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Lagrange and the calculus of variations

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Abstract This paper gives a simple presentation in modern language of the theory of calculus of variations as invented by Euler and Lagrange, as well as an account of the history of its invention. The discussion will show how it serves to solve simple optimization problems and how it has influenced mathematics, physics and related fields up to the present day.

Keywords Lagrange · Euler · Calculus of variations · Mechanics · Partial differential equations

1 Introduction

Together with Euler, Lagrange is the inventor of the calculus of variations, a simple and elegant idea that revolutionised the way of solving problems of optimisation, the formulation of classical physics, and had an enormous influence on how partial derivatives equations are viewed. Used today almost as much as ordinary differential calculus, with all sorts of domains of application, the calculus of variations forms the basis of the mechanics known as Lagrangian, without which modern physics could not exist.

Here we will look at how Lagrange was led to his interest in these problems, discuss the simplest elements and principles of his discovery, and finally, show the repercussions they have had up to the present day.

The history of the calculus of variations and Lagrange's contribution to it is well documented. A good point of departure is the work of Catherine Goldstein [6]. The reader can also consult [2–4] for a more advanced epistemological analysis, as well as [9] regarding biographical elements.

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2 The first steps

In 1754, at the age of eighteen, Lagrange read the article "Une méthode pour trouver des lignes courbes jouissant de propriétés de maximum ou de minimum" [5] by the great Euler. Inspired by this lesson, he obtained his first original mathematical result, and dared to communicate it by letter to Euler, already at the time a leading figure in science. His letter remained unanswered.

Lagrange, however, was undeterred, and continued to reflect on Euler's article. In 1755 he wrote a second letter to Euler in which he described the new method that he had developed, that is, his own manner for dealing with the problem examined by Euler. That method would be named by Euler himself in one of his letters: "calculus of variations". This time, Euler replied to Lagrange, in terms of praise:

> Votre solution du problème des isopérimètres ne laisse rien à désirer, et je me réjouis que ce sujet, dont je m'étais presque seul occupé depuis les premières tentatives, ait été porté par vous au plus haut degré de perfection. L'importance de la matière m'a excité à en tracer, à l'aide de vos lumières, une solution analytique à laquelle je ne donnerai aucune publicité jusqu'à ce que vous-même ayez publié la suite de vos recherches, pour ne vous enlever aucune partie de la gloire qui vous est due.

> (Your solution to the isoperimetric problem leaves nothing to be desired, and I rejoice that this subject,

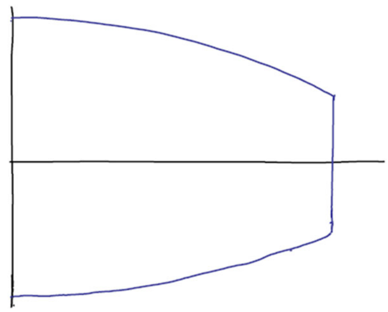

Fig. 1 Newton's geometric solution to the problem of the solid of revolution offering the least resistance to a fluid

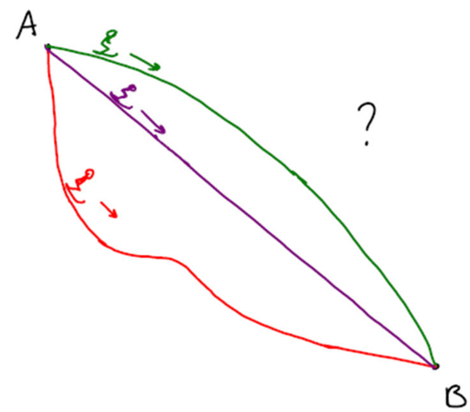

Fig. 2 The "ski slope" of the brachistochrone problem

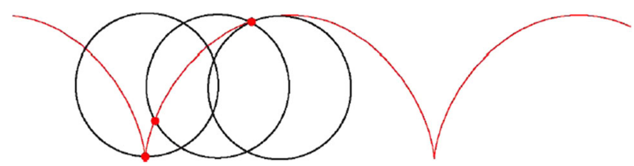

Fig. 3 The cycloid

> of which I was almost the only one who dealt with it since the first attempts, has been taken by you to the highest degree of perfection. The importance of the matter has led me to outline, with the aid of your light, an analytical solution to which I will give no publicity until you yourself have published the whole of your research, so that I do not take away any part of the glory that is due to you.)

This letter was enough of a recommendation to secure Lagrange a position as a teacher at the Royal School of Artillery in Turin.

The debut of the young Lagrange was a period of great activity. In 1758 he co-founded what would become the Academy of Sciences in Turin. He published at that time numerous articles in the *Miscellanea Taurinensia*, the first one of which, in 1762, was entitled "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies" [7]. It is this article that interests us the most because it already contains the foundations of the calculus of variations and the methods of multipliers called "Lagrange multipliers", ideas that would both be developed over the course of his career. However, Lagrange had a broad mathematical range and also wrote other articles at that time on different topics, including the vibrating string and differential equations.

3 Lagrange's contribution to the calculus of variations

The calculus of variations is a fundamental instrument for analysing the problems posed as problems of optimisation, and as such intervenes in numerous disciplines: geometry, physics, economy, engineering, etc. In geometry, the simplest, most classic question is without a doubt that of geodetics: how to find the shortest route from one point to another on the earth, or more generally, on a surface. In physics, the search for the trajectories that "minimise action"; in engineering the optimal shape of an object—such as the wing of a plane—to encounter the least possible resistance; in economics, the search to maximise a "utility function", and so forth. Examples from the very ancient to the most contemporary abound.

In general the history of science traces the genesis of the calculus of variations to the problem that Newton posed in 1685: find the shape of a solid of revolution offering the least resistance (in the direction of its axis) to a fluid. Newton himself proposed a purely geometric solution of what this shape looked like (Fig. 1); for some very recent developments on this problem the reader can consult [1].

The second problem that genuinely enthralled the mathematicians, and which was the true birth of the calculus of variations, is that of the *brachistochrone* (Greek: *brachis* = short, *brachiston* = the shortest, *chrone* = time).

This was a challenge (with the promise of a prize!) launched in 1699 by Johann Bernoulli[1]: find among all of the curves connecting two points A and B the one along which a particle falling from A and gliding under the effect of gravity arrives at B in the shortest time. It is thus asking us to determine, among all possible shapes of the ski slopes (for example) connecting points A and B, which one will permit the fastest run (ideally, without friction) (Fig. 2).

The greatest mathematicians of the day, that is Johann Bernoulli, his brother Jacob, Newton, then Leibniz, Euler and finally Lagrange, attacked the problem and gave solutions to it.

[1] Bernoulli realised afterwards that the problem had already been raised and examined, in an incomplete way, by Galileo.

The solution is an (inverse) cycloid, that is, a curve described by a fixed point on a circle that rolls without slipping on a straight line (Fig. 3).

Before Euler, the solutions were all of a geometric nature, and very ad hoc.

In [5], Euler is the first to propose a systematic treatment of this kind of problem: instead of concerning himself only with the problem of the brachistochrone, he seeks a method to find a curve that minimises or maximises any quantity expressed by an integral, and to derive the equation that must be satisfied by the minima. That equation, which has become known as the Euler–Lagrange equation, takes for example the form (in the notation of physics):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1}$$

We will discuss this in more detail equation below. Euler is also the first to transform this question of Bernoulli's into a domain of mathematics and to formulate the "principle of least action" (although the concept had already been introduced by Maupertuis).

More precisely, he wrote the problem as that of minimising an integral quantity of the form:

$$\int Z\left(x, u, Du, D^2u\right) dx$$

where $u(x)$ represents the curve, and Du and D^2u are the first and second derivatives of u (he even considers the case of dependencies and of derivatives of arbitrary order). He writes:

$$dZ = Mdx + Ndu + Pd(Du) + Qd(D^2u),$$

calculating the variation of Z when one moves the graph of u from $u(x)$ to $u(x)$ $+du(x)$, he obtains that the relation $N = \frac{dP}{dx}$ must be satisfied by a minimum. In other words, and since $N = \frac{\partial Z}{\partial u}$, he obtains the equation

$$\frac{\partial Z}{\partial u_i} = \frac{\partial}{\partial x}\frac{\partial Z}{\partial (Du)},$$

which is Eq. (1). Below we will see a more precise way to derive it.

Even though Euler's treatment is systematic for the first time, his argumentation for deriving the equation called "Euler–Lagrange" still remains somewhat geometric in spirit. This leaves Lagrange dissatisfied, and pushes him to improve the method.

Lagrange, who at the time was 19 years old, was greatly influenced by this work of Euler's. In his article [7], he described it as:

> ouvrage original et qui brille partout d'une profonde science de calcul. Cependant, quelque ingénieuse et féconde que soit sa méthode, il faut avouer qu'elle n'a pas toute la simplicité qu'on peut désirer dans un ouvrage de pure analyse. L'auteur lui-même le fait sentir par ces paroles: "il semble désirable de trouver une méthode indépendante de la géométrie"

> (an original work and one throughout which there shines a profound knowledge of calculus. However, as ingenious and fertile as its method is, we must admit that it has not all the simplicity that one might wish in a work of pure analysis. The author himself seems to feel this, by his words: "it seems desirable to find a method that is independent of geometry").

Lagrange's accomplishment was that he was able to find anew the results of Euler while freeing himself from geometric intuition (displacing the graph of the function), and replacing it with a "machinery" of operations of calculus. He had seen that Euler's calculus led to defining a new type of differential calculus, in which the objects are no longer functions of real variables, but functions of functions (today called functionals). This crucial conceptual leap (seeing the functions themselves as variables) is truly due to Lagrange, and can be seen as one of his fundamental contributions.

The new calculus of Lagrange consists in defining a new notion of derivative or of differential, this time for an integral expression on curves, which are none other than the particular case of functions of functions. This new differential, he denotes δ (and his notion of differential is thus called the "δ-calculus of Lagrange"), in order to distinguish it from the ordinary derivative of the differential calculus of Newton or Leibniz; for example, in his notation the differential of Z is denoted as δZ. Here is what he says in the *Essai* [7]:

> Maintenant voici une méthode qui ne demande qu'un usage fort simple des principes du calcul différentiel et intégral; mais avant tout je dois avertir que, comme cette méthode exige que les mêmes quantités varient de deux manières différentes, pour ne pas confondre ces variations, j'ai introduit dans mes calculs une nouvelle caractéristique δ. Ainsi δ exprimera une différence de Z qui ne sera pas la même que dZ, mais qui sera cependant formée par les mêmes règles…

> (Now here is a method that requires only a very simple use of the principles of differential and integral calculus, but first of all I must remark that, as this method requires that the same quantities vary in two different ways, so as not to confuse these variations, I introduced in my calculations a new character δ. Thus δ expresses a difference of Z which is not the same as dZ, but will be however formed by the same rules…).

Instead of the "curves of comparison" that Euler was very specialised in, Lagrange is free to make comparisons that are completely general. He also simplified all of Euler's proofs, and found his results anew, notably Eq. (1).

To prove that Lagrange had truly freed himself of geometrical intuition, one can read what he says in the preface to the *Méchanique analitique* [8]:

> On ne trouvera point de figures dans cet ouvrage. Le méthodes que j'y expose ne demandant ni constructions ni raisonnements géométriques, mais seulement des opérations algébriques assujetties à une marche régulière et uniforme

> (One will find no figures in this work. The method here expounded demands neither constructions nor geometric arguments, but only algebraic operations subject to a regular and uniform procedure).[2]

His method is systematic and sets out to reduce everything to the rules of calculus, essentially centred around his new symbol δ (this was for him an essential point of his contribution): δ commutes with d^m, δ commutes with $\int$, the integration by parts, etc., rules that were never rigorously justified, as was most often the case in the mathematics texts of that period. This aspect must nevertheless be qualified by the appearance of a preoccupation with this later in Lagrange's career, when he was teaching at the École Polytechnique (see [2]).

Also impressive is the generality that Lagrange sought and the number of examples that he dealt with. His *Essai* [7] treated variations that were completely general: where the extremities of the curves could vary, and deriving the conditions of the associated boundary, he treats the case with integral dependencies, of dependencies on x, of dependencies on higher derivatives. The treatment of the case with constraints: he already introduces the "Lagrange multipliers", which will become a fundamental notion. He proposes numerous applications; for example, he already derives for the first time the equation of the minimal surface for what came to be called the Plateau problem, or the question of finding a polygon of maximum area among all those with a given number of sides (it is the one inscribed in a circle) (Fig. 4).

As we have seen, Euler's reaction to Lagrange's progress was enthusiastic, and it was Euler who coined the term "calculus of variations" to denote the new method. Euler and Lagrange had corresponded long before the publication of Lagrange's first important result. As we saw previously, Euler had indicated that he would allow Lagrange the time to finalise and write up his method, so that all the glory would be his. He had, however, made

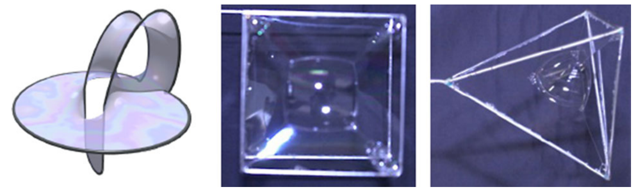

Fig. 4 Plateau's problem of the minimal surface

several communications (oral and written) without waiting for Lagrange's ideas, one to the Berlin Academy and the other to that of St. Petersburg, before the publication of Lagrange's memoir.

We can therefore trace back to Lagrange the importance given to the simple, unifying ideas and the primacy given to analysis, considered as the most accomplished form and superior to all other intuition, with an analytical approach that was almost "algebraic". With the *Méchanique analitique*, Lagrange himself said that his ambition was that of reducing mechanics to a branch of analysis!

4 A modern description of the method

Ordinary differential calculus, developed by Fermat and Newton and then finalised by Leibniz, makes it possible to say that if a function f of a variable x reaches its minimum in x, then the derivative of f must be null in x, that is, $f'(x) = 0$ in modern notation (or $\frac{df}{dx}(x) = 0$ in the notation of Leibniz). This is obtained by writing that:

$$f(x+h) \approx f(x) + hf'(x)$$

when h is small. The fact that $f(x + h) \geq f(x)$ for all small h implies that

$$f'(x) \cong \frac{f(x+h) - f(x)}{h} \geq 0 \text{ if } h \geq 0 \tag{2}$$

$$f'(x) \cong \frac{f(x+h) - f(x)}{h} \leq 0 \text{ if } h < 0 \tag{3}$$

In making h tend to 0, we thus obtain $f'(x) = 0$. Obviously this condition is not in fact exclusive to the minima of f; it is applied to the maxima, to local maxima and minima, and more generally to all critical points.

Let us now describe the class of problems that Euler and Lagrange studied. The brachistochrone problem can be written in modern notation: minimise in the graphs of $y(x)$ the quantity:

$$F(y) = \int_{xA}^{x_B} \sqrt{\frac{1 + (y'(x))^2}{2g(y_A - y(x))}}\, dx,$$

which can shown to closely corresponds to the time of the descent along the curve: the variation of potential energy

[2] For more on this, see [4].

$2\,g(y_A - y(x))$ corresponds to the square of the velocity, and $\sqrt{1+(y'(x))^2}$ is the length travelled along the curve; length by velocity gives a time.

The problem of geodetics corresponds to minimising the length of a curve that joins A and B, or if the curve is a graph $(x,f(x))$:

$$\int_{x_0}^{x_1} \sqrt{1+(f'(x))^2}dx.$$

In all these problems what is sought is to minimise an expression that depends on the curve, is the "Lagrangian". One sees that one can always put this expression in the form $L(u,u')$ for a function $(z, p) \to L(z, p)$, while trying to minimise:

$$u \to F(u) = \int_{x_0}^{x_1} L(u(x), u'(x))\,dx. \tag{4}$$

For the sake of simplicity, we will examine here only the simplest case where L depends only on u and u', since the dependencies on x and on derivatives higher than u can in fact be treated in the same way. Thus, for the brachistochrone, for example, the Lagrangian is $L(z,p) = \sqrt{\frac{1+p^2}{2g(y_A-z)}}$, while for geodetics it is $L(z,p) = \sqrt{1+p^2}$. The question dealt with by the calculus of Lagrange is thus to give a meaning to the equation $F'(u) = 0$, in analogy with ordinary differential calculus. The fundamental idea of Lagrange is to make, in analogy to the variation $x \to x + h$ used in differential calculus (see above), the "variations on the function u" itself, transforming $u(x)$ into $u(x) + hv(x)$, where v is another function and h a small number. If one wants the ends of the curve to be fixed, it is sufficient to require that the edge conditions

$$v(x_0) = 0 \quad \text{and} \quad v(x_1) = 0 \tag{5}$$

are satisfied. The optimal curve $y = u(x)$ can thus be compared to the curve after the small variation $y = u(x) + hv(x)$. For Lagrange, the function hv is a δu. Lagrange remarks that the derivative of the variation hv is equal to hv', that is, it is the variation of the derivative. He writes this as the rule:
$d\delta = \delta d$.

If u is a minimising function of F (4), then we have
$F(u+hv) \geq F(u)$

for all functions v and all numbers h. Dividing by h, we obtain:

$$\frac{F(u+hv)-F(u)}{h} \geq 0 \quad \text{for} \quad h > 0;$$
$$\frac{F(u+hv)-F(u)}{h} \leq 0 \quad \text{for} \quad h < 0,$$

that is,

$$\lim_{h\to 0}\frac{F(u+hv)-F(u)}{h} = 0,$$

if that limit exists. This is the equivalent, in Lagrange's notation, of:

$$\frac{\delta F}{\delta u}(u) = 0.$$

It remains to calculate, at least formally, this limit. In using the rules of "ordinary differential calculus" for several variables, one obtains that when $h \to 0$,

$$\begin{aligned} &F(u+hv) - F(u) \\ &= \int_{x_0}^{x_1} L(u(x)+hv(x), u'(x)+hv'(x)) - L(u(x),u'(x))\,dx \\ &\cong \int_{x_0}^{x_1}\left(\frac{\partial L}{\partial z}(u(x),u'(x))\cdot hv(x) - \frac{\partial L}{\partial p}(u(x),u'(x))\cdot hv'(x)\right)dx \end{aligned} \tag{6}$$

and integrating by parts, and using the edge conditions (5), we obtain:

$$\begin{aligned} &F(u+hv) - F(u) \\ &= h\int_{x_0}^{x_1}\left(\frac{\partial L}{\partial z}(u(x),u'(x)) - \left(\frac{\partial L}{\partial p}(u(x),u'(x))\right)'\right) \end{aligned} \tag{7}$$
$v(x)\,dx$ when $h \to 0$

Thus, if u minimises F, the fact that $\lim_{h\to 0}\frac{F(u+hv)-F(u)}{h} = 0$ makes it so that for all functions v that are very regular and satisfy (5), we have

$$\int_{x_0}^{x_1}\left(\frac{\partial L}{\partial z}(u(x),u'(x)) - \left(\frac{\partial L}{\partial p}(u(x),u'(x))\right)'\right)v(x)\,dx = 0.$$

Now one can prove the following.

Lemma 3.1 If $\int_{x_0}^{x_1} f(x)v(x)\,dx = 0$ *for all functions* $v \in C^1([x_0,x_1])$ that satisfy (5), then $f(x) = 0$ on $[x_0, x_1]$.

Obviously, the way in which we understand $f(x) = 0$ on $[x_0, x_1]$ depends on the regularity of f (in the worst case it is in the sense of distributions) but this is not the place to go into subtleties. The foregoing thus implies:

Theorem 1 *If* u *is a minimising function (or critical point) of* $F(u) = \int_{x_0}^{x_1} L(u(x),u'(x))\,dx$ *with fixed ends, then*

$$\frac{\partial L}{\partial z}(u(x),u'(x)) - \left(\frac{\partial L}{\partial p}(u(x),u'(x))\right)' = 0 \quad \text{on} \quad [x_0,x_1]$$

where ′ indicates the derivative with respect to the spatial variable x*, that is, that the Euler–Lagrange equation associated with the problem of minimisation is satisfied.*

It is important to underline that this equation is not equivalent to the fact that a minimising function exists; rather, it characterises the fact that there exists a critical point of F. Euler and Lagrange do not appear to be too concerned about making this distinction, nor the need to underline this point.

Replacing in our notation x for t, z for q and p for $\dot{q}$, we find once again the expression (1), the most used in physics.

In fact, the method proceeds by the function of several variables: in lieu of varying the curve, one varies the "surfaces".

Lagrange remarks that this method can be applied whether the ends are fixed or not. Let us now see the application of this result to the examples given previously. For the problem of geodesy, applying the results to $L(z,p) = \sqrt{1+|p|^2}$, since $\frac{\partial L}{\partial p} = \frac{p}{\sqrt{1+|p|^2}}$, we find that the associated Euler–Lagrange equation is:

$$\frac{u'}{\sqrt{1+(u')}^2} = const,$$

which implies that $u' = const$. We find that in Euclidean space the curve that gives the shortest path must be a straight line!

This works in higher dimensions by the area of a graph, and we obtain

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0,$$

which is the equation for minimal surfaces, derived for the first time by Lagrange in [7]! It is the equation associated with Plateau's problem of finding the surface of minimal area that has a given boundary.

In the case of the brachistochrone, reprising the equation $L(z,p) = \sqrt{\frac{1+p^2}{-2gz}}$, we calculate $\frac{\partial L}{\partial p} = \frac{p}{\sqrt{-2gz(1+p^2)}}$. Moreover, we can see that the Euler–Lagrange equation implies that

$$L(u(x), u'(x)) - u'(x)\frac{dL}{dp}(u(x), u'(x)) = const.$$

(It is sufficient to derive this expression with respect to x and to verify that the derivative is null). Inserting the expression for L and $\frac{\partial L}{\partial p}$ and supposing for the sake of simplicity that $y_A = 0$, we find

$$\sqrt{\frac{1+y'^2}{-2gy}} - \frac{(y'^2)}{\sqrt{-2gy(1+(y\prime)^2)}} = const.$$

Multiplying by $\sqrt{-2gy(1+(y')^2)}$ and simplifying, we obtain

$$y(1+(y')^2 = const,$$

the differential equation of an inverted cycloid, generated by a circle of diameter *const.*

We shall conclude with a fundamental example, because it is the one that led to the whole development of the *Méchanique analitique* [8]: that of the principle of least action. The action along a path $X(t)$ for a particle in a potential V is defined by

$$F(X(t)) = \int_{t_0}^{t_1} \frac{1}{2}m|X'|^2(t) - V(X(t))\,dt \cdot L(z,p)$$
$$= \frac{1}{2}m|p|^2 - V(z)$$

where we recognise the integral of the difference between a kinetic energy and a potential energy. Since $\frac{\partial L}{\partial z} = -V'(z)$ and $\frac{\partial L}{\partial' e} = mp$, the Euler–Lagrange equation is

$$mX''(t) = -V'(X(t)),$$

which is none other than Newton's law $\vec{F} = m\vec{\gamma}$ with $\vec{F}$ the force, m the mass, and $\vec{\gamma}$ the acceleration. Here we find a first form of the principle of least action, which says that the particles follow the course that minimises the action. Incidentally, this point is not clear, because we recall that the Euler–Lagrange equation does not necessarily characterize minimizers; we should say rather that the particles that obey Newton's law follow the trajectories that are *critical points* of the action.

4.1 The extensions due to Lagrange

In his article "Essai d'une nouvelle methode pour determiner les maxima et les minima des formules integrals indefinies" [7], Lagrange also introduced the method of multipliers—today called Lagrange multipliers—which makes it possible to treat problems with constraints. An example of such a problem is the isoperimetric problem: find a curve of given length that delimits a domain of maximum area (in the plane); the solution to this classic problem is the circle. Another example, a variant of the geodetic problem, is the geodetic problem with an obstacle: find the shortest path from A to B that goes around an obstacle, a bounded set Ω in space. Lagrange's method of variation tells us that the geodesic "takes off" from the obstacle in such a way that it remains tangent to it. This is a particular case of a variational problem that was well-studied in the twentieth century, called the "obstacle problem" (Fig. 5).

Examples in economics of the problem under constraints abound: maximising the utility function under constraints of resources, etc.

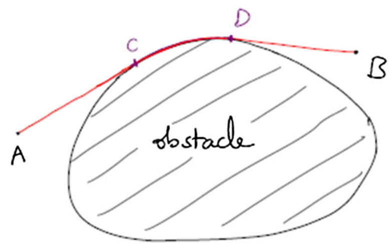

Fig. 5 The obstacle problem

The general framework is as follows. We try to minimise a function of the type:

$$F(u) = \int_{x_0}^{x_1} L(u(x), u'(x))\,dx$$

under an integral constraint of the type:

$$G(u) = \int_{x_0}^{x_1} K(u(x), u'(x))\,dx.$$

One example might be the constraint of total mass $\int_{x_0}^{x_1} u^2(x)\,dx = 1$. The idea is to authorise the variations $u(x) + hv(x)$ that do not affect (in first order) the constraint, that is, such that $\frac{\delta G}{\delta u}(v) = 0$. Carrying out the calculation as above, we find that there is a constant $\lambda \in$ P such that

$$\frac{\delta F}{\delta u}(u) = \lambda \frac{\delta G}{\delta u}(u)$$

in Lagrange's notation. The number λ is called the "Lagrange multiplier". The method never gives its value, but this can be found by indirect means.

5 Later developments

5.1 Hamiltonian and quantum mechanics

Lagrange invented the "Lagrangian" formulation of Newtonian mechanics. In the nineteenth century, Hamilton proposed the "Hamiltonian" formulation, which is a re-writing of Lagrangian mechanics, a kind of a dual point of view. The "Hamiltonian" is in fact the Legendre transform of the Lagrangian formulation. Even though the equations of dynamics associated are the same [still (1)], in discovering the symmetry between the coordinates of position and impulsion and symplectic structure to which it is associated, the Hamiltonian formulation of mechanics paves the way in its turn for quantum mechanics. It can be said without any doubt that without having passed through Lagrange, the mathematical formulation of quantum mechanics would not have been possible.

5.2 The most rigorous versions

We have already mentioned that the proofs in the articles by Euler and Lagrange are far from satisfactory according to contemporary standards in matters of rigour. In fact, such standards arose with the axiomatisation of analysis undertaken by Weierstrass in the nineteenth century.

It should be noted in particular that neither Euler nor Lagrange had taken care to prove the existence of a minimising function. The discussion of this aspect would have to await the work of David Hilbert around 1900. The question (with which they never concerned themselves) of guaranteeing that the conditions found give a minimum, rather than a maximum or a critical point, was addressed shortly after Lagrange by Legendre. He defined the conditions such that a solution of an equation called Euler–Lagrange is a stable critical point. This work would be extended by that of Jacobi in the nineteenth century.

(Translated from the French by Kim Williams)

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