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On exceptional zeros of Garrett–Hida *p*-adic *L*-functions

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To Bernadette Perrin-Riou on the occasion of her 65th birthday.

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Abstract
This article proves a case of the p-adic Birch and Swinnerton-Dyer conjecture for Garrett p-adic L-functions of [6], in the exceptional zero setting of extended analytic rank 2.

Résumé
Cet article prouve un cas de la conjecture p-adique de Birch et Swinnerton-Dyer pour les fonctions L p-adiques de Garrett formulée dans [6], dans le cadre de zéros exceptionnels de rang analytique étendu égal à 2.

Keywords Birch and Swinnerton-Dyer Conjecture · p-adic L-functions · Exceptional zeros

Mathematics Subject Classification 11F67 (11G40 11G35)

Introduction

Let A be an elliptic curve defined over $\mathbf{Q}$, having ordinary reduction at a rational prime $p > 3$. Let ϱ_1 and ϱ_2 be odd, irreducible, two-dimensional Artin representations of the absolute Galois group of $\mathbf{Q}$, which are unramified at p and satisfy the self-duality condition

$$\det(\varrho_1) = \det(\varrho_2)^{-1}.$$

By modularity, the triple $(A, \varrho_1, \varrho_2)$ arises from a triple (f, g, h) of cuspidal p-ordinary newforms of weights $w_o = (2, 1, 1)$. Let f_α be the ordinary p-stabilisation of f, and fix

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p-stabilisations g_α and h_α of g and h respectively. Set $\varrho = \varrho_1 \otimes \varrho_2$. In the recent paper [6] we proposed a p-adic analogue of the Birch and Swinnerton-Dyer conjecture for the leading term at w_o of the 3-variable Garrett–Hida p-adic L-function $L_p^{\alpha\alpha}(A, \varrho) = L_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ associated with the triple $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ of Hida families specialising to $(f_\alpha, g_\alpha, h_\alpha)$ at w_o. In this article we verify our conjecture in the analytic rank-zero exceptional cases, viz. when the complex Garrett L-function $L(A, \varrho, s) = L(f \otimes g \otimes h, s)$ does not vanish at $s = 1$ and $L_p^{\alpha\alpha}(A, \varrho)$ has an exceptional zero at w_o in the sense of Mazur–Tate–Teitelbaum (cf. Theorem 2.1 and Sect. 2.1 below). Moreover, when $L(A, \varrho, 1) = 0$ and $L_p^{\alpha\alpha}(A, \varrho)$ has an exceptional zero, we propose a conjecture relating the value at w_o of the fourth partial derivative of $L_p^{\alpha\alpha}(A, \varrho)$ along the $\boldsymbol{f}$-direction to the p-adic logarithms of two global points on A rational over the number field cut out by ϱ (cf. Conjecture 2.3).

1 Setting and notations

Fix algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$ respectively, and field embeddings $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ and $i_\infty : \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$. With the notations of the Introduction, let

$$\xi = \sum_{n\geq 1} a_n(\xi) \cdot q^n \in S_u(N_\xi, \chi_\xi)_{\bar{\mathbf{Q}}}$$

denote one of the cuspidal newforms f, g and h. Here u and N_ξ are the weight and the conductor of ξ respectively, and $S_u(N_\xi, \chi_\xi)_F$ is the space of cuspidal modular forms of level $\Gamma_1(N_\xi)$, weight u, character χ_ξ and Fourier coefficients in the subfield F of $\bar{\mathbf{Q}}_p$. Fix a number field $\mathbf{Q}(\varrho)$ containing for any ξ the Fourier coefficients $a_n(\xi)$, as well as the roots α_ξ and β_ξ of the pth Hecke polynomials $P_{\xi,p} = X^2 - a_p(\xi) \cdot X + \chi_\xi(p) \cdot p$. Let V_{ϱ_i} be a two-dimensional $\mathbf{Q}(\varrho)$-vector space affording the representation ϱ_i, and let K_ϱ be a Galois number field such that ϱ_i factors through $\mathrm{Gal}(K_\varrho/\mathbf{Q})$. Set

$$V_\varrho = V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_2} \quad \text{and} \quad V_p(A, \varrho) = V_p(A) \otimes_{\mathbf{Q}} V_\varrho,$$

where $V_p(A) = H^1_{\acute{e}t}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$ is the p-adic Tate module of A with $\mathbf{Q}_p$-coefficients. Throughout this note we make the following

Assumption 1.1 1. (Self-duality) The characters χ_g and χ_h are inverse to each other.
2. (Local signs) The conductors N_g and N_h are coprime to $p \cdot N_f$.
3. (Étaleness) The forms g and h are cuspidal, p-regular and do not have RM by a real quadratic field in which p splits.

The first condition is a reformulation of the self-duality condition mentioned in the Introduction, namely $\det(\varrho_1) = \det(\varrho_2)^{-1}$. Recall that the form ξ is *p-regular* if $P_{\xi,p}$ has distinct roots. Moreover, one says that a weight-one eigenform has *RM (real multiplication)* if it is the theta series associated with a ray class character of a real quadratic field. Assumption 1.1.3 is equivalent to require that V_{ϱ_i} is irreducible, not isomorphic to $\mathrm{Ind}_K^{\mathbf{Q}}\chi$ for a finite order character $\chi : G_K \longrightarrow \mathbf{Q}(\varrho)^*$ of a real quadratic field K in which p splits, and that an arithmetic Frobenius at p acts on V_{ϱ_i} with distinct eigenvalues. For $\xi = g, h$, this assumption guarantees that the p-adic Coleman–Mazur–Buzzard eigencurve of tame level N_ξ is étale over the weight space at the points corresponding to the p-stabilisations of ξ (cf. [2]). It is used in [6] to construct the Garrett–*Nekovář* height $\langle\langle \cdot, \cdot \rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ which appears in the main result of this note. To explain the relevance of Assumptions 1.1.1 and 1.1.2, let α_f be the unit root of $P_{f,p}$ and fix roots α_g and α_h of $P_{g,p}$ and $P_{h,p}$ respectively. Fix a finite extension

L of $\mathbf{Q}_p$ containing $\mathbf{Q}(\varrho)$ and the roots of unity of order $\mathrm{lcm}(N_f, N_g, N_h)$. Let ξ be one of f, g and h, and let u_o be the weight of ξ. According to the results of [2,10,18], there exists a unique Hida family

$$\boldsymbol{\xi}_\alpha = \sum_{n\geq 1} a_n(\boldsymbol{\xi}_\alpha)\cdot q^n \in \mathscr{O}_{\boldsymbol{\xi}}[\![q]\!]$$

which specialises at u_o to the p-stabilised newform

$$\xi_\alpha = \xi(q) - \frac{\chi_\xi(p)p^{u-1}}{\alpha_\xi}\cdot \xi(q^p) \in S_{u_o}(p\cdot M_\xi, \chi_\xi)_L.$$

Here $M_\xi = N_\xi/p^{\mathrm{ord}_p(N_\xi)}$ is the tame level of ξ (so that $M_\xi = N_\xi$ if $\xi = g, h$), and $\mathscr{O}_{\boldsymbol{\xi}}$ is the ring of bounded analytic functions on a (sufficiently small) connected open disc $U_{\boldsymbol{\xi}}$ in the p-adic weight space over L. For each classical weight u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geq 3}$, the weight-$u$ specialisation $\boldsymbol{\xi}_{\alpha,u} = \sum_{n\geq 1} a_n(\boldsymbol{\xi}_\alpha)(u)\cdot q^n \in L[\![q]\!]$ of $\boldsymbol{\xi}_\alpha$ is the q-expansion of the ordinary p-stabilisation of a newform ξ_u in $S_u(M_\xi, \chi_\xi)_L$. Since f has a unique p-ordinary p-stabilisation f_α, we simply write $\boldsymbol{f}$ for $\boldsymbol{f}_\alpha$.

Assumption 1.1.1 guarantees that for each classical triple $w = (k, l, m)$ in the set

$$\Sigma = U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3_{\geq 1}$$

the complex Garrett L-function $L(f_k \otimes g_l \otimes h_m, s)$ admits an analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation relating its values at s and $k+l+m-2-s$, with root number $\varepsilon(w) = \prod_{\ell\leq\infty} \varepsilon_\ell(w)$ equal to $+1$ or to -1. Assumption 1.1.2 implies that all the local signs $\varepsilon_\ell(w)$ are equal to $+1$ for every w in the *f-unbalanced region* $\Sigma_f = \{w = (k, l, m) \in \Sigma : k \geq l + m\}$ (cf. [11]). Under these assumptions, [12] associates with $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ an analytic function

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$$

in the ring $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}}\hat{\otimes}_L\mathscr{O}_{\boldsymbol{g}}\hat{\otimes}_L\mathscr{O}_{\boldsymbol{h}}$, whose square

$$L_p^{\alpha\alpha}(A, \varrho) = L_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^2$$

satisfies the following interpolation property. For each $w = (k, l, m)$ in Σ_f, the value of $L_p^{\alpha\alpha}(A, \varrho)$ at w is an explicit non-zero complex multiple of

$$\left(1 - \frac{\beta_k\alpha_l\alpha_m}{p^{c_w}}\right)^2\left(1 - \frac{\beta_k\beta_l\alpha_m}{p^{c_w}}\right)^2\left(1 - \frac{\beta_k\alpha_l\beta_m}{p^{c_w}}\right)^2\left(1 - \frac{\beta_k\beta_l\beta_m}{p^{c_w}}\right)^2\cdot L(f_k\otimes g_l\otimes h_m, c_w). \tag{1}$$

Here $c_w = \frac{k+l+m-2}{2}$, and for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ one denotes by α_u the unit root of $P_{\xi_u,p}$ and sets $\beta_u\cdot\alpha_u = \chi'_\xi(p)\cdot p^{u-1}$, where χ'_ξ is the prime-to-p part of χ_ξ (so that $\chi'_\xi = \chi_\xi$ for $\xi = g, h$, and χ'_f is the trivial character modulo M_f). We refer to Theorem A of loc. cit. for the precise interpolation formula. We call $L_p^{\alpha\alpha}(A, \varrho) = L_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ the *Garrett–Hida p-adic L-function* associated with (A, ϱ) (or with $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$).

2 Exceptional zero formulae

The p-adic variant of the Birch and Swinnerton-Dyer conjecture formulated in [6] predicts that the leading term of $L_p^{\alpha\alpha}(A, \varrho)$ at $w_o = (2, 1, 1)$ is encoded by the discriminant of the

Garrett– Nekovář height pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha} : A^\dagger(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2 \tag{2}$$

constructed in Section 2 of loco citato, where $\mathscr{I}$ is the ideal of functions in $\mathscr{O}_{\boldsymbol{fgh}}$ which vanish at w_o and the p-extended Mordell–Weil group $A^\dagger(K_\varrho)^\varrho$ is defined as follows. When A has good reduction at p, one sets $A^\dagger(K_\varrho)^\varrho = A(K_\varrho)^\varrho$, where $A(K_\varrho)^\varrho$ is a shorthand for the $\mathrm{Gal}(K_\varrho/\mathbf{Q})$-invariants of $A(K_\varrho) \otimes_{\mathbf{Z}} V_\varrho$. If A has multiplicative reduction at p, then $\alpha_f = a_p(f) = \pm 1$ and the maximal p-unramified quotient $V_p(A)^-$ of $V_p(A)$ is a 1-dimensional $\mathbf{Q}_p$-vector space on which an arithmetic Frobenius acts as multiplication by α_f. Let q_A in $p\mathbf{Z}_p$ be the p-adic Tate period of the base change $A_{\mathbf{Q}_p}$ of A to $\mathbf{Q}_p$ (cf. Chapter V of [15]), and let $\mathbf{Q}_{p^2}$ be the quadratic unramified extension of $\mathbf{Q}_p$. The Tate uniformisation yields a rigid analytic morphism

$$\wp_{\mathrm{Tate}} : \mathbf{G}^{rig}_{m,\mathbf{Q}_{p^2}} \longrightarrow A_{\mathbf{Q}_{p^2}}$$

with kernel $q_A^{\mathbf{Z}}$ and unique up to sign. Set

$$q(A) = p^-\big((\wp_{\mathrm{Tate}}(\sqrt[p^n]{q_A}))_{n\geq 1}\big) \in V_p(A)^-,$$

where p^- denotes the projection $V_p(A) \longrightarrow V_p(A)^-$ and $(\sqrt[p^n]{q_A})_{n\geq 1}$ is any compatible system of p^n-th roots of q_A, and define

$$A^\dagger(K_\varrho)^\varrho = A(K_\varrho)^\varrho \oplus \mathscr{Q}_p(A,\varrho)$$

to be the direct sum of $A(K_\varrho)^\varrho$ and the $\mathbf{Q}(\varrho)$-submodule

$$\mathscr{Q}_p(A,\varrho) = H^0(\mathbf{Q}_p, \mathbf{Q}(\varrho)\cdot q(A) \otimes_{\mathbf{Q}(\varrho)} V_\varrho)$$

of $H^0(\mathbf{Q}_p, V_p(A)^- \otimes_{\mathbf{Q}} V_\varrho)$. The Garrett–*Nekovář* height $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha}$ depends on the choice of suitably normalised $G_{\mathbf{Q}}$-equivariant embeddings

$$\gamma_g : V_{\varrho_1} \hookrightarrow V(g) \quad \text{and} \quad \gamma_h : V_{\varrho_2} \hookrightarrow V(h), \tag{3}$$

where $V(\xi) = V(\boldsymbol{\xi}_\alpha) \otimes_1 L$ (for $\xi = g, h$) is the weight-one specialisation of the big Galois representation $V(\boldsymbol{\xi}_\alpha)$ associated with $\boldsymbol{\xi}_\alpha$. (We refer to Sect. 3.1 below for precise definitions.) More precisely, denote by $V(f)$ the f_α-isotypic component of the cohomology group $H^1_{\acute{e}t}(X_1(N_f,p)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$, where $X_1(N_f,p)_{\bar{\mathbf{Q}}}$ is the base change to $\bar{\mathbf{Q}}$ of the compact modular curve $X_1(N_f,p)$ of level $\Gamma_1(N_f)\cap\Gamma_0(p)$ over $\mathbf{Q}$, and set

$$V(f,g,h) = V(f) \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h).$$

Section 2 of [6] constructs a *canonical* Garrett–*Nekovář* p-adic height pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha} : \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h)) \otimes_L \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathscr{I}/\mathscr{I}^2 \tag{4}$$

on the naive extended Selmer group of $V(f,g,h)$ over $\mathbf{Q}$, defined as the direct sum of the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ of $V(f,g,h)$ over $\mathbf{Q}$ and the module $H^0(\mathbf{Q}_p, V(f,g,h)^-)$ of $G_{\mathbf{Q}_p}$-invariants of the maximal p-unramified quotient $V(f,g,h)^-$ of $V(f,g,h)$. (The definition of $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha}$ is briefly recalled in Sect. 3.2.3 below.) Fix a modular parametrisation $\wp_\infty : X_1(N_f,p) \longrightarrow A$, under which one identifies $V(f)$ and $V_p(A)$. The embeddings γ_g and γ_h and the global Kummer map on $A(K_\varrho)$ then induce an embedding $\gamma_{gh} : A^\dagger(K_\varrho)^\varrho \hookrightarrow \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h))$. The pairing (2) is defined to be composition of the canonical Garrett–*Nekovář* height and $\gamma_{gh}^{\otimes 2}$. The pairings (2) and (4) are skew-symmetric, and the discriminant of (2) in $(\mathscr{I}^{r^\dagger(A,\varrho)}/\mathscr{I}^{r^\dagger(A,\varrho)+1})/\mathbf{Q}(\varrho)^{*2}$, where

$r^{\dagger}(A,\varrho) = \dim_{\mathbf{Q}(\varrho)} A^{\dagger}(K_{\varrho})^{\varrho}$, is independent of the choice of $\wp_{\infty}$, γ_g and γ_h. We refer to [6] for more details.

If ξ denotes either g or h, then the restriction to $G_{\mathbf{Q}_p}$ of the Artin representation $V(\xi)$ is the direct sum of the submodules $V(\xi)_{\alpha}$ and $V(\xi)_{\beta}$ on which an arithmetic Frobenius acts as multiplication by α_{ξ} and β_{ξ} respectively (cf. Assumption 1.1.3). The $G_{\mathbf{Q}_p}$-representation $V(f,g,h)^{-}$ then decomposes as the direct sum of the subspaces

$$V(f)_{ij}^{-} = V(f)^{-} \otimes_{\mathbf{Q}_p} V(g)_i \otimes_L V(h)_j,$$

where (i,j) is a pair of elements of $\{\alpha,\beta\}$. If ξ denotes either g or h, Sect. 3.1.1 below recalls the definition of canonical *weight-one differentials*

$$\omega_{\xi_{\alpha}} \in (V(\xi)_{\alpha} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}} \quad \text{and} \quad \eta_{\xi_{\alpha}} \in (V(\xi)_{\beta} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}, \tag{5}$$

where $\mathbf{Q}_p^{\mathrm{nr}}$ is the maximal unramified extension of $\mathbf{Q}_p$. If A is multiplicative at p, set

$$q(f) = \wp_{\infty}^{-1}(q(A)) \in V(f)^{-},$$

where one denotes again by $\wp_{\infty} : V(f)^{-} \simeq V_p(A)^{-}$ the isomorphism arising form the fixed modular parametrisation $\wp_{\infty} : X_1(N_f, p) \longrightarrow A$.

Under the running assumptions, the $\mathbf{Q}(\varrho)$-module $\mathcal{Q}_p(A,\varrho)$ (resp., the L-module $H^0(\mathbf{Q}_p, V(f,g,h)^{-})$) is non-zero precisely A is multiplicative at p and

$$\alpha_f = \alpha_g \cdot \alpha_h \quad \text{or} \quad \alpha_f = \beta_g \cdot \alpha_h,$$

in which case it has dimension 2 and one says that (A,ϱ) is *exceptional at* p. More precisely, note that $\alpha_g \neq \beta_g$ by Assumptions 1.1.3, hence only one of the previous identities can be satisfied. Moreover $\alpha_f = \alpha_g \cdot \alpha_h$ (resp., $\alpha_f = \beta_g \cdot \alpha_h$) if and only if $\alpha_f = \beta_g \cdot \beta_h$ (resp., $\alpha_f = \alpha_g \cdot \beta_h$) by Assumption 1.1.1. Fix an auxiliary integer m_p such that p splits (resp., is inert) in $\mathbf{Q}\left[\sqrt{m_p}\right]$ if $\alpha_f = +1$ (resp., $\alpha_f = -1$), so that $G_{\mathbf{Q}_p}$ acts trivially on $\sqrt{m_p} \cdot q(f)$ in $V(f)^{-} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{nr}}$. If $\alpha_f = \alpha_g \cdot \alpha_h$, then $G_{\mathbf{Q}_p}$ acts trivially on $V(f)_{\alpha\alpha}^{-}$ and $V(f)_{\beta\beta}^{-}$, hence the p-adic periods

$$q_{\alpha\alpha} = \sqrt{m_p} \cdot q(f) \otimes \omega_{g_{\alpha}} \otimes \omega_{h_{\alpha}} \quad \text{and} \quad q_{\beta\beta} = \sqrt{m_p} \cdot q(f) \otimes \eta_{g_{\alpha}} \otimes \eta_{h_{\alpha}}$$

can naturally be viewed as elements of $V(f)_{\alpha\alpha}^{-}$ and $V(f)_{\beta\beta}^{-}$ respectively, which generate $H^0(\mathbf{Q}_p, V(f,g,h)^{-})$. Similarly, if $\alpha_f = \beta_g \cdot \alpha_h$, then the periods

$$q_{\alpha\beta} = \sqrt{m_p} \cdot q(f) \otimes \omega_{g_{\alpha}} \otimes \eta_{g_h} \quad \text{and} \quad q_{\beta\alpha} = \sqrt{m_p} \cdot q(f) \otimes \eta_{g_{\alpha}} \otimes \omega_{h_{\alpha}}$$

can naturally be viewed as generators of $H^0(\mathbf{Q}_p, V(f,g,h)^{-})$.

Equation (1) shows that the value of the square-root Garrett–Hida L-function $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ at w_o is a non-zero multiple of

$$\left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right)\left(1 - \frac{\beta_g \alpha_h}{\alpha_f}\right)\left(1 - \frac{\alpha_g \beta_h}{\alpha_f}\right)\left(1 - \frac{\beta_g \beta_h}{\alpha_f}\right) \cdot \sqrt{L(A,\varrho,1)},$$

where $L(A,\varrho,s) = L(f \otimes g \otimes h, s)$. The previous discussion then shows that (A,ϱ) is exceptional at p precisely if one of the Euler factors which appear in the previous expression is zero, id est if $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ (or $L_p^{\alpha\alpha}(A,\varrho)$) has an exceptional zero in the sense of Mazur–Tate–Teitelbaum [13]. In this case Lemma 9.8 of [7] proves that the restriction $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)|_{\mathsf{L}}$ of $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ to the *improving line* L defined by the equations $\boldsymbol{m} = 1$ and $\boldsymbol{k} = l + 1$ admits the factorisation

$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho)|_{\mathsf{L}} = \mathscr{E}_f \cdot \mathscr{E}_g \cdot \mathscr{L}_p^{\alpha\alpha}(A,\varrho)^{\star}$$

in the ring $\mathcal{O}(\mathsf{L})$ of analytic functions on L, where

$$\mathscr{E}_f = 1 - \frac{a_p(\boldsymbol{f})}{a_p(\boldsymbol{g}_\alpha)\cdot a_p(\boldsymbol{h}_\alpha)}\Big|_{\mathsf{L}} \quad \text{and} \quad \mathscr{E}_g = 1 - \chi_h(p)\cdot \frac{a_p(\boldsymbol{g}_\alpha)}{a_p(\boldsymbol{f})\cdot a_p(\boldsymbol{h}_\alpha)}\Big|_{\mathsf{L}}.$$

Moreover, the value at w_o of the *improved* p-adic L-function $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)^\star$ is an explicit algebraic number in $\mathbf{Q}(\varrho)$, equal to zero precisely if $L(A,\varrho,s)$ vanishes at $s=1$. We refer to the proof of Proposition 8.3 of [12] for details.

The following is the main result of this note.

Theorem 2.1 *Assume that (A,ϱ) is exceptional at p. Let $(q_\flat, q_\natural)$ denote either the pair $(q_{\alpha\alpha}, q_{\beta\beta})$ or $(q_{\alpha\beta}, q_{\beta\alpha})$, depending on whether $\alpha_f = \alpha_g\cdot\alpha_h$ or $\alpha_f = \beta_g\cdot\alpha_h$ respectively. Then the following equality holds in $\mathscr{I}/\mathscr{I}^2$ up to sign.*

$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2} = \frac{\deg(\wp_\infty)\cdot(1-\beta_h/\alpha_h)}{m_p\cdot ord_p(q_A)}\cdot \mathscr{L}_p^{\alpha\alpha}(A,\varrho)^\star(w_o)\cdot \langle\!\langle q_\flat, q_\natural\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$$

Theorem 2.1 is proved in Sect. 4 below. More precisely, Sects. 3.3 and 3.4 below prove that the following equality holds in $\mathscr{I}/\mathscr{I}^2$ up to sign:

$$\frac{2\cdot\deg(\wp_\infty)}{m_p\cdot \mathrm{ord}_p(q_A)}\cdot \langle\!\langle q_\flat, q_\natural\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \left(\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{g}_\alpha}^{\mathrm{an}}\right)\cdot(\boldsymbol{l}-1) + \varepsilon\cdot\left(\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{h}_\alpha}^{\mathrm{an}}\right)\cdot(\boldsymbol{m}-1), \tag{6}$$

where $\varepsilon = +1$ if $\alpha_f = \alpha_g\cdot\alpha_h$ and $\varepsilon = -1$ if $\alpha_f = \beta_g\cdot\beta_h$, and where

$$-\frac{1}{2}\cdot\mathfrak{L}_{\boldsymbol{\xi}}^{\mathrm{an}} = d\log a_p(\boldsymbol{\xi})_{\boldsymbol{u}=u_o} \tag{7}$$

is the value at the centre u_o of $U_{\boldsymbol{\xi}}$ of the logarithmic derivative of the p-th Fourier coefficient of the Hida family $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$. In Sect. 4 we then deduce Theorem 2.1 from Eq. (6) and the study carried out in [7, Section 9] of the linear term of $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ at w_o in the exceptional case.

It should be possible to extend Theorem 2.1 (and Conjecture 2.3 below) to the case of p-new eigenforms of even weight $k\geq 2$ and trivial character (cf. Section 1.1 of [6]). We have not checked the details.

2.1 The rank-zero exceptional case of [6, Conjecture 1.1]

Assume in this section that (A,ϱ) is exceptional at p, and that the Garrett complex L-function $L(A,\varrho,s) = L(f\otimes g\otimes h, s)$ does not vanish at $s=1$:

$$L(A,\varrho,1)\neq 0.$$

According to the main result of [8] (see also Theorem B of [3]), one has

$$A(K_\varrho)^\varrho = 0,$$

hence $A^\dagger(K_\varrho)^\varrho = \mathcal{Q}_p(A,\varrho)$. The *Garrett– Nekovář p-adic regulator* $R_p^{\alpha\alpha}(A,\varrho)$, viz. the discriminant of the p-adic height $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ on $A^\dagger(K_\varrho)^\varrho$, is then given by

$$R_p^{\alpha\alpha}(A,\varrho) = \det\left(\langle\!\langle q_i, q_j\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\right)_{1\leq i,j\leq 2} = \langle\!\langle q_1, q_2\rangle\!\rangle^2_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$$

in $(\mathscr{I}^2/\mathscr{I}^3)/\mathbf{Q}(\varrho)^{*2}$, where (q_1,q_2) is a $\mathbf{Q}(\varrho)$-basis of $\mathcal{Q}_p(A,\varrho)$.

Let $\gamma_{gh} : V(A,\varrho)^- \hookrightarrow V(f,g,h)^-$ be the $G_{\mathbf{Q}}$-equivariant embedding defined by the tensor product of the isomorphism $V_p(A)^- \simeq V(f)^-$ induced by $\wp_\infty, \gamma_g$ and γ_h (cf. Eq. (3)). The normalisation imposed on the embeddings γ_g and γ_h (and described in Sect. 3.1.1 below) implies that the matrix M in $\mathrm{GL}_2(L)$ defined by the identity $(q_\flat\ q_\natural)\cdot M = (\gamma_{gh}(q_1)\ \gamma_{gh}(q_2))$ has determinant in $\mathbf{Q}(\varrho)^*$. In light of the above discussion, Theorem 2.1 then proves the following corollary, which together with Eq. (6) establishes [6, Conjecture 1.1] in the present setting.

Corollary 2.2 *If $L(A,\varrho,s)$ does not vanish at $s=1$, then $A^\dagger(K_\varrho)^\varrho = \mathcal{Q}_p(A,\varrho)$ and the following equality holds in the quotient of $\mathscr{I}^2/\mathscr{I}^3$ by the action of $\boldsymbol{Q}(\varrho)^{*2}$.*

$$L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^3} = R_p^{\alpha\alpha}(A,\varrho)$$

2.2 Exceptional zeros and rational points (cf. [14])

Assume in this section that (A,ϱ) is exceptional at p, and that the Garrett complex L-function $L(A,\varrho,s)$ vanishes at the central critical point $s=1$:

$$L(A,\varrho,1)=0.$$

Set $\{\flat,\natural\}=\{\alpha\alpha,\beta\beta\}$ of $\{\flat,\natural\}=\{\alpha\beta,\beta\alpha\}$, depending on whether

$$\alpha_f=\alpha_g\cdot\alpha_h \mathrm{or} \alpha_f=\beta_g\cdot\alpha_h.$$

The p-adic L-function $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^2$ (cf. Theorem 2.1) and Conjecture 2.3 of [6] predicts that its image in $(\mathscr{I}^2/\mathscr{I}^3)/\mathbf{Q}(\varrho)^*$ equals

$$\langle\!\langle q_\flat,q_\natural\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\langle\!\langle P,Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} - \langle\!\langle q_\flat,P\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\langle\!\langle q_\natural,Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} + \langle\!\langle q_\flat,Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\langle\!\langle q_\natural,P\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$$

for two rational points P and Q in $A(K_\varrho)^\varrho$. (Recall that the p-adic height $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, hence the previous expression is a square root of its discriminant on the $\mathbf{Q}(\varrho)$-submodule of $A^\dagger(K_\varrho)^\varrho$ generated by $q_\flat$, $q_\natural$, P and Q.) One has

$$\langle\!\langle q_\flat,q_\natural\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(\boldsymbol{k},1,1)=0$$

by Eq. (6). Moreover, Sect. 3.5 below proves that

$$\langle\!\langle q_\natural,x\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(\boldsymbol{k},1,1)=\frac{1}{2}\cdot\log_\flat(\mathrm{res}_p(x))\cdot(\boldsymbol{k}-2) \tag{8}$$

for each Selmer class x in $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$, where

$$\log_\flat=\langle\log_p(\cdot),q_\natural\rangle_{fgh}: H^1_{\mathrm{fin}}(\mathbf{Q}_p,V(f,g,h))\longrightarrow L.$$

Here $\log_p : H^1_{\mathrm{fin}}(\mathbf{Q}_p,V(f,g,h))\simeq D_{\mathrm{dR}}(V(f,g,h))/\mathrm{Fil}^0$ is the Bloch–Kato p-adic logarithm (cf. Lemma 9.1 of [7]), and $\langle\cdot,\cdot\rangle_{fgh} : D_{\mathrm{dR}}(V(f,g,h))^{\otimes 2}\longrightarrow L$ is the pairing induced by the natural Kummer duality $\pi_{fgh} : V(f,g,h)^{\otimes 2}\longrightarrow L(1)$ defined in Sect. 3.1.1 below (cf. Eq. (11)). We are then led to the following

Conjecture 2.3 Assume that $A(K_\varrho)^\varrho$ is a 2-dimensional $\mathbf{Q}(\varrho)$-vector space. Then for any $\mathbf{Q}(\varrho)$-basis (P,Q) of $A(K_\varrho)^\varrho$, the equality

$$\frac{\partial^2\mathscr{L}_p^{\alpha\alpha}(A,\varrho)}{\partial\boldsymbol{k}^2}(w_o)=\log_\flat(P)\cdot\log_\natural(Q)-\log_\natural(P)\cdot\log_\flat(Q)$$

holds in L up to multiplication by a non-zero scalar in $\mathbf{Q}(\varrho)^*$.

As explained in [5], the main result of [1] can be used to prove cases of Conjecture 2.3 when g and h are theta series associated with certain ray class characters of the same imaginary quadratic field in which p is inert (and P and Q are Heegner points). By combining this with an extension of the height computations carried out in [16,17], the article [4] proves instances of Conjecture 1.1 of [6] in this setting.

Remark 2.4 In light of the aforementioned results of [5], Rivero proposes in [14, Conjecture 4.5] a variant of Conjecture 2.3. He also asks (cf. Question 5.3 of [14]) if one can expect a similar description of $\frac{\partial^2 \mathscr{L}_p^{\alpha\alpha}(A,\varrho)}{\partial \boldsymbol{k}^2}(w_o)$ when A has good reduction at p. The previous discussion places Rivero's conjecture within a conceptual framework and sheds some light on this question.

3 Height computations

Throughout the rest of this note we assume that (A,ϱ) is exceptional at p. In particular A has multiplicative reduction at p, id est p divides exactly N_f.

3.1 Setting and notations

This subsection briefly recalls the needed definitions and notations from our previous articles [6,7].

3.1.1 Galois representations

Set $N = \mathrm{lcm}(N_f, N_g, N_h)$ and let $G_{\mathbf{Q},N}$ be the Galois group of the maximal extension of $\mathbf{Q}$ contained in $\bar{\mathbf{Q}}$ and unramified outside $N\infty$. If $\boldsymbol{\xi}$ denotes one of $\boldsymbol{f}, \boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$, let $V(\boldsymbol{\xi})$ be the big Galois representation associated with $\boldsymbol{\xi}$ (cf. Section 5 of [7]). It is a free $\mathscr{O}_{\boldsymbol{\xi}}$-module of rank two, equipped with a continuous linear action $G_{\mathbf{Q},N}$. For each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geq 2}$ the base change $V(\boldsymbol{\xi}) \otimes_u L$ of $V(\boldsymbol{\xi})$ along evaluation at u on $\mathscr{O}_{\boldsymbol{\xi}}$ is canonically isomorphic to the homological p-adic Deligne representation of $\boldsymbol{\xi}_u$ with coefficients in L (cf. loco citato for more details). In particular if $\boldsymbol{\xi} = \boldsymbol{f}$ and $u = 2$ there is a natural *specialisation isomorphism* $\rho_2 : V(\boldsymbol{f}) \otimes_2 L \simeq V(f)$. If $\boldsymbol{\xi} = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ and $u = 1$ set $V(\xi) = V(\boldsymbol{\xi}) \otimes_1 L$ (cf. Sect. 1). It is a two-dimensional L-vector space affording the dual of the p-adic Deligne–Serre representation of $\xi = g, h$ with coefficients in L. In order to have a uniform notation, in this case one defines $\rho_1 : V(\boldsymbol{\xi}) \otimes_1 L \longrightarrow V(\xi)$ to be the identity.

The restriction of $V(\boldsymbol{\xi})$ to $G_{\mathbf{Q}_p}$ (via the embedding i_p fixed at the outset) fits into a short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules $V(\boldsymbol{\xi})^+ \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^-$ with $V(\boldsymbol{\xi})^\pm$ free of rank one over $\mathscr{O}_{\boldsymbol{\xi}}$. More precisely, let $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^*$ be the p-adic cyclotomic character, and let $\breve{a}_p(\boldsymbol{\xi}) : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ be the unramified character sending an arithmetic Frobenius to the p-th Fourier coefficients $a_p(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$. Then

$$V(\boldsymbol{\xi})^+ \simeq \mathscr{O}_{\boldsymbol{\xi}}\big(\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \chi_{\xi} \breve{a}_p(\boldsymbol{\xi})^{-1}\big) \quad \text{and} \quad V(\boldsymbol{\xi})^- \simeq \mathscr{O}_{\boldsymbol{\xi}}(\breve{a}_p(\boldsymbol{\xi})), \tag{9}$$

where $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ satisfies $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1}(\sigma)(u) = \chi_{\mathrm{cyc}}(\sigma)^{u-1}$ for each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}$. (The freeness of $V(\boldsymbol{\xi})^\pm$ is guaranteed by Assumption 1.1.3, cf. Section 5 of [7].) If $\boldsymbol{\xi} = \boldsymbol{f}$ and $u = 2$ the specialisation isomorphism ρ_2 identifies $V(\boldsymbol{f})^- \otimes_2 L$ with the maximal unramified quotient $V(f)^-$ of $V(f)$. If $\boldsymbol{\xi} = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ and $u = 1$ we set $V(\xi)_\beta = V(\boldsymbol{\xi})^+ \otimes_1 L$

and $V(\xi)_\alpha = V(\boldsymbol{\xi})^- \otimes_1 L$. One has $V(\xi) = V(\xi)_\alpha \oplus V(\xi)_\beta$, where $V(\xi)_\gamma = V(\xi)^{\mathrm{Frob}_p=\gamma_\xi}$ for $\gamma = \alpha, \beta$ is the submodule of $V(\xi)$ on which an arithmetic Frobenius Frob_p acts as multiplication by $\gamma_\xi = \alpha_\xi, \beta_\xi$ (cf. Assumption 1.1.3).

There is a natural $G_{\mathbf{Q}}$-equivariant skew-symmetric perfect pairing

$$\pi_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(\chi_\xi \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1}),$$

inducing perfect dualities $\pi_{\boldsymbol{\xi}} : V(\boldsymbol{\xi})^{\pm} \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi})^{\mp} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(\chi_\xi \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1})$. (See Section 5 cf. [7] for the definitions).

Denote by $\Xi_{\boldsymbol{fgh}} = \chi_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$ the character whose composition with evaluation at (k, l, m) in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3$ on $\mathscr{O}_{\boldsymbol{fgh}}$ equals $\chi_{\mathrm{cyc}}^{(4-k-l-m)/2}$. If $\cdot$ denotes one of the symbols $\emptyset$, $+$ and $-$, define

$$\boldsymbol{V}^{\cdot} = V(\boldsymbol{f})^{\cdot} \hat{\otimes}_L V(\boldsymbol{g}_\alpha) \hat{\otimes} V(\boldsymbol{h}_\alpha) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

Then $\boldsymbol{V} = V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$, resp. $\boldsymbol{V}^{\pm} = V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^{\pm}$ is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module of rank 8, resp. 4, equipped with a continuous action of $G_{\mathbf{Q},N}$, resp. $G_{\mathbf{Q}_p}$. As $\chi_g \cdot \chi_h = 1$ (cf. Assumption 1.1), the product of the perfect dualities $\pi_{\boldsymbol{\xi}}$, for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$, yields a perfect skew-symmetric Kummer duality $\boldsymbol{\pi} : \boldsymbol{V} \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \boldsymbol{V} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1)$, inducing a perfect local Kummer duality $\boldsymbol{\pi} : \boldsymbol{V}^{\pm} \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \boldsymbol{V}^{\mp} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1)$. After setting

$$V^{\cdot} = V(f, g, h)^{\cdot} = V(f)^{\cdot} \otimes_L V(g) \otimes_L V(h)$$

and $w_o = (2, 1, 1)$, the product $\rho_{w_o} = \rho_2 \hat{\otimes} \rho_1 \hat{\otimes} \rho_1$ gives natural isomorphisms

$$\rho_{w_o} : \boldsymbol{V}^{\cdot} \otimes_{w_o} L \simeq V^{\cdot} \tag{10}$$

(where $\cdot \otimes_{w_o} L$ denotes the base change along evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$). Let

$$\pi_{fgh} : V \otimes_L V \longrightarrow L(1) \tag{11}$$

be the specialisation of $\boldsymbol{\pi}$ via ρ_{w_o}, and define $\pi : V^{\pm} \otimes_L V^{\mp} \longrightarrow L(1)$ similarly.

Weight one differentials Define $D(\boldsymbol{\xi})^- = H^0(\mathbf{Q}_p, V(\boldsymbol{\xi})^- \hat{\otimes}_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}})$, where $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ is the p-adic completion of the maximal unramified extension of $\mathbf{Q}_p$ (and as usual $\boldsymbol{\xi}$ denotes one of $\boldsymbol{f}$, $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$). For each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geq 2}$ there is a natural comparison isomorphism between $D(\boldsymbol{\xi})^- \otimes_u L$ and the $\boldsymbol{\xi}_u$-isotypic component of the space of cuspidal modular forms of weight u, level $\Gamma_1(N_\xi p)$ and Fourier coefficients in L. Assumption 1.1.3 guarantees that $D(\boldsymbol{\xi})^-$ is free (of rank one) over $\mathscr{O}_{\boldsymbol{\xi}}$, and admits a basis $\omega_{\boldsymbol{\xi}}$ whose image in $D(\boldsymbol{\xi})^- \otimes_u L$ corresponds to $\boldsymbol{\xi}_u$ under the aforementioned comparison isomorphism, for each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geq 2}$. (We refer to Section 3.1 of [6] and the references therein for more details.)

For $\boldsymbol{\xi} = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$, the *holomorphic weight-one differential*

$$\omega_{\xi_\alpha} \in (V(\xi)_\alpha \otimes_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}$$

mentioned in Eq. (5) is defined to be the weight-one specialisation of $\omega_{\boldsymbol{\xi}}$, viz. the image of $\omega_{\boldsymbol{\xi}}$ in the quotient $D(\boldsymbol{\xi})^- \otimes_1 L = D(\xi)_\alpha$. The weight-one specialisation of $\pi_{\boldsymbol{\xi}}$ yields a perfect $G_{\mathbf{Q}}$-equivariant skew-symmetric pairing

$$\pi_\xi : V(\xi) \otimes_L V(\xi) \longrightarrow L(\chi_\xi).$$

Let c be the common conductor of χ_g and χ_h, and identify $(L(\chi_\xi) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}$ with L via the Gauß sum $G(\chi_\xi) = (-c)^{i_\xi} \sum_{a \in (\mathbf{Z}/c\mathbf{Z})^*} \chi_\xi(a)^{-1} \otimes e^{2\pi i a/c}$, where $i_g = 0$ and $i_h = 1$ (so that $G(\chi_g)\cdot G(\chi_h)=1$ by Assumption 1.1.1). The pairing π_ξ then induces a perfect duality $\langle\cdot,\cdot\rangle_\xi : D(\xi)_\alpha\otimes_L D(\xi)_\beta \longrightarrow L$, where $D(\xi)_\gamma=(V(\xi)_\gamma\otimes_{\mathbf{Q}_p}\mathbf{Q}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}$. One defines the *antiholomorphic* weight-one differential (cf. Eq. (5))

$$\eta_{\xi_\alpha}\in (V(\xi)_\beta\otimes_{\mathbf{Q}_p}\mathbf{Q}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}$$

to be the dual of ω_{ξ_α} under $\langle\cdot,\cdot\rangle_\xi$, viz. the element satisfying $\left\langle\omega_{\xi_\alpha},\eta_{\xi_\alpha}\right\rangle_\xi=1$.

The embeddings $\boldsymbol{\gamma}_g$ and $\boldsymbol{\gamma}_h$ With the notations of Sect. 1, set $V_g=V_{\varrho 1}$ and $V_h=V_{\varrho 2}$. Let ξ denote either g or h. As recalled above, the Artin representation $V(\xi)=V(\boldsymbol{\xi})\otimes_1 L$ affords the dual of the p-adic Deligne representation of ξ with coefficients in L, id est is isomorphic to $V_\xi\otimes_{\mathbf{Q}(\varrho)} L$. Enlarging L if necessary, we normalise the $G_{\mathbf{Q}}$-equivariant embedding $\gamma_\xi : V_\xi\longrightarrow V(\xi)$ (introduced in Eq. (3)) by requiring that the composition $\pi_\xi\circ(\gamma_\xi\otimes\gamma_\xi)$ takes values in the number field $\mathbf{Q}(\varrho)$ (via the embedding $i_p:\bar{\mathbf{Q}}\hookrightarrow\bar{\mathbf{Q}}_p$ fixed at the outset).

3.1.2 Selmer complexes

Let $\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},V)$ be the *Nekovář* Selmer complex associated with (V,V^+) (cf. Section 2.2 of [6]). It is an element of the derived category $\mathrm{D}^b_{\mathrm{ft}}(L)$ of cohomologically bounded complexes of L-modules with cohomology of finite type over L, sitting is an exact triangle

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_{\mathbf{Q},N},V)\xrightarrow{p^-\circ\mathrm{res}_p}\mathbf{R}\Gamma_{\mathrm{cont}}(G_{\mathbf{Q}_p},V^-)\longrightarrow\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},V)[1], \tag{12}$$

where $\mathbf{R}\Gamma_{\mathrm{cont}}(G,\cdot)$ is the complex of continuous non-homogeneous cochains of G with values in $\cdot$, res_p is the restriction map (induced by the embedding $i_p:\bar{\mathbf{Q}}\hookrightarrow\bar{\mathbf{Q}}_p$ fixed at the outset) and p^- is the map induced by the projection $V\longrightarrow V^-$. Denote by

$$\tilde{H}^{\cdot}_f(\mathbf{Q},V)=H^{\cdot}(\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},V))$$

the cohomology of $\mathbf{R}\tilde{\Gamma}(\mathbf{Q},V)$, let $\mathrm{Sel}(\mathbf{Q},V)$ be the Bloch–Kato Selmer group of V over $\mathbf{Q}$, and let $i^+:V^+\longrightarrow V$ be the natural inclusion. Then there is a commutative and exact diagram of L-vector spaces (cf. loc. cit.)

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbf{Q}_p,V^-) \arrow[r, "\jmath"] & \tilde{H}^1_f(\mathbf{Q},V) \arrow[r] \arrow[d, "\iota^+"'] & \mathrm{Sel}(\mathbf{Q},V) \arrow[r] \arrow[d, "\mathrm{res}_p"] & 0 \\
& & H^1(\mathbf{Q}_p,V^+) \arrow[r, "i^+"] & H^1(\mathbf{Q}_p,V) &
\end{tikzcd} \tag{13}$$

where the first line arises from the exact triangle (12). In addition there is a unique section $\iota_{\mathrm{ur}}:\mathrm{Sel}(\mathbf{Q},V)\longrightarrow\tilde{H}^1_f(\mathbf{Q},V)$ of the above projection such that $\iota_{\mathrm{ur}}(x)^+$ belongs to the Bloch–Kato finite subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p,V^+)$ for each x in $\mathrm{Sel}(\mathbf{Q},V)$. We often use $\jmath$ and ι_{ur} to identify *Nekovář*'s extended Selmer group $\tilde{H}^1_f(\mathbf{Q},V)$ with the naive extended Selmer group $\mathrm{Sel}^\dagger(\mathbf{Q},V)=H^0(\mathbf{Q}_p,V^-)\oplus\mathrm{Sel}(\mathbf{Q},V)$ (cf. Sect. 1).

One similarly associates with $(\boldsymbol{V},\boldsymbol{V}^+)$ a Selmer complex

$$\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},\boldsymbol{V})\in\mathrm{D}^b_{\mathrm{ft}}(\mathscr{O}_{\boldsymbol{fgh}})$$

sitting in an exact triangle analogous to (12). (We refer to loc. cit. for more details.)

3.2 Preliminary lemmas

This section gives a concrete description of the functionals $\langle\!\langle q, \cdot \rangle\!\rangle_{\boldsymbol{f g_\alpha h_\alpha}} : \mathrm{Sel}^{\dagger}(\mathbf{Q}, V) \longrightarrow L$ for q in $H^0(\mathbf{Q}_p, V^-)$ (cf. Lemma 3.4 below).

3.2.1 Bockstein maps

Let $(\boldsymbol{\mathcal{C}}, \mathcal{C})$ denote one of the pairs

$$(\mathbf{R}\Gamma_p(\boldsymbol{V}^-), \mathbf{R}\Gamma_p(V^-)), (\mathbf{R}\Gamma(\boldsymbol{V}), \mathbf{R}\Gamma(V)) \quad \text{and} \quad (\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, \boldsymbol{V}), \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V)),$$

where $\mathbf{R}\Gamma_p(\cdot)$ and $\mathbf{R}\Gamma(\cdot)$ are shorthands for $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, \cdot) = \mathbf{R}\Gamma_{\mathrm{cont}}(G_{\mathbf{Q}_p}, \cdot)$ and $\mathbf{R}\Gamma_{\mathrm{cont}}(G_{\mathbf{Q},N}, \cdot)$ respectively (cf. Sect. 3.1.2). The specialisation maps ρ_{w_o} (cf. Eq. (10)) induce isomorphisms

$$\rho_{w_o} : \boldsymbol{\mathcal{C}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L \simeq \mathcal{C} \quad \text{and} \quad \rho_{w_o} \otimes \mathrm{id} : \boldsymbol{\mathcal{C}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{I}/\mathscr{I}^2[1] \simeq \mathcal{C} \otimes_L \mathscr{I}/\mathscr{I}^2[1]. \tag{14}$$

Applying $\boldsymbol{\mathcal{C}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \cdot$ to the exact triangle

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}/\mathscr{I}^2 \longrightarrow L \longrightarrow \mathscr{I}/\mathscr{I}^2[1]$$

(arising from evaluation on w_o) then yields a *derived Bockstein map*

$$\boldsymbol{\beta}_{\boldsymbol{\mathcal{C}}/\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C} \otimes_L \mathscr{I}/\mathscr{I}^2[1],$$

which in turn induces in cohomology a *Bockstein map*

$$\beta_{\boldsymbol{\mathcal{C}}/\mathcal{C}} : H^i(\mathcal{C}) \longrightarrow H^{i+1}(\mathcal{C}) \otimes_L \mathscr{I}/\mathscr{I}^2.$$

If no risk of confusion arises, we simply write β for $\beta_{\boldsymbol{\mathcal{C}}/\mathcal{C}}$. Let

$$\jmath : H^i(\mathbf{Q}_p, V^-) \longrightarrow \tilde{H}^{i+1}_f(\mathbf{Q}, V)$$

be the maps arising from the exact triangle (12).

Lemma 3.1 *The following diagram commutes.*

$$\begin{CD}
H^0(\boldsymbol{Q}_p, V^-) @>{\beta}>> H^1(\boldsymbol{Q}_p, V^-) \otimes_L \mathscr{I}/\mathscr{I}^2\\
@V{\jmath}VV @VV{\jmath \otimes \mathscr{I}/\mathscr{I}^2}V\\
\tilde{H}^1_f(\boldsymbol{Q}, V) @>{\beta}>> \tilde{H}^2_f(\boldsymbol{Q}, V) \otimes_L \mathscr{I}/\mathscr{I}^2
\end{CD}$$

Proof For $M = V, \boldsymbol{V}$ one has an exact triangle (cf. Equation (12))

$$\Delta_M : \mathbf{R}\Gamma_{\mathrm{cont}}(G_{\mathbf{Q},N}, M)[-1] \overset{p^-\mathrm{ores}_p}{\longrightarrow} \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, M^-)[-1] \xrightarrow{\jmath_M} \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, M).$$

Moreover Δ_V is obtained by applying $\cdot \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L$ to $\Delta_{\boldsymbol{V}}$ (cf. Eq. (14)). It follows from the definition of the derived Bockstein maps $\boldsymbol{\beta}^-$ and $\boldsymbol{\beta}$ on $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, V^-)$ and $\mathbf{R}\tilde{\Gamma}(\mathbf{Q}, V)$ respectively that $\boldsymbol{\jmath}_V \otimes \mathscr{I}/\mathscr{I}^2[1] \circ \boldsymbol{\beta}^-$ is equal to $\boldsymbol{\beta} \circ \boldsymbol{\jmath}_V$. Since by definition the maps $\jmath$ are the ones induced in cohomology by $\boldsymbol{\jmath}_V$, the lemma follows. $\square$

The following lemma gives a concrete description of $\beta_{\boldsymbol{\mathcal{C}}/\mathcal{C}}$.

Lemma 3.2 *Let $(\boldsymbol{\mathcal{C}}, \mathcal{C})$ be as above, let z be a 1-cocycle in $\mathcal{C}$, let Z be a 1-cochain in $\boldsymbol{\mathcal{C}}$, and let $Z_{\boldsymbol{k}}$, Z_l and $Z_{\boldsymbol{m}}$ be 2-cochains in $\boldsymbol{\mathcal{C}}$ such that*

$$\rho_{w_o}(Z) = z \ \text{ and } \ dZ = Z_{\boldsymbol{k}} \cdot (\boldsymbol{k} - 2) + Z_l \cdot (l - 1) + Z_{\boldsymbol{m}} \cdot (\boldsymbol{m} - 1).$$

Then $z_\cdot = \rho_{w_o}(Z_\cdot)$ is a 2-cocycle for $\cdot = \boldsymbol{k}, l, \boldsymbol{m}$, and one has the equality

$$-\beta_{\boldsymbol{\mathcal{C}}/\mathcal{C}}(cl(z)) = cl(z_{\boldsymbol{k}}) \cdot (\boldsymbol{k} - 2) + cl(z_l) \cdot (l - 1) + cl(z_{\boldsymbol{m}}) \cdot (\boldsymbol{m} - 1)$$

in $H^2(\mathcal{C}) \otimes_L \mathscr{I}/\mathscr{I}^2$, where $cl(\cdot)$ is the class in $H^i(\mathcal{C})$ represented by the i-cocycle $\cdot$.

Proof The proof is very similar to that of [16, Lemma 5.5]. We omit it. □

3.2.2 Local and global duality

Nekovář's generalised Poitou–Tate duality associates with the perfect duality π_{fgh} introduced in Eq. (11) a global cup-product pairing (cf. Section 2.4 of [6])

$$\langle \cdot, \cdot \rangle_{\mathrm{Nek}} : \tilde{H}^2_f(\mathbf{Q}, V) \otimes_L \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow L. \tag{15}$$

The pairing π_{fgh} induces a Kummer duality $V^- \otimes_L V^+ \longrightarrow L(1)$ and we denote by

$$\langle \cdot, \cdot \rangle_{\mathrm{Tate}} : H^1(\mathbf{Q}_p, V^-) \otimes_L H^1\left(\mathbf{Q}_p, V^+\right) \longrightarrow L \tag{16}$$

the induced local Tate duality pairing. Recall finally the map

$$\cdot^+ : \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_p, V^+)$$

introduced in diagram (13).

Lemma 3.3 *For each ζ in $H^1(\boldsymbol{Q}_p, V^-)$ and ξ in $\tilde{H}^1_f(\boldsymbol{Q}, V)$ one has*

$$\langle \jmath(\zeta), \xi \rangle_{Nek} = \langle \zeta, \xi^+ \rangle_{Tate}.$$

Proof This is proved as in [16, Lemma 5.7]. □

3.2.3 The Garrett–*Nekovář* *p*-adic height pairing

Set

$$\tilde{\beta}_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \beta_{\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},\boldsymbol{V})/\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},V)} : \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow \tilde{H}^2_f(\mathbf{Q}, V) \otimes_L \mathscr{I}/\mathscr{I}^2.$$

After identifying $\tilde{H}^1_f(\mathbf{Q}, V)$ with $\mathrm{Sel}^\dagger(\mathbf{Q}, V)$ (cf. Sect. 3.1.2), the canonical height $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ introduced in Sect. is defined by (cf. [6, Section 2])

$$\langle\!\langle x, y \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \left\langle \tilde{\beta}_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(x), y \right\rangle_{\mathrm{Nek}}$$

for each x and y in $\tilde{H}^1_f(\mathbf{Q}, V)$, where we write again $\langle \cdot, \cdot \rangle_{\mathrm{Nek}}$ for the $\mathscr{I}/\mathscr{I}^2$-base change of *Nekovář*'s cup-product (15). Lemmas 3.1 and 3.3 give the following

Lemma 3.4 *For each q in $H^0(\boldsymbol{Q}_p, V^-)$ one has*

$$\langle\!\langle \jmath(q), \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \left\langle \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q), \cdot^+ \right\rangle_{Tate}$$

as $\mathscr{I}/\mathscr{I}^2$-valued maps on $\tilde{H}^1_f(\boldsymbol{Q}, V)$, where $\beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \beta_{\boldsymbol{R}\Gamma_p(\boldsymbol{V}^-)/\boldsymbol{R}\Gamma_p(V^-)}$ (and we write again $\langle \cdot, \cdot \rangle_{Tate}$ for the $\mathscr{I}/\mathscr{I}^2$-base change of the local Tate pairing (16)).

3.3 Computation of $\langle\!\langle \boldsymbol{q}_{\beta\beta}, \boldsymbol{q}_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha}$

Assume in this subsection $\alpha_f = \alpha_g \cdot \alpha_h$, so that $H^0(\mathbf{Q}_p, V^-)$ is generated over L by the periods

$$q_{\alpha\alpha} = \sqrt{m_p} \cdot q(f) \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \quad \text{and} \quad q_{\beta\beta} = \sqrt{m_p} \cdot q(f) \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}.$$

Recall that $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^*$ denotes the p-adic cyclotomic character. Fix a lift $\boldsymbol{q}_{\beta\beta}$ in $\boldsymbol{V}^-$ of $q_{\beta\beta}$ under ρ_{w_o}. Since (cf. Sect. 3.1.1)

$$q_{\beta\beta} \in V(f)^- \otimes_{\mathbf{Q}_p} V(g)_\beta \otimes_L V(h)_\beta \hookrightarrow V^-$$

and $V(\xi)_\beta = V(\boldsymbol{\xi}_\alpha)^+ \otimes_1 L$ for $\xi = g, h$, we can choose $\boldsymbol{q}_{\beta\beta}$ in the $G_{\mathbf{Q}_p}$-submodule

$$V(\boldsymbol{f})^- \hat{\otimes}_L V(\boldsymbol{g})^+ \hat{\otimes}_L V(\boldsymbol{h})^+ \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}} \hookrightarrow \boldsymbol{V}^-$$

(cf. Sect. 3.1.1). By Eq. (9) one has

$$d\boldsymbol{q}_{\beta\beta} = \Phi \cdot \boldsymbol{q}_{\beta\beta}, \tag{17}$$

where d denotes the differentials of the complex $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)$ and

$$\Phi = \frac{\breve{a}_p(\boldsymbol{f})}{\breve{a}_p(\boldsymbol{g}_\alpha) \cdot \breve{a}_p(\boldsymbol{h}_\alpha)} \cdot \chi_{\mathrm{cyc}}^{(\boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k})/2} - 1 : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}.$$

The assumption $\alpha_f = \alpha_g \cdot \alpha_h$ implies that Φ takes value in $\mathscr{I}$, and that its composition Φ' with the projection $\mathscr{I} \longrightarrow \mathscr{I}/\mathscr{I}^2$ is of the form

$$\Phi' = \varphi_{\boldsymbol{k}} \cdot (\boldsymbol{k} - 2) + \varphi_{\boldsymbol{l}} \cdot (\boldsymbol{l} - 1) + \varphi_{\boldsymbol{m}} \cdot (\boldsymbol{m} - 1)$$

with $\varphi_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ for $\boldsymbol{u} = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$. Identify $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ with the $\mathbf{Q}_p$-vector space $\mathrm{Hom}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ of continuous morphisms of groups from $\mathbf{Q}_p^*$ to $\mathbf{Q}_p$ via the local reciprocity map $\mathrm{rec}_p : \mathbf{Q}_p^* \longrightarrow G_{\mathbf{Q}_p}^{\mathrm{ab}}$, normalised by requiring $\mathrm{rec}_p(p^{-1})$ to be an arithmetic Frobenius. By local class field theory, for each p-adic unit u one has

$$\varphi_{\boldsymbol{k}}(u) = \frac{\partial}{\partial \boldsymbol{k}} \left(\langle u \rangle^{(\boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k})/2} - 1 \right)\Big|_{w_o} = -\frac{1}{2} \cdot \log_p(u),$$

where $\langle \cdot \rangle : \mathbf{Z}_p^* \longrightarrow 1 + p\mathbf{Z}_p$ denotes the projection to principal units, and

$$\varphi_{\boldsymbol{k}}(p) = \frac{\partial}{\partial \boldsymbol{k}} \left(\frac{a_p(\boldsymbol{g}_\alpha) \cdot a_p(\boldsymbol{h}_\alpha)}{a_p(\boldsymbol{f})} - 1 \right)\Big|_{w_o} = \frac{1}{2} \cdot \mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}$$

(cf. Eq. (7)). As a consequence $-2 \cdot \varphi_{\boldsymbol{k}}$ is equal to

$$\log_{\boldsymbol{f}} = \log_p - \mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} \cdot \mathrm{ord}_p \in H^1(\mathbf{Q}_p, \mathbf{Q}_p)$$

(where the p-adic valuation $\mathrm{ord}_p : \mathbf{Q}_p^* \longrightarrow \mathbf{Q}_p$ is normalised by $\mathrm{ord}_p(p) = 1$). Similarly one shows that $2 \cdot \varphi_{\boldsymbol{l}}$ and $2 \cdot \varphi_{\boldsymbol{m}}$ are equal to the logarithms $\log_{\boldsymbol{g}_\alpha} = \log_p - \mathfrak{L}_{\boldsymbol{g}_\alpha}^{\mathrm{an}} \cdot \mathrm{ord}_p$ and $\log_{\boldsymbol{h}_\alpha} = \log_p - \mathfrak{L}_{\boldsymbol{g}_\alpha}^{\mathrm{an}} \cdot \mathrm{ord}_p$. It then follows from Eq. (17) and Lemma 3.2 that

$$2 \cdot \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha}(q_{\beta\beta}) = \Big(\log_{\boldsymbol{f}} \cdot (\boldsymbol{k} - 2) - \log_{\boldsymbol{g}_\alpha} \cdot (\boldsymbol{l} - 1) - \log_{\boldsymbol{h}_\alpha} \cdot (\boldsymbol{m} - 1) \Big) \otimes q_{\beta\beta} \tag{18}$$

in $H^1(\mathbf{Q}_p, V^-) \otimes_L \mathscr{I}/\mathscr{I}^2$, where (with the notations introduced in Sect. 3.2.1) one writes $\beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha}$ for the Bockstein map $\beta_{\boldsymbol{C}/C}$ associated with $\boldsymbol{C} = \mathbf{R}\Gamma_p(\boldsymbol{V}^-)$. Note that

$$V(f)^-_{\beta\beta} = V(f)^- \otimes_{\mathbf{Q}_p} V(g)_\beta \otimes_L V(h)_\beta$$

is an $L[G_{\mathbf{Q}_p}]$-direct summand of V^- on which $G_{\mathbf{Q}_p}$ acts trivially, so that $\log_{\boldsymbol{\xi}} \otimes q_{\beta\beta}$ (for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$) belongs to the direct summand

$$H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta}) = H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V(f)^-_{\beta\beta}$$

of the local cohomology group $H^1(\mathbf{Q}_p, V^-)$. Similarly

$$V(f)^+_{\alpha\alpha} = V(f)^+ \otimes_{\mathbf{Q}_p} V(g)_\alpha \otimes_L V(h)_\alpha$$

is an $L[G_{\mathbf{Q}_p}]$-direct summand of V^+ isomorphic to $\mathbf{Q}_p(1)$, hence

$$H^1(\mathbf{Q}_p, V(f)^+_{\alpha\alpha}) = H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} V(f)^+_{\alpha\alpha}(-1) \tag{19}$$

is a direct summand of $H^1(\mathbf{Q}_p, V^+)$. The local Tate pairing $\langle\cdot,\cdot\rangle_{\mathrm{Tate}}$ introduced in Sect. 3.2.2 induces a perfect duality (denoted by the same symbol) between $H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta})$ and $H^1(\mathbf{Q}_p, V(f)^+_{\alpha\alpha})$, and identifying $H^1(\mathbf{Q}_p, \mathbf{Z}_p(1))$ with the p-adic completion $\hat{\mathbf{Q}}_p^*$ of $\mathbf{Q}_p^*$ via the local Kummer map, local class field theory gives

$$\langle \varphi \otimes v^-, u \otimes v^+ \rangle_{\mathrm{Tate}} = \varphi(u) \cdot \pi_{fgh}(-1)(v^+ \otimes v^-) \tag{20}$$

for each φ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$, u in $H^1(\mathbf{Q}_p, \mathbf{Q}_p(1))$, v^- in $V(f)^-_{\beta\beta}$ and v^+ in $V(f)^+_{\alpha\alpha}$. Here

$$\pi_{fgh}(-1) : V(f)^+_{\alpha\alpha}(-1) \otimes_L V(f)^-_{\beta\beta} \longrightarrow L$$

is the composition of $\pi_{fgh} \otimes \mathbf{Q}_p(-1)$ with the evaluation pairing $L(1) \otimes_L L(-1) \longrightarrow L$.

Recall that we identify $H^0(\mathbf{Q}_p, V^-)$ with a submodule of $\tilde{H}^1_f(\mathbf{Q}, V)$ via the embedding $\jmath$ introduced in Diagram (13). Lemma 3.4 and Eqs. (18) and (20) give

$$\begin{aligned} 2 \cdot \langle\!\langle q_{\beta\beta}, z \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} &\overset{\text{Lemma 3.8}}{=} 2 \cdot \langle \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q_{\beta\beta}), z^+ \rangle_{\mathrm{Tate}} \\ &\overset{\text{Equation (18)}}{=} \sum_{\boldsymbol{\xi}} (-1)^{u_o} \cdot \langle \log_{\boldsymbol{\xi}} \otimes q_{\beta\beta}, z^+ \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o) \\ &\overset{\text{Equation (20)}}{=} \sum_{\boldsymbol{\xi}} (-1)^{u_o} \cdot \log_{\boldsymbol{\xi}}(z^+_{\alpha\alpha}) \cdot (\boldsymbol{u} - u_o) \end{aligned} \tag{21}$$

for each z in $\tilde{H}^1_f(\mathbf{Q}, V)$, where $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$, $u_o = 2, 1, 1$ is the centre of $U_{\boldsymbol{\xi}}$, and

$$z^+_{\alpha\alpha} \in H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) = \hat{\mathbf{Q}}_p^* \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

is defined as follows. Let $\mathrm{pr}_{\alpha\alpha}$ denote the projection onto the direct summand $H^1(\mathbf{Q}_p, V(f)^+_{\alpha\alpha})$ of the local cohomology group $H^1(\mathbf{Q}_p, V^+)$, and let $q^*_{\beta\beta}$ be the generator of $V(f)^+_{\alpha\alpha}(-1)$ dual to $q_{\beta\beta}$ under $\pi_{fgh}(-1)$, namely satisfying

$$\pi_{fgh}(-1)(q^*_{\beta\beta} \otimes q_{\beta\beta}) = 1.$$

Then $z^+_{\alpha\alpha}$ is defined (via the natural isomorphism (19)) by the identity

$$\mathrm{pr}_{\alpha\alpha}(z^+) = z^+_{\alpha\alpha} \otimes q^*_{\beta\beta}. \tag{22}$$

We now determine $z^+_{\alpha\alpha}$ for $z = \jmath(q_{\alpha\alpha})$. By definition $\jmath(q_{\alpha\alpha})$ is represented by

$$c_{\alpha\alpha} = (0, d\tilde{q}_{\alpha\alpha}, \tilde{q}_{\alpha\alpha}) \in \tilde{\mathrm{C}}^1_f(\mathbf{Q}, V),$$

where $\tilde{q}_{\alpha\alpha}$ in V is a lift of $q_{\alpha\alpha}$ under the the projection $V \longrightarrow V^-$, and where

$$d\tilde{q}_{\alpha\alpha} : G_{\mathbf{Q}_p} \longrightarrow V^+$$

is its image under the differential in $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, V)$. By construction $d\tilde{q}_{\alpha\alpha}$ represents the class $q^+_{\alpha\alpha} = \jmath(q_{\alpha\alpha})^+$ in $H^1(\mathbf{Q}_p, V^+)$. Since $V(\xi)$ is the direct sum of $V(\xi)_\alpha$ and $V(\xi)_\beta$ for $\xi = g, h$, we can (and will) choose $\tilde{q}_{\alpha\alpha}$ of the form

$$\tilde{q}_{\alpha\alpha} = \sqrt{m_p} \cdot \tilde{q}(f) \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}$$

for a lift $\tilde{q}(f)$ of $q(f)$ under the projection $V(f) \longrightarrow V(f)^-$, so that $d\tilde{q}_{\alpha\alpha}$ represents the image of $q_{\alpha\alpha}$ under the connecting morphism

$$\delta_{\alpha\alpha} : V(f)^-_{\alpha\alpha} \longrightarrow H^1(\mathbf{Q}_p, V(f)^+_{\alpha\alpha})$$

arising from the short exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0 \longrightarrow V(f)^+_{\alpha\alpha} \longrightarrow V(f)_{\alpha\alpha} \longrightarrow V(f)^-_{\alpha\alpha} \longrightarrow 0,$$

where $V(f)^{\cdot}_{\alpha\alpha}$ is the $L[G_{\mathbf{Q}_p}]$-direct summand $V(f)^{\cdot} \otimes_{\mathbf{Q}_p} V(g)_\alpha \otimes_L V(h)_\alpha$ of $V^{\cdot}$. Let q_A in $p\mathbf{Z}_p$ be the Tate period of $A_{\mathbf{Q}_p}$. Tate's theory gives a rigid analytic isomorphisms between the base change $E_{\mathbf{Q}_p^2}$ of the Tate curve $E = \mathbf{G}^{rig}_{m,\mathbf{Q}_p}/q_A^{\mathbf{Z}}$ to the quadratic unramified extension $\mathbf{Q}_{p^2}$ of $\mathbf{Q}_p$ and $A_{\mathbf{Q}_{p^2}}$. Set $V_p(E) = H^1_{\text{ét}}(E_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p(1))$ and let $\wp_{\mathrm{Tate}} : V_p(E) \simeq V_p(A)$ be the isomorphisms of $G_{\mathbf{Q}_{p^2}}$-modules induced by the Tate uniformisation. There is a short exact sequence of $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow \mathbf{Q}_p(1) \xrightarrow{a} V_p(E) \xrightarrow{b} \mathbf{Q}_p \longrightarrow 0, \tag{23}$$

where $a(\zeta_{p^\infty}) = (\zeta_{p^n} \cdot q_A^{\mathbf{Z}})_{n\geq 1}$ for each compatible system $\zeta_{p^\infty} = (\zeta_{p^n})_{n\geq 1}$ of p^n-th roots of unity, and b is the $\mathbf{Q}_p$-linear extension of the inverse limit of (canonical) maps

$$b_n : E(\bar{\mathbf{Q}}_p)_{p^n} = (\bar{\mathbf{Q}}^*_p/q_A^{\mathbf{Z}})_{p^n} \longrightarrow \mathbf{Z}/p^n\mathbf{Z}$$

defined by $b_n(x \cdot q_A^{\mathbf{Z}}) = \frac{p^n \cdot \mathrm{ord}_p(x)}{\mathrm{ord}_p(q_A)} + p^n \cdot \mathbf{Z}$. By definition $q(A) = \wp^-_{\mathrm{Tate}}(1)$, where $\wp^-_{\mathrm{Tate}} \circ b$ is the composition of $\wp_{\mathrm{Tate}}$ and the projection $V_p(A) \longrightarrow V_p(A)^-$ onto the maximal $G_{\mathbf{Q}_p}$-unramified quotient, and

$$\tilde{q}(f) = \wp_\infty^{-1} \circ \wp_{\mathrm{Tate}}(\sqrt[p^\infty]{q_A})$$

is the image of a compatible system $\sqrt[p^\infty]{q_A}$ of p^n-th roots of the Tate period q_A under the composition of $\wp_{\mathrm{Tate}}$ and the inverse of the isomorphism $\wp_\infty : V(f) \simeq V_p(A)$ induced by the fixed modular parametrisation $\wp_\infty : X_1(N_f) \longrightarrow A$. As a consequence 1 in $\mathbf{Q}_p$ maps to $q_A\hat{\otimes}1$ under the connecting map $\mathbf{Q}_p \longrightarrow H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) = \mathbf{Q}^*_p\hat{\otimes}\mathbf{Q}_p$ associated with the short exact sequence (23), hence

$$\jmath(q_{\alpha\alpha})^+ = cl(d\tilde{q}_{\alpha\alpha}) = \delta_{\alpha\alpha}(q_{\alpha\alpha}) = \sqrt{m_p} \cdot (\wp^{-1}_{\infty *} \circ \wp_{\mathrm{Tate}})^+_*(q_A\hat{\otimes}1) \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \tag{24}$$

in

$$H^1(\mathbf{Q}_p, V(f)^+_{\alpha\alpha}) = H^0\big(\mathrm{Gal}(\mathbf{Q}_{p^2}/\mathbf{Q}), H^1(\mathbf{Q}_{p^2}, V(f)^+) \otimes_{\mathbf{Q}_p} V(g)_\alpha \otimes_L V(h)_\alpha\big),$$

where

$$(\wp_\infty^{-1} \circ \wp_{\mathrm{Tate}})^+_* : \mathbf{Q}^*_{p^2}\hat{\otimes}\mathbf{Q}_p \simeq H^1(\mathbf{Q}_{p^2}, V(f)^+)$$

is the map induced in cohomology by the composition of $\wp_\infty^{-1}$ and

$$\wp_{\mathrm{Tate}}^{+} = \wp_{\mathrm{Tate}} \circ a.$$

If $\mathcal{A}$ denotes either A or E, denote by

$$\pi_{\mathcal{A}} : V_p(\mathcal{A})(-1) \otimes_{\mathbf{Q}_p} V_p(\mathcal{A}) \longrightarrow \mathbf{Q}_p$$

the composition of the evaluation pairing $\mathbf{Q}_p(1) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(-1) \longrightarrow \mathbf{Q}_p$ with the base change of the Weil pairing on $V_p(\mathcal{A})$ by $\mathbf{Q}_p(-1)$. Set

$$q(A)^* = \wp_{\mathrm{Tate}}^{+}(\zeta_{p^\infty}) \otimes \zeta_{p^\infty}^* \in V_p(A)^+(-1),$$

where ζ_{p^∞} is a generator of $\mathbf{Q}_p(1)$ and $\zeta_{p^\infty}^*$ in $\mathbf{Q}_p(-1)$ is its dual basis, and set

$$q(f)^* = \deg(\wp_\infty) \cdot \wp_\infty^{-1}(q(A)^*) \in V(f)^+(-1).$$

As $\pi_E((a(y) \otimes z) \otimes x) = b(x) \cdot z(y)$ for each x in $V_p(E)$, y in $\mathbf{Q}_p(1)$ and z in $\mathbf{Q}_p(-1)$, the functoriality of the Poincaré duality under finite morphisms yields

$$\pi_f(q(f)^* \otimes q(f)) = \pi_A(q(A)^* \otimes q(A)) = \pi_E\big((a(\zeta_{p^\infty}) \otimes \zeta_{p^\infty}^*) \otimes \sqrt[p^\infty]{q_A}\big) = 1,$$

then (by the definition of the weight-one differentials η_{ξ_α}, cf. Sect. 3.1.1)

$$q_{\beta\beta}^* = \frac{1}{\sqrt{m_p}} \cdot q(f)^* \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}.$$

Together with Eq. (24) this gives

$$\jmath(q_{\alpha\alpha})^+ = \frac{m_p}{\deg(\wp_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q_{\beta\beta}^*, \tag{25}$$

id est

$$\jmath(q_{\alpha\alpha})_{\alpha\alpha}^+ = \frac{m_p}{\deg(\wp_\infty)} \cdot q_A \hat{\otimes} 1. \tag{26}$$

According to Theorem 3.18 of [9] $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} = \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$, so that

$$-\frac{2 \cdot \deg(\wp_\infty)}{m_p \cdot \mathrm{ord}_p(q_A)} \cdot \langle\!\langle q_{\beta\beta}, q_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{g}_\alpha}^{\mathrm{an}}) \cdot (\boldsymbol{l} - 1) + (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{h}_\alpha}^{\mathrm{an}}) \cdot (\boldsymbol{m} - 1) \tag{27}$$

by Eqs. (21) and (26).

3.4 Computation of $\langle\!\langle q_{\alpha\beta}, q_{\beta\alpha} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$

Assume in this subsection $\alpha_f = \beta_g \cdot \alpha_h$, so that $H^0(\mathbf{Q}_p, V^-)$ is generated by the p-adic periods

$$q_{\alpha\beta} = \sqrt{m_p} \cdot q(f) \otimes \omega_{g_\alpha} \otimes \eta_{h_\alpha} \quad \text{and} \quad q_{\beta\alpha} = \sqrt{m_p} \cdot q(f) \otimes \eta_{g_\alpha} \otimes \omega_{h_\alpha}.$$

For $\gamma\delta = \alpha\beta, \beta\alpha$ and $\cdot = \emptyset, \pm$, define $V(f)^{\cdot}_{\gamma\delta} = V(f)^{\cdot} \otimes_{\mathbf{Q}_p} V(g)_\gamma \otimes V(h)_\delta$. Then

$$H^0(\mathbf{Q}_p, V^-) = V(f)^-_{\alpha\beta} \oplus V(f)^-_{\beta\alpha},$$

$G_{\mathbf{Q}_p}$ acts on $V(f)^+_{\alpha\beta}$ and $V(f)^+_{\beta\alpha}$ via the p-adic cyclotomic character, and the local Tate pairing $\langle\cdot,\cdot\rangle_{\mathrm{Tate}}$ introduced in Sect. 3.2.2 induces a perfect duality (denoted by the same symbol) between $H^1(\mathbf{Q}_p, V(f)^-_{\alpha\beta})$ and $H^1(\mathbf{Q}_p, V(f)^+_{\beta\alpha})$. The argument of the proof of Eq. (25) shows that

$$\jmath(q_{\beta\alpha})^+ = \frac{m_p}{\deg(\wp_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q^*_{\alpha\beta} \tag{28}$$

in the direct summand $H^1(\mathbf{Q}_p, V(f)^+_{\beta\alpha}) = \mathbf{Q}^*_p \hat{\otimes} V(f)^+_{\beta\alpha}(-1)$ of $H^1(\mathbf{Q}_p, V^+)$, where

$$q^*_{\alpha\beta} = \frac{1}{\sqrt{m_p}} \cdot q(f)^* \otimes \eta_{g_\alpha} \otimes \omega_{h_\alpha} \ \text{ satisfies } \ \pi_{fgh}(-1)(q^*_{\alpha\beta} \otimes q_{\alpha\beta}) = 1. \tag{29}$$

Let $\mathrm{pr}_{\alpha\beta} : H^1(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V(f)^-_{\alpha\beta}$ denote the projection, and write

$$\mathrm{pr}_{\alpha\beta} \otimes \mathscr{I}/\mathscr{I}^2 \circ \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q_{\alpha\beta}) = \sum_{\boldsymbol{u}} \gamma_{\boldsymbol{u}} \otimes q_{\alpha\beta} \cdot (\boldsymbol{u} - u_o) \tag{30}$$

with $\gamma_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p) = \mathrm{Hom}(\mathbf{Q}^*_p, \mathbf{Q}_p)$ for $\boldsymbol{u} = \boldsymbol{k}, l, \boldsymbol{m}$, where (with the notations introduced in Sect. 3.2.1) $\beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is a shorthand for

$$\beta_{\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)/\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, V^-)} : H^0(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V^-) \otimes_L \mathscr{I}/\mathscr{I}^2,$$

and $u_o = 2$ if $\boldsymbol{u} = \boldsymbol{k}$ and $u_o = 1$ if $\boldsymbol{u} = l, \boldsymbol{m}$. Then (cf. Eq. (21))

$$\begin{aligned}
\langle\!\langle q_{\alpha\beta}, q_{\beta\alpha} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} &\overset{\text{Lemma 3.4}}{=} \langle \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q_{\alpha\beta}), \jmath(q_{\beta\alpha})^+ \rangle_{\mathrm{Tate}} \\
&\overset{\text{Eqs. (28) and (30)}}{=} \frac{m_p}{\deg(\wp_\infty)} \cdot \sum_{\boldsymbol{u}} \langle \gamma_{\boldsymbol{u}} \otimes q_{\alpha\beta}, (q_A \hat{\otimes} 1) \otimes q^*_{\alpha\beta} \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o) \\
&= \frac{m_p}{\deg(\wp_\infty)} \cdot \sum_{\boldsymbol{u}} \gamma_{\boldsymbol{u}}(q_A) \cdot (\boldsymbol{u} - u_o),
\end{aligned} \tag{31}$$

where the last equality follows from Eq. (29) and the analogue of Eq. (20) obtained by replacing $\alpha\alpha$ and $\beta\beta$ with $\beta\alpha$ and $\alpha\beta$ respectively. It then remains to compute $\gamma_{\boldsymbol{u}}$ for $\boldsymbol{u}$ equal to $\boldsymbol{k}$, l and $\boldsymbol{m}$.

For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$, fix $\mathscr{O}_{\boldsymbol{\xi}}$-bases $b^\pm_{\boldsymbol{\xi}}$ of $V(\boldsymbol{\xi})^\pm$. After identifying $V(\boldsymbol{\xi})$ with $\mathscr{O}_{\boldsymbol{\xi}} \oplus \mathscr{O}_{\boldsymbol{\xi}}$ via the $\mathscr{O}_{\boldsymbol{\xi}}$-basis $(b^+_{\boldsymbol{\xi}}, b^-_{\boldsymbol{\xi}})$, the action of $G_{\mathbf{Q}_p}$ on $V(\boldsymbol{\xi})$ is given by (cf. Eq. (9))

$$\begin{pmatrix} \chi_{\boldsymbol{\xi}} \cdot \breve{a}_p(\boldsymbol{\xi})^{-1} \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} & c_{\boldsymbol{\xi}} \\ 0 & \breve{a}_p(\boldsymbol{\xi}) \end{pmatrix} : G_{\mathbf{Q}_p} \longrightarrow \mathrm{GL}_2(\mathscr{O}_{\boldsymbol{\xi}})$$

for a continuous map $c_{\boldsymbol{\xi}} : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}$. Without loss of generality, assume that

$$\boldsymbol{q}_{\alpha\beta} = b^-_{\boldsymbol{f}} \hat{\otimes} b^-_{\boldsymbol{g}_\alpha} \hat{\otimes} b^+_{\boldsymbol{h}_\alpha} \otimes 1$$

in $\boldsymbol{V}^- = V(\boldsymbol{f})^- \hat{\otimes}_L V(\boldsymbol{g}_\alpha) \hat{\otimes}_L V(\boldsymbol{h}_\alpha) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}$ maps to

$$q_{\alpha\beta} \in V(f)^-_{\alpha\beta} = V(f)^- \otimes_{\mathbf{Q}_p} V(g)_\alpha \otimes_L V(h)_\beta$$

under $\rho_w : \boldsymbol{V}^- \longrightarrow V^-$. (Recall that $V(\xi) = V(\boldsymbol{\xi}_\alpha) \otimes_1 L$ is the direct sum of the modules $V(\xi)_\alpha = V(\boldsymbol{\xi}_\alpha)^- \otimes_1 L$ and $V(\xi)_\beta = V(\boldsymbol{\xi}_\alpha)^+ \otimes_1 L$ for $\xi = g, h$, cf. Sect. 3.1.1.) Then

$$d\boldsymbol{q}_{\alpha\beta} = \Gamma \cdot \boldsymbol{q}_{\alpha\beta} + \Delta \cdot \boldsymbol{q}_{\beta\beta}, \tag{32}$$

where $\boldsymbol{q}_{\beta\beta} = b_{\boldsymbol{f}}^{-} \hat{\otimes} b_{\boldsymbol{g}_\alpha}^{+} \hat{\otimes} b_{\boldsymbol{h}_\alpha}^{+} \otimes 1$, where

$$\Gamma = \frac{\breve{a}_p(\boldsymbol{f}) \cdot \breve{a}_p(\boldsymbol{g}_\alpha)}{\breve{a}_p(\boldsymbol{h}_\alpha)} \cdot \chi_h \cdot \chi_{\mathrm{cyc}}^{(\boldsymbol{m}-\boldsymbol{k}-l+2)/2} - 1$$

and where

$$\Delta = \breve{a}_p(\boldsymbol{f}) \cdot \breve{a}_p(\boldsymbol{h}_\alpha)^{-1} \cdot \chi_h \cdot \chi_{\mathrm{cyc}}^{(\boldsymbol{m}-\boldsymbol{k}-l+2)/2} \cdot c_{\boldsymbol{g}_\alpha}.$$

The exceptional zero condition $\alpha_f = \beta_g \cdot \alpha_h$ and the self duality condition $\chi_g \cdot \chi_h = 1$ imply that Γ takes values in $\mathscr{I}$. Moreover, since the $G_{\mathbf{Q}_p}$-module $V(g) = V(\boldsymbol{g}_\alpha) \otimes_1 L$ splits as the direct sum of $V(g)_\beta = V(\boldsymbol{g}_\alpha)^+ \otimes_1 L$ and $V(g)_\alpha = V(\boldsymbol{g}_\alpha)^- \otimes_1 L$, the map $c_{\boldsymbol{g}_\alpha}$ takes values in $(l-1) \cdot \mathscr{O}_{\boldsymbol{g}}$, hence Δ takes values in $\mathscr{I}$. Because by construction $\boldsymbol{q}_{\beta\beta}$ maps to an element of $V(f)_{\beta\beta}^-$ under the specialisation map $\rho_{w_o} : \boldsymbol{V}^- \longrightarrow V^-$, Lemma 3.2 and Eqs. (30) and (32) yield the identities

$$\gamma_{\boldsymbol{u}} = -\frac{\partial}{\partial \boldsymbol{u}} \Gamma(\cdot)(w_o),$$

hence (as in the previous subsection) a direct computation gives

$$\gamma_{\boldsymbol{k}} = \frac{1}{2} \cdot \log_{\boldsymbol{f}}, \quad \gamma_l = \frac{1}{2} \cdot \log_{\boldsymbol{g}_\alpha} \quad \text{and} \quad \gamma_{\boldsymbol{m}} = -\frac{1}{2} \cdot \log_{\boldsymbol{h}_\alpha}. \tag{33}$$

Recalling that $\log_{\boldsymbol{f}}(q_A) = 0$ by [9, Theorem 3.18], Eq. (31) finally proves

$$\frac{2 \cdot \deg(\wp_\infty)}{m_p \cdot \mathrm{ord}_p(q_A)} \cdot \langle\!\langle q_{\alpha\beta}, q_{\beta\alpha} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{g}_\alpha}^{\mathrm{an}}) \cdot (l-1) - (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{h}_\alpha}^{\mathrm{an}}) \cdot (\boldsymbol{m}-1). \tag{34}$$

3.5 Proof of equation (8)

Assume in this subsection that (A, ϱ) is exceptional at p, and fix a Selmer class x in $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$. Let

$$\tilde{x} = \imath_{\mathrm{ur}}(x) \in \tilde{H}_f^1(\mathbf{Q}, V(f, g, h))$$

be the corresponding extended Selmer class (cf. Sect. 3.1.2). By construction $\tilde{x}^+$ belongs to the finite subspace of $H^1(\mathbf{Q}_p, V^+)$, and its image under the natural map $i^+ : H_{\mathrm{fin}}^1(\mathbf{Q}_p, V^+) \longrightarrow H_{\mathrm{fin}}^1(\mathbf{Q}_p, V)$ equals the restriction of x at p:

$$\mathrm{res}_p(x) = i^+(\tilde{x}^+). \tag{35}$$

The Galois group $G_{\mathbf{Q}_p}$ acts on $V(f)_\natural^+$ via the p-adic cyclotomic character, hence

$$H_{\mathrm{fin}}^1(\mathbf{Q}_p, V(f)_\natural^+) = \mathbf{Z}_p^* \otimes_{\mathbf{Z}_p} V(f)_\natural^+(-1)$$

by Kummer theory. If $q_\flat^*$ in $V(f)_\natural^+$ denotes (as in the previous subsections) the dual basis of $q_\flat$ in $V(f)_\natural^-$ under the pairing π_{fgh}, and if one writes

$$\mathrm{pr}_\natural(\tilde{x}^+) = \tilde{x}_\natural^+ \otimes q_\flat^* \in H_{\mathrm{fin}}^1(\mathbf{Q}_p, V(f)_\natural^+)$$

for some $\tilde{x}_\natural^+$ in $\mathbf{Z}_p^* \otimes_{\mathbf{Z}_p} L$, then Eq. (35) yields the equality

$$\log_\natural(\mathrm{res}_p(x)) = \langle \log_p^+(\tilde{x}^+), q_\flat \rangle_{fgh} = \langle \log_p(\tilde{x}_\natural^+) \otimes q_\flat^*, q_\flat \rangle_{fgh} = \log_p(\tilde{x}_\natural^+), \tag{36}$$

where $\log_p^+ : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V^+) \simeq D_{\mathrm{dR}}(V^+)$ is the Bloch–Kato logarithm and (with a slight abuse of notation) we denote again by $\log_p : \mathbf{Z}_p^* \otimes_{\mathbf{Z}_p} L \longrightarrow L$ the L-linear extension of the p-adic logarithm. In the previous equation we used the functoriality of the Bloch–Kato logarithm and the fact that (by construction) the linear form $\langle \cdot, q_\flat \rangle_{fgh}$ on $D_{\mathrm{dR}}(V^+)$ factors through the projection onto $D_{\mathrm{dR}}(V(f)^+_\natural) = V(f)^+_\natural(-1)$.

Assume ($\alpha_f = \alpha_g \cdot \alpha_h$ and) $q_\flat = q_{\beta\beta}$. According to Eqs. (21) and (36)

$$2 \cdot \langle\!\langle q_{\beta\beta}, x \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \log_{\alpha\alpha}(\mathrm{res}_p(x)) \cdot (\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m}), \tag{37}$$

thus proving Eq. (8) in this case.

Assume $q_\flat = q_{\alpha\beta}$. Since (with the notations of Section 3.4) Δ takes values in $(\boldsymbol{l}-1)\cdot\mathscr{O}_{\boldsymbol{fgh}}$, it follows from Lemma 3.2 and Eqs. (32) and (33) that

$$2 \cdot \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q_{\alpha\beta}) = \sum_{\boldsymbol{\xi}} \varepsilon_{\boldsymbol{\xi}} \cdot \log_{\boldsymbol{\xi}} \otimes q_{\alpha\beta} \cdot (\boldsymbol{u} - u_o) + \vartheta \cdot (\boldsymbol{l} - 1) \tag{38}$$

for some cohomology class ϑ in $H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta})$, where $\varepsilon_{\boldsymbol{h}_\alpha} = -1$ and $\varepsilon_{\boldsymbol{\xi}} = +1$ for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha$. One has then

$$\begin{aligned}
\langle\!\langle q_{\alpha\beta}, x \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(\boldsymbol{k}, 1, 1) &\overset{\text{Lemma 3.4}}{=} \langle \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q_{\alpha\beta}), \tilde{x}^+ \rangle_{\mathrm{Tate}}(\boldsymbol{k}, 1, 1) \\
&\overset{\text{Equation (38)}}{=} \frac{1}{2} \cdot \langle \log_{\boldsymbol{f}} \otimes q_{\alpha\beta}, \tilde{x}^+_{\beta\alpha} \otimes q^*_{\alpha\beta} \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{k} - 2) \\
&= \frac{1}{2} \cdot \log_{\boldsymbol{f}}(\tilde{x}^+_{\alpha\beta}) \cdot \pi_{fgh}(q_{\alpha\beta} \otimes q^*_{\alpha\beta}) \cdot (\boldsymbol{k} - 2) \\
&\overset{\text{Equation (36)}}{=} \frac{1}{2} \cdot \log_{\alpha\beta}(\mathrm{res}_p(x)) \cdot (\boldsymbol{k} - 2),
\end{aligned} \tag{39}$$

thus proving Eq. (8) when $q_\flat = q_{\alpha\beta}$. Switching the roles of the Hida families $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$, this also proves Eq. (8) when $q_\flat = q_{\beta\alpha}$.

Assume finally $q_\flat = q_{\alpha\alpha}$. With the notations of Sect. 3.4, let $(b^+_{\boldsymbol{\xi}}, b^-_{\boldsymbol{\xi}})$ be $\mathscr{O}_{\boldsymbol{\xi}}$-bases of $V(\boldsymbol{\xi})$ such that $\boldsymbol{q}_{\alpha\alpha} = b^-_{\boldsymbol{f}} \hat{\otimes} b^-_{\boldsymbol{g}_\alpha} \hat{\otimes} b^-_{\boldsymbol{h}_\alpha} \otimes 1$ is a lift of $q_{\alpha\alpha}$ under the specialisation map $\rho_{w_o} : \boldsymbol{V}^- \longrightarrow V^-$. Since $c_{\boldsymbol{\xi}}$ takes values in $(\boldsymbol{u} - u_o) \cdot \mathscr{O}_{\boldsymbol{\xi}}$ for $\boldsymbol{\xi} = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$, one has

$$d\boldsymbol{q}_{\alpha\alpha} \equiv \Big(\chi_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2} \cdot \prod_{\boldsymbol{\xi}} \breve{a}_p(\boldsymbol{\xi}) - 1 \Big) \cdot \boldsymbol{q}_{\alpha\alpha} \ \big(\mathrm{mod}\ (\boldsymbol{l} - 1, \boldsymbol{m} - 1) \cdot \mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)\big),$$

hence Lemma 3.2 and a direct computation give

$$2 \cdot \beta^-_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(q_{\alpha\alpha}) = \log_{\boldsymbol{f}} \otimes q_{\alpha\alpha} \cdot (\boldsymbol{k} - 2) + \vartheta \cdot (\boldsymbol{l} - 1) + \vartheta' \cdot (\boldsymbol{m} - 1) \tag{40}$$

for some local cohomology classes ϑ and ϑ' in $H^1(\mathbf{Q}_p, V^-)$. As in (39) one deduces Eq. (8) for $q_\flat = q_{\alpha\alpha}$ from Lemma 3.4 and Eqs. (36) and (40).

4 Proof of theorem 2.1

Let Π_f, Π_g and Π_h be the *improving* planes in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ defined respectively by the equations $\boldsymbol{k} = \boldsymbol{l} + \boldsymbol{m}$, $\boldsymbol{k} = \boldsymbol{l} - \boldsymbol{m} + 2$ and $\boldsymbol{k} = \boldsymbol{m} - \boldsymbol{l} + 2$. For $\xi = f, g, h$ define

$$\mathcal{E}_\xi = 1 - \bar{\chi}_\xi(p) \cdot \frac{a_p(\boldsymbol{\xi})}{a_p(\boldsymbol{\xi}') \cdot a_p(\boldsymbol{\xi}'')}$$

in $\mathscr{O}_{\boldsymbol{fgh}}$, where $\{\boldsymbol{\xi}, \boldsymbol{\xi}', \boldsymbol{\xi}''\} = \{\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha\}$. Lemma 9.8 of [7] implies that

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho)|_{\Pi_\xi} = \mathcal{E}_\xi|_{\Pi_\xi} \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho)_\xi^\star \tag{41}$$

for an improved p-adic L-function $\mathscr{L}_p^{\alpha\alpha}(A, \varrho)_\xi^\star$ in $\mathcal{O}(\Pi_\xi)$. Indeed loc. cit. (together with its analogue obtained by switching the roles of g and h) proves that the meromorphic function $\mathscr{L}_p^{\alpha\alpha}(A, \varrho)_\xi^\star$ on Π_ξ defined by the previous equation is (bounded, hence) regular at w_o. Shrinking the discs $U_{\boldsymbol{\xi}}$ if necessary, we then conclude that the improved p-adic L-function $\mathscr{L}_p^{\alpha\alpha}(A, \varrho)_\xi^\star$ is analytic on Π_ξ, as claimed.

Assume first $\alpha_f = \alpha_g \cdot \alpha_h$, so that

$$2 \cdot \mathcal{E}_f \pmod{\mathscr{I}^2} = \mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} \cdot (\boldsymbol{k} - 2) - \mathfrak{L}_{\boldsymbol{g}_\alpha}^{\mathrm{an}} \cdot (l - 1) - \mathfrak{L}_{\boldsymbol{h}_\alpha}^{\mathrm{an}} \cdot (\boldsymbol{m} - 1). \tag{42}$$

According to Theorem A and Proposition 9.3 of [7], the partial derivative of $\mathscr{L}_p^{\alpha\alpha}(A, \varrho)$ with respect to $\boldsymbol{k}$ vanishes at w_o, hence

$$2 \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^2}$$

is equal to

$$\left((\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{g_\alpha}^{\mathrm{an}}) \cdot (l - 1) + (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{h}_\alpha}^{\mathrm{an}}) \cdot (\boldsymbol{m} - 1)\right) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho)_f^\star(w_o)$$

by Eqs. (41) and (42). Moreover, with the notations introduced before the statement of Theorem 2.1, one has $\mathsf{L} = \Pi_f \cap \Pi_g$ and $\mathscr{E}_f = \mathcal{E}_f|_{\mathsf{L}}$, thus

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho)_f^\star(w_o) = \mathscr{E}_g(w_o) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho)^\star(w_o).$$

Noting that $\mathscr{E}_g(w_o) = 1 - \beta_h/\alpha_h$ (when $\alpha_f = \alpha_g \cdot \alpha_h$), the previous discussion and Eq. (27) conclude the proof of Theorem 2.1 when $\alpha_f = \alpha_g \cdot \alpha_h$.

Assume now $\alpha_f = \beta_g \cdot \alpha_h$. In this case, for $\xi = g, h$, one has

$$2 \cdot \mathcal{E}_\xi \pmod{\mathscr{I}^2} = \mathfrak{L}_{\boldsymbol{\xi}_\alpha}^{\mathrm{an}} \cdot (\boldsymbol{u} - 1) - \mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} \cdot (\boldsymbol{k} - 2) - \mathfrak{L}_{\boldsymbol{\xi}'_\alpha}^{\mathrm{an}} \cdot (\boldsymbol{u}' - 1), \tag{43}$$

where $\{(\boldsymbol{\xi}_\alpha, \boldsymbol{u}), (\boldsymbol{\xi}'_\alpha, \boldsymbol{u}')\} = \{(\boldsymbol{g}_\alpha, l), (\boldsymbol{h}_\alpha, \boldsymbol{m})\}$, and

$$-\mathscr{L}_p^{\alpha\alpha}(A, \varrho)_h^\star(w_o) = \mathscr{L}_p^{\alpha\alpha}(A, \varrho)_g^\star(w_o) = \mathscr{E}_f(w_o) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho)^\star(w_o). \tag{44}$$

The second equality in the previous equation follows as above from the definitions, according to which $\mathsf{L} = \Pi_f \cap \Pi_g$ and $\mathscr{E}_g = \mathcal{E}_g|_{\mathsf{L}}$. The first equality follows by noting that the restrictions of $\mathcal{E}_g$ and $\mathcal{E}_h$ to the line $\Pi_g \cap \Pi_h$ satisfy

$$\mathcal{E}_g|_{\Pi_g \cap \Pi_h} = -\left.\frac{\bar{\chi}_g(p) \cdot a_p(\boldsymbol{g}_\alpha)}{a_p(\boldsymbol{f}) \cdot a_p(\boldsymbol{h}_\alpha)}\right|_{\Pi_g \cap \Pi_h} \cdot \mathcal{E}_h|_{\Pi_g \cap \Pi_h}$$

(as $a_p(\boldsymbol{f})|_{\Pi_g \cap \Pi_h} = \alpha_f = \alpha_f^{-1}$ and $\chi_g \cdot \chi_h = 1$ by Assumption 1.1.1) with

$$-\frac{\bar{\chi}_g(p) \cdot a_p(\boldsymbol{g}_\alpha)}{a_p(\boldsymbol{f}) \cdot a_p(\boldsymbol{h}_\alpha)}(w_o) = -1.$$

(In other words $\mathcal{E}_g|_{\Pi_g \cap \Pi_h}$ and $-\mathcal{E}_h|_{\Pi_g \cap \Pi_h}$ have the same leading term at w_o, which together with the equality $\mathcal{E}_g \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho)_g^\star|_{\Pi_g \cap \Pi_h} = \mathcal{E}_h \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho)_h^\star|_{\Pi_g \cap \Pi_h}$ implies the first identity in Eq. (44).) Write

$$2 \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^2} = a \cdot (\boldsymbol{k} - 2) + b \cdot (l - 1) + c \cdot (\boldsymbol{m} - 1)$$

with a, b and c in L. Equations (41) and (43) with $\xi = g$ and Eq. (44) give

$$a + b = \mathscr{E}_f(w_o) \cdot \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}_\alpha} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big) \cdot \mathscr{L}^\star_p(w_o) \quad \text{and} \quad c - a = \mathscr{E}_f(w_o) \cdot \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}_\alpha}\big) \cdot \mathscr{L}^\star_p(w_o),$$

where $\mathscr{L}^\star_p$ is a shorthand for $\mathscr{L}^{\alpha\alpha}_p(A, \varrho)^\star$. Similarly

$$b - a = \mathscr{E}_f(w_o) \cdot \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}_\alpha} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big) \cdot \mathscr{L}^\star_p(w_o) \quad \text{and} \quad a + c = \mathscr{E}_f(w_o) \cdot \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}_\alpha}\big) \cdot \mathscr{L}^\star_p(w_o)$$

by Eqs. (41) and (43) with $\xi = h$ and Eq. (44). As a consequence

$$-2 \cdot \mathscr{L}^{\alpha\alpha}_p(A, \varrho) \pmod{\mathscr{I}^2}$$

equals

$$\mathscr{E}_f(w_o) \cdot \big((\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}_\alpha}) \cdot (\boldsymbol{l} - 1) - (\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}_\alpha}) \cdot (\boldsymbol{m} - 1)\big) \cdot \mathscr{L}^{\alpha\alpha}_p(A, \varrho)^\star(w_o).$$

Noting that $\mathscr{E}_f(w_o) = 1 - \frac{\beta_h}{\alpha_h}$ (when $\alpha_f = \beta_g \cdot \alpha_h$), the previous discussion and Eq. (34) prove Theorem 2.1 when $\alpha_f = \beta_g \cdot \alpha_h$.

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