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RESEARCH ARTICLE

Smooth rational surfaces violating Kawamata–Viehweg vanishing

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Abstract We show that over any algebraically closed field of positive characteristic, there exists a smooth rational surface which violates Kawamata–Viehweg vanishing.

Keywords Rational surfaces · Kawamata–Viehweg vanishing theorem · Positive characteristic

Mathematics Subject Classification 14E30 · 14J26

1 Introduction

It is a well-known fact that Kodaira vanishing fails in positive characteristic [23]. Nevertheless, it has often been believed that a stronger version, namely Kawamata–Viehweg vanishing, holds over a smooth rational surface (e.g. see [32,33]). In this note, we show that this is in fact not true:

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Theorem 3.1 *Let k be a field of positive characteristic. Then there exist a smooth projective rational surface X over k, a Cartier divisor D, and a $\mathbb{Q}$-divisor $\Delta \geqslant 0$ such that*

- (X, Δ) *is klt,*
- $D - (K_X + \Delta)$ *is nef and big, and*
- $H^1(X, \mathcal{O}_X(D)) \neq 0$.

To prove Theorem 3.1, we use some surfaces constructed by Langer [18]. If $k = \mathbb{F}_p$, then X can be obtained by taking the blowup of $\mathbb{P}^2_{\mathbb{F}_p}$ along all the $\mathbb{F}_p$-rational points. Since the proper transforms $L'_1, \ldots, L'_{p^2+p+1}$ of the $\mathbb{F}_p$-lines $L_1, \ldots, L_{p^2+p+1}$ are pairwise disjoint, we can contract all these curves and obtain a birational morphism $g\colon X \to Y$ onto a klt surface Y such that $\rho(Y) = 1$ (cf. Lemma 2.4). Note that $-K_Y$ is ample if and only if $p = 2$ (cf. Lemma 2.4). Further, we show:

- For any $p > 0$, Y is obtained as a purely inseparable cover of $\mathbb{P}^2$ (cf. Theorem 4.1). If $p = 2$, then the morphism $Y \to \mathbb{P}^2$ is induced by the anti-canonical linear system $|-K_Y|$ (cf. Remark 4.2).
- If $p = 2$, then the Kleimann–Mori cone $\mathrm{NE}(X)$ is generated by exactly 14 curves (cf. Theorem 5.4).
- If $p = 2$, then X is isomorphic to a surface constructed by Keel–M$^{\mathrm{c}}$Kernan (cf. Proposition 6.4).

Related results. After Raynaud constructed the first counter-example to Kodaira vanishing in positive characteristic [23], several other people studied this problem (e.g. see [3,4,6], [15, Section 2.6], [21,26]). In particular, Fano varieties are known to violate Kawamata–Viehweg vanishing. As far as the authors know, the examples constructed by Lauritzen and Rao [19] (of dimension at least 6) are the only ones over an algebraically closed field. If we admit imperfect fields, then Schröer and Maddock constructed log del Pezzo surfaces with $H^1(X, \mathcal{O}_X) \neq 0$ [20,24]. In [2], the authors and Witaszek showed that Kawamata–Vieweg vanishing holds for klt del Pezzo surfaces in large characteristic. On the other hand, if $p = 2$, then the surface mentioned above is a smooth weak del Pezzo surface (cf. Lemma 2.4), hence our result cannot be extended to characteristic two (see also Proposition 7.1).

2 Preliminaries

2.1 Notation

We say that X is a *variety* over a field k if X is an integral scheme which is separated and of finite type over k. A *curve* (respectively *surface*) is a variety of dimension one (respectively two). We say that two schemes X and Y over a field k are k*-isomorphic* if there exists an isomorphism $\theta\colon X \to Y$ of schemes such that both θ and θ^{-1} commute with the structure morphisms: $X \to \operatorname{Spec} k$ and $Y \to \operatorname{Spec} k$. Given a proper morphism $f\colon X \to Y$ between normal varieties, we say that two $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors D_1, D_2 on X are *numerically equivalent over* Y, denoted $D_1 \equiv_f D_2$, if their difference is numerically trivial on any fibre of f.

We refer to [17, Section 2.3] or [16, Definition 2.8] for the classical definitions of singularities (e.g. *klt*) appearing in the minimal model programme. Note that we always assume that for any klt pair (X, Δ), the $\mathbb{Q}$-divisor Δ is effective.

2.2 Construction by Langer

We now recall the construction of a rational surface due to Langer [18] (see also [11, Exercise III.10.7]). A similar method was used to construct also some K3 surfaces and Calabi–Yau threefolds (cf. [5,12]).

Notation 2.1 Let $q = p^e$, where p is a prime number and e is a positive integer. Let $P_1^{(0)}, \ldots, P_{q^2+q+1}^{(0)}$ be the $\mathbb{F}_q$-rational points on $\mathbb{P}^2_{\mathbb{F}_q}$, and let $L_1^{(0)}, \ldots, L_{q^2+q+1}^{(0)}$ be the $\mathbb{F}_q$-lines on $\mathbb{P}^2_{\mathbb{F}_q}$, i.e. the lines which are defined over $\mathbb{F}_q$. Let

$$f^{(0)} \colon X^{(0)} \to \mathbb{P}^2_{\mathbb{F}_q}$$

be the blowup along all the $\mathbb{F}_q$-points $P_1^{(0)}, \ldots, P_{q^2+q+1}^{(0)}$. For any $i = 1, \ldots, q^2+q+1$, let $E_i^{(0)}$ be the $f^{(0)}$-exceptional prime divisor lying over $P_i^{(0)}$, hence $E_i^{(0)} \xrightarrow{\simeq} \mathbb{P}^1_{\mathbb{F}_q}$. The proper transforms $L_1'^{(0)}, \ldots, L_{q^2+q+1}'^{(0)}$ of the $\mathbb{F}_q$-lines are disjoint with each other and satisfy $(L_i'^{(0)})^2 = -q$ for any $i = 1, \ldots, q^2+q+1$. Let

$$g^{(0)} \colon X^{(0)} \to Y^{(0)}$$

be the birational morphism contracting all of the curves $L_1'^{(0)}, \ldots, L_{q^2+q+1}'^{(0)}$. We define

$$(E_i^Y)^{(0)} = g_*^{(0)} E_i^{(0)}.$$

Let k be a field containing $\mathbb{F}_q$ and let

$$f \colon X \to \mathbb{P}^2_k, \qquad g \colon X \to Y$$

be the base changes of $f^{(0)}$ and $g^{(0)}$ induced by $(-) \times_{\mathbb{F}_q} k$. We denote by P_i, L_i, E_i, L_i' and E_i^Y the inverse images of $P_i^{(0)}$, $L_i^{(0)}$, $E_i^{(0)}$, $L_i'^{(0)}$ and $(E_i^Y)^{(0)}$, respectively. We fix an arbitrary line $H \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ defined over k. By abuse of notation, each P_i (respectively L_i) is also called an $\mathbb{F}_q$-*point* (respectively an $\mathbb{F}_q$-*line*), although these depend on the choice of the homogeneous coordinates.

Notation 2.2 We use the same notation as in Notation 2.1 but we assume that $q = 2$, i.e. $p = 2$ and $e = 1$.

Remark 2.3 The configuration of the $\mathbb{F}_q$-points and the $\mathbb{F}_q$-lines on $\mathbb{P}^2_{\mathbb{F}_q}$ satisfies the following properties:

- For any $\mathbb{F}_q$-line L on $\mathbb{P}^2_{\mathbb{F}_q}$, the number of the $\mathbb{F}_q$-points contained in L is equal to $q+1$.
- For any $\mathbb{F}_q$-point P on $\mathbb{P}^2_{\mathbb{F}_q}$, the number of the $\mathbb{F}_q$-lines passing through P is equal to $q+1$.

If $q=2$, then the picture of the configuration is classically known as Fano plane (e.g. see [22, Subsection 3.1.1]).

2.3 Basic properties

We now summarise some basic properties of the surfaces X and Y constructed in Notation 2.1.

Lemma 2.4 *We use Notation 2.1. The following hold:*

(i) $\rho(Y)=1$.
(ii) *Y is klt.*
(iii) *Y has at most canonical singularities if and only if $q=2$.*
(iv) *If $q>2$, then K_Y is ample.*
(v) *If $q=2$, then $-K_Y$ is ample.*
(vi) *If $q=2$, then $-K_X$ is nef and big.*

Proof (i) follows immediately by the construction. Further, we have

$$g^*K_Y = K_X + \left(1-\frac{2}{q}\right)\sum_{i=1}^{q^2+q+1} L'_i.$$

Thus, (ii) and (iii) hold.

We now show (iv) and (v). Since $K_X = f^*K_{\mathbb{P}^2} + \sum_i E_i \sim -3f^*H + \sum_i E_i$ and

$$(q^2+q+1)f^*H \sim f^*\left(\sum_{i=1}^{q^2+q+1} L_i\right) = \sum_{i=1}^{q^2+q+1} L'_i + (q+1)\sum_{i=1}^{q^2+q+1} E_i,$$

we have

$$\begin{aligned}(q^2+q+1)K_X &\sim -3(q^2+q+1)f^*H + (q^2+q+1)\sum_{i=1}^{q^2+q+1} E_i \\ &\sim -3\sum_{i=1}^{q^2+q+1} L'_i + (q^2-2q-2)\sum_{i=1}^{q^2+q+1} E_i.\end{aligned}$$

Taking the push-forward g_*, we get

$$(q^2+q+1)K_Y \sim (q^2-2q-2)\sum_{i=1}^{q^2+q+1} E_i^Y.$$

Therefore, if $q = 2$ (respectively $q > 2$), then $-K_Y$ (respectively K_Y) is ample. Thus, (iv) and (v) hold. (vi) follows directly from (iii) and (v). $\square$

Lemma 2.5 *We use Notation 2.1. We assume that $k = \mathbb{F}_q$. For any $\mathbb{F}_q$-point $P_i \in \mathbb{P}^2_{\mathbb{F}_q}(\mathbb{F}_q)$, let $L_{j_1}, \ldots, L_{j_{q+1}}$ be the $\mathbb{F}_q$-lines passing through P_i. Then $\mathbb{P}^2_{\mathbb{F}_q}(\mathbb{F}_q) = L_{j_1}(\mathbb{F}_q) \cup \cdots \cup L_{j_{q+1}}(\mathbb{F}_q)$.*

Proof Since we have $L_{j_\alpha} \cap L_{j_\beta} = P_i$ for any $1 \leqslant \alpha < \beta \leqslant q+1$, the claim follows by counting the number of $\mathbb{F}_q$-rational points (cf. Remark 2.3):

$$\#(L_{j_1} \cup \cdots \cup L_{j_{q+1}})(\mathbb{F}_q) = q(q+1)+1 = q^2+q+1 = \mathbb{P}^2_{\mathbb{F}_q}(\mathbb{F}_q). \qquad \square$$

3 Counter-examples to Kawamata–Viehweg vanishing

In this section, we construct some counter-examples to Kawamata–Viehweg vanishing on a family of smooth rational surfaces.

Theorem 3.1 *We use Notation 2.1. We consider the following $\mathbb{Q}$-divisors on X:*

- $\Delta = q/(q+1) \cdot \sum_{i=1}^{q^2+q+1} L'_i$, *and*
- $B = (q^2+1) f^*H - q \sum_{i=1}^{q^2+q+1} E_i$.

Then the following hold:

(i) (X, Δ) *is klt.*
(ii) $B - \Delta$ *is nef and big.*
(iii) $h^1(X, \mathcal{O}_X(K_X + B)) \geqslant (q^2 - q)/2$.

In particular, Kawamata–Viehweg vanishing fails on X.

Proof Since $L'_1, \ldots, L'_{q^2+q+1}$ are pairwise disjoint, (i) follows immediately. We now show (ii). We have

$$(q^2+q+1) f^*H \sim f^*\left(\sum_{i=1}^{q^2+q+1} L_i\right) = \sum_{i=1}^{q^2+q+1} L'_i + (q+1) \sum_{i=1}^{q^2+q+1} E_i.$$

It follows that

$$B = (q^2+1) f^*H - q \sum_{i=1}^{q^2+q+1} E_i \sim_{\mathbb{Q}} \frac{1}{q+1} f^*H + \frac{q}{q+1} \sum_{i=1}^{q^2+q+1} L'_i.$$

Thus, (ii) holds.

We now show (iii). By Riemann–Roch, it follows that

$$\chi(X, \mathcal{O}_X(K_X + B)) = 1 + \frac{1}{2}(B^2 + B \cdot K_X).$$

Since

$$B^2 = \left((q^2+1)f^*H - q\sum_{i=1}^{q^2+q+1} E_i\right)^2 = (q^2+1)^2 - q^2(q^2+q+1)$$
$$= -q^3 + q^2 + 1$$

and

$$B \cdot K_X = \left((q^2+1)f^*H - q\sum_{i=1}^{q^2+q+1} E_i\right) \cdot \left(-3f^*H + \sum_{i=1}^{q^2+q+1} E_i\right)$$
$$= -3(q^2+1) + q(q^2+q+1) = q^3 - 2q^2 + q - 3,$$

we have

$$\chi(X, K_X+B) = 1 + \frac{1}{2}\left((-q^3+q^2+1) + (q^3-2q^2+q-3)\right) = \frac{1}{2}(-q^2+q).$$

Thus, (iii) holds. □

Remark 3.2 We do not know whether there exist a klt del Pezzo surface X and a nef and big Cartier divisor A on X such that $H^1(X, \mathcal{O}_X(A)) \neq 0$.

As an application, we now show that the pair $\left(X, \sum E_i + \sum L'_j\right)$ is not liftable to $W_2(k)$. Note that, a similar result was proven in [18, Proposition 8.4].

Corollary 3.3 *We use Notation 2.1. Assume that k is perfect. If $p \geqslant 3$, then*

$$\left(X, \sum_{i=1}^{q^2+q+1} E_i + \sum_{j=1}^{q^2+q+1} L'_j\right)$$

is not liftable to $W_2(k)$.

Proof We use the same notation as in Theorem 3.1. As in the proof of Theorem 3.1, it follows that $B - \Delta - \sum \epsilon_i E_i$ is ample for some $\epsilon_i > 0$. Thus, Theorem 3.1 and [10, Corollary 3.8] imply the claim. □

4 Purely inseparable morphisms to $\mathbb{P}^2$

The main purpose of this section is to show that the surface Y, as in Notation 2.1, can be obtained as a purely inseparable cover of $\mathbb{P}^2$ (cf. Theorem 4.1). Moreover if $q = 2$, then the morphism $Y \to \mathbb{P}^2$ is induced by the anti-canonical linear system (cf. Remark 4.2).

We also show that the complete linear system $|M|$, appearing in Theorem 4.1, does not have any smooth element (cf. Proposition 4.3), even though it is base point free and big. We were not able to find a similar example in the literature (cf. [11, Theorem II.8.18 and Corollary III.10.9]).

Theorem 4.1 *We use Notation 2.1. Let*

$$M = (q+1)f^*H - \sum_{i=1}^{q^2+q+1} E_i.$$

Then the following hold:

(i) *$|M|$ is base point free.*
(ii) *$M \cdot L'_j = 0$ for any $j = 1, \ldots, q^2+q+1$.*
(iii) *$M^2 = q$.*
(iv) *Given the natural injective k-linear map*

$$\iota\colon H^0(X, \mathcal{O}_X(M)) \hookrightarrow H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(q+1)),$$

the following holds:

$$\iota(H^0(X, \mathcal{O}_X(M))) = k\cdot(x^q y - xy^q) + k\cdot(y^q z - yz^q) + k\cdot(z^q x - zx^q).$$

(v) *There exists a Cartier divisor M_Y on Y such that $M = g^*M_Y$.*
(vi) *The morphism induced by the complete linear system $|M_Y|$*

$$\varphi = \Phi_{|M_Y|}\colon Y \to \mathbb{P}^2_k$$

is a finite universal homeomorphism of degree q.

Proof We may assume that $k = \mathbb{F}_q$. We first show (i). Given a $\mathbb{F}_q$-point P_i on $\mathbb{P}^2_{\mathbb{F}_q}$, we denote by $L_{j_1}, \ldots, L_{j_{q+1}}$ the $\mathbb{F}_q$-lines passing through P_i. Then Lemma 2.5 implies that

$$M = (q+1)f^*H - \sum_{r=1}^{q^2+q+1} E_r \sim \sum_{\alpha=1}^{q+1} f^*L_{j_\alpha} - \sum_{r=1}^{q^2+q+1} E_r = qE_i + \sum_{\alpha=1}^{q+1} L'_{j_\alpha}.$$

Thus, $|M|$ is base point free by symmetry and (i) holds.

(ii) and (iii) are simple calculations, and (iv) follows from [27,28] (see also [13, Proposition 2.1][1]). Further, $g\colon X \to Y$ is the Stein factorisation of $\psi = \Phi_{|M|}\colon X \to \mathbb{P}^2_k$. Thus, (v) holds.

We now show (vi). Since $M = g^*M_Y$, (i) implies that $|M_Y|$ is base point free and (v) implies that $h^0(Y, \mathcal{O}_Y(M_Y)) = 3$. Since M_Y is ample, it follows that φ is a finite surjective morphism. By (iii), the degree of φ is equal to q.

[1] Note that we cite the arXiv version, as the published version omits the proof of [13, Proposition 2.1].

It is enough to show that φ is a purely inseparable morphism. To this end, we may assume that $k = \overline{\mathbb{F}}_q$. By (iv), we have that

$$\psi \circ f^{-1} \colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k, \qquad [x:y:z] \mapsto [x^q y - xy^q : y^q z - yz^q : z^q x - zx^q].$$

Generically, the rational map $\psi \circ f^{-1}$ can be written by

$$\Psi \colon \mathbb{A}^2_k \setminus \bigcup_{i=1}^{q+1} \widetilde{L}_i \to \mathbb{A}^2_k, \qquad (u, v) \mapsto \left(\frac{v^q - v}{u^q v - uv^q}, \frac{u - u^q}{u^q v - uv^q} \right),$$

where $\widetilde{L}_1, \dots, \widetilde{L}_{q+1}$ are the affine lines passing through the origin with coefficients in $\mathbb{F}_q$, and in particular $\bigcup_{i=1}^{q+1} \widetilde{L}_i = \{u^q v - uv^q = 0\}$. Fix a general closed point $(\alpha, \beta) \in \mathbb{A}^2_k$. It is enough to show that its fibre $\Psi^{-1}((\alpha, \beta))$ consists of one point. Let $(u, v) \in \mathbb{A}^2_k \setminus \bigcup_{i=1}^{q+1} \widetilde{L}_i$ be such that $\Psi(u, v) = (\alpha, \beta)$. Since (α, β) is chosen to be general, we can assume that the denominators of the fractions appearing in the following calculation are always nonzero. We have

$$\alpha(u^q v - uv^q) = v^q - v, \qquad \beta(u^q v - uv^q) = u - u^q,$$

which implies

$$\alpha(u^q - uv^{q-1}) = v^{q-1} - 1, \tag{1}$$

and

$$\beta(u^{q-1} v - v^q) = 1 - u^{q-1}. \tag{2}$$

By (1), we have

$$v^{q-1} = \frac{\alpha u^q + 1}{\alpha u + 1}. \tag{3}$$

Substituting (3) to (2), we get

$$v = \frac{1}{\beta} \frac{1 - u^{q-1}}{u^{q-1} - v^{q-1}} = \frac{1}{\beta} \frac{1 - u^{q-1}}{u^{q-1} - (\alpha u^q + 1)/(\alpha u + 1)} = -\frac{\alpha u + 1}{\beta}. \tag{4}$$

Substituting (4) to (3), it follows that

$$\alpha u^q + 1 = (\alpha u + 1) v^{q-1} = (\alpha u + 1) \left(-\frac{\alpha u + 1}{\beta} \right)^{q-1} = \frac{(-1)^{q-1}(\alpha^q u^q + 1)}{\beta^{q-1}},$$

which implies that

$$u^q = \frac{-\beta^{q-1} + (-1)^{q-1}}{\alpha\beta^{q-1} - (-1)^{q-1}\alpha^q}.$$

Hence u is uniquely determined by (α, β), and so is v by (4). Thus, (vi) holds. □

Remark 4.2 Using the same notation as in Theorem 4.1, if $q = 2$, then $M = -K_X$ and $M_Y = -K_Y$. This can be considered as an analogue of the fact that a smooth del Pezzo surface S with $K_S^2 = 2$ is a double cover of $\mathbb{P}^2$ which is induced by the anti-canonical system $|-K_X|$. Indeed, both X and S are obtained by taking blowups along seven points.

Proposition 4.3 *We use Notation 2.1. Let*

$$M = (q+1)f^*H - \sum_{i=1}^{q^2+q+1} E_i.$$

Then the following hold:

(i) *If $k = \mathbb{F}_q$, then for any element $D \in |M|$, there exists a unique $\mathbb{F}_q$-point P_i on $\mathbb{P}^2_{\mathbb{F}_q}$ such that*

$$D = qE_i + \sum_{\alpha=1}^{q+1} L'_{j_\alpha},$$

where $L_{j_1}, \ldots, L_{j_{q+1}}$ are the $\mathbb{F}_q$-lines passing through P_i.

(ii) *If k is an algebraically closed field, then a general member of $|M|$ is integral.*

(iii) *Any element of $|M|$ is not smooth.*

Proof Note that for each $\mathbb{F}_q$-point P_i on $\mathbb{P}^2_{\mathbb{F}_q}$, the divisor $D = qE_i + \sum_{\alpha=1}^{q+1} L'_{j_\alpha}$, as in (i), is an element of $|M|$. Thus, there are $q^2 + q + 1$ of such divisors. On the other hand, (iv) of Theorem 4.1 implies

$$\#|M| = \frac{q^3 - 1}{q - 1}.$$

Thus, (i) holds (see also [13, Proposition 2.3]).

We now show (ii) and (iii). To this end, we may assume that k is algebraically closed. We set $M_Y = g_*M$. By (i), there exists an irreducible divisor in $|M_Y|$. Thus, any general element of $|M_Y|$ is irreducible.

Since, by Theorem 4.1, $|M_Y|$ is base point free, if $D \in |M|$ is a general element, then D is irreducible. By Theorem 4.1, we may write

$$f_*D = \left\{\gamma(x^q y - xy^q) + \alpha(y^q z - yz^q) + \beta(z^q x - zx^q) = 0\right\}$$

for some $(\alpha, \beta, \gamma) \in k^3 \setminus \{(0, 0, 0)\}$. By the Jacobian criterion for smoothness, it follows that $[\alpha^{1/q} : \beta^{1/q} : \gamma^{1/q}]$ is a unique singular point of f_*D. Since f_*D is smooth outside $[\alpha^{1/q} : \beta^{1/q} : \gamma^{1/q}]$, we see that f_*D is reduced. Since α, β, γ are chosen to be general, it follows that $[\alpha^{1/q} : \beta^{1/q} : \gamma^{1/q}]$ is not an $\mathbb{F}_q$-point. Thus, D is the proper transform of f_*D, hence D is integral. Thus, (ii) holds. Since f_*D has a singular point outside $f(\mathrm{Ex}(f))$, it follows that D is not smooth. Thus, (iii) holds. □

5 The Kleimann–Mori cone

The main result of this section is Theorem 5.4 which determines the generators of the Kleimann–Mori cone of X as in Notation 2.2. To this end, we classify the curves whose self-intersection numbers are negative (cf. Proposition 5.3).

Lemma 5.1 *We use Notation 2.2. The following hold:*

(i) *If C is a curve on X which satisfies $C^2 = -1$ and differs from any of $E_1, \dots, E_7$, then* $\deg f_*(C) \leqslant 3$.
(ii) *If C is a curve on X with $C^2 = -2$, then* $\deg f_*(C) \leqslant 2$.

Proof We show (i). We have

$$C \sim a f^* \mathcal{O}_{\mathbb{P}^2}(1) + \sum_{i=1}^{7} b_i E_i,$$

where $a = \deg f_*(C) > 0$ and $b_1, \dots, b_7 \in \mathbb{Z}$. Since $q = 2$, Lemma 2.4 implies that C is a (-1)-curve. Thus, we have

$$\begin{aligned} -1 = C^2 = a^2 - \sum_{i=1}^{7} b_i^2 - 1 = K_X \cdot C \\ = \left(-3 f^* H + \sum_{i=1}^{7} E_i\right) \cdot \left(a f^* H + \sum_{i=1}^{7} b_i E_i\right) = -3a - \sum_{i=1}^{7} b_i. \end{aligned}$$

By Schwarz's inequality, we obtain

$$(3a-1)^2 = \left(\sum_{i=1}^{7} b_i\right)^2 \leqslant 7 \sum_{i=1}^{7} b_i^2 = 7(a^2+1),$$

which implies $a^2 - 3a - 3 \leqslant 0$. Thus, (i) holds. The proof of (ii) is similar. □

Lemma 5.2 *We use Notation 2.2. Let C be a curve on X such that $C_0 = f(C)$ is a conic or a cubic. Then $C^2 \geqslant 0$.*

Proof First, we assume that C_0 is conic. Suppose that C_0 passes through five of the $\mathbb{F}_2$-points, say $P_1, \dots, P_5$. Let us derive a contradiction. Let P_6 and P_7 be the remaining two $\mathbb{F}_2$-points. Since there are exactly three $\mathbb{F}_2$-lines passing through P_6 (respectively P_7), we can find an $\mathbb{F}_2$-line L_i such that $P_6 \notin L_i$ and $P_7 \notin L_i$. In particular, $C_0 \cap L_i$ contains at least three points, within $P_1, \dots, P_5$. This contradicts the fact that $C_0 \cdot L_i = 2$.

Now, we assume that C_0 is cubic. If C_0 is smooth, then $C^2 \geqslant C_0^2 - 7 = 2$. Thus, we may assume that C_0 is singular and $C^2 < 0$. It follows that C_0 must pass through all the $\mathbb{F}_2$-points $P_1, \dots, P_7$ and the unique singular point of C_0 is an $\mathbb{F}_2$-point, say P_1. Let L_j be an $\mathbb{F}_2$-line passing through P_1. Since $C_0 \cap L_j$ contains at least three

$\mathbb{F}_2$-rational points $P_1, P_i, P_{i'}$, we have that $C_0 \cdot L_j \geqslant 4$. This contradicts the fact that $C_0 \cdot L_j = 3$. Thus, the claim follows. □

Proposition 5.3 *We use Notation 2.2. Let C be a curve on X with $C^2 < 0$. Then C is equal to one of the curves $E_1, \ldots, E_7, L'_1, \ldots, L'_7$.*

Proof Assume that $C \notin \{E_1, \ldots, E_7\}$. Let $C_0 = f_*C$. Since $-K_X$ is nef and big, we have that $C^2 \geqslant -2$. Lemma 5.1 implies that $\deg C_0 \leqslant 3$. By Lemma 5.2, we have that $\deg C_0 = 1$, hence C_0 is a line. Then C_0 passes through at least two of the $\mathbb{F}_2$-points. It follows that C_0 is equal to some L_i, hence $C = L'_i$, as desired. □

Theorem 5.4 *We use Notation 2.2. Then*

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_{i=1}^{7} \mathbb{R}_{\geqslant 0}[E_i] + \sum_{j=1}^{7} \mathbb{R}_{\geqslant 0}[L'_j].$$

Proof Since there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample, the cone theorem [30, Theorem 1.7] implies that $\mathrm{NE}(X)$ is closed and generated by the extremal rays spanned by curves. By [31, Theorem 4.3], any extremal ray of $\mathrm{NE}(X)$ is generated by a curve C whose self-intersection number is negative. Thus, the claim follows from Proposition 5.3. □

6 Relation to Keel–M$^{\mathrm{c}}$Kernan surfaces

The goal of this section is to prove Proposition 6.4 which shows that the surface X, constructed in Notation 2.2, is isomorphic to some surface obtained by Keel–M$^{\mathrm{c}}$Kernan [14, end of Section 9].

We first recall their construction. Let k be a field of characteristic two. We fix a k-rational point in $\mathbb{P}^2_k$ and a conic over k as follows:

$$Q = [0:0:1] \in \mathbb{P}^2_k, \qquad C = \{xy + z^2 = 0\} \subset \mathbb{P}^2_k.$$

Note that any line through Q is tangent to C. Let $\varphi_0 \colon S_0 \to \mathbb{P}^2_k$ be the blowup at Q. We choose k-rational points $P_1, \ldots, P_d$ at $\varphi_0^{-1}(C)$. We first consider the blowup along these points $\psi \colon S'_0 \to S_0$ and then we take the blowup $S \to S'_0$ along the intersection $\mathrm{Ex}(\psi) \cap \psi_*^{-1}(\varphi^{-1}(C))$, where $\psi_*^{-1}(\varphi_0^{-1}(C))$ is the proper transform of $\varphi^{-1}(C)$. Note that the intersection $\mathrm{Ex}(\psi) \cap \psi_*^{-1}(\varphi^{-1}(C))$ is a collection of k-rational points. We call S a *Keel–M^{c}Kernan surface* of degree d over k.

Let us recall a well-known result on the theory of Severi–Brauer varieties.

Lemma 6.1 *Let X be a projective scheme over $\mathbb{F}_q$. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. If the base change $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is $\overline{\mathbb{F}}_q$-isomorphic to $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$, then X is $\mathbb{F}_q$-isomorphic to $\mathbb{P}^n_{\mathbb{F}_q}$.*

Proof See, for example, [25, Chapter X, Sections 5–7]. As an alternative proof, one can conclude the claim from [7, Corollary 1.2] and Châtelet's theorem [9, Theorem 5.1.3]. □

The following two lemmas may be well-known, however we include proofs for the sake of completeness.

Lemma 6.2 *Let k be a field. Take k-rational points $P_1, \ldots, P_4, Q_1, \ldots, Q_4 \in \mathbb{P}^2_k$. Assume that no three of $P_1, \ldots, P_4$ (respectively $Q_1, \ldots, Q_4$) lie on a single line of $\mathbb{P}^2_k$. Then there exists a k-automorphism $\sigma : \mathbb{P}^2_k \to \mathbb{P}^2_k$ such that $\sigma(P_i) = Q_i$ for any $i \in \{1, 2, 3, 4\}$.*

Proof We may assume that

$$P_1 = [1{:}0{:}0], \qquad P_2 = [0{:}1{:}0], \qquad P_3 = [0{:}0{:}1], \qquad P_4 = [1{:}1{:}1].$$

For each $i \in \{1, 2, 3, 4\}$, we write $Q_i = [a_i{:}b_i{:}c_i]$ for some $a_i, b_i, c_i \in k$. Consider the matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Since Q_1, Q_2, Q_3 do not lie on a line, it follows that $\det M \neq 0$. Let $\tau : \mathbb{P}^2_k \to \mathbb{P}^2_k$ be the k-automorphism induced by M. In particular,

$$\tau([1{:}0{:}0]) = Q_1, \qquad \tau([0{:}1{:}0]) = Q_2, \qquad \tau([0{:}0{:}1]) = Q_3.$$

We may write $\tau^{-1}(Q_4) = [d{:}e{:}f]$ for some $d, e, f \in k$. Again by the assumption, we have that $d, e, f \neq 0$. Then the k-automorphism

$$\rho : \mathbb{P}^2_k \to \mathbb{P}^2_k, \qquad [x{:}y{:}z] \mapsto [dx{:}ey{:}fz]$$

satisfies

$$\rho([1{:}0{:}0]) = [1{:}0{:}0], \qquad \rho([0{:}1{:}0]) = [0{:}1{:}0],$$
$$\rho([0{:}0{:}1]) = [0{:}0{:}1], \qquad \rho([1{:}1{:}1]) = [d{:}e{:}f].$$

Thus, the k-automorphism $\sigma = \tau \circ \rho$ satisfies $\sigma(P_i) = Q_i$ for any $i \in \{1, 2, 3, 4\}$. □

Lemma 6.3 *Let k be a field of characteristic two. Let C_1 and C_2 be smooth conics in $\mathbb{P}^2_k$. Assume that there exist distinct four k-rational points P_1, P_2, P_3, Q of $\mathbb{P}^2_k$ such that $\{P_1, P_2, P_3\} \subset C_1 \cap C_2$ and the tangent line T_{C_i, P_j} of C_i at P_j passes through Q for any $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Then $C_1 = C_2$.*

Proof By Lemma 6.2, we may assume that

$$P_1 = [1{:}0{:}0], \qquad P_2 = [0{:}1{:}0], \qquad P_3 = [1{:}1{:}1], \qquad Q = [0{:}0{:}1].$$

It is well known that C_1 and C_2 are strange curves (e.g. see [8, Theorem 1.1]). [8, Proposition 2.1] implies that for each $i \in \{1, 2\}$, C_i is defined by a quadric homogeneous polynomial:

$$a_i x^2 + b_i xy + c_i y^2 + d_i z^2 \in k[x, y, z].$$

Since $P_1, P_2, P_3 \in C_i$, we get $a_i = c_i = 0$ and $b_i = d_i$. In particular, both of C_1 and C_2 are defined by the same polynomial $xy + z^2$. □

Proposition 6.4 *Let k be a field of characteristic two. Then any Keel–M^c^Kernan surface S of degree 3 over k is k-isomorphic to the surface X constructed in Notation 2.2.*

Proof We use the same notation as above. Let $\pi : S_0 \to \mathbb{P}^1$ be the induced $\mathbb{P}^1$-fibration. We divide the proof into two steps.

Step 1. In this step, we show that any two Keel–McKernan surfaces S and S' of degree 3 over k are isomorphic over k.

There are three k-rational points $P_1, P_2, P_3 \in C$ (respectively $P'_1, P'_2, P'_3 \in C$) such that S (respectively S') is the blowup of S_0 twice along $P_1 \cup P_2 \cup P_3$ (respectively $P'_1 \cup P'_2 \cup P'_3$). Thanks to Lemma 6.2, there is a k-automorphism $\sigma : \mathbb{P}^2_k \to \mathbb{P}^2_k$ such that $\sigma(Q) = Q$ and $\sigma(P_i) = P'_i$ for $i = 1, 2$ and 3. Lemma 6.3 implies that $\sigma(C) = C$ and, in particular, σ induces a k-isomorphism $\widetilde{\sigma} : S \xrightarrow{\simeq} S'$, as desired.

Step 2. In this step, we assume that $k = \mathbb{F}_2$. Note that C has exactly three $\mathbb{F}_2$-rational points:

$$Q_1 = [1{:}0{:}0], \qquad Q_2 = [0{:}1{:}0], \qquad Q_3 = [1{:}1{:}1].$$

Let

$$P_1 = \varphi_0^{-1}(Q_1), \qquad P_2 = \varphi_0^{-1}(Q_2), \qquad P_3 = \varphi_0^{-1}(Q_3),$$

and S be the Keel–McKernan surface of degree 3 over $\mathbb{F}_2$ as above. We now show that S is $\mathbb{F}_2$-isomorphic to $X^{(0)}$ defined in Notation 2.2.

There are pairwise disjoint (-1)-curves $E_1, \ldots, E_7$ on S over $\mathbb{F}_2$, i.e. for any $i = 1, \ldots, 7$, E_i is $\mathbb{F}_2$-isomorphic to $\mathbb{P}^1_{\mathbb{F}_2}$ and satisfies $K_S \cdot E_i = E_i^2 = -1$. Indeed, we can check that the following seven curves listed below satisfy these properties.

- The exceptional curve over Q is a (-1)-curve over $\mathbb{F}_2$.
- For any $i = 1, 2, 3$, the exceptional curve over Q_i obtained by the second blowup is a (-1)-curve over $\mathbb{F}_2$.
- For any $1 \leqslant i < j \leqslant 3$, the proper transform of the $\mathbb{F}_2$-line, passing through Q_i and Q_j, is a (-1)-curve over $\mathbb{F}_2$.

Let $\psi : S \to T$ be the birational morphism with $\psi_* \mathcal{O}_S = \mathcal{O}_T$ that contracts $E_1, \ldots, E_7$. Since T is a projective scheme over $\mathbb{F}_2$ whose base change to $\overline{\mathbb{F}}_2$ is a projective plane, it follows that T is $\mathbb{F}_2$-isomorphic to $\mathbb{P}^2_{\mathbb{F}_2}$ by Lemma 6.1. Thus, S is obtained by the blowup along all the $\mathbb{F}_2$-rational points of $\mathbb{P}^2_{\mathbb{F}_2}$ which implies $S \simeq X^{(0)}$ (cf. Notation 2.2), as desired.

By Steps 1 and 2, we are done. □

7 Appendix: Kawamata–Viehweg vanishing for smooth del Pezzo surfaces

By Theorem 3.1, there exists a smooth weak del Pezzo surface of characteristic 2 which violates Kawamata–Viehweg vanishing. We now show that Kawamata–Viehweg vanishing holds on smooth del Pezzo surfaces.

Proposition 7.1 *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective surface over k such that $-K_X$ is ample and let (X, Δ) be a klt pair for some effective $\mathbb{Q}$-divisor Δ. Let D be a Cartier divisor such that $D-(K_X+\Delta)$ is nef and big. Then $H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 0$.*

Proof After perturbing Δ, we may assume that $D - (K_X+\Delta)$ is ample. We define $A = D - (K_X+\Delta)$. We run a $(\Delta + A)$-MMP $f : X \to Y$. Since $-K_X$ is ample, Y is also a smooth del Pezzo surface. Moreover, this MMP can be considered as a $(K_X+\Delta + A)$-MMP. By the Kawamata–Viehweg vanishing theorem for birational morphisms (cf. [16, Theorem 10.4], [29, Theorem 2.12]), it follows that

$$H^i(X, \mathcal{O}_X(D)) \simeq H^i(Y, f_*\mathcal{O}_X(D)) \simeq H^i(Y, \mathcal{O}_Y(f_*D))$$

for any i, where the latter isomorphism follows from the fact that f is obtained by running a D-MMP.

Therefore, after replacing X by Y, we may assume that $\Delta+A$ is nef. Thus, $D-K_X$ is nef and big. In this case, it is well-known that $H^i(X, \mathcal{O}_X(D)) = 0$ (e.g. see [21, Proposition 3.2] or [1, Proposition 3.3]). □

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