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RESEARCH ARTICLE

Symplectic instanton bundles on $\mathbb{P}^3$ and 't Hooft instantons

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Abstract We study the moduli space $I_{n,r}$ of rank-$2r$ symplectic instanton vector bundles on $\mathbb{P}^3$ with $r \geqslant 2$ and second Chern class $n \geqslant r+1$, $n-r \equiv 1 \pmod 2$. We introduce the notion of tame symplectic instantons by excluding a kind of pathological monads and show that the locus $I^*_{n,r}$ of tame symplectic instantons is irreducible and has the expected dimension equal to $4n(r+1)-r(2r+1)$. The proof is inherently based on a relation between the spaces $I^*_{n,r}$ and the moduli spaces of 't Hooft instantons.

Keywords Vector bundles · Symplectic bundles · Instantons · Moduli space

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1 Introduction

A *symplectic instanton vector bundle* of rank-$2r$ and charge n on the projective 3-space $\mathbb{P}^3$ is an algebraic vector bundle $E = E_{2r}$ of rank-$2r$ on $\mathbb{P}^3$ which is equipped with a symplectic structure $\phi\colon E \xrightarrow{\sim} E^\vee$, $\phi^\vee = -\phi$ and satisfies the vanishing conditions $h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$. The Chern classes $c_1(E)$ and $c_3(E)$ vanish, and we also assume $c_2(E) = n \geqslant 1$. We shall denote by $I_{n,r}$ the moduli space of symplectic (n, r)-instantons.

Rank-r symplectic instantons on $\mathbb{P}^3$ relate in a natural manner with "physical" $\mathbf{Sp}(r)$ instantons on the four-sphere S^4, i.e., connections on principal $\mathbf{Sp}(r)$-bundles on S^4 with self-dual curvature [1]; the moduli spaces of the former are in a sense a complexification of the moduli spaces of the latter. This relation is expressed by the so-called Atiyah–Ward correspondence [1,3], which relies on the fact that the projective space $\mathbb{P}^3$ is the twistor space of the four-sphere S^4. The present paper and its companion [7] are the first to study the geometry of the moduli spaces $I_{n,r}$. While [7] studied the case $n \equiv r \pmod 2$, with $n \geqslant r$, the present paper deals with the other case, $n \equiv r + 1 \pmod 2$, with $n \geqslant r + 1$. The main result of this paper is that a component $I^*_{n,r}$ of $I_{n,r}$ that is singled out by a certain open condition (which rules out some "badly behaved" monads) is irreducible.

We exploit as usual the monad method [2,4–6,8,11,12], which allows one to study instantons by means of hyperwebs of quadrics. Namely, we realize $I_{n,r}$ as the quotient space of a principal $\mathrm{GL}(H_n)/\{\pm \mathrm{id}\}$-bundle $\pi_{n,r}\colon MI_{n,r} \to I_{n,r}$, where $MI_{n,r}$ is a locally closed subset of the vector space $\mathbf{S}_n$ of hyperwebs of quadrics (precise definitions will be given later on). The tame locus $I^*_{n,r}$ being open in $I_{n,r}$, its irreducibility is equivalent to that of $MI^*_{n,r} = \pi^{-1}_{n,r}(I^*_{n,r})$. The key ingredient of our approach is the reduction of the last problem to that of certain sets Z_{n-r+1} (see Sect. 3). The sets Z_i as locally closed subsets of some vector spaces related to $\mathbf{S}_n$ were first defined in [9]. It is shown in [9, Section 9] that Z_i can be interpreted essentially as open subsets of certain affine bundles over the monad spaces M^{tH}_{2i-1} of 't Hooft rank-2 mathematical instantons of charge $2i - 1$—see more details in Sect. 3.2. Thus the irreducibility of Z_{n-r+1}, hence that of $I^*_{n,r}$, is reduced to the irreducibility of the moduli spaces of 't Hooft instantons of fixed charge, which is well known; see references in [9]. This nontrivial relation between the spaces $I^*_{n,r}$ and the moduli of 't Hooft instantons is crucial for the results in this paper. Note that this process of reduction from $I^*_{n,r}$ to the moduli of 't Hooft instantons somewhat resembles Barth's approach in [4] to the proof of irreducibility of the moduli space I_4 of instantons of charge 4. In that paper, Barth reduces the problem to the irreducibility of the space Q_n of commuting pairs of (good in some sense) pencils of quadrics for $n = 4$. In our case the role of spaces Q_n is played by the moduli spaces of 't Hooft instantons.

Notation and conventions. Throughout this paper, we consider an algebraically closed base field $\Bbbk$ of characteristic 0. All schemes will be Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme $\mathcal{X}$ we mean a closed point of a dense open subset of $\mathcal{X}$. An irreducible scheme is generically reduced if it is reduced at all general points. We follow the notation of [9]. So, we fix an integer $n \geqslant 1$, and denote by H_n and V fixed vector spaces over $\Bbbk$ of dimension n and 4, respectively, and set $\mathbb{P}^3 = P(V)$. Furthermore, $\mathbf{S}_n$ (the *space of hyperwebs of quadrics*) will denote the vector space $S^2H_n^\vee \otimes \wedge^2 V^\vee$. A hyperweb of quadrics $A \in \mathbf{S}_n$ is a skew-symmetric homomorphism $A\colon H_n \otimes V \to H_n^\vee \otimes V^\vee$, and we denote by W_A the vector space $H_n \otimes V/\ker A$ and by c_A the canonical epimorphism $H_n \otimes V \twoheadrightarrow W_A$. A choice of A induces a skew-symmetric isomorphism $q_A\colon W_A \xrightarrow{\sim} W_A^\vee$, and A is the composition

$$H_n \otimes V \overset{c_A}{\twoheadrightarrow} W_A \underset{\sim}{\xrightarrow{q_A}} W_A^\vee \overset{c_A^\vee}{\rightarrowtail} H_n^\vee \otimes V^\vee.$$

For any morphism of $\mathcal{O}_X$-sheaves $f\colon \mathcal{F} \to \mathcal{F}'$ we denote by the same letter f the induced morphism $\mathrm{id} \otimes f\colon U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$, and analogously, for any homomorphism $f\colon U \to U'$ of $\Bbbk$-vector spaces, the induced morphism $f \otimes \mathrm{id}\colon U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$. For $A \in \mathbf{S}_n$ we denote by a_A the composition

$$H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3},$$

where u is the tautological subbundle morphism. By abuse of notation, we denote by the same symbol a $\Bbbk$-vector space, say U, and the associated affine space $\mathbf{V}(U^\vee) = \mathrm{Spec}(\mathrm{Sym}^* U^\vee)$.

2 Explicit construction of symplectic instantons

In this section we provide some examples and recall some facts about $MI_{n,r}$, in particular, its relation with the moduli space $I_{n,r}$ of symplectic instantons, see [7, Section 3]. Let us consider the *set of* (n, r)*-instanton hyperwebs of quadrics*

$$MI_{n,r} = \left\{ A \in \mathbf{S}_n : \begin{array}{rl} \text{(i)} & \mathrm{rk}(A\colon H_n \otimes V \to H_n^\vee \otimes V^\vee) = 2n + 2r, \\ \text{(ii)} & \text{the morphism } a_A^\vee\colon W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ is surjective}, \\ \text{(iii)} & h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) = \ker(a_A^\vee \circ q_A)/\mathrm{im}\, a_A \end{array} \right\}. \tag{1}$$

Theorem 2.1 (i) *For each $n \geqslant 1$, the space $MI_{n,r}$ of (n, r)-instanton nets of quadrics is a locally closed subscheme of the vector space $\mathbf{S}_n$, given locally at any point $A \in MI_{n,r}$ by*

$$\binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1) \tag{2}$$

equations obtained as the rank condition (i) *in* (1).

(ii) *The natural morphism*

$$\pi_{n,r}\colon MI_{n,r} \to I_{n,r}, \qquad A \mapsto [E_{2r}(A)],$$

is a principal $\mathrm{GL}(H_n)/\{\pm\mathrm{id}\}$*-bundle in the étale topology. Hence* $I_{n,r}$ *is a quotient stack* $MI_{n,r}/(\mathrm{GL}(H_n)/\{\pm\mathrm{id}\})$*, and is therefore an algebraic space.*

The fibre $F_{[E]} = \pi_n^{-1}([E])$ over a point $[E] \in I_{n,r}$ is a principal homogeneous space of $\mathrm{GL}(H_n)/\{\pm\mathrm{id}\}$, so that the irreducibility of $(I_{n,r})_{\mathrm{red}}$ amounts to the irreducibility of the scheme $(MI_{n,r})_{\mathrm{red}}$. Besides, (2) yields

$$\begin{aligned} \dim_A MI_{n,r} &\geqslant \dim \mathbf{S}_n - \bigl(2n^2 - n(4r+1) + r(2r+1)\bigr) \\ &= n^2 + 4n(r+1) - r(2r+1) \end{aligned} \tag{3}$$

at all points $A \in MI_{n,r}$. Thus, $\dim_{[E]} I_{n,r} \geqslant 4n(r+1) - r(2r+1)$ at all points $[E] \in I_{n,r}$, as $MI_{n,r} \to I_{n,r}$ is an étale principal $\mathrm{GL}(H_n)/\{\pm\mathrm{id}\}$-bundle.

2.1 Symplectic $(n+1, n)$-instantons

We give a construction of symplectic $(n+1, n)$-instantons and describe their relation to usual rank-2 instantons with second Chern class $c_2 = 2n$. This will be established at the level of spaces of hyperwebs of quadrics $MI_{n+1,n}$ and $MI_{2n,1}$, regarded as spaces of monads.

Denote by $\mathrm{Isom}_{n+1,n-1}$ the set of all isomorphisms

$$\zeta\colon H_{n+1} \oplus H_{n-1} \xrightarrow{\sim} H_{2n}. \tag{4}$$

This is the principal homogeneous space of the group $\mathrm{GL}(2n)$. Moreover, for any $\zeta \in \mathrm{Isom}_{n+1,n-1}$, let $p_\zeta\colon \mathbf{S}_{2n} \twoheadrightarrow \mathbf{S}_{n+1}$ be the induced epimorphism, and, for any monomorphism $i\colon H_n \hookrightarrow H_{n+1}$, let $\mathrm{pr}_{(i)}\colon \mathbf{S}_{n+1} \to \mathbf{S}_n$ be the induced epimorphism.

Note that $MI_{2n,1}$ is irreducible [10, Theorem 1.1], and one has the following result [10, Theorem 3.1].

Theorem 2.2 *There exists a dense open subset* $MI^*_{2n,1}$ *of* $MI_{2n,1}$ *such that for any hyperweb* $A \in MI^*_{2n,1}$ *and a general* $\zeta \in \mathrm{Isom}_{n+1,n-1}$ *the rank of the homomorphism* $B = p_\zeta(A)\colon H_{n+1} \otimes V \to H^\vee_{n+1} \otimes V^\vee$ *coincides with the rank of* $A\colon H_{2n} \otimes V \to H^\vee_{2n} \otimes V^\vee$*:*

$$\mathrm{rk}\, B = \mathrm{rk}\, A = 4n + 2. \tag{5}$$

Set $W_{4n+2} = H_{2n} \otimes V/\ker A$ and define the skew-symmetric isomorphism $q_A\colon W_{4n+2} \xrightarrow{\sim} W^\vee_{4n+2}$ and the morphism of sheaves $a_A\colon H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3}$ with H_{2n} and W_{4n+2} taken instead of H_n and W_A, respectively. The morphism a_A and its transpose ${}^t a_A = a_A^\vee \circ q_A\colon W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \to H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ yield a monad

$$\mathcal{M}_A\colon\ 0 \to H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_A} H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with the cohomology sheaf $E_2(A)$, $[E_2(A)] \in I_{2n,1}$, see Theorem 2.1.

Let $i_\zeta: H_{n+1} \hookrightarrow H_{2n}$ be the monomorphism defined by the isomorphism (4). The composition

$$a_B: \ H_{n+1}\otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i_\zeta} H_{2n}\otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_{4n+2}\otimes \mathcal{O}_{\mathbb{P}^3}$$

and its transpose ${}^t a_B = a_B^\vee \circ q_A$ yield a monad

$$\mathcal{M}_B: \ 0 \to H_{n+1}\otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_B} W_{4n+2}\otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_B} H_{n+1}^\vee\otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with the cohomology sheaf

$$E_{2n}(B) = \ker {}^t a_B/\operatorname{im} a_B, \quad c_2(E_{2n}(B)) = n+1.$$

The symplectic isomorphism $q_A: W_{4n+2} \xrightarrow{\sim} W_{4n+2}^\vee$ induces a symplectic structure on $E_{2n}(B)$,

$$\phi_B: E_{2n}(B) \xrightarrow{\sim} E_{2n}(B)^\vee. \tag{6}$$

Moreover, (5) implies an isomorphism $H_{n+1}\otimes V/\ker B \simeq W_{4n+2}$, hence a monomorphism of spaces of sections

$$h^0({}^t a_B): \ W_{4n+2}\otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_B} H_{n+1}^\vee\otimes V^\vee$$

in the monad $\mathcal{M}_B$. Hence for this monad one has $h^0(E_{2n}(B)) = 0$. This together with (6) means that $E_{2n}(B)$ is a symplectic instanton

$$[E_{2n}(B)] \in I_{n+1,n}.$$

Note that, by construction, the monads $\mathcal{M}_A$ and $\mathcal{M}_B$ fit into the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_{n+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_B} & W_{4n+2}\otimes\mathcal{O}_{\mathbb{P}^3} & \underset{\cong}{\xrightarrow{q_A}} & W_{4n+2}^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_B^\vee} & H_{n+1}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
 & & \Big\downarrow i_\zeta & & \cong \Big\| & & w^\vee \Big\| \cong & & \Big\uparrow i_\zeta^\vee & & \\
0 & \longrightarrow & H_{2n}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_A} & W_{4n+2}\otimes\mathcal{O}_{\mathbb{P}^3} & \underset{\cong}{\xrightarrow{q_A}} & W_{4n+2}^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_A^\vee} & H_{2n}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0.
\end{array} \tag{7}$$

In view of (6) and the canonical isomorphism $H_{2n}/i_\zeta(H_{n+1}) \simeq H_{n-1}$, this diagram yields the quotient monad

$$\mathcal{M}_{A,B}: \ 0 \to H_{n-1}\otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_{A,B}} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi_B}} E_{2n}(B)^\vee \xrightarrow{a_{A,B}^\vee} H_{n-1}^\vee\otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

whose cohomology sheaf is $E_2(A) = \ker\left(a_{A,B}^\vee \circ \phi_B\right)/\operatorname{im} a_A$.

2.2 A special family of symplectic $(2n-r+1, r)$-instantons

For any integer r, $2 \leqslant r \leqslant n$, with $n \geqslant 2$, consider a monomorphism

$$\tau : H_{2n-r+1} \hookrightarrow H_{2n}$$

such that

$$\tau(H_{2n-r+1}) \supset i_\zeta(H_{n+1}). \tag{8}$$

The image of $A \in MI_{2n,1}$ under the projection $\mathbf{S}_{2n} \twoheadrightarrow \mathbf{S}_{2n-r+1}$ induced by τ produces a hyperweb of quadrics $A_\tau \in \mathbf{S}_{2n-r+1}$. This corresponds to a monad

$$\begin{aligned}\mathcal{M}_\tau :\ 0 \to H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_\tau} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \\ \xrightarrow{a_\tau^\vee \circ q_A} H^\vee_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,\end{aligned}$$

whose cohomology is the rank-$2r$ bundle

$$E_{2r}(A_\tau) = \ker\left(a_\tau^\vee \circ q_A\right)/\operatorname{im} a_\tau, \tag{9}$$

where $a_\tau = a_A \circ \tau$. The bundle $E_{2r}(A_\tau)$ has a natural symplectic structure

$$\phi_r : E_{2r}(A_\tau) \xrightarrow{\sim} E_{2r}(A_\tau)^\vee \tag{10}$$

induced by the antiselfduality of the monad $\mathcal{M}_\tau$. Moreover, by (8), the monad $\mathcal{M}_\tau$ can be included into diagram (7) as a middle row, thus obtaining a three-row commutative, anti-self-dual diagram. Thus, in addition to the monad $\mathcal{M}_{A,B}$, we also have the monads

$$\mathcal{M}'_\tau :\ 0 \to H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a'_\tau} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{a'^\vee_\tau} H^\vee_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

with the cohomology $E_{2r}(A_\tau) = \ker\left(a'^\vee_\tau \circ \phi\right)/\operatorname{im} a'_\tau$, and

$$\begin{aligned}\mathcal{M}''_\tau :\ 0 \to H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a''_\tau} E_{2r}(A_\tau) \underset{\simeq}{\xrightarrow{\phi_\tau}} E_{2r}(A_\tau)^\vee \\ \xrightarrow{a''^\vee_\tau} H^\vee_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,\end{aligned}$$

with the cohomology $E_2(A) = \ker\left(a''^\vee_\tau \circ \phi_\tau\right)/\operatorname{im} a''_\tau$.

Since $E_{2n}(B)$ is a symplectic instanton, $h^0(E_{2n}(B)) = h^i(E_{2n}(B)(-2)) = 0$, and the monad $\mathcal{M}'_\tau$ yields

$$h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \quad i \geqslant 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r + 1.$$

This, together with (10), means that $[E_{2r}(A_\tau)] \in I_{2n-r+1,r}$.

Remark 2.3 The maps τ lie in the set

$$N_{n,r} = \left\{\tau \in \operatorname{Hom}(H_{2n-r+1}, H_{2n}) : \tau \text{ is injective and } \operatorname{im}\tau \supset \operatorname{im} i_\zeta\right\}$$

which, for fixed $A \in MI_{2n,1}(\zeta)$, parameterizes a family of hyperwebs A_τ from $MI_{2n-r+1,r}$. Now, $N_{n,r}$ is a principal $\mathrm{GL}(H_{2n-r+1})$-bundle over an open subset of the Grassmannian $\mathrm{Gr}(n-r, n-1)$, so it is irreducible. As a result, the family of the three-row extensions of diagram (7) is parameterized by the irreducible variety $MI_{2n,1}(\zeta) \times N_{n,r}$. This in turn implies that the family $D_{n,r}$ of isomorphism classes of symplectic rank-$2r$ bundles obtained from these diagrams by (9) is an irreducible, locally closed subset of $I_{2n-r+1,r}$. It is not clear a priori if the closure of $D_{n,r}$ in $I_{2n-r+1,r}$ is an irreducible component of $I_{2n-r+1,r}$.

Let $2 \leqslant r \leqslant n$. For every monomorphism $i\colon H_n \hookrightarrow H_{2n-r+1}$, denote by $B(A, i)$ the image of $A \in MI_{2n-r+1,r}$ under the projection $\mathbf{S}_{2n-r+1} \twoheadrightarrow \mathbf{S}_n$ induced by i. It may be regarded as a homomorphism $B(A, i)\colon H_n \otimes V \to H_n^\vee \otimes V^\vee$.

Definition 2.4 We say that $A \in MI_{2n-r+1,r}$ *satisfies property* $(*)$ if there exists a monomorphism $i\colon H_n \hookrightarrow H_{2n-r+1}$ such that $B(A, i)$ is invertible.

This is an open condition on A. By Theorem 2.1, $\pi_{2n-r+1,r}\colon MI_{2n-r+1,r} \to I_{2n-r+1,r}$ is a principal bundle, so that, if an element $A \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies $(*)$, then any other point $A' \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies $(*)$. A symplectic instanton E_{2r} from $I_{2n-r+1,r}$ is said to be *tame* if some (hence all) $A \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies property $(*)$. This is an open condition on $[E_{2r}] \in I_{2n-r+1,r}$.

Remark 2.5 Using (8), we see that any $[E_{2r}] \in D_{n,r}$ is tame. We define

$$I^*_{2n-r+1,r} = I_{(1)} \cup \cdots \cup I_{(k)},$$

where $I_{(1)}, \ldots, I_{(k)}$ are the irreducible components of $I_{2n-r+1,r}$ whose general points are tame symplectic instantons. As $D_{n,r} \subset I^*_{2n-r+1,r}$ by definition, $I^*_{2n-r+1,r}$ is nonempty. If we define $MI^*_{2n-r+1,r} = \pi^{-1}_{2n-r+1,r}\left(I^*_{2n-r+1,r}\right)$, then the map $\pi_{2n-r+1,r}\colon MI^*_{2n-r+1,r} \to I^*_{2n-r+1,r}$ is a principal $\mathrm{GL}(H_{2n-r+1})/\{\pm 1\}$-bundle.

3 Irreducibility of $I^*_{2n-r+1,r}$

3.1 A dense open subset of $MI^*_{2n-r+1,r}$

We want to obtain the irreducibility of $I^*_{n,r}$ by reducing it to that of $X_{n,r}$, a dense open subset of $MI^*_{2n-r+1,r}$. The subset $X_{n,r}$ is a locally closed subset of the product of an affine space and an affine cone over a Grassmannian. Given an integer $n \geqslant 1$, we define the following dense open subset of $\mathbf{S}_n$:

$$\mathbf{S}_n^0 = \left\{A \in \mathbf{S}_n : A\colon H_n \otimes V \to H_n^\vee \otimes V^\vee \text{ an invertible map}\right\}.$$

We need some more notation. By definition, an element $B \in \mathbf{S}_n^0$ is an invertible anti-self-dual map $H_n \otimes V \to H_n^\vee \otimes V^\vee$. Its inverse $B^{-1}\colon H_n^\vee \otimes V^\vee \to H_n \otimes V$ is also anti-self-dual. Consider the vector space $\mathbf{\Sigma}_{n,r} = H_{n-r+1}^\vee \otimes H_n^\vee \otimes \wedge^2 V^\vee$. An element $C \in \mathbf{\Sigma}_{n,r}$ can be viewed as a linear map $C\colon H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee$, and its dual $C^\vee\colon H_n \otimes V \to H_{n-r+1}^\vee \otimes V^\vee$. As the composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, we can consider it as an element of $\wedge^2(H_{n-r+1}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-r+1} \oplus \wedge^2 H_{n-r+1}^\vee \otimes S^2 V^\vee$. Thus the condition

$$D - C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r+1}, \quad D \in \wedge^2\big(H_{n-r+1}^\vee \otimes V^\vee\big),$$

makes sense.

Under an arbitrary direct sum decomposition

$$\xi\colon H_n \oplus H_{n-r+1} \xrightarrow{\sim} H_{2n-r+1} \tag{11}$$

we can represent the hyperweb $A \in \mathbf{S}_{2n-r+1}$, regarded as a homomorphism

$$A\colon\ H_n \otimes V \oplus H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee \oplus H_{n-r+1}^\vee \otimes V^\vee,$$

as the $(8n-4r+4) \times (8n-4r+4)$-matrix of homomorphisms

$$A = \begin{pmatrix} A_1(\xi) & A_2(\xi) \\ -A_2(\xi)^\vee & A_3(\xi) \end{pmatrix}, \tag{12}$$

where

$$A_1(\xi) \in \mathbf{S}_n, \quad A_2(\xi) \in \mathbf{\Sigma}_{n,r} = \operatorname{Hom}\big(H_n, H_{n-r+1}^\vee\big) \otimes \wedge^2 V^\vee, \quad A_3(\xi) \in \mathbf{S}_{n-r+1}.$$

With this notation, decomposition (11) induces an isomorphism

$$\widetilde{\xi}\colon \mathbf{S}_{2n-r+1} \xrightarrow{\sim} \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} \oplus \mathbf{S}_{n-r+1}, \quad A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{13}$$

Let $\mathrm{Isom}_{n,r}$ be the set of all isomorphisms ξ in (11). According to Definition 2.4, there exists $\xi \in \mathrm{Isom}_{n,r}$ such that the set

$$\begin{aligned} &MI^*_{2n-r+1,r}(\xi) \\ &\quad = \big\{A \in MI_{2n-r+1,r} : A \text{ satisfies property } (*) \text{ for the monomorphism} \\ &\qquad\qquad i_\xi : H_n \hookrightarrow H_{2n-r+1} \text{ determined by } \xi\big\} \end{aligned}$$

is a dense open subset of $MI^*_{2n-r+1,r}$. Now take $A \in MI^*_{2n-r+1,r}(\xi)$ and consider A as a matrix of homomorphisms as in (12). By definition, the submatrix $A_1(\xi)$ is invertible. By a suitable elementary transformation we reduce the matrix A to an equivalent matrix $\widetilde{A}$ of the form

$$\widetilde{A} = \begin{pmatrix} \mathrm{id}_{H_n \otimes V} & A_1(\xi)^{-1} \circ A_2(\xi) \\ 0 & A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi) \end{pmatrix}.$$

Since $\operatorname{rk}\widetilde{A} = \operatorname{rk} A = 2(2n - r + 1) + 2r = 4n + 2$, we obtain the following relation between the matrices $A_1(\xi)$, $A_2(\xi)$ and $A_3(\xi)$:

$$\operatorname{rk}\left(A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi)\right) = 2. \tag{14}$$

Consider the embedding of the Grassmannian

$$G = \operatorname{Gr}\left(2, H_{n-r+1}^\vee \otimes V^\vee\right) \hookrightarrow P\left(\wedge^2\left(H_{n-r+1}^\vee \otimes V^\vee\right)\right),$$

and let $KG \subset \wedge^2(H_{n-r+1}^\vee \otimes V^\vee)$ be the affine cone over G. Set $KG^* = KG \setminus \{0\}$. We can now rewrite (14) as

$$A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi) \in KG^*, \tag{15}$$

where

$$A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) \in \wedge^2\left(H_{n-r+1}^\vee \otimes V^\vee\right), \quad A_3(\xi) \in \mathbf{S}_{n-r+1}. \tag{16}$$

Now consider the set

$$\widetilde{X}_{n,r} = \left\{(B, C, D) \in \mathbf{S}_n^0 \times \mathbf{\Sigma}_{n,r} \times KG^* : D - C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r+1}\right\}. \tag{17}$$

Since for an arbitrary point $y = (B, C, D) \in \widetilde{X}_n$ the point $\widetilde{\xi}^{-1}(B, C, D - C^\vee \circ B^{-1} \circ C)$ lies in $\mathbf{S}_{2n-r+1}$, it may be considered as a homomorphism $A_y : H_{2n-r+1} \otimes V \to H_{2n-r+1}^\vee \otimes V^\vee$ of rank $4n + 2$, and we have a well-defined $(4n+2)$-dimensional vector space $W_{4n+2}(y) = H_{2n-r+1} \otimes V / \ker A_y$ together with a canonical epimorphism $c_y : H_{2n-r+1} \otimes V \twoheadrightarrow W_{4n+2}(y)$ and an induced skew-symmetric isomorphism $q_y : W_{4n+2}(y) \xrightarrow{\sim} W_{4n+2}(y)^\vee$ such that $A_y = c_y^\vee \circ q_y \circ c_y$. Now, similarly to the morphism $a_A : H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3}$ (see Sect. 2.1), a morphism of sheaves

$$a_y = c_y \circ u : H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2}(y) \otimes \mathcal{O}_{\mathbb{P}^3}$$

is defined, together with its transpose ${}^t a_y = a_y^\vee \circ q_y : W_{4n+2}(y) \otimes \mathcal{O}_{\mathbb{P}^3} \to H_{2n-r+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$. We now introduce the following open subset $X_{n,r}$ of the set $\widetilde{X}_{n,r}$:

$$X_{n,r} = \left\{ y \in \widetilde{X}_{n,r} : \begin{array}{l} \text{(i)} \ \ {}^t a_y \text{ is epimorphic,} \\ \text{(ii)} \ \ [\ker {}^t a_y / \operatorname{im} a_y] \in I_{2n-r+1,r}^* \end{array} \right\}. \tag{18}$$

Since conditions (i) and (ii) on a point $y \in \widetilde{X}_{n,r}$ in (18) are open, from (15) and (16) we obtain the following result.

Proposition 3.1 *There exist a decomposition* $\xi \in \operatorname{Isom}_{n,r}$, *a dense open subset* $MI_{2n-r+1,r}^*(\xi)$ *of* $MI_{2n-r+1,r}^*$ *and an isomorphism of reduced schemes*

$$f_{n,r} : MI_{2n-r+1,r}^*(\xi) \xrightarrow{\sim} X_{n,r}, \quad A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).$$

The inverse isomorphism is given by the formula

$$f_{n,r}^{-1}\colon X_{n,r} \xrightarrow{\sim} MI^*_{2n-r+1,r}(\xi), \quad (B, C, D) \mapsto \widetilde{\xi}^{-1}\big(B, C, D - C^\vee \circ B^{-1} \circ C\big),$$

where $\widetilde{\xi}$ *is defined in* (13).

The following theorem will be proved in Sect. 3.2.

Theorem 3.2 *The set* $X_{n,r}$ *is irreducible of dimension* $(2n - r + 1)^2 + 4(2n - r + 1)(r + 1) - r(2r + 1)$.

Proposition 3.1 and Theorem 3.2 imply that $MI^*_{2n-r+1,r}$ is irreducible of dimension $(2n - r + 1)^2 + 4(2n - r + 1)(r + 1) - r(2r + 1)$ for any $n \geqslant 2$ and $2 \leqslant r \leqslant n$. Thus, for these values of n and r, the space $I^*_{2n-r+1,r}$ is irreducible and has dimension $4(2n - r + 1)(r + 1) - r(2r + 1)$. Substituting $2n - r + 1 \mapsto n$, we obtain the main result of this paper.

Theorem 3.3 *For any integer* $r \geqslant 2$ *and for any integer* $n \geqslant r + 1$ *such that* $n \equiv r + 1 \pmod 2$, *the moduli space* $I^*_{n,r}$ *of tame symplectic instantons is an open subset of an irreducible component of* $I_{n,r}$ *of dimension* $4n(r + 1) - r(2r + 1)$.

3.2 Proof of irreducibility of $X_{n,r}$

We prove now Theorem 3.2. Consider the set $\widetilde{X}_{n,r}$ defined in (17). Since $X_{n,r}$ is an open subset of $\widetilde{X}_{n,r}$, it is enough to prove the irreducibility of $\widetilde{X}_{n,r}$. In view of the isomorphism $\mathbf{S}_n^0 \xrightarrow{\sim} (\mathbf{S}_n^\vee)^0\colon B \mapsto B^{-1}$, we rewrite $\widetilde{X}_{n,r}$ as

$$\widetilde{X}_{n,r} = \big\{(B, C, D) \in (\mathbf{S}_n^\vee)^0 \times \mathbf{\Sigma}_{n,r} \times KG^* : D - C^\vee \circ B \circ C \in \mathbf{S}_{n-r+1}\big\}.$$

If a direct sum decomposition

$$H_n \xrightarrow{\sim} H_{n-r+1} \oplus H_{r-1}$$

has been fixed, any linear map

$$C \in \mathbf{\Sigma}_{n,r} = \mathrm{Hom}\big(H_{n-r+1}, H_n^\vee \otimes \wedge^2 V^\vee\big), \quad C\colon H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee,$$

can be represented as a homomorphism

$$C\colon\ H_{n-r+1} \otimes V \to H_{n-r+1}^\vee \otimes V^\vee \oplus H_{r-1}^\vee \otimes V^\vee,$$

and also as a block matrix

$$C = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{19}$$

with

$$\phi \in \mathrm{Hom}\big(H_{n-r+1}, H_{n-r+1}^{\vee}\big) \otimes \wedge^2 V^{\vee} = \mathbf{\Phi}_{n-r+1},$$
$$\psi \in \mathbf{\Psi}_{n,r} = \mathrm{Hom}\big(H_{n-r+1}, H_{r-1}^{\vee}\big) \otimes \wedge^2 V^{\vee}.$$

In the same way, any $B \in (\mathbf{S}_n^{\vee})^0 \subset \mathbf{S}_n^{\vee} = S^2 H_n \otimes \wedge^2 V \subset \mathrm{Hom}\big(H_n^{\vee} \otimes V^{\vee}, H_n \otimes V\big)$ can be represented as

$$B = \begin{pmatrix} B_1 & \lambda \\ -\lambda^{\vee} & \mu \end{pmatrix}, \tag{20}$$

with

$$\begin{aligned} B_1 \in \mathbf{S}_{n-r+1}^{\vee} \subset \mathrm{Hom}\big(H_{n-r+1}^{\vee} \otimes V^{\vee}, H_{n-r+1} \otimes V\big), \\ \lambda \in \mathbf{L}_{n,r} = \mathrm{Hom}\big(H_r^{\vee}, H_{n-r+1}\big) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_{r-1} = S^2 H_{r-1} \otimes \wedge^2 V. \end{aligned} \tag{21}$$

By (19) and (20), the composition

$$C^{\vee} \circ B \circ C \colon H_{n-r+1} \otimes V \to H_{n-r+1}^{\vee} \otimes V^{\vee}, \quad C^{\vee} \circ B \circ C \in \wedge^2\big(H_{n-r+1}^{\vee} \otimes V^{\vee}\big),$$

can be written in the form

$$C^{\vee} \circ B \circ C = \phi^{\vee} \circ B_1 \circ \phi + \phi^{\vee} \circ \lambda \circ \psi - \psi^{\vee} \circ \lambda^{\vee} \circ \phi + \psi^{\vee} \circ \mu \circ \psi. \tag{22}$$

In view of (19)–(21), we have

$$\mathbf{S}_n^{\vee} \times \mathbf{\Sigma}_{n,r} = \mathbf{S}_{n-r+1}^{\vee} \times \mathbf{\Phi}_{n-r+1} \times \mathbf{\Psi}_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_{r-1},$$

well-defined morphisms

$$\widetilde{p} \colon \widetilde{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_r \times KG, \quad (B_1, \phi, \psi, \lambda, \mu, D) \mapsto (\lambda, \mu, D),$$

and

$$p = \widetilde{p}|\overline{X}_{n,r} \colon \overline{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG.$$

Here $\overline{X}_{n,r}$ is the closure of $\widetilde{X}_{n,r}$ in $(\mathbf{S}_n^{\vee})^0 \times \mathbf{\Sigma}_{n,r} \times KG$. Moreover, we have

Proposition 3.4 *Let $n \geqslant 2$. For any $B \in (\mathbf{S}_n^{\vee})^0$ and for a general choice of the decomposition $H_n \simeq H_{n-r+1} \oplus H_{r-1}$, the block B_1 of B in (20) is nondegenerate.*

Proof By applying [9, Proposition 7.3], r times, one obtains a decomposition $H_n \xrightarrow{\sim} H_{n-r+1} \oplus H_{r-1}$ such that $B_1 \colon H_{n-r+1}^{\vee} \otimes V^{\vee} \to H_{n-r+1} \otimes V$ in (20) is nondegenerate, that is, $B_1 \in (\mathbf{S}_{n-r+1}^{\vee})^0$. □

If $\mathcal{X}$ is any irreducible component of $X_{n,r}$, taken with its reduced structure, and $\overline{\mathcal{X}}$ is its closure in $\overline{X}_{n,r}$, we pick up a point $z = (B_1, \phi, \psi, \lambda, \mu, D) \in \mathcal{X}$ not lying in the components of $X_{n,r}$ different from $\mathcal{X}$, and such that the decomposition $H_n \simeq$

$H_{n-r+1} \oplus H_{r-1}$ is general. Then, by Proposition 3.4, $B_1 \in (\mathbf{S}^\vee_{n-r+1})^0$. Consider the morphism

$$f\colon \mathbb{A}^1 \to \overline{\mathcal{X}}, \quad t \mapsto \big(B_1, t^2\phi, t\psi, t\lambda, t^2\mu, t^4 D\big), \qquad f(1) = z.$$

This is well defined as a consequence of (22). The point $f(0) = (B_1, 0, 0, 0, 0, 0)$ lies in the fibre $p^{-1}(0, 0, 0)$, so that $p^{-1}(0, 0, 0) \cap \overline{\mathcal{X}} \neq \varnothing$. In different terms,

$$\rho^{-1}(0, 0, 0) \neq \varnothing, \quad \text{where} \quad \rho = p|\overline{\mathcal{X}}. \tag{23}$$

By (22) and the definition of $\widetilde{X}_{n,r}$, one has

$$\widetilde{p}^{-1}(0, 0, 0) = \big\{(B_1, \phi, \psi) \in (\mathbf{S}^\vee_{n-r+1})^0 \times \mathbf{\Phi}_{n-r+1} \times \mathbf{\Psi}_{n,r} : \\ \phi^\vee \circ B_1 \circ \phi \in \mathbf{S}_{n-r+1}\big\}. \tag{24}$$

Now for each $i \geqslant 1$ consider the set Z_i mentioned in the introduction. This set Z_i is defined in [9, Section 7] as

$$Z_i = \big\{(B, \phi) \in (\mathbf{S}^\vee_i)^0 \times \mathbf{\Phi}_i : \phi^\vee \circ B \circ \phi \in \mathbf{S}_i\big\}, \tag{25}$$

and has a natural structure of closed subscheme of $(\mathbf{S}^\vee_i)^0 \times \mathbf{\Phi}_i$. The key point in the sequel is the fact that Z_i is an integral scheme of dimension $4i(i+2)$—see [9, Theorem 7.2]. This statement is based on the following relation between Z_i for $i \geqslant 2$ and the moduli space of 't Hooft instantons of charge $2i - 1$. Fix a monomorphism $j\colon H_{i-1} \hookrightarrow H_i$. For an arbitrary point $z = (B, \phi) \in Z_i$, let E_{2i} be a symplectic vector bundle of rank-$2i$ defined as a cokernel of a morphism of sheaves $\widetilde{B}\colon H_i \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_i^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ naturally induced by B. Let $s(z)\colon H_i \to H^0(E_{2i}(1))$ be the composition of ϕ understood as a homomorphism $H_i \to H_i^\vee \otimes \wedge^2 V^\vee$ and of the evaluation map $H_i^\vee \otimes \wedge^2 V^\vee \to H^0(E_{2i}(1))$, and let s_z be the composition

$$s_z\colon H_i \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s(z)} H^0(E_{2i}(1)) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\text{ev}} E_{2i},$$

where ev is the evaluation morphism. Using the symplecticity of E_{2i}, one obtains an antiselfdual monad

$$M(z)\colon\ 0 \to H_{i-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s_z \circ j} E_{2i} \xrightarrow{{}^t(s_z \circ j)} H_{i-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with a rank-2 cohomology vector bundle $E_2(z)$ with $c_1 = 0$ and $c_2 = 2i - 1$. A standard diagram chase yields a monomorphism $H_i/j(H_{i-1}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_2(z)$ showing that $h^0(E_2(z)(1)) \neq 0$, i.e., that $E_2(z)$ is a 't Hooft instanton vector bundle. Thus the association $z \rightsquigarrow M(z)$ yields a morphism of Z_i to the space M^{tH}_{2i-1} of the 't Hooft monads, which is irreducible since the moduli space of 't Hooft instantons of charge $2i - 1$ is known to be irreducible. It is shown in [9, Section 9] that this morphism $Z_i \to M^{\mathrm{tH}}_{2i-1}$ is a composition of a dense open embedding and the structure map of an affine bundle over M^{tH}_{2i-1}. This implies the irreducibility of Z_i.

Now, comparing (25) for $i = n - r + 1$ with (24), we obtain scheme-theoretic inclusions

$$\rho^{-1}(0,0,0) \subset p^{-1}(0,0,0) \subset \widetilde{p}^{-1}(0,0,0) = Z_{n-r+1} \times \boldsymbol{\Psi}_{n,r}. \tag{26}$$

By the above, Z_{n-r+1} is an integral scheme of dimension $4(n-r+1)(n-r+3)$. This together with (26) implies that

$$\begin{aligned} \dim \rho^{-1}(0,0,0) &\leqslant \dim p^{-1}(0,0,0) \leqslant \dim Z_{n-r+1} + \dim \boldsymbol{\Psi}_{n,r} \\ &= 4(n-r+1)(n-r+3) + 6(r-1)(n-r+1) \\ &= (n-r+1)(4n+2r+6). \end{aligned} \tag{27}$$

Hence, in view of (23),

$$\begin{aligned} \dim \overline{\mathcal{X}} &\leqslant \dim \rho^{-1}(0,0,0) + \dim \mathbf{L}_{n,r} + \dim \mathbf{M}_{r-1} + \dim KG \\ &\leqslant (n-r+1)(4n+2r+6) + 6(r-1)(n-r+1) \\ &\qquad + 3(r-1)r + (8n-8r+5) \\ &= (2n-r+1)^2 + 4(2n-r+1)(r+1) - r(2r+1). \end{aligned} \tag{28}$$

On the other hand, formula (3)—with n replaced by $2n-r+1$—and Proposition 3.1 show that, for any point $x \in \mathcal{X}$ such that $A = f_{n,r}^{-1}(x) \in MI^*_{2n-r+1,r}(\xi)$,

$$\begin{aligned} (2n-r+1)^2 + 4(2n-r+1)(r+1) - r(2r+1) &\leqslant \dim_A MI^*_{2n-r+1,r}(\xi) \\ &= \dim \overline{\mathcal{X}}. \end{aligned} \tag{29}$$

Comparing (28) with (29), we see that all inequalities in (27)–(29) are equalities. In particular,

$$\dim \rho^{-1}(0,0) = \dim (Z_{n-r+1} \times \boldsymbol{\Psi}_{n,r}) = \dim \overline{\mathcal{X}} - \dim (\mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG). \tag{30}$$

Since, by [9, Theorem 7.2], the scheme Z_{n-r+1} is integral and so $Z_{n-r+1} \times \boldsymbol{\Psi}_{n,r}$ is integral as well, (26) and (30) yield the coincidence of the integral schemes

$$\rho^{-1}(0,0,0) = p^{-1}(0,0,0) = \widetilde{p}^{-1}(0,0,0) = Z_{n-r+1} \times \boldsymbol{\Psi}_{n,r}. \tag{31}$$

We need now the following easy lemma, which is a slight generalization of [9, Lemma 7.4].

Lemma 3.5 *Let $f : X \to Y$ be a morphism of reduced schemes, with Y an integral scheme. Assume that there exists a closed point $y \in Y$ such that, for any irreducible component X' of X,*

(a) $\dim f^{-1}(y) = \dim X' - \dim Y$,

(b) *the scheme-theoretic inclusion of fibres* $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ *is an isomorphism of integral schemes.*

Then

(i) *there exists an open subset* U *of* Y *containing* y *such that the morphism* $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ *is flat, and*
(ii) X *is integral.*

By applying this lemma to $X = X_{n,r}$, $X' = \mathcal{X}$, $Y = \mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG$, $y = (0, 0)$, $f = p$, also in view of (30) and (31), one obtains that $X_{n,r}$ is integral and is of dimension

$$(2n - r + 1)^2 + 4(2n - r + 1)(r + 1) - r(2r + 1).$$

Theorem 3.2 is thus proved.

References

1. Atiyah, M.F.: Geometry of Yang–Mills Fields. Scuola Normale Superiore, Pisa (1979)
2. Atiyah, M.F., Hitchin, N.J., Drinfeld, V.G.: Manin, Yu.I: Construction of instantons. Phys. Lett. A **65**(3), 185–187 (1978)
3. Atiyah, M.F., Ward, R.S.: Instantons and algebraic geometry. Comm. Math. Phys. **55**(2), 117–124 (1977)
4. Barth, W.: Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2 = 4$. Math. Ann. **258**(1), 81–106 (1981)
5. Barth, W.: Lectures on mathematical instanton bundles. In: Dita, P., Georgescu, V., Purice, R. (eds.) Gauge Theories. Progress in Physics, vol. 5, pp. 177–206. Birkhäuser, Boston (1982)
6. Barth, W., Hulek, K.: Monads and moduli of vector bundles. Manuscripta Math. **25**(4), 323–347 (1978)
7. Bruzzo, U., Markushevich, D., Tikhomirov, A.S.: Moduli of symplectic instanton vector bundles of higher rank on projective space $\mathbb{P}^3$. Cent. Eur. J. Math. **10**(4), 1232–1245 (2012)
8. Horrocks, G.: Vector bundles on the punctured spectrum of a local ring. Proc. London Math. Soc. **s3–14**(4), 689–713 (1964)
9. Tikhomirov, A.S.: Moduli of mathematical instanton vector bundles with odd c_2 on projective space. Izv. Math. **76**(5), 991–1073 (2012)
10. Tikhomirov, A.S.: Moduli of mathematical instanton vector bundles with even c_2 on projective space. Izv. Math. **77**(6), 1195–1223 (2013)
11. Tyurin, A.N.: On the superpositions of mathematical instantons. In: Arithmetic and Geometry, II. Progress in Mathematics, vol. 36, pp. 430–450. Birkhäuser, Boston (1983)
12. Tyurin, A.N.: The structure of the variety of pairs of commuting pencils of symmetric matrices. Math. USSR-Izv. **20**(2), 391–410 (1983)