting perturbations:

Example 3.5 Let $Q \in L(X)$ denote an injective quasi-nilpotent operator (for instance the Volterra integral operator defined on the space of all continuous function $C[a, b]$). Then $\Delta_a(Q) = \sigma_a(Q) \setminus \sigma_{uw}(Q) = \emptyset$. Since $\alpha(Q) = 0$, the point 0 cannot be a pole of the resolvent, i.e. $\sigma_d(Q) = \{0\}$, otherwise the condition $p(Q) = q(Q) < \infty$ would implies $\alpha(Q) = \beta(Q) = 0$ and hence $0 \notin \sigma(Q)$). Therefore, Q has property $(UW\Pi)$. On the other hand, $0 = Q - Q$ does not have property $(UW\Pi)$ since $\Delta_a(0) = \emptyset$, while $\sigma_d(0) = \emptyset$ and $\sigma(0) = \{0\}$, hence $\Pi(0) = \{0\}$.

Theorem 3.6 *Suppose that $T \in L(X)$ has property $(UW\Pi)$. If $\mathrm{iso}\,\sigma_b(T) = \emptyset$, or $\mathrm{iso}\,\sigma_{ub}(T) = \emptyset$, then $T + Q$ has property $(UW\Pi)$ for every commuting quasi-nilpotent operator Q.*

Proof The spectra $\sigma(T)$, $\sigma_a(T)$ and $\sigma_{uw}(T)$ are invariant under a commuting quasi-nilpotent perturbation Q, so, if T has property $(UW\Pi)$, then $T + Q$ has property $(UW\Pi)$ exactly when $\sigma_d(T + Q) = \sigma_d(T)$. Each one of the assumptions $\mathrm{iso}\,\sigma_b(T) = \emptyset$, or $\mathrm{iso}\,\sigma_{ub}(T) = \emptyset$, entails the equality $\sigma_d(T + Q) = \sigma_d(T)$, see [11, Proposition 2.7]. □

The result of Corollary 3.2 may be strongly improved if we consider injective quasi-nilpotent perturbations. To see this we need a preliminary lemma.

Lemma 3.7 *Let $T \in L(X)$ be such that $\alpha(T) < \infty$. Suppose that there exists an injective quasi-nilpotent operator $Q \in L(X)$ such that $TQ = QT$. Then T is injective.*

Proof Set $Y := \ker T$. Clearly, Y is invariant under Q and the restriction $(\lambda I - Q)|Y$ is injective for all $\lambda \neq 0$. Since Y is finite-dimensional then $(\lambda I - Q)|Y$ is also surjective for all $\lambda \neq 0$, thus $\sigma(Q|Y) \subseteq \{0\}$. On the other hand, from assumption we know that $Q|Y$ is injective and hence $Q|Y$ is surjective, so $\sigma(Q|Y) = \emptyset$, from which we conclude that $Y = \{0\}$. □

Recall that $T \in L(X)$ is said to be *finite-isoloid* (respectively, *finite a-isoloid*) if every isolated point of $\sigma(T)$ (respectively, $\sigma_a(T)$) is an eigenvalue having finite rank. It is easily seen that if T is finite-isoloid then $\sigma_b(T) = \sigma_d(T)$. Indeed, if $\lambda \notin \sigma_d(T)$ then $\lambda \in \mathrm{iso}\,\sigma(T)$, so $\alpha(\lambda I - T) < \infty$. Since $p(\lambda I - T) = q(\lambda I - T) < \infty$, then $\lambda I - T$ is Browder, by [1, Chapter 1], so $\lambda \notin \sigma_b(T)$. Therefore, $\sigma_b(T) \subseteq \sigma_d(T)$, and being the opposite inclusion true for every operator, we have $\sigma_b(T) = \sigma_d(T)$.

Theorem 3.8 *Suppose that $T \in L(X)$ is finite-isoloid. If Q is an injective quasi-nilpotent operator commuting with T, then both T and $T + Q$ have property $(UW\Pi)$.*

Proof Let $\lambda \in \Delta_a(T) = \sigma_a(T)\setminus\sigma_{uw}(T)$. Since $\lambda I - T \in W_+(X)$ then $\alpha(\lambda I - T) < \infty$ and $\lambda I - T$ has closed range. Since $\lambda I - T$ commutes with Q it then follows, by Lemma 3.7, that $\lambda I - T$ is injective, so $\lambda \notin \sigma_a(T)$, a contradiction. Therefore, $\Delta_a(T)$ is empty. Also $\Pi(T)$ is empty. Indeed, suppose that $\lambda \in \Pi(T) = \sigma(T) \setminus \sigma_d(T)$. Then λ is an isolated point of $\sigma(T)$, and since T is finite-isoloid then $0 < \alpha(\lambda I - T) < \infty$. But by Lemma 3.7 we have $\alpha(\lambda I - T) = 0$, a contradiction. Therefore $\Pi(T) = \emptyset$, and hence T has property $(UW\Pi)$.

To show that $T + Q$ has property $(UW\Pi)$, observe first that

$$\Delta_a(T + Q) = \sigma_a(T + Q) \setminus \sigma_{uw}(T + Q) = \sigma_a(T) \setminus \sigma_{uw}(T) = \Delta_a(T),$$

so, from the first part, $\Delta_a(T + Q) = \emptyset$. Also $\Pi(T + Q)$ is empty. Indeed, suppose that there exists $\lambda \in \Pi(T + Q)$. Then $\lambda \in \operatorname{iso}\sigma(T + Q) = \operatorname{iso}\sigma(T)$, and hence, since T is finite-isoloid, $0 < \alpha(\lambda I - T) < \infty$. Again by Lemma 3.7 we have $\alpha(\lambda I - T) = 0$, a contradiction. □

The assumption that T is finite-isoloid in Theorem 3.8 is crucial. For instance, if T is an injective quasi-nilpotent operator Q then T is not finite-isoloid, and, as observed before, T has property $(UW\Pi)$, while $T - Q = 0$ does not have property $(UW\Pi)$.

In the next result, we give a necessary and sufficient condition under which property $(UW\Pi)$ is preserved under commuting Riesz perturbations.

Theorem 3.9 *Let $T \in L(X)$ and let R be a Riesz operator commuting with T. If T satisfies property $(UW\Pi)$ then the following assertions are equivalent:*

(i) *$T + R$ satisfies property $(UW\Pi)$;*
(ii) *$p_{00}^a(T + R) = \Pi(T + R)$.*

Proof (i)⇒ (ii) Suppose that $T + R$ satisfies property $(UW\Pi)$. By Theorem 2.3 we then have $p_{00}^a(T + R) = \Pi(T + R)$.
(ii) ⇒ (i) Assume hat $p_{00}^a(T + R) = \Pi(T + R)$. Since T satisfies property $(UW\Pi)$ then T satisfies a-Browder's theorem. Hence $\sigma_{uw}(T) = \sigma_{ub}(T)$. Since $\sigma_{uw}(T+R) = \sigma_{uw}(T)$ and $\sigma_{ub}(T + R) = \sigma_{ub}(T)$, then so $\sigma_{ub}(T + R) = \sigma_{uw}(T + R)$, and $T + R$ satisfies a-Browder's theorem i.e $\Delta_a(T + R) = p_{00}^a(T + R)$. Since $p_{00}^a(T + R) = \Pi(T+R)$, then $\Delta_a(T+R) = \Pi(T+R)$. Therefore $T+R$ satisfies property $(UW\Pi)$.□

Property $(UW\Pi)$) is transmitted under commuting Riesz perturbations in a very special case. Recall that an operator $T \in L(X)$ is said to be *finitely a-polaroid* if every isolated point of $\sigma_a(T)$ is a pole of T having finite rank. Recall that, in general, the equality $\sigma_a(T + R) = \sigma_a(T)$, for a Riesz commuting perturbation, does not hold, even if T is a finite-rank operator.

Theorem 3.10 *Suppose that $T \in L(X)$ is finitely a-polaroid and let $R \in L(X)$ be a commuting Riesz operator such that $\sigma_a(T + R) = \sigma_a(T)$. If T has property $(UW\Pi)$) then $T + R$ has property $(UW\Pi)$.*

Proof By Theorem 3.9 it suffices to show the equality $\Pi(T + R) = p_{00}^a(T + R)$. Let $\lambda \in \Pi(T+R)$. Then $\lambda \in \operatorname{iso}\sigma_a(T+R) = \operatorname{iso}\sigma_a(T)$. Since T is finitely a-polaroid then

$\lambda I - T$ is Drazin invertible and $\alpha(\lambda I - T) < \infty$. Then $\lambda I - T \in B(X)$, see [1, Chapter 1], in particular is upper semi-Browder. This implies that $\lambda I - (T + R) \in B_+(X)$, so $\lambda \in \sigma_a(T + R) \setminus \sigma_{ub}(T + R) = p_{00}^a(T + R)$.

Conversely, suppose that $\lambda \in p_{00}^a(T + R)$. Then

$$\begin{aligned} \lambda &\in \sigma_a(T + R) \setminus \sigma_{ub}(T + R) \subseteq \sigma_a(T + R) \setminus \sigma_{uw}(T + R) \\ &= \sigma_a(T) \setminus \sigma_{uw}(T) = \Delta_a(T) = \Pi(T). \end{aligned}$$

In particular, λ is an isolated point of $\sigma_a(T)$, and consequently, a pole of T having finite rank. This implies that $\lambda I - T \in B(X)$, see [1, Chapter 3], and hence $\lambda I - (T + R) \in B(X)$, so $\lambda \in \Pi(T + R)$. □

4 Property $(UW\Pi)$ under functional calculus

In this section we study the preservation of property $(UW\Pi)$ under functional calculus. Furthermore, we show that this property is transferred from a Drazin invertible operator to its Drazin inverse. Let $\mathcal{H}(\sigma(T))$ be the set of all analytic functions defined on a neighborhood of $\sigma(T)$, and for every $f \in \mathcal{H}(\sigma(T))$ let $f(T)$ be defined by means of the classical functional calculus. The spectral theorem for the spectrum asserts that $\sigma(f(T)) = f(\sigma(T))$ for every $f \in \mathcal{H}(\sigma(T))$ and a similar equality holds for the approximate point spectrum.

Lemma 4.1 *Let $T \in L(X)$ and let $\{\lambda_1, \cdots, \lambda_k\}$ be a finite subset of $\mathbb{C}$ such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Assume that $\{\nu_1, \cdots, \nu_n\} \subset \mathbb{N}$ and set $h(\lambda) := \prod_{i=1}^n (\lambda_i - \lambda)^{\nu_i}$. Then, for the operator $h(T) := \prod_{i=1}^n (\lambda_i I - T)^{\nu_i}$ we have*

$$\ker h(T) = \bigoplus_{i=1}^{n} \ker\, (\lambda_i I - T)^{\nu_i} \tag{5}$$

and

$$h(T)(X) = \bigcap_{i=1}^{n} (\lambda_i I - T)^{\nu_i}(X). \tag{6}$$

Furthermore,

$$0 \in \Pi(h(T)) \Leftrightarrow \lambda_i \in \Pi(T) \text{ for all } i = 1, \ldots, n.$$

Proof A proof of the equalities (5) and (6) may be found in [1]. Suppose that $\lambda_i \in \Pi(T)$ for all $i = 1, \ldots, n$, i.e., $\lambda_i \in \sigma(T)$ and $\lambda_i I - T$ Drazin invertible, for all $i = 1, \ldots, n$. Then, for k large enough, we have $\ker(\lambda_i I - T)^{\nu_i k} = \ker(\lambda_i I - T)^{\nu_i (k+1)}$ and $(\lambda_i - T)^{\nu_i k}(X) = (\lambda_i - T)^{\nu_i (k+1)}(X)$. This implies, by (5) and (6), that $\ker\, (h(T)^k) = \ker\, (h(T)^{k+1})$ and $h(T)^k(X) = h(T)^{k+1}(X)$, so $h(T)$ is Drazin invertible. Since $0 = h(\lambda_i) \in h(\sigma(T)) = \sigma(h(T))$ it then follows that $0 \in \Pi(h(T))$.

Conversely, assume that $\lambda_j \notin \Pi(T)$ for some $j \in \{1, \dots, n\}$. Let p_j and q_j denote the ascent and the descent of $\lambda_j I - T$, respectively. We have either $\lambda_j \notin \sigma(T)$ or $\lambda_j \in \sigma(T)$. In the case where $\lambda_j \notin \sigma(T)$, then $0 = h(\lambda_j) \notin h(\sigma(T)) = \sigma(h(T))$, so $0 \notin \Pi(T)$. Consider the other case $\lambda_j \in \sigma(T)$. Since $\lambda_j \notin \Pi(T)$ then $\lambda_j I - T$ is not Drazin invertible, so either $p_j = \infty$, i.e. $\ker(\lambda_j I - T)^k$ is properly contained in $\ker(\lambda_j I - T)^{k+1}$ for every $k \in \mathbb{N}$, or $q_j = \infty$, i.e., $(\lambda_j I - T)^{k+1}(X)$ is properly contained in $(\lambda_j I - T)^k(X)$ for every $k \in \mathbb{N}$. If $p_j = \infty$, being $\ker(\lambda_i I - T)^{\nu_i} \cap \ker(\lambda_j I - T)^{\nu_j} = \emptyset$ for all $i \neq j$, it then follows that $p(h(T)) = \infty$. Analogously, if $q_j = \infty$ then $q(h(T)) = \infty$. Therefore, also in this case $0 \notin \Pi(T)$. □

Remark 4.2 If $T \in L(X)$ is invertible and $S \in L(X)$ commutes with T then $N(S^n) = N((TS))^n$ and $(TS)^n(X) = S^n(X)$ for all $n \in \mathbb{N}$. Consequently, TS is Drazin invertible if and only if S is Drazin invertible, while $0 \in \Pi(TS)$ if and only if $0 \in \Pi(S)$.

In the sequel we need the following lemma.

Lemma 4.3 *For every $T \in L(X)$, X a Banach space, and $f \in \mathcal{H}(\sigma(T))$ we have*

$$\sigma_a(f(T)) \setminus \Pi(f(T)) \subseteq f\left(\sigma_a(T) \setminus \Pi(T)\right). \tag{7}$$

Furthermore, if T is a-isoloid then

$$\sigma_a(f(T)) \setminus \Pi(f(T)) = f\left(\sigma_a(T) \setminus \Pi(T)\right). \tag{8}$$

Proof Suppose that $\lambda_0 \in \sigma_a(f(T)) \setminus \Pi(f(T))$. We consider two cases.

Case (I). Suppose first that $\lambda_0 \notin \operatorname{iso} f(\sigma_a(T)) = \operatorname{iso} \sigma_a(f(T))$. In this case $\lambda_0 \notin \Pi(f(T))$, since each point of $\Pi(f(T))$ is an isolated point of $\sigma(f(T))$ and hence an isolated point of $\sigma_a(f(T))$. Let $(\lambda_n) \subset f(\sigma_a(T)) = \sigma_a(f(T))$ be a sequence such that $\lambda_n \to \lambda_0$ and let $(\mu_n) \in \sigma_a(T)$ be such that $f(\mu_n) = \lambda_n$. Then (μ_n) admits a subsequence which converge to some $\mu_0 \in \sigma_a(T))$. There is no harm if we assume that $\mu_n \to \mu_0$. Then $f(\mu_n) \to f(\mu_0) = \lambda_0$. Evidently, $\mu_0 \notin \Pi(T)$, since μ_0 is a cluster point of $\sigma_a(T)$. Hence

$$\lambda_0 = f(\mu_0) \in f\left(\sigma_a(T) \setminus \Pi(T)\right).$$

Case (II). Suppose that $\lambda \in \operatorname{iso} f(\sigma_a(T)) = \operatorname{iso} \sigma_a(f(T))$. Define $g(\lambda) := \lambda_0 - f(\lambda)$, Since $g(\lambda)$ is analytic then g has only a finite number of zeros in $\sigma(T)$, say $\{\lambda_1, \dots, \lambda_n\}$. Write

$$g(\lambda) = h(\lambda)k(\lambda) \quad \text{and} \quad h(\lambda) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{\nu_i}, \tag{9}$$

where $k(\lambda)$ has no zero in $\sigma(T)$. Then

$$g(T) = \lambda_0 I - f(T) = h(T)k(T),$$

where $k(T)$ invertible, and $h(T) := \prod_{i=1}^{k}(\lambda_i I - T)^{\nu_i}$.

Now, $\lambda_0 \notin \Pi(f(T)$ implies that $\lambda_0 I - f(T)$ is invertible, or $\lambda_0 I - f(T)$ is not Drazin invertible. In the first case $g(T)$ is invertible, so $0 \notin \Pi(g(T))$. If $\lambda_0 I - f(T)$ is not Drazin invertible, we have either $p(g(T)) = p(\lambda_0 I - f(T)) = \infty$ or $q(g(T)) = q(\lambda_0 I - f(T)) = \infty$, so also in this case $0 \notin \Pi(g(T))$. According Remark 4.2 then $0 \notin \Pi(h(T))$. By Lemma 4.1 it then follows that $\lambda_j \notin \Pi(T)$ for some j. Thus, $\lambda_j \in \sigma_a(T) \setminus \Pi(T)$, and hence $\lambda_0 = f(\lambda_j) \in f(\sigma_a(T) \setminus \Pi(T))$.

To show the equality (8), assume that T is a-isoloid.

To show the equality (8) we need only to prove the inclusion $\supseteq$. Let $\lambda_0 \in f(\sigma_a(T) \setminus \Pi(T))$ be arbitrary given. Since $\lambda_0 \in f(\sigma_a(T)) = \sigma_a(f(T)) \subseteq \sigma(f(T))$ it then suffices to prove that $\lambda_0 \notin \Pi(f(T))$. To do this, suppose that $\lambda_0 \in \Pi(f(T))$. Then $0 \in \Pi(h(T)$, and by Lemma 4.1 it then follows that $\lambda_i \in \Pi(T)$ for all $i = 1, \dots, n$. Thus, $\lambda_i \notin \sigma_a(T) \setminus \Pi(T)$ for all i. Since $0 = g(\lambda_i) = \lambda_0 - f(\lambda_i)$, we have $\lambda_0 = f(\lambda_i) \notin f(\sigma_a(T) \setminus \Pi(T))$, a contradiction. Therefore, $\lambda_0 \notin \Pi(f(T))$, and hence

$$\lambda_0 \in \sigma_a(f(T)) \setminus \Pi(f(T)),$$

as desired. □

Recall that, in general, the spectral theorem does not hold for $\sigma_{uw}(T)$, see [1, Chapter 3].

Theorem 4.4 *Let $T \in L(X)$ be a-isoloid and $f \in \mathcal{H}(\sigma(T))$. If T satisfies property $(UW\Pi))$ then the following are equivalent:*

(i) *$f(T)$ satisfies property $(UW\Pi)$;*
(ii) *$f(\sigma_{uw}(T)) = \sigma_{uw}(f(T))$.*

Proof (i) $\to$ (ii) Observe first that the spectral mapping theorem holds for $\sigma_{\text{ub}}(T)$, i.e., $f(\sigma_{\text{ub}}(T)) = \sigma_{\text{ub}}(f(T))$ for every $f \in \mathcal{H}(\sigma(T))$, see [1, Chapter 3]. Since $f(T)$ satisfies property $(UW\Pi)$, then both T and $f(T)$ satisfies a-Browder's theorem, by Theorem 2.3, so $\sigma_{\text{uw}}(T) = \sigma_{\text{ub}}(T)$ and $\sigma_{\text{uw}}(f(T)) = \sigma_{\text{ub}}(f(T))$. Then we have

$$f(\sigma_{\text{uw}}(T)) = f(\sigma_{\text{ub}}(T)) = \sigma_{\text{ub}}(f(T)) = \sigma_{\text{uw}}(f(T)).$$

(ii) $\to$ (i) Since T satisfies property $(UW\Pi))$ we have $\sigma_{uw}(T) = \sigma_a(T) \setminus \Pi(T)$. By Lemma 4.3 it then follows that

$$\sigma_{uw}(f(T)) = f(\sigma_{uw}(T)) = f(\sigma_a(T) \setminus \Pi(T)) = \sigma_a(f(T)) \setminus \Pi(f(T)),$$

so $f(T)$ satisfies property $(UW\Pi)$. □

It is known that the spectral theorem for $\sigma_{uw}(T)$ holds if T or T^* has SVEP ([1, Chapter 3], or if f is injective ([18]. Consequently, property $(UW\Pi))$ is transmitted from T to $f(T)$ if T is a-isoloid and T or T^* has SVEP. In the case, that T^* has SVEP, then we can require that T is isoloid, since in this case $\sigma_a(T) = \sigma(T)$ and hence the properties of being a-isoloid and isoloid coincide.

Lemma 4.5 *Suppose that for a bounded operator $T \in L(X)$ there exists $\lambda_0 \in \mathbb{C}$ such that $K(\lambda_0 I - T) = \{0\}$ and $\ker(\lambda_0 I - T) = \{0\}$. Then $\sigma_p(T) = \emptyset$.*

Proof For all complex $\lambda \neq \lambda_0$ we have $\ker(\lambda I - T) \subseteq K(\lambda_0 I - T)$, so that $\ker(\lambda I - T) = \{0\}$, for $\lambda \neq \lambda_0$. Since $\ker(\lambda_0 I - T) = \{0\}$ we then conclude that $\ker(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$. □

Theorem 4.6 *Let $T \in L(X)$ be such that there exists $\lambda_0 \in \mathbb{C}$ such that*

$$K(\lambda_0 I - T) = \{0\} \quad \textit{and} \quad \ker(\lambda_0 I - T) = \{0\}. \tag{10}$$

Then property $(UW\Pi)$ holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(f(T))$.

Proof We know from Lemma 4.5 that $\sigma_p(T) = \emptyset$, so T has SVEP. We show that also $\sigma_p(f(T)) = \emptyset$. Let $\mu \in \sigma(f(T))$ and write $\mu - f(\lambda) = p(\lambda)g(\lambda)$, where g is analytic on an open neighborhood $\mathcal{U}$ containing $\sigma(T)$ and without zeros in $\sigma(T)$, p a polynomial of the form $p(\lambda) = \Pi_{k=1}^n (\lambda_k - \lambda)^{\nu_k}$, with distinct roots $\lambda_1, \ldots, \lambda_n$ lying in $\sigma(T)$. Then

$$\mu I - f(T) = \Pi_{k=1}^n (\lambda_k I - T)^{\nu_k} g(T).$$

Since $g(T)$ is invertible, $\sigma_{\mathrm{p}}(T) = \emptyset$ implies that $\ker(\mu I - f(T)) = \{0\}$ for all $\mu \in \mathbb{C}$, so $\sigma_p(f(T)) = \emptyset$. Since T has SVEP then $f(T)$ has SVEP, see [1, Chapter 2], so that a-Browder's theorem holds for $f(T)$ ([1, Chapter 5]). To prove that property $(UW\Pi)$ holds for $f(T)$, by Theorem 2.3 it then suffices to prove that

$$p_{00}^a(f(T)) = \Pi(f(T)).$$

Obviously, the condition $\sigma_p(f(T)) = \emptyset$ entails that $\Pi(f(T)) = \emptyset$, since every point of $\Pi(f(T))$ is an eigenvalue of $f(T)$. On the other hand, if $\lambda \in p_{00}^a(f(T))$ then $\lambda \in \sigma_a(f(T))$ and $\lambda \notin \sigma_{ub}(f(T)$, so $\lambda I - f(T)$ has closed range, since it is a upper semi-Browder operator. This implies that $\alpha(\lambda I - f(T)) > 0$ and we get a contradiction, since $\sigma_p(f(T)) = \emptyset$. □

The conditions of Theorem 4.6 are satisfied by any injective operator for which the *hyperrange* $T^\infty(X) := \bigcap T^n(X)$ is $\{0\}$. In fact, $K(T) \subseteq T^\infty(X)$ for all $T \in L(X)$, so that $K(T) = \{0\}$. In particular, the conditions of Theorem 4.6 are satisfied by a *semi-shift* T, i.e. T is an isometry for which $T^\infty(X) = \{0\}$, see [15] for details on this class of operators. Clearly, a semi-shift T on a non-trivial Banach space is a non-invertible isometry.

The following result applies to several operator, in particular to shift operators.

Lemma 4.7 *Suppose that for $T \in L(X)$ we have $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) = \emptyset$. If $K \in L(X)$ commutes with T and K^n is finite-dimensional for some $n \in \mathbb{N}$, then $\mathrm{iso}\,\sigma_{\mathrm{a}}(T + K) = \emptyset$. Consequently, $\sigma_{\mathrm{a}}(T + K) = \sigma_{\mathrm{a}}(T)$.*

Proof By Lemma 4.7 we have

$$\begin{aligned}\sigma_{\mathrm{a}}(T) &= \mathrm{iso}\,\sigma_{\mathrm{a}}\,(T) \cup \mathrm{acc}\,\sigma_{\mathrm{a}}\,(T) = \mathrm{acc}\,\sigma_{\mathrm{a}}\,(T)\\ &= \mathrm{acc}\,\sigma_{\mathrm{a}}\,(T+K) \subseteq \sigma_{\mathrm{a}}(T+K).\end{aligned}$$

On the other hand, if $\sigma_{\mathrm{a}}(K) = \{\lambda_1, \lambda_2, \dots \lambda_n\}$ we have

$$\mathrm{iso}\,\sigma_{\mathrm{a}}(T+K) \subseteq \mathrm{iso}\,(\sigma_{\mathrm{a}}(T) + \sigma_{\mathrm{a}}(K)) = \mathrm{iso}\,\bigcup_{k=1}^{n}(\lambda_k + \sigma_{\mathrm{a}}(T)) = \emptyset,$$

hence,

$$\begin{aligned}\sigma_{\mathrm{a}}(T+K) &= \mathrm{iso}\,\sigma_{\mathrm{a}}(T+K) \cup \mathrm{acc}\,\sigma_{\mathrm{a}}(T+K) = \mathrm{acc}\,\sigma_{\mathrm{a}}(T+K)\\ &= \mathrm{acc}\,\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T),\end{aligned}$$

so $\sigma_{\mathrm{a}}(T+K) = \sigma_{\mathrm{a}}(T)$ holds.

□

Theorem 4.8 *Let $T \in L(X)$ be a semi-shift and suppose that $K \in L(X)$ is such that K^n is finite-dimensional for some $n \in \mathbb{N}$ and that $TK = KT$. Then property $(UW\Pi)$ holds for $f(T) + K$ for all $f \in \mathcal{H}(\sigma(f(T))$.*

Proof Since T is a non-invertible isometry, the aproximate point spectrum $\sigma_{\mathrm{a}}(T)$ is the closed unit circle of $\mathbb{C}$, see [15, Proposition 1.6.2]. Hence $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) = \emptyset$. From Lemma 4.7 and Theorem 3.3 then property $(UW\Pi)$ holds for $f(T) + K$. □

Recall that an operator $T \in L(X)$ is Drazin invertible if and only if there exists an operator $S \in L(X)$ and $n \in \mathbb{N}$ such that

$$TS = ST, \quad STS = S, \quad T^n ST = T^n, \tag{11}$$

see [16, Chap.3, Theorem 10]. The operator S is called the *Drazin inverse* of T. From [16, Chap. 3, Theorem 10] we also know that if $T \in L(X)$ is Drazin invertible if and only if there exist two closed invariant subspaces Y and Z such that $X = Y \oplus Z$ and, with respect to this decomposition,

$$T = T_1 \oplus T_2, \quad \text{with } T_1 := T|Y \text{ nilpotent and } T_2 := T|Z \text{ invertible.} \tag{12}$$

Note that the Drazin inverse S of an operator, if it exists, is uniquely determined ([12]), and with respect to the decomposition $X = Y \oplus Z$, the Drazin inverse S may be represented as the directed sum

$$S := 0 \oplus S_2 \quad \text{with } S_2 := {T_2}^{-1}. \tag{13}$$

From the decomposition (13) it is obvious that the Drazin inverse S is also Drazin invertible. Evidently, if T is invertible then $S = T^{-1}$, while $0 \in \sigma(T)$ if and only if $0 \in \sigma(S)$.

The decompositions (12) and (13) are very useful in order to study the spectral properties of a Drazin invertible operator, see [3] and [4], in particular the decomposition (13) shows that the Drazin inverse S is itself Drazin invertible, since is the direct sum of the nilpotent operator 0 and the invertible operator S_2. It should be noted that if $0 \in \sigma(T)$ then 0 is a pole of the first order of the resolvent of S, see [18]. Furthermore, the following relationship of reciprocity holds for the spectra of S and T:

$$\sigma(S) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T) \setminus \{0\} \right\}, \tag{14}$$

see [1, Chapter 1].

We also have,

$$\sigma_a(S) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_a(T) \setminus \{0\} \right\},$$

and

$$\sigma_{uw}(S) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_{uw}(T) \setminus \{0\} \right\},$$

see [4]. By [4, Theorem 2.8] we also have

$$\Pi(S) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \Pi(T) \setminus \{0\} \right\}.$$

Lemma 4.9 *Suppose that $T \in L(X)$ is Drazin invertible with Drazin inverse S. Then $T \in W_+(X)$ if and only if $S \in W_+(X)$.*

Proof (i) If $0 \notin \sigma(T)$ then T is invertible and the Drazin inverse is $S = T^{-1}$ so the assertion is trivial in this case. Suppose that $0 \in \sigma(T)$ and that T is upper semi-Weyl. Since $p(\lambda I - T) = q(\lambda I - T) < \infty$ then, see [1, Chapter 1], T is Browder. Then 0 is a pole of the resolvent of T and is also a pole (of the first order) of the resolvent of S. Let $X = Y \oplus Z$ such that $T = T_1 \oplus T_2$, $T_1 = T|Y$ nilpotent and $T_2 = T|Z$ invertible. Observe that

$$\ker\, T = \ker\, T_1 \oplus \ker T_2 = \ker T_1 \oplus \{0\}, \tag{15}$$

and, analogously, since $S = 0 \oplus S_2$ with $S_2 = {T_2}^{-1}$, we have

$$\ker\, S = \ker\, 0 \oplus \ker S_2 = Y \oplus \{0\}. \tag{16}$$

Since T is upper semi-Weyl we have $\alpha(T) = \dim \ker\, T < \infty$, and from the inclusion $\ker\, T_1 \subseteq \ker\, T$ we obtain $\alpha(T_1) < \infty$. Consequently, $\alpha(T_1^n) < \infty$ for all $n \in \mathbb{N}$. Let $T_1^\nu = 0$. Since $Y = \ker\, T_1^\nu$ the subspace Y is finite-dimensional, and hence $\ker\, S = Y \oplus \{0\}$ is finite-dimensional, i.e. $\alpha(S) < \infty$. Now, S is Drazin invertible, so $p(S) = q(S) < \infty$ and hence, see [1, Chapter 1], $\alpha(S) = \beta(S) < \infty$. Hence S is Browder, and in particular upper semi-Weyl.

Conversely, suppose that S is upper semi-Weyl. Then $\alpha(S) < \infty$ and hence by (16) the subspace Y is finite-dimensional, from which it follows that also $\ker T_1 = \ker T|Y$ is finite-dimensional. From (15) we then have that $\alpha(T) < \infty$ and since $p(T) = q(T) < \infty$ we then conclude that $\alpha(T) = \beta(T)$, see [1, Chapter 1]. Therefore, T is a Browder operator, in particular upper semi-Weyl. □

In [4] it is shown that several Browder type theorems and Weyl type theorems are transferred from a Drazin invertible operator to its Drazin inverse. We show now that the same happens for property ($UW\Pi$).

Theorem 4.10 *Suppose that $T \in L(X)$ is Drazin invertible with Drazin inverse S. Then T satisfies property ($UW\Pi$) if and only if S satisfies property ($UW\Pi$).*

Proof Suppose that T satisfies property ($UW\Pi$). Consider first the case that T is invertible. Then $S = T^{-1}$. Suppose that $\lambda \in \sigma_a(T^{-1})$. Then $\lambda \neq 0$, and hence $\frac{1}{\lambda} \in \sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi(T)$. If $\frac{1}{\lambda} \in \sigma_{uw}(T)$ then $\lambda \in \sigma_{uw}(T^{-1}$, while if $\frac{1}{\lambda} \in \Pi(T)$ then $\lambda \in \sigma_{uw}(T^{-1}$. This shows that $\sigma_a(T^{-1}) \subseteq \sigma_{uw}(T^{-1}) \bigsqcup \Pi(T^{-1})$, and since the opposite inclusion always holds we then have

$$\sigma_a(T^{-1}) = \sigma_{uw}(T^{-1}) \bigsqcup \Pi(T^{-1}),$$

i.e., property ($UW\Pi$) holds for T^{-1}. The proof that property ($UW\Pi$) for T^{-1} implies property ($UW\Pi$) for T is similar.

Consider the case that T is not invertible. Then $0 \in \sigma(T)$, as well as $0 \in \sigma(S)$, and in this case 0 is a pole of both T and S. Let $\lambda \in \sigma_a(S)$. If $\lambda = 0$ we have already observed that $0 \in \Pi(S)$ and $0 \in \Pi(T)$. Since T has property ($UW\Pi$) then $\sigma_a(T) = \sigma_{uw}(T) \bigsqcup \Pi(T)$, so $0 \notin \sigma_{uw}(T)$. But $p(T) = q(T) < \infty$, so, by [1, Chapter 1], T is Browder, and in particular is upper semi-Weyl. By Lemma 4.9, S is upper semi-Weyl, so $0 \notin \sigma_{uw}(S)$. Therefore, $0 \in \sigma_{uw}(T) \bigsqcup \Pi(T)$. Suppose that $\lambda \neq 0$. Then $\frac{1}{\lambda} \in \sigma_a(T)$, and hence $\frac{1}{\lambda} \in \sigma_{uw}(T) \bigsqcup \Pi(T)$. This implies that either $\lambda \in \sigma_{uw}(S)$ or $\lambda \in \Pi(S)$, so $\lambda \in \sigma_{uw}(T) \bigsqcup \Pi(T)$. We have shown that $\sigma_a(S) \subseteq \sigma_{uw}(T) \bigsqcup \Pi(T)$ and since the reverse inclusion holds we then have $\sigma_a(S) \subseteq \sigma_{uw}(T) \bigsqcup \Pi(T)$.

Conversely, suppose that S has property ($UW\Pi$). The Drazin inverse of S is $U := T^2S = TST$ and the Drazin inverse of U is T, see [1, Chapter 1]. Therefore, from the first part, U inherits property ($UW\Pi$) from S and property ($UW\Pi$) is then transferred from U to T. □

References

1. Aiena, P.: Fredholm and local spectral theory II, with application to Weyl-type theorems. Springer Lecture Notes of Math **2235**, (2018)
2. Aiena, P., Kachad, M.: Property $(UW\Pi)$ and localized SVEP. Acta Sci. Math. (Szeged) **84**(3–4), 555–571 (2018)
3. Aiena, P., Triolo, S.: Local spectral theory for Drazin invertible operators. J. Math. Anal. Appl. **435**, 414–424 (2016)
4. Aiena, P., Triolo, S.: Fredholm spectra and Weyl type theorems for Drazin invertible operators. Mediterr. J. Math. **13**(6), 4385–4400 (2016)
5. Amouch, M., Zquitti, H.: On the equivalence of Browder's and generalized Browder's theorem. Glasgow Math. J. **48**, 179–185 (2006)
6. Berkani, M.: On a class of quasi-Fredholm operators. Integr. Equ. Oper. Theory **34**(2), 244–249 (1999)
7. Berkani, M.: Restriction of an operator to the range of its powers. Stud. Math. **140**(2), 163–75 (2000)
8. Berkani, M.: Index of B-Fredholm operators and generalization of a Weyl's theorem. Proc. Am. Math. Soc. **34**(6), 1717–1723 (2001)
9. Berkani, M., Kachad, M.: New Browder–Weyl type theorems. Bull. Korean. Math. Soc. **49**(2), 1027–1040 (2015)
10. Berkani, M., Sarih, M.: On semi B-Fredholm operators. Glasgow Math. J. **43**, 457–465 (2001)
11. Berkani, M., Zariuoh, H.: B-Fredholm spectra and Riesz perturbations. Matematici Vesnik **67**(3), 155–165 (2015)
12. Caradus, S. R.: Operator theory of the pseudo inverse. Queen's Papers Pure Appl. Math. **38** Queen's University, Kingston, Ontario (1974)
13. Harte, R., Lee, W.Y.: Another note on Weyl's theorem. Trans. Am. Math. Soc. **349**, 2115–2124 (1997)
14. Heuser, H.: Functional analysis. Wiley, New York (1982)
15. Laursen, K.B., Neumann, M.M.: An introduction to local spectral theory. Clarendon Press, Oxford (2000)
16. Müller, V.: Spectral theory of linear operators and spectral systems on Banach algebras. Birkhäuser Verlag, Berlin (2003)
17. Schechter, M., Whitley, R.: Best Fredholm perturbation theorems. Stud. Math. **90**(1), 175–190 (1988)
18. Schmoeger, C.: On operators T such that Weyl's theorem holds for $f(T)$. Extracta Math. **13**(1), 23–33 (1998)
19. Zariouh, H.: On the property Z_{E_a}. Rend. Circ. Mat. Palermo **65**(2), 175–184 (2016)
20. Zeng, Q., Zhong, H., Yan, K.: An extension of a result of Djordjevid and its applications. Linear Multilinear Algebra **64**, 247–257 (2015)
21. Zeng, Q.P., Jiang, Q.F., Zhong, H.J.: Spectra originating from semi-B-Fredholm theory and commuting perturbations. Stud. Math. **219**(1), 1–18 (2013)