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A note on Kuttler–Sigillito's inequalities

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Abstract
We provide several inequalities between eigenvalues of some classical eigenvalue problems on compact Riemannian manifolds with C^2 boundary. A key tool in the proof is the generalized Rellich identity on a Riemannian manifold. Our results in particular extend some inequalities due to Kuttler and Sigillito from subsets of $\mathbb{R}^2$ to the manifold setting.

Keywords Steklov eigenvalue problems · Eigenvalue bounds · Rellich identity

Mathematics Subject Classification 35P15 · 58C40 · 58J50

Rèsumè
On donne plusieurs inégalités concernant les valeurs propres dans certains problèmes classiques des valeurs propres sur des variétés riemanniennes compactes à bord C^2. Comme méthode centrale de la preuve, on utilise l'identité généralisée de Rellich sur une variété riemannienne. En particulier, nos résultats étendent au cas des variétés certaines inégalités établies par Kuttler et Sigillito sur des sous-domaines de $\mathbb{R}^2$.

1 Introduction

The objective of this manuscript is to establish several inequalities between eigenvalues of the classical eigenvalue problems mentioned below. Let (M^n, g) be a compact and connected Riemannian manifold of dimension $n \geq 2$ with nonempty C^2 boundary ∂M. The eigenvalue problems we consider include the Neumann and Dirichlet eigenvalue problems on M:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } M, \\ u = 0 & \text{on } \partial M, \end{cases} \qquad \text{Dirichlet eigenvalue problem}, \tag{1.1}$$

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } M, \\ \partial_\nu u = 0 & \text{on } \partial M, \end{cases} \qquad \text{Neumann eigenvalue problem}, \tag{1.2}$$

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where $\Delta = \mathrm{div}\nabla$ is the Laplace–Beltrami operator, ν is the unit outward normal vector on ∂M, and ∂_ν denotes the outward normal derivative. The Dirichlet eigenvalues describe the fundamental modes of vibration of an idealized drum, and for $n = 2$, the Neumann eigenvalues appear naturally in the study of the vibrations of a free membrane; see e.g. [3,6].

We also consider the Steklov eigenvalue problem, which is an eigenvalue problem with the spectral parameter in the boundary conditions:

$$\begin{cases} \Delta u = 0 & \text{in } M, \\ \partial_\nu u = \sigma u & \text{on } \partial M, \end{cases} \qquad \text{Steklov eigenvalue problem .} \tag{1.3}$$

The Steklov eigenvalues encode the squares of the natural frequencies of vibration of a thin membrane with free frame, whose mass is uniformly distributed at the boundary; see the recent survey paper [11] and references therein.

The last set of eigenvalue problems we consider are the so-called Biharmonic Steklov problems:

$$\begin{cases} \Delta^2 u = 0 & \text{in } M, \\ u = \Delta u - \eta \partial_\nu u = 0 & \text{on } \partial M, \end{cases} \qquad \text{Biharmonic Steklov problem I;} \tag{1.4}$$

$$\begin{cases} \Delta^2 u = 0 & \text{in } M, \\ \partial_\nu u = \partial_\nu \Delta u + \xi u = 0 & \text{on } \partial M, \end{cases} \qquad \text{Biharmonic Steklov problem II.} \tag{1.5}$$

The eigenvalues problems (1.4) and (1.5) for example play an important role in elastic mechanics. We refer the reader to [5,9,17,18] for some recent results on eigenvalue estimates of problem (1.4). Moreover, a physical interpretation of problem (1.4) can be found in [9,17]. Problem (1.5) was first studied in [12,13] where the main focus was on the first nonzero eigenvalue, which appears as an optimal constant in a priori inequality; see [12] for more details.

It is well-known that the spectra of the eigenvalue problems (1.2)–(1.5) are discrete and non-negative, see e.g. [2,6,9,10,12,17]. We thus arrange their eigenvalues in increasing order, where we repeat an eigenvalue as often as its multiplicity requires. The k-th eigenvalue of one of the above eigenvalue problems will be denoted by the corresponding letter for the eigenvalue with a subscript k, e.g. the k-th Neumann eigenvalue will be denoted by μ_k. Note that $\mu_1 = \sigma_1 = \xi_1 = 0$.

There is a variety of literature on the study of bounds on the eigenvalues of each problem mentioned above in terms of the geometry of the underlying space [11,15,17,22]. However, instead of studying each eigenvalue problem individually, it is also interesting to explore relationships and inequalities between eigenvalues of different eigenvalue problems. Among this type of results, one can mention the relationships between the Laplace and Steklov eigenvalues studied in [14,21,24], and various inequalities between the first nonzero eigenvalue of problems (1.2)–(1.5) on bounded domains of $\mathbb{R}^2$ obtained by Kuttler and Sigillito in [13]; see Table 1 (Note that there was a misprint in Inequality VI in [13]. The correct version of the inequality is stated in Table 1.).

We extend Kuttler–Sigillito's results in two ways. Firstly, we consider compact manifold M with C^2 boundary of any dimension $n \geq 2$. Secondly, we also prove inequalities between higher-order eigenvalues.

Our first theorem provides lower bounds for ξ_k in terms of Neumann and Steklov eigenvalues.

Theorem 1.1 *Let (M^n, g) be a compact manifold of dimension $n \geq 2$ with C^2 boundary. For every $k \in \mathbb{N}$ we have (a) $\mu_k \sigma_2 \leq \xi_k$, and (b) $\mu_2 \sigma_k \leq \xi_k$.*

Table 1 Inequalities obtained by Kuttler and Sigillito in [13]

Inequalities	Conditions on $M \subset \mathbb{R}^2$	Special case of
$\mu_2\sigma_2 \leq \xi_2$		Theorem 1.1
$\mu_2 h_{\min}/(1+\mu_2^{1/2} r_{\max}) \leq 2\sigma_2$	Star-shaped with respect to a point	Theorem 1.3
$\eta_1 \leq \frac{1}{2}\lambda_1 h_{\max}$	Star-shaped with respect to a point	Theorem 1.4 (i)
$\lambda_1^{1/2} \leq 2\eta_1 r_{\max}/h_{\min}$	Star-shaped with respect to a point	Theorem 1.4 (i)
$\xi_2 \leq \mu_2^2 h_{\max}$	Star-shaped with respect to its centroid	Theorem 1.4 (ii)

Compared to inequality (b), inequality (a) gives a better lower bound for ξ_k for large k. For $k = 2$ and $M \subset \mathbb{R}^2$, Theorem 1.1 was previously proved in [13]. Kuttler in [12] also obtained an inequality between some higher order eigenvalues ξ_k and μ_k for a rectangular domain in $\mathbb{R}^2$ using symmetries of the eigenfunctions.

In order to state our next results, we need to introduce some notation first. For any given $p \in M$, consider the distance function

$$d_p : M \to [0, \infty), \quad d_p(x) := d(p, x),$$

and one half of the square of the distance function,

$$\rho_p(x) := \frac{1}{2} d_p(x)^2.$$

Furthermore, we set

$$r_{\max} := \max_{x\in M} d_p(x) = \max_{x\in\partial M} d_p(x),$$
$$h_{\max} := \max_{x\in\partial M} \langle \nabla\rho_p, \nu\rangle, \quad \text{and} \quad h_{\min} := \min_{x\in\partial M} \langle \nabla\rho_p, \nu\rangle,$$

where we borrowed the notation from [13].

Remark 1.2 Note that ρ_p is not necessarily differentiable on the cut locus of p. However, the direction derivative denoted by $\langle \nabla\rho_p(x), \zeta\rangle$, $\zeta \in T_x M$ always exists and is given by

$$\langle \nabla\rho_p(x), \zeta\rangle := \inf\{-\langle v, \zeta\rangle : v \in T_x M \text{ is the unit tangent vector of a geodesic joining } x \text{ to } p\}.$$

We shall see that under the assumption of a lower Ricci curvature bound, there exists a lower bound on the first nonzero Steklov eigenvalue σ_2 in terms of μ_2 on star shaped manifolds. A manifold M with C^2 boundary is called a star shaped manifold if there exist $p \in M$ and a star shaped domain Ω in $\mathbb{R}^n \cong T_p M$ such that $\exp_p$ is defined on Ω and $\exp_p(\Omega) = M$. This implies that $\langle \nabla\rho_p(x), \nu(x)\rangle \geq 0$ for every $x \in \partial M$.

Theorem 1.3 *Let (M^n, g) be a compact, star shaped Riemannian manifold whose Ricci curvature Ric_g satisfies $\mathrm{Ric}_g \geq (n-1)\kappa$. Then we have*

$$\sigma_2 \geq \frac{h_{\min}\mu_2}{2r_{\max}\mu_2^{1/2} + C_0}, \tag{1.6}$$

where $C_0 := C_0(n, \kappa, r_{\max})$ is a positive constant depending only on n, κ and $r_{\max}$.

When M is a subdomain of $\mathbb{R}^n$, inequality (1.6) was stated in [13] with $C_0 = 2$.

In the following theorem we provide several inequalities for eigenvalues of (1.2)–(1.5) on star shaped manifolds under the assumption of bounded sectional curvature. Here and hereafter, we make use of the notation

$$A \vee B := \max\{A, B\} \qquad \text{for all } A, B \in \mathbb{R},$$

and the convention $c/0 = +\infty$, $c \in \mathbb{R} \setminus \{0\}$.

Theorem 1.4 *Let (M^n, g) be a compact, star shaped Riemannian manifold of dimension n whose sectional curvature K_g satisfies $\kappa_1 \leq K_g \leq \kappa_2$. Moreover, assume that there exists $p \in M$ such that M is star shaped with respect to p and the cut locus of p in M is the empty set. Then there exist constants $C_i := C_i(n, \kappa_1, \kappa_2, r_{\max})$, $i = 1, 2$, depending only on n, κ_1, κ_2 and $r_{\max}$ and $C_3 = C_3(n, \kappa_1, r_{\max})$ such that*

(i) $C_1\eta_m/h_{\max} \leq \lambda_k \leq \left(4r_{\max}^2\eta_k^2 - 2C_2h_{\min}\eta_k\right)/h_{\min}^2$,
(ii) $\xi_{m+1} \leq h_{\max}\mu_k^2/\Big((C_3 - n^{-1}\mathrm{vol}(M)^{-1}\mu_k \int_M d_p^2\, dv_g) \vee 0\Big)$, *provided* $\kappa_2 \leq 0$.

Here, m is the multiplicity of λ_k.

Note that the constants C_i, $i = 1, 2, 3$ are not positive in general. However, there exists $r_0 := r_0(n, \kappa_1, \kappa_2) > 0$ such that for $r_{\max} \leq r_0$ these constants are positive; see Sect. 4 for details. In inequality (ii), we have a non trivial upper bound only if

$$\mu_k < nC_3\mathrm{vol}(M)\left(\int_M d_p^2\, dv_g\right)^{-1}.$$

When M is a domain in $\mathbb{R}^n$, the quantity $\int_M d_p^2 dv_g$ is called the second moment of inertia; see Example 4.2. The proof of Theorem 1.4 also leads to a non-sharp lower bound on η_1

$$\eta_1 \geq \frac{h_{\min}C_2}{r_{\max}^2}.$$

This in particular shows that the right-hand side of the inequality in part $i)$ is always positive.

The proof of Theorem 1.1 is based on using the variational characterization of the eigenvalues and alternative formulations thereof. Apart from the Laplace and Hessian comparison theorems, and the variational characterization of the eigenvalues, the key tool in the proof of Theorems 1.3 and 1.4 is a generalization of the classical Rellich identity to the manifold setting. This is the content of the next theorem. Let us denote $M \setminus \partial M$ by M°.

Theorem 1.5 (Generalized Rellich identity) *Let $F : M \to TM$ be a Lipschitz vector field on M. Then for every $w \in C^2(M^\circ) \cap C^1(M)$ we have*

$$\begin{aligned}\int_M (\Delta w + \lambda w)\langle F, \nabla w\rangle dv_g &= \int_{\partial M} \partial_\nu w \langle F, \nabla w\rangle ds_g - \frac{1}{2}\int_{\partial M} |\nabla w|^2 \langle F, \nu\rangle ds_g \\ &+ \frac{\lambda}{2}\int_{\partial M} w^2\langle F, \nu\rangle ds_g + \frac{1}{2}\int_M \mathrm{div} F |\nabla w|^2 dv_g - \int_M DF(\nabla w, \nabla w) dv_g \\ &- \frac{\lambda}{2}\int_M w^2 \mathrm{div} F\, dv_g,\end{aligned}$$

where ν denotes the outward pointing normal and $\langle\cdot,\cdot\rangle = g(\cdot,\cdot)$.

The classical Rellich identity was first stated by Rellich in [23]. A special case of Theorem 3.1, called the generalized Pohozaev identity, was proved in [21,25] in order to get some spectral inequalities between the Steklov and Laplace eigenvalues.

The paper is structured as follows. In Sect. 2, we recall tools needed in later sections, namely the Hessian and Laplace comparison theorems. Moreover, we give variational characterizations and alternative representations for the eigenvalues of problems (1.2)–(1.5). Sect. 3 contains the deduction of the Rellich identity on manifolds, as well as several applications thereof. Finally, we prove the main theorems in Sect. 4.

2 Preliminaries

In this section we provide the basic tools needed in later sections. Namely, we give the variational characterizations and alternative representations of the eigenvalues of problems (1.2)–(1.5) in the first subsection. In the second subsection, we recall the Hessian and Laplace comparison theorems.

2.1 Variational characterization and alternative representations

Below, we list the variational characterization of eigenvalues of (1.1)–(1.5) and their alternative representations. We refer to [2,6] for the variational characterization of (1.1)–(1.3), and to Appendix for (1.4) and (1.5). For the special case of the first nonzero eigenvalues of (1.1)–(1.5), their alternative representations are contained in [13]. The general proofs of their alternative representations follow along the same lines of the proofs in [13] and are therefore omitted.

Dirichlet eigenvalues:

$$\begin{aligned}\lambda_k &= \inf_{\substack{V\subset H_0^1(M)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_M |\nabla u|^2\,dv_g}{\int_M u^2\,dv_g}\\ &= \inf_{\substack{V\subset H^2(M)\cap H_0^1(M)\\ \dim V=k}} \sup_{\substack{u\in V\\ \nabla u\neq 0}} \frac{\int_M (\Delta u)^2\,dv_g}{\int_M |\nabla u|^2\,dv_g}.\end{aligned} \tag{2.1}$$

Neumann eigenvalues:

$$\begin{aligned}\mu_k &= \inf_{\substack{V\subset H^1(M)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_M |\nabla u|^2\,dv_g}{\int_M u^2\,dv_g}\\ &= \inf_{\substack{V\subset H^2(M)\\ \partial_\nu u=0 \text{ on } \partial M\\ \dim V=k}} \sup_{\substack{u\in V\\ \nabla u\neq 0}} \frac{\int_M (\Delta u)^2\,dv_g}{\int_M |\nabla u|^2\,dv_g}.\end{aligned} \tag{2.2}$$

Steklov eigenvalues:

$$\begin{aligned}\sigma_k &= \inf_{\substack{V\subset H^1(M)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_M |\nabla u|^2\,dv_g}{\int_{\partial M} u^2\,dv_g}\\ &= \inf_{\substack{V\subset \mathcal{H}(M)\\ \dim V=k}} \sup_{\substack{u\in V\\ \nabla u\neq 0}} \frac{\int_{\partial M} (\partial_\nu u)^2\,ds_g}{\int_M |\nabla u|^2\,dv_g},\end{aligned} \tag{2.3}$$

where $\mathcal{H}(M)$ is the space of harmonic functions on M.

Biharmonic Steklov I eigenvalues:

$$\eta_k = \inf_{\substack{V\subset H^2(M)\cap H_0^1(M)\\ \dim(V/H_0^2(M))=k}} \sup_{\substack{u\in V\\ u\in V\setminus H_0^2(M)}} \frac{\int_M |\Delta u|^2\, dv_g}{\int_{\partial M} (\partial_\nu u)^2\, ds_g}. \tag{2.4}$$

Biharmonic Steklov II eigenvalues:

$$\xi_k = \inf_{\substack{V\subset H_N^2(M)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_M |\Delta u|^2\, dv_g}{\int_{\partial M} u^2\, ds_g}, \tag{2.5}$$

where $H_N^2(M) := \{u \in H^2(M) \,:\, \partial_\nu u = 0 \ \text{ on } \partial M\}$.

2.2 Hessian and Laplace comparison theorems

The idea of comparison theorems is to compare a given geometric quantity on a Riemannian manifold with the corresponding quantity on a model space. Below we recall the Hessian and Laplace comparison theorems. For more details we refer the reader to [4,7,20].

For any $\kappa \in \mathbb{R}$, denote by $H_\kappa : [0,\infty) \to \mathbb{R}$ the function satisfying the Riccati equation

$$H_\kappa' + H_\kappa^2 + \kappa = 0, \quad \text{with} \quad \lim_{r\to 0} \frac{r H_\kappa(r)}{n-1} = 1.$$

Clearly, we have

$$H_\kappa(r) = \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0,\\ \frac{n-1}{r} & \kappa = 0,\\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0. \end{cases}$$

With this preparation at hand we can now state the Hessian comparison theorem.

Theorem 2.1 (Hessian comparison theorem) *Let (M^n, g) be a complete Riemannian manifold. Let $\gamma : [0, L] \to M$ be a minimizing geodesic starting from $p \in M$, such that its image is disjoint from the cut locus of p. Assume furthermore that*

$$\kappa_1 \le K_g(X, \dot\gamma(t)) \le \kappa_2$$

for all $t \in [0, L]$ and $X \in T_{\gamma(t)}M$ perpendicular to $\dot\gamma(t)$. Then

(a) d_p satisfies the inequalities

$$\nabla^2 d_p(X,X) \le \frac{H_{\kappa_1}(t)}{n-1} g(X,X), \qquad \forall t \in [0,L], \quad X \in \langle\dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M,$$
$$\nabla^2 d_p(X,X) \ge \frac{H_{\kappa_2}(t)}{n-1} g(X,X), \qquad \forall t \in \left[0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}\right], \quad X \in \langle\dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M.$$

Furthermore, we have

$$\nabla^2 d_p(\dot\gamma(t), \dot\gamma(t)) = 0, \qquad \forall t \in [0, L].$$

Here $A \wedge B := \min\{A, B\}$ and $A \vee B := \max\{A, B\}$ for $A, B \in \mathbb{R}$.

(b) ρ_p *satisfies the inequalities*

$$\nabla^2\rho_p(X,X) \le \frac{tH_{\kappa_1}(t)}{n-1}g(X,X), \qquad \forall t \in [0,L], \quad X \in \langle\dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M,$$
$$\nabla^2\rho_p(X,X) \ge \frac{tH_{\kappa_2}(t)}{n-1}g(X,X), \qquad \forall t \in \left[0, L\wedge\frac{\pi}{2\sqrt{\kappa_2\vee 0}}\right], \quad X \in \langle\dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M,$$

and

$$\nabla^2\rho_p(\dot\gamma(t),\dot\gamma(t)) = 1, \qquad \forall t \in [0,L].$$

Next, we state the Laplace comparison theorem.

Theorem 2.2 *(Laplace comparison theorem) Let* (M^n, g) *be a complete Riemannian manifold. The distance function* d_p *and the squared distance function satisfy the following.*

(a) Let $\mathrm{Ric}_g \ge (n-1)\kappa$, $\kappa \in \mathbb{R}$. *Then for every* $p \in M$ *the inequalities*

$$\Delta d_p(x) \le H_\kappa(d_p(x)), \quad \text{and} \quad \Delta\rho_p(x) \le 1 + d_p(x)H_\kappa(d_p(x))$$

hold at smooth points of d_p. *Moreover the above inequalities hold on the whole manifold in the sense of distribution.*

(b) Under the same assumption and notations of Theorem 2.1, the following inequalities hold.

(i) For every $t \in [0,L]$

$$\Delta d_p(\gamma(t)) \le H_{\kappa_1}(t), \quad \text{and} \quad \Delta\rho_p(\gamma(t)) \le 1 + tH_{\kappa_1}(t);$$

(ii) For every $t \in [0, L\wedge\frac{\pi}{2\sqrt{\kappa_2\vee 0}}]$

$$\Delta d_p(\gamma(t)) \ge H_{\kappa_2}(t), \quad \text{and} \quad \Delta\rho_p(\gamma(t)) \ge 1 + tH_{\kappa_2}(t).$$

Notice that part (b) in the above theorems is an immediate consequence of part (a), since the distance function d_p and one half of the square of the distance function ρ_p satisfy

$$\nabla^2\rho_p = d_p\nabla^2 d_p + \nabla d_p \otimes \nabla d_p, \qquad \Delta\rho_p = |\nabla d_p|^2 + d_p\Delta d_p.$$

Remark 2.3 Theorems 2.1 and 2.2 hold for a star shaped manifold M, when M is star shaped with respect to the point p given in these theorems.

3 Generalized Rellich identity

An important identity which is used in the study of eigenvalue problems is the Rellich identity. To our knowledge it was first stated and used by Rellich [23] in the study of the Dirichlet eigenvalue problem. Some versions of the Rellich identity are also referred to as the Pohozaev identity; see [8,21,25] for more details and its applications. In this section, we provide the generalized Rellich identity on Riemannian manifolds, i.e. Theorem 1.5, and its higher order version. Some applications of this result can be found in the last subsection and in Sect. 4.

3.1 Rellich identity on manifolds

The next theorem states the Rellich identity on Riemannian manifolds.

Theorem 3.1 *(Generalized Rellich identity for manifolds) Let* (M, g) *be a compact Riemannian manifold with* C^2*–smooth boundary. Let* $F : M \to TM$ *be a Lipschitz vector field on* M*. Then for every* $w \in C^2(M^\circ) \cap C^1(M)$ *we have*

$$\int_M (\Delta w + \lambda w)\langle F, \nabla w\rangle dv_g = \int_{\partial M} \partial_\nu w \langle F, \nabla w\rangle ds_g - \frac{1}{2}\int_{\partial M} |\nabla w|^2 \langle F, \nu\rangle ds_g$$
$$+\frac{\lambda}{2}\int_{\partial M} w^2 \langle F, \nu\rangle ds_g + \frac{1}{2}\int_M \mathrm{div} F |\nabla w|^2 dv_g - \int_M DF(\nabla w, \nabla w) dv_g$$
$$-\frac{\lambda}{2}\int_M w^2 \mathrm{div} F\, dv_g,$$

where ν *denotes the outward pointing normal and* $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$.

In [21,25], the authors proved the above identity when w is harmonic and $\lambda = 0$. The proof of the general version follows the same line of argument. For the sake of completeness we give the whole argument.

Proof of Theorem 3.1 We calculate $\int_M \Delta w \langle F, \nabla w\rangle dv_g$ and $\int_M \lambda w \langle F, \nabla w\rangle dv_g$ separately. In order to calculate the latter, we apply the divergence theorem to obtain

$$\int_{\partial M} w^2 \langle F, \nu\rangle\, ds_g = \int_M \mathrm{div}(w^2 F) dv_g = \int_M \big(2w\langle F, \nabla w\rangle + w^2 \mathrm{div} F\big)\, dv_g.$$

Thus, we get

$$\int_M \lambda w \langle F, \nabla w\rangle dv_g = \frac{\lambda}{2}\left(\int_{\partial M} w^2 \langle F, \nu\rangle ds_g - \int_M w^2 \mathrm{div} F\, dv_g\right).$$

For the other term, using integration by parts, we obtain

$$\begin{aligned}\int_M \Delta w \langle F, \nabla w\rangle dv_g &= \int_{\partial M} \langle F, \nabla w\rangle \partial_\nu w ds_g - \int_M \langle \nabla\langle F, \nabla w\rangle, \nabla w\rangle dv_g \\ &= \int_{\partial M} \langle F, \nabla w\rangle \partial_\nu w ds_g - \int_M \langle \nabla_{\nabla w} F, \nabla w\rangle dv_g \\ &\quad - \int_M \langle \nabla_{\nabla w} \nabla w, F\rangle dv_g \\ &= \int_{\partial M} \langle F, \nabla w\rangle \partial_\nu w ds_g - \int_M DF(\nabla w, \nabla w) dv_g \\ &\quad - \int_M \nabla^2 w(\nabla w, F) dv_g. \end{aligned} \tag{3.1}$$

For further simplification, we observe that

$$2\int_M \nabla^2 w(\nabla w, F) dv_g = \int_M \mathrm{div}(F|\nabla w|^2) dv_g - \int_M \mathrm{div} F |\nabla w|^2 dv_g$$
$$= \int_{\partial M} |\nabla w|^2 F ds_g - \int_M \mathrm{div} F |\nabla w|^2 dv_g.$$

Plugging this identity into (3.1) we get

$$\int_M \Delta w\langle F,\nabla w\rangle dv_g=\int_{\partial M}\partial_\nu w\langle F,\nabla w\rangle ds_g-\frac{1}{2}\int_{\partial M}|\nabla w|^2\langle F,\nu\rangle ds_g\\+\frac{1}{2}\int_M \mathrm{div}F|\nabla w|^2dv_g-\int_M DF(\nabla w,\nabla w)dv_g.$$

This completes the proof. □

3.2 Higher order Rellich identities

In this section, we provide a higher order Rellich identity. Throughout the section, M is a compact Riemannian manifold with nonempty C^2 boundary.

The following preparatory lemma is a simple consequence from Theorem 3.1. For the special case $M\subset\mathbb{R}^n$, the identity stated in the lemma was first proven by Mitidieri in [19].

Lemma 3.2 *For $v,w\in C^2(M^\circ)\cap C^1(M)$ we have*

$$\int_M \Delta w\langle F,\nabla v\rangle+\Delta v\langle F,\nabla w\rangle dv_g=\int_{\partial M}\{\partial_\nu w\langle F,\nabla v\rangle+\partial_\nu v\langle F,\nabla w\rangle\}ds_g\\-\int_{\partial M}\langle\nabla w,\nabla v\rangle\langle F,\nu\rangle ds_g+\int_M \mathrm{div}F\langle\nabla w,\nabla v\rangle dv_g-2\int_M DF(\nabla w,\nabla v)dv_g.$$

Proof Replacing w by $w+v$ in Theorem 3.1 and set $\lambda=0$ we get the identity. □

The following theorem states the higher order Rellich identity.

Theorem 3.3 *Let the boundary of M be C^2 smooth. Then for $w\in C^4(M^\circ)\cap C^3(M)$ we have*

$$\int_M(\Delta^2w+\lambda\Delta w)\langle F,\nabla w\rangle dv_g=\frac{1}{2}\int_M \mathrm{div}F(\Delta w)^2dv_g-\frac{1}{2}\int_{\partial M}(\Delta w)^2\langle F,\nu\rangle dv_g\\+\int_{\partial M}\{\partial_\nu w\langle F,\nabla\Delta w\rangle+\partial_\nu\Delta w\langle F,\nabla w\rangle\}ds_g-\int_{\partial M}\langle\nabla w,\nabla\Delta w\rangle\langle F,\nu\rangle ds_g\\+\int_M \mathrm{div}F\langle\nabla w,\nabla\Delta w\rangle dv_g-2\int_M DF(\nabla w,\nabla\Delta w)dv_g+\lambda\int_{\partial M}\partial_\nu w\langle F,\nabla w\rangle ds_g\\-\frac{\lambda}{2}\int_{\partial M}|\nabla w|^2\langle F,\nu\rangle ds_g+\frac{\lambda}{2}\int_M \mathrm{div}F|\nabla w|^2dv_g-\lambda\int_M DF(\nabla w,\nabla w)dv_g.$$

Proof If we choose $v=\Delta w$ in Lemma 3.2, we obtain

$$\int_M \Delta^2 w\langle F,\nabla w\rangle dv_g=-\int_M \Delta w\langle F,\nabla\Delta w\rangle dv_g\\+\int_{\partial M}\{\partial_\nu w\langle F,\nabla\Delta w\rangle+\partial_\nu\Delta w\langle F,\nabla w\rangle\}ds_g\\-\int_{\partial M}\langle\nabla w,\nabla\Delta w\rangle\langle F,\nu\rangle ds_g+\int_M \mathrm{div}F\langle\nabla w,\nabla\Delta w\rangle dv_g\\-2\int_M DF(\nabla w,\nabla\Delta w)dv_g.$$

By the divergence theorem we have

$$\int_M \Delta w \langle F, \nabla \Delta w\rangle dv_g = \frac{1}{2}\int_M \langle F, \nabla(\Delta w)^2\rangle dv_g$$
$$= -\frac{1}{2}\int_M \operatorname{div} F(\Delta w)^2 dv_g + \frac{1}{2}\int_{\partial M} (\Delta w)^2 \langle F, \nu\rangle dv_g,$$

which together with Theorem 3.1 establishes the claim. □

For the special case $M \subset \mathbb{R}^n$ and $\lambda = 0$, the statement of Theorem 3.3 is contained in [19].

3.3 Applications of the Rellich identities

In 1940, Rellich [23] dealt with the Dirichlet eigenvalue problem on sets $M \subset \mathbb{R}^n$. For this special case he used the identity derived in Theorem 3.1 to express the Dirichlet eigenvalues in terms of an integral over the boundary. One decade ago, Liu [16] extended Rellich's result to the Neumann eigenvalue problem, the clamped plate eigenvalue problem and the buckling eigenvalue problem, each on sets $M \subset \mathbb{R}^n$. In the latter two cases Liu (implicitly) applied the higher order Rellich identity.

Recall that for any compact Riemannian manifold M with C^2 boundary ∂M, the clamped plate eigenvalue problem and the buckling eigenvalue problem are given by

$$\begin{cases} \Delta^2 u + \Lambda \Delta u = 0 & \text{in } M, \\ u = \partial_\nu u = 0 & \text{on } \partial M; \end{cases} \qquad \text{Buckling problem}, \tag{3.2}$$

$$\begin{cases} \Delta^2 u - \Gamma^2 u = 0 & \text{in } M, \\ u = \partial_\nu u = 0 & \text{on } \partial M; \end{cases} \qquad \text{Clamped plate}, \tag{3.3}$$

respectively.

Below we reprove the result of Liu for the case of the buckling eigenvalue problem. Note there is no new idea for the proof, however, our proof is shorter and clearer since we do not carry out the calculations in coordinates. One can proceed similarly for the clamped plate eigenvalue problem.

Lemma 3.4 *([16]) Let $M \subset \mathbb{R}^n$ be a bounded domain with C^2 smooth boundary.*

(i) Let w be an eigenfunction corresponding to the eigenvalue Λ of the buckling eigenvalue problem. Then we have

$$\Lambda = \frac{\int_{\partial M} (\partial^2_{\nu\nu} w)^2 \partial_\nu (r^2) ds_g}{4\int_M |\nabla w|^2 dv_g},$$

where $r^2 = x_1^2 + \cdots + x_n^2$ and x_i are Euclidean coordinates.

(ii) Let w be an eigenfunction corresponding to the eigenvalue Γ of the clamped plate eigenvalue problem. Then we have

$$\Gamma = \frac{\int_{\partial M} (\partial^2_{\nu\nu} w)^2 \partial_\nu (r^2) ds_g}{8\int_M w^2 dv_g}.$$

Proof In order to prove (i) we apply Theorem 3.3 for the special case $M \subset \mathbb{R}^n$ and where F is given by the gradient of the distance function. In this case we have $DF(\cdot,\cdot) = g(\cdot,\cdot)$ and $\operatorname{div} F = n$. Note furthermore that $w_{|\partial M} = 0$ implies $\nabla w = \partial_\nu w \nu$ on ∂M. Since we have $\partial_\nu w_{|\partial M} = 0$ by assumption, ∇w vanishes along the boundary of M.

Plugging the above information into Theorem 3.3 we get

$$0=\int_M(\Delta^2 w+\lambda\Delta w)\langle F,\nabla w\rangle dv_g=\frac{n}{2}\int_M(\Delta w)^2dv_g-\frac{1}{2}\int_{\partial M}(\Delta w)^2\langle F,\nu\rangle dv_g$$
$$+(n-2)\int_M\langle\nabla w,\nabla\Delta w\rangle dv_g+\Lambda\left(\frac{n}{2}-1\right)\int_M|\nabla w|^2dv_g.$$

Applying the divergence theorem once more, we thus obtain

$$\Lambda\left(\frac{n}{2}-1\right)\int_M|\nabla w|^2dv_g=\frac{1}{2}\int_{\partial M}(\Delta w)^2\langle F,\nu\rangle ds_g-\left(2-\frac{n}{2}\right)\int_M(\Delta w)^2dv_g.$$

The variational characterization of Λ asserts that for an eigenfunction w corresponding to Λ we have

$$\int_M(\Delta w)^2dv_g-\Lambda\int_M|\nabla w|^2dv_g=0. \tag{3.4}$$

Furthermore, the identities

$$\langle F,\nu\rangle=\sum_{i=1}^n x_i\partial_\nu x_i=\frac{1}{2}\partial_\nu(r^2)$$

and $\Delta w=\partial^2_{\nu\nu}w$ hold on the boundary of M. Thus the claim is established.

The proof of (ii) is omitted since it is similar to the one of (i). □

Remark 3.5 In Lemma 3.4 (i), when normalizing the eigenfunction w such that $\int_M|\nabla w|^2dv_g = 1$, we obtain

$$\Lambda=\frac{1}{4}\int_{\partial M}(\partial^2_{\nu\nu}w)^2\partial_\nu(r^2)ds_g;$$

i.e. Λ is expressed in terms of an integral over the boundary. A similar remark holds for Lemma 3.4 (ii).

Finally we use the Rellich identities to get some estimates on eigenvalues. Note that from now on we do not assume anymore that M is a subset of the Euclidean space. However, we assume that M is a manifold with C^2 smooth boundary and that there exists a Lipschitz vector field F on M satisfying the following properties:

(A) There exist some positive constants $c_1, c_2\in\mathbb{R}_+$ such that

$$0<c_1\le \operatorname{div}F\le c_2,$$

wherever F is differentiable.

(B) There exists a positive constant $\alpha\in\mathbb{R}_+$ such that

$$DF(X,X)\ge\alpha g(X,X),$$

wherever F is differentiable.

Remark 3.6 Domains in Hadamard manifolds, and free boundary minimal hypersurfaces in the unit ball in $\mathbb{R}^{n+1}$ provide examples for which conditions A and B for the gradient of the distance function on M are satisfied. For the latter, see Example 4.3 in which condition A with $c_1=c_2$ holds.

The following lemma is an easy consequence of Theorems 3.1 and 3.3, respectively. It establishes upper estimates for eigenvalues in terms of integrals over the boundary ∂M and α.

Lemma 3.7 *Let M be a manifold with C^2 smooth boundary. Assume that there exists a Lipschitz vector field F on M satisfying properties A and B above. Then*

(i) the eigenvalue λ corresponding to eigenfunction w of the Dirichlet eigenvalue problem satisfies

$$\lambda \leq \frac{\int_{\partial M} (\partial_\nu w)^2 \langle F, \nu \rangle ds_g}{(2\alpha + c_1 - c_2) \int_M w^2 dv_g};$$

(ii) the eigenvalue Λ corresponding to eigenfunction w of the buckling eigenvalue problem satisfies

$$\frac{\int_{\partial M} (\Delta w)^2 \langle F, \nu \rangle dv_g}{2\alpha \int_M |\nabla w|^2 dv_g} \leq \Lambda$$

provided $c_1 = c_2 =: c$ in property A.

Proof We start by proving (i). Theorem 3.1 and Condition A imply

$$\begin{aligned} 0 = \int_M (\Delta w + \lambda w) \langle F, \nabla w \rangle dv_g &\leq \int_{\partial M} \partial_\nu w \langle F, \nabla w \rangle ds_g - \frac{1}{2} \int_{\partial M} |\nabla w|^2 \langle F, \nu \rangle ds_g \\ &+ \frac{c_2}{2} \int_M |\nabla w|^2 dv_g - \int_M DF(\nabla w, \nabla w) dv_g - \frac{\lambda c_1}{2} \int_M w^2\, dv_g. \end{aligned}$$

Since $w \equiv 0$ on ∂M we have $\nabla w = \partial_\nu w \nu$ on ∂M. Combining this with Condition B we obtain

$$\frac{\lambda c_1}{2} \int_M w^2 dv_g \leq \frac{1}{2} \int_{\partial M} (\partial_\nu w)^2 \langle F, \nu \rangle ds_g + \left(\frac{\lambda c_2}{2} - \alpha\lambda \right) \int_M w^2\, dv_g.$$

The latter inequality implies the claim.

Below, we prove (ii). Theorem 3.3 implies

$$\begin{aligned} 0 &\leq \frac{c}{2} \int_M (\Delta w)^2 dv_g - \frac{1}{2} \int_{\partial M} (\Delta w)^2 \langle F, \nu \rangle dv_g + c \int_M \langle \nabla w, \nabla \Delta w \rangle dv_g \\ &\quad -2 \int_M DF(\nabla w, \nabla \Delta w) dv_g + \frac{c\Lambda}{2} \int_M |\nabla w|^2 dv_g - \Lambda \int_M DF(\nabla w, \nabla w) dv_g \\ &\leq \left(2\alpha - \frac{c}{2} \right) \int_M (\Delta w)^2 dv_g - \frac{1}{2} \int_{\partial M} (\Delta w)^2 \langle F, \nu \rangle dv_g + \left(\frac{c\Lambda}{2} - \Lambda\alpha \right) \int_M |\nabla w|^2 dv_g. \end{aligned}$$

Here, we made use of

$$\int_M \langle \nabla w, \nabla \Delta w \rangle dv_g = -\int_M (\Delta w)^2 dv_g,$$

which is a consequence of the divergence theorem. Applying (3.4) yields

$$0 \leq -\frac{1}{2} \int_{\partial M} (\Delta w)^2 \langle F, \nu \rangle dv_g + \Lambda\alpha \int_M |\nabla w|^2 dv_g,$$

and thus the claim is established. □

4 Proof of the Main Theorems

In this section, we prove the main theorems. The key ingredients of the proof are the comparison theorems and the Rellich identity.

Proof of Theorem 1.1 Inequalities (a) and (b) are an immediate consequence of the variational characterizations of μ_k, σ_k and ξ_k given in (2.2), (2.3) and (2.5). Indeed, let V be the space generated by eigenfunctions associated with $\xi_2, \ldots, \xi_k$. Then by the variational characterization (2.2) we get

$$\begin{aligned}\mu_k &\leq \sup_{0\neq u\in V} \frac{\int_M (\Delta u)^2\, dv_g}{\int_M |\nabla u|^2\, dv_g} \leq \xi_k \sup_{0\neq u\in V} \frac{\int_{\partial M} u^2\, dv_g}{\int_M |\nabla u|^2\, dv_g}\\ &= \xi_k \left(\inf_{0\neq u\in V} \frac{\int_M |\nabla u|^2\, dv_g}{\int_{\partial M} u^2\, dv_g}\right)^{-1} \leq \frac{\xi_k}{\sigma_2}.\end{aligned}$$

The proof of part (b) is similar. Let V be given as in part (a). By the variational characterization, we obtain

$$\begin{aligned}\sigma_k &\leq \sup_{0\neq u\in V} \frac{\int_M |\nabla u|^2\, dv_g}{\int_{\partial M} u^2\, dv_g} \leq \xi_k \sup_{0\neq u\in V} \frac{\int_M |\nabla u|^2\, dv_g}{\int_M |\Delta u|^2\, dv_g}\\ &= \xi_k \left(\inf_{0\neq u\in V} \frac{\int_M |\Delta u|^2\, dv_g}{\int_M |\nabla u|^2\, dv_g}\right)^{-1} \leq \frac{\xi_k}{\mu_2}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3 Let $p \in M$ be a point such that M is star shaped centered at p. We use the following identity

$$\frac{1}{2}\int_{\partial M} w^2 \langle \nu, \nabla \rho_p\rangle ds_g = \int_M w\langle \nabla w, \nabla \rho_p\rangle dv_g + \frac{1}{2}\int_M w^2 \Delta\rho_p dv_g$$

which follows easily from integration by parts. Using the Laplace comparison theorem, we thus get

$$\frac{1}{2}\int_{\partial M} w^2 \langle \nu, \nabla \rho_p\rangle ds_g \leq \int_M w\langle \nabla w, \nabla \rho_p\rangle dv_g + \frac{1}{2}\max_{x\in M}(1 + d_p(x) H_{\kappa_1}(d_p(x))) \int_M w^2 dv_g. \tag{4.1}$$

The Cauchy Schwarz inequality yields

$$\left(\int_M w\langle \nabla w, \nabla \rho_p\rangle dv_g\right)^2 \leq r_{\max}^2 \int_M w^2 dv_g \int_M |\nabla w|^2 dv_g.$$

Assuming $\int_M w dv_g = 0$ and using the variational characterization of μ_2 we get

$$\int_M w\langle \nabla w, \nabla \rho_p\rangle dv_g \leq r_{\max}\mu_2^{-1/2} \int_M |\nabla w|^2 dv_g.$$

Thus, from inequality (4.1), we get

$$\frac{1}{2}\int_{\partial M} w^2 \langle \nu, \nabla \rho_p\rangle ds_g \leq \left(r_{\max}\mu_2^{-1/2} + \frac{1}{2}\max_{x\in M}(1 + d_p(x) H_\kappa(d_p(x)))\mu_2^{-1}\right) \int_M |\nabla w|^2 dv_g. \tag{4.2}$$

Let u be an eigenfunction associated to the eigenvalue σ_2 and choose w to be

$$w := u - \mathrm{vol}(M)^{-1} \int_M u dv_g.$$

Then we have

$$\int_M |\nabla w|^2 dv_g = \int_M |\nabla u|^2 dv_g = \sigma_2 \int_{\partial M} u^2 ds_g \leq \sigma_2 \int_{\partial M} w^2 ds_g.$$

Combining this inequality with (4.2), we finally get

$$\begin{aligned}\frac{1}{2} h_{\min} \int_{\partial M} w^2 ds_g &\leq \frac{1}{2} \int_{\partial M} w^2 \langle \nu, \nabla \rho_p \rangle ds_g \\ &\leq \left(r_{\max} \mu_2^{-1/2} + \frac{1}{2} \max_{x \in M} (1 + d_p(x) H_\kappa(d_p(x))) \mu_2^{-1} \right) \int_M |\nabla w|^2 dv_g \\ &\leq \left(r_{\max} \mu_2^{-1/2} + \frac{1}{2} \max_{x \in M} (1 + d_p(x) H_\kappa(d_p(x))) \mu_2^{-1} \right) \sigma_2 \int_{\partial M} w^2 ds_g.\end{aligned}$$

Setting

$$C_0 := \max_{x \in M} (1 + d_p(x) H_\kappa(d_p(x)))$$

establishes the claim. □

Proof of Theorem 1.4 Throughout the proof we repeatedly use the Hessian and Laplace comparison theorems as well as the generalized Rellich identity, i.e. Theorem 3.1.

(i) We start by proving the first inequality in (i), namely $C_1 \eta_m / h_{\max} \leq \lambda_k$. Let E_k be the eigenspace associated with λ_k and let $u_1, \ldots, u_m$ be an orthonormal basis for E_k.
We first show that $\partial_\nu u_1, \ldots, \partial_\nu u_m$ are linearly independent functions on ∂M. We prove it by contradiction. Let assume that there exists $u \in \mathrm{Span}(\partial_\nu u_1, \cdots, \partial_\nu u_m)$ such that $\partial_\nu u = 0$. Let $\tilde{M}$ be a Riemannian manifold such that M admits an isometric embedding. Let N be a Riemannian manifold obtained by doubling $\tilde{M}$ along its boundary (if $\partial \tilde{M} \neq \emptyset$), endowed with the induced metric from $\tilde{M}$. More precisely, $N \cong \tilde{M} \sqcup \tilde{M}/\sim$, where $\sim$ identifies the two boundaries by the identity map. We smooth out the metric along the image of $\partial \tilde{M}$ without changing the metric on the two copies of M in N. Then we define

$$v(x) = \begin{cases} u(x) & \text{if } x \in M, \\ 0 & \text{if } x \in N \setminus M. \end{cases}$$

Clearly, we have $v \in C^1(N)$. Furthermore, v satisfies the identity $\Delta v = \lambda_k v$ on N in the distribution sense, i.e. v is the weak of solution of $\Delta v = \lambda_k v$ on N. Therefore, it is also a strong solution. Since $v \equiv 0$ on $N \setminus M$, we get $v \equiv 0$ on N by the unique continuation theorem. This in particular shows that $\dim(E_k / H_0^2(M)) = k$. Thus, we can consider E_k as a test functional space in (2.4).
Let $h_{\max} = \sup_{x \in \partial M} \langle \nabla \rho_p, \nu \rangle$. Since $0 < \frac{1}{h_{\max}} \langle \nabla \rho_p, \nu \rangle \leq 1$, we get

$$\eta_m \leq \sup_{u \in E_k} \frac{\int_M |\Delta u|^2 \, dv_g}{\int_{\partial M} (\partial_\nu u)^2 \, ds_g} \leq h_{\max} \lambda_k^2 \sup_{u \in E_k} \frac{\int_M u^2 \, dv_g}{\int_{\partial M} \langle \nabla \rho_p, \nu \rangle (\partial_\nu u)^2 \, ds_g}.$$

Next we bound the denominator from below. Applying Theorem 3.1 with $\lambda = 0$ and $F = \nabla \rho_p$ yields

$$\int_{\partial M} \langle \nabla \rho_p, \nu \rangle (\partial_\nu u)^2 \, ds_g = 2 \int_M \Delta u \langle \nabla \rho_p, \nabla u \rangle dv_g - \int_M \Delta \rho_p |\nabla u|^2 dv_g + 2 \int_M \nabla^2 \rho_p (\nabla u, \nabla u) dv_g,$$

for any $u \in E_k$. Using integration by parts we get

$$2 \int_M \Delta u \langle \nabla \rho_p, \nabla u \rangle dv_g = -\lambda_k \int_M \langle \nabla \rho_p, \nabla u^2 \rangle dv_g = \lambda_k \int_M u^2 \Delta \rho_p dv_g.$$

Consequently, we have

$$\begin{aligned} \int_{\partial M} \langle \nabla \rho_p, \nu \rangle (\partial_\nu u)^2 \, ds_g &= \lambda_k \int_M u^2 \Delta \rho_p dv_g - \int_M \Delta \rho_p |\nabla u|^2 dv_g \\ &\quad + 2 \int_M \nabla^2 \rho_p (\nabla u, \nabla u) dv_g \\ &\geq \lambda_k \left(1 + \min_{x \in M} d_p(x) H_{\kappa_2}(d_p(x)) \right) \int_M u^2 dv_g \\ &\quad - \left(1 + \max_{x \in M} d_p(x) H_{\kappa_1}(d_p(x)) \right) \int_M |\nabla u|^2 dv_g \\ &\quad + 2 \min_{x \in M} \frac{d_p(x) H_{\kappa_2}(d_p(x))}{n-1} \int_M |\nabla u|^2 dv_g \\ &= \lambda_k C_1 \int_M u^2 dv_g. \end{aligned}$$

In the second line we used the Hessian and Laplace comparison theorems; see Sect. 2. Here C_1 is

$$C_1 := \left(1 + \frac{2}{n-1} \right) \min_{r \in [0, r_{\max})} r H_{\kappa_2}(r) - \max_{r \in [0, r_{\max})} r H_{\kappa_1}(r). \tag{4.3}$$

Therefore, we get

$$C_1 \eta_m \leq h_{\max} \lambda_k.$$

We conclude the proof of the first inequality with a remark on the sign of C_1. The function $r H_\kappa(r)$ is constant if $\kappa = 0$, increasing on $[0, \infty)$ if $\kappa < 0$, and decreasing on $[0, \infty)$ if $\kappa > 0$. Thus we calculate C_1 considering the following different cases:

(a) If $\kappa_1 = \kappa_2 = 0$, then $C_1 = 2$.
(b) If $\kappa_1 \leq \kappa_2 \leq 0$, then $C_1 = n + 1 - r_{\max} H_{\kappa_1}(r_{\max})$.
(c) If $0 \leq \kappa_1 \leq \kappa_2$, then $C_1 = \left(1 + \frac{2}{n-1} \right) r_{\max} H_{\kappa_2}(r_{\max}) - (n-1)$.
(d) If $\kappa_1 \leq 0 \leq \kappa_2$, then $C_1 = \left(1 + \frac{2}{n-1} \right) r_{\max} H_{\kappa_2}(r_{\max}) - r_{\max} H_{\kappa_1}(r_{\max})$.

Of course when $C_1 \leq 0$, we only get a trivial bound. However, depending on κ_1 and κ_2, in all cases, there exists $r_0 \in (0, \infty]$ such that for $r_{\max} < r_0$, C_1 is positive.

We proceed with the proof of the second inequality of part (i). Let $u_1, \ldots, u_k \in H^2(M)$ be a family of eigenfunctions associated to $\eta_1, \ldots, \eta_k$. We can choose $u_1, \ldots, u_k$ such that $\partial_\nu u_1, \ldots, \partial_\nu u_k$ are orthonormal in $L^2(\partial M)$. Then, due to (2.1) and (2.4), we have

$$\lambda_k \leq \eta_k \sup_{u \in E_k} \frac{\int_{\partial M} (\partial_\nu u)^2 \, ds_g}{\int_M |\nabla u|^2 \, dv_g}, \tag{4.4}$$

where $E_k := \mathrm{Span}(u_1, \ldots, u_k)$. Applying Theorem 3.1 with $\lambda = 0$ and $F = \nabla\rho_p$ we get

$$\begin{aligned}\int_{\partial M} \langle \nabla\rho_p, \nu\rangle (\partial_\nu u)^2\, ds_g &= 2\int_M \Delta u \langle \nabla\rho_p, \nabla u\rangle dv_g - \int_M \Delta\rho_p |\nabla u|^2 dv_g \\ &\quad +2\int_M \nabla^2\rho_p(\nabla u, \nabla u) dv_g \\ &\le 2\max_{x\in M}|\nabla\rho_p| \left(\int_M (\Delta u)^2 dv_g \int_M |\nabla u|^2 dv_g\right)^{1/2} \\ &\quad + \left(-1 - \min_{x\in M} d_p(x) H_{\kappa_2}(d_p(x)) + 2\max_{x\in M}\frac{d_p(x)H_{\kappa_1}(d_p(x))}{n-1}\right) \\ &\quad \times \int_M |\nabla u|^2 dv_g \\ &\le 2r_{\max}\eta_k^{\frac{1}{2}}\left(\int_{\partial M}(\partial_\nu u)^2\, ds_g \int_M |\nabla u|^2 dv_g\right)^{1/2} - C_2\int_M |\nabla u|^2 dv_g,\end{aligned}$$

where

$$C_2 := 1 + \min_{x\in M} d_p(x) H_{\kappa_2}(d_p(x)) - 2\max_{x\in M}\frac{d_p(x)H_{\kappa_1}(d_p(x))}{n-1}. \tag{4.5}$$

Let $A^2 := \frac{\int_{\partial M}(\partial_\nu u)^2\, ds_g}{\int_M |\nabla u|^2 dv_g}$. From the above inequality, A satisfies

$$h_{\min}A^2 \le 2r_{\max}\eta_k^{\frac{1}{2}}A - C_2.$$

This implies

$$r_{\max}^2\eta_k - h_{\min}C_2 \ge 0.$$

Remark that since this is true for every k, we get in particular

$$\eta_1 \ge \frac{h_{\min}C_2}{r_{\max}^2}. \tag{4.6}$$

We now obtain the following upper bound on A^2

$$A^2 \le \frac{\left(r_{\max}\eta_k^{\frac{1}{2}} + \sqrt{r_{\max}^2\eta_k - C_2h_{\min}}\right)^2}{h_{\min}^2} \le \frac{4r_{\max}^2\eta_k - 2C_2h_{\min}}{h_{\min}^2}.$$

Replacing in (4.4) we conclude

$$\lambda_k \le \frac{4r_{\max}^2\eta_k^2 - 2C_2h_{\min}\eta_k}{h_{\min}^2}.$$

Remark 4.1 The function $rH_\kappa(r)$ is constant if $\kappa = 0$, increasing on $[0,\infty)$ if $\kappa < 0$, and decreasing on $[0,\infty)$ if $\kappa > 0$. We calculate C_2 considering different cases:

(a) If $\kappa_1 = \kappa_2 = 0$, then $C_2 = n-2$.
(b) If $\kappa_1 \le \kappa_2 \le 0$, then $C_2 = n - 2\frac{r_{\max}H_{\kappa_1}(r_{\max})}{n-1}$.
(c) $0 \le \kappa_1 \le \kappa_2$. Then $C_2 = r_{\max}H_{\kappa_2}(r_{\max}) - 1$.
(d) $\kappa_1 \le 0 \le \kappa_2$. Then $C_2 = 1 + r_{\max}H_{\kappa_2}(r_{\max}) - 2\frac{r_{\max}H_{\kappa_1}(r_{\max})}{n-1}$.

Depending on κ_1 and κ_2, in all cases , there exists $r_0 \in (0,\infty]$ so that when $r_{\max} < r_0$, then C_2 is positive.

ii) Let $\phi > 0$ be a continuous function on ∂M. For every $l \in \mathbb{N}$ set

$$\xi_{l+1}(\phi) := \inf_{\substack{V\subset \tilde{H}^2_{N,\phi}(M)\\ \dim V=l}} \sup_{u\in V} \frac{\int_M |\Delta u|^2\, dv_g}{\int_{\partial M} u^2\phi\, ds_g}, \qquad \xi_1(\phi)=0,$$

where $\tilde{H}^2_{N,\phi}(M) := \{u \in H^2(M) : \partial_\nu u = 0 \text{ on} \partial M \text{and} \int_{\partial M} \phi u ds_g = 0\}$. The following relation between ξ_l and $\xi_l(\phi)$ holds:

$$\xi_l \le \|\phi\|_\infty \xi_l(\phi). \tag{4.7}$$

Indeed, let $V = \text{Span}(v_1, \cdots, v_l)$ be a subspace of $\tilde{H}^2_{N,\phi}(M)$ of dimension l. The functional space $W = \text{Span}(w_1, \cdots, w_l)$, where $w_j = v_j - \frac{1}{\text{vol}(\partial M)} \int v_j\, ds_g$, is an l-dimensional subspace of $\tilde{H}^2_N(M) := \{u \in H^2(M) : \partial_\nu u = 0 \text{ on} \partial M \text{and} \int_{\partial M} u ds_g = 0\}$ since $1 \notin V$. It is easy to check that for every $v \in \tilde{H}^2_{N,\phi}(M)$ and $w = v - \frac{1}{\text{vol}(\partial M)} \int v\, ds_g$ we have

$$\frac{\int_M |\Delta w|^2\, dv_g}{\|\phi\|_\infty \int_{\partial M} w^2\, ds_g} \le \frac{\int_M |\Delta v|^2\, dv_g}{\int_{\partial M} v^2\phi\, ds_g},$$

and inequality (4.7) follows. Later on we take $\phi := \langle \nabla \rho_p, \nu\rangle$. Thus, it is enough to show that

$$\xi_{m+1}(\phi) \le \frac{\mu_k^2}{(C_3 - n^{-1}\mu_k r_{\text{in}}^2) \vee 0},$$

for some constants C_3. Let E_k be the eigenspace associated with μ_k, $k \ge 2$, and $u_1, \cdots, u_m$ be an orthonormal basis for E_k. Let F be a vector field on M satisfying properties A and B on page 10. Consider

$$v_j := u_j - \frac{1}{\int_M \text{div} F\, dv_g} \int_{\partial M} u_j \langle F, \nu\rangle ds_g, \qquad j = 1, \cdots, m.$$

The functional space $V = \text{Span}(v_1, \ldots, v_m)$ forms an m-dimensional subspace of $\tilde{H}^2_{N,\phi}(M)$, where $\phi := \langle F, \nu\rangle$.

$$\begin{aligned}\xi_{m+1}(\phi) &\le \sup_{v\in V} \frac{\int_M |\Delta v|^2\, dv_g}{\int_{\partial M} v^2 \langle F, \nu\rangle\, ds_g}\\ &= \sup_{u\in E_k} \frac{\mu_k^2 \int_M u^2\, dv_g}{\int_{\partial M} u^2 \langle F, \nu\rangle\, ds_g - (\int_M \text{div} F\, dv_g)^{-1} \left(\int_{\partial M} u\langle F, \nu\rangle ds_g\right)^2}.\end{aligned}$$

By the Green formula and Theorem 3.1, we get

$$\begin{aligned}\int_{\partial M} u^2 \langle F, \nu\rangle\, ds_g &= 2\int_M u\langle \nabla u, F\rangle dv_g + \int_M u^2 \text{div} F dv_g\\ &= 2\mu_k^{-1} \int_M \Delta u \langle \nabla u, F\rangle dv_g + \int_M u^2 \text{div} F dv_g\\ &= \mu_k^{-1} \left(\int_{\partial M} |\nabla u|^2 \langle F, \nu\rangle ds_g - \int_M \text{div} F |\nabla u|^2 dv_g + 2\int_M DF(\nabla u, \nabla u) dv_g \right)\\ &\quad + \int_M u^2 \text{div} F dv_g\end{aligned}$$

$$\geq \mu_k^{-1} \int_{\partial M} |\nabla u|^2 \langle F, \nu \rangle ds_g + (c_1 - c_2 + 2\alpha) \int_M u^2 \, dv_g$$
$$\geq (c_1 - c_2 + 2\alpha) \int_M u^2 \, dv_g.$$

We also have

$$\left(\int_{\partial M} u \langle F, \nu \rangle ds_g \right)^2 = \left(\int_M \langle F, \nabla u \rangle \, dv_g \right)^2 \leq \int_M |F|^2 \, dv_g \int_M |\nabla u|^2 \, dv_g$$
$$= \mu_k \int_M |F|^2 \, dv_g \int_M u^2 \, dv_g.$$

Therefore,

$$\xi_{m+1}(\phi) \leq \frac{\mu_k^2}{((c_1 - c_2 + 2\alpha) - c_1^{-1} \mathrm{vol}(M)^{-1} \mu_k \int_M |F|^2 \, dv_g) \vee 0}.$$

Thanks to the Laplace and Hessian comparison theorem, the vector field $F = \nabla \rho_p$ satisfies properties A and B (see page 10) on M with $\alpha = 1$, and

$$c_1 = n, \qquad c_2 = 1 + \max_{r \in [0, r_{\max})} r H_\kappa(r) = 1 + r_{\max} H_\kappa(r_{\max}).$$

Taking

$$C_3 := n + 1 - r_{\max} H_\kappa(r_{\max}), \tag{4.8}$$

we get

$$\xi_{m+1}(\phi) \leq \frac{\mu_k^2}{(C_3 - n^{-1} \mathrm{vol}(M)^{-1} \mu_k \int_M d_p^2 \, dv_g) \vee 0}$$

which completes the proof. □

Finally, we provide examples for Theorem 1.4 (ii) in which vector fields satisfying conditions A and B arise naturally. The first example is just a special case of Theorem 1.4 (ii).

Example 4.2 Let M be a star-shaped domain in $\mathbb{R}^n$ with respect to the origin. Thus $F(x) = x$ satisfies properties A and B on M for $\alpha = 1$ and $c_1 = c_2 = n$. Then by Theorem 1.4 (ii) we have

$$\xi_{m+1} \leq \frac{\max_{x \in \partial M} \langle x, \nu \rangle \mu_k^2}{(2 - n^{-1} \mathrm{vol}(M)^{-1} \mu_k I_2(M)) \vee 0},$$

where m is the multiplicity of μ_k and $I_2(M) = \int_M |x|^2 \, dv_g$ is the second moment of inertia. If in addition the origin is also the centroid of M, i.e. $\int_M x dv_g = 0$, then we have

$$\xi_{m_0+1} \leq \max_{x \in \partial M} \langle x, \nu \rangle \mu_2^2,$$

where m_0 denotes the multiplicity of μ_2. Combining this inequality with Theorem 1.1 (b) we get

$$\sigma_{m_0+1} \leq \max_{x \in \partial M} \langle x, \nu \rangle \mu_2.$$

These two last inequalities has been previously obtained in [13] for the special case $n = 2$.

Example 4.3 Let $\mathbf{B^{n+1}}$ be the unit ball in $\mathbb{R}^{n+1}$ centered at the origin, and M be a free boundary minimal hypersurface in $\mathbf{B^{n+1}}$. Consider $F(x) = x$, or equivalently $\rho_0(x) = \rho(x) = \frac{|x|^2}{2}$. It is well-known that the coordinate functions of $\mathbb{R}^{n+1}$ are harmonic on M. Hence

$$\mathrm{div} F = \Delta\rho = n.$$

Thus, condition A on page 10 is satisfied. Also, by the definition of a free boundary minimal hypersurface, we have $\langle \nabla\rho, \nu\rangle = 1$ on ∂M. To verify condition B, one can show that the eigenvalues of $\nabla^2\rho$ at point $x \in M$ are given by $1 - \kappa_i\langle x, N(x)\rangle$, $i = 1, \ldots, n$, where $N(x)$ is the unit normal to the M such that $N|_{\partial M} = \nu$, and κ_i are principal curvatures. Indeed, let $X, Y \in T_x M$. Then we have

$$\begin{aligned}
\nabla^2\rho(x)(X, Y) &= X \cdot (Y \cdot \rho(x)) - \nabla_X Y \cdot \rho(x)\\
&= X\langle x, Y\rangle - \langle x, \nabla_X Y\rangle\\
&= \langle X, Y\rangle + \langle x, D_X Y\rangle - \langle x, \nabla_X Y\rangle\\
&= \langle X, Y\rangle - \langle x, \langle S(X), Y\rangle N(x)\rangle\\
&= \langle X - S(X), Y\rangle\langle x, N(x)\rangle,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidian inner product, ∇ is the induced connection on M, D is the Euclidean connection (or simply the differentiation) on $\mathbb{R}^{n+1}$, and $S(x)$ is the shape operator

$$S : T_x M \to T_x M, \quad X \mapsto \nabla_X N.$$

Then the eigenvalues of $\nabla^2\rho(x)$ are of the form $1 - \kappa_i(x)\langle x, N(x)\rangle$, $i = 1, \ldots, n$. Define

$$\alpha := \min_{\substack{i=1,\ldots,n\\ x\in M}} (1 - \kappa_i\langle x, N(x)\rangle).$$

When $\alpha > 0$, then M with vector field F as above satisfies properties A and B on page 10. Moreover, $\langle F, \nu\rangle = 1$. Thus, following the proof of Theorem 1.4 ii, we get

$$\xi_{m+1} \le \frac{\mu_k^2}{(2\alpha - n^{-1}\mathrm{vol}(M)^{-1}\mu_k \int_M |x|^2\, dv_g) \vee 0}.$$

In dimension two, $\alpha > 0$ is equivalent to $|\kappa_i|\langle x, N(x)\rangle < 1$. By results in [1], if $|\kappa_i|\langle x, N(x)\rangle < 1$ then $\langle x, N(x)\rangle \equiv 0$ on M, and M is the equilateral disk. Hence, there is no nontrivial 2-dimensional minimal surface satisfying Properties A and B. It is an intriguing question whether there are non-trivial minimal hypersurfaces with $\alpha > 0$ in higher dimensions.

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Appendix

In this section, we prove the variational characterization for the biharmonic Steklov problems. It directly follows from the results in [9,10]. Since we could not locate a detailed proof of the variational characterization for the biharmonic Steklov problems in the literature, we include a proof for the reader's convenience.

We start by providing the variational characterization for the eigenvalues η_k of the biharmonic Steklov problem I, see (1.4).

Theorem 4.4 *For every $k \in \mathbb{N}$, we have*

$$\eta_k = \inf_{\substack{V \subset H^2(M) \cap H_0^1(M) \\ \dim(V/H_0^2(M))=k}} \sup_{\substack{u \in V \\ u \in V \setminus H_0^2(M)}} \frac{\int_M |\Delta u|^2 \, dv_g}{\int_{\partial M} (\partial_\nu u)^2 \, ds_g}.$$

Denote by $\mathcal{V}$ the completion of the space

$$Z := \left\{ v \in C^\infty(M) : \Delta^2 v = 0, \text{ in } M \text{ and } v = 0 \text{ on } \partial M \right\},$$

with respect to the inner product

$$(f, g) = \int_M \Delta f \Delta g. \tag{4.9}$$

Observe that Z is a subspace of the Hilbert space $H^2(M) \cap H_0^1(M)$.

Theorem 4.5 [10, Theorem 3.18] *Let M be a manifold*[1] *with C^2 boundary. Then the spectrum of eigenvalue problem* (1.4) *consists of a countable set of non-negative eigenvalues $\{\eta_k\}$ with finite multiplicities, and the corresponding eigenfunctions $\{\phi_k\}$ form a complete orthogonal system for $\mathcal{V}$.*

One can consider another inner product on Z as follows:

$$(f, g)_{\mathcal{W}} := \int_{\partial M} \partial_\nu f \partial_\nu g \, ds_g. \tag{4.10}$$

Let $\mathcal{W}$ be the completion of Z with respect to this new inner product. Then $\mathcal{V} \subset \mathcal{W}$ and the embedding is compact, see [10, Page 85]. We now assume that $\{\phi_k\}$ is an orthonormal system of $\mathcal{V}$ with respect to the inner product (4.10). Notice that the orthogonality of the eigenfunctions is preserved when changing the inner product from(4.9) to (4.10).
We need the following key lemma for the proof of Theorem 4.4.

Lemma 4.6 *For every $k \in \mathbb{N}$ we have*

$$\eta_k = \inf_{\substack{V \subset \mathcal{V} \\ \dim V = k}} \sup_{0 \neq v \in V} \frac{\int_M |\Delta v|^2 \, dv_g}{\int_{\partial M} (\partial_\nu v)^2 \, ds_g}. \tag{4.11}$$

Proof Let $\{\phi_1, \ldots, \phi_{k-1}\}$ be the first $k-1$ eigenfunctions which are chosen to be orthonormal with respect to inner product (4.10). Further, let $0 \neq v \perp \phi_i, i = 1, \ldots, k-1$. Then we have

$$v = \sum_{i=k}^{\infty} \alpha_i \phi_i,$$

[1] Note that [10, Theorem 3.19] is stated for domains in $\mathbb{R}^n$. But the proof can be extended to the manifold setting.

where $\alpha_i = (v, \phi_i)_{\mathcal{W}}$. Note that for every $N \in \mathbb{N}$ we have

$$0 \le \int_M \left|\Delta\left(v - \sum_{i=k}^{N} \alpha_i \phi_i\right)\right|^2 = \int_M |\Delta v|^2 - \sum_{i=k}^{N} \eta_i \alpha_i^2.$$

Thus, the sum $\sum_{i=k}^{\infty} \alpha_i^2 \eta_i$ is finite and we get

$$\eta_k \sum_{i=k}^{\infty} \alpha_i^2 \le \sum_{i=k}^{\infty} \alpha_i^2 \eta_i \le \int_M |\Delta v|^2.$$

Therefore, we obtain the inequality

$$\eta_k \le \frac{\int_M |\Delta v|^2 \, dv_g}{\int_{\partial M} (\partial_\nu v)^2 \, ds_g}.$$

This in particular proves that

$$\eta_k = \inf_{\substack{v \in \mathcal{V} \\ v \perp \mathrm{span}(Œ_1, \cdots, Œ_{k-1})}} \frac{\int_M |\Delta v|^2 \, dv_g}{\int_{\partial M} (\partial_\nu v)^2 \, ds_g}. \tag{4.12}$$

Let V be a k-dimensional subspace of $\mathcal{V}$. It is easy to show that there exists $v \in V$ such that $v \perp \mathrm{span}(Œ_1, \ldots, Œ_{k-1})$. Therefore, by (4.12), for every k-dimensional subspace of $\mathcal{V}$ we have

$$\eta_k \le \sup_{0 \ne v \in V} \frac{\int_M |\Delta v|^2 \, dv_g}{\int_{\partial M} (\partial_\nu v)^2 \, ds_g}.$$

This completes the proof. □

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4 By Lemma 4.6, it is clear that

$$\eta_k \ge A_k := \inf_{\substack{V \subset H^2(M) \cap H_0^1(M) \\ \dim(V/H_0^2(M)) = k}} \sup_{\substack{u \in V \\ u \in V \setminus H_0^2(M)}} \frac{\int_M |\Delta u|^2 \, dv_g}{\int_{\partial M} (\partial_\nu u)^2 \, ds_g}.$$

It remains to prove the reverse inequality. By [10, Theorem 3.19], the space $H^2(M) \cap H_0^1(M)$ admits the following orthogonal decomposition with respect to inner product (4.9).

$$H^2(M) \cap H_0^1(M) = \mathcal{V} \oplus H_0^2(M).$$

Hence, for every $v \in \mathcal{V}$ and $w \in H_0^2(M)$ we have

$$\int_M |\Delta(v + w)|^2 \, dv_g = \int_M |\Delta v|^2 \, dv_g + \int_M |\Delta w|^2 \, dv_g.$$

Therefore, for every subspace $V \subset H^2(M) \cap H_0^1(M)$, we have

$$\sup_{0 \ne v \in \bar{V}} \frac{\int_M |\Delta v|^2 \, dv_g}{\int_{\partial M} (\partial_\nu v)^2 \, ds_g} \le \sup_{\substack{u \in V \\ u \in V \setminus H_0^2(M)}} \frac{\int_M |\Delta u|^2 \, dv_g}{\int_{\partial M} (\partial_\nu u)^2 \, ds_g},$$

where $\bar{V} \subset \mathcal{V}$ is the projection of V on $\mathcal{V}$. This finally gives us $\eta_k \le A_k$. □

We proceed with a discussion on the proof of the variational characterization for biharmonic Steklov problem II, see (1.5).

Theorem 4.7 *For every $k \in \mathbb{N}$ we have*

$$\xi_k = \inf_{\substack{V \subset H^2_N(M) \\ \dim V = k}} \sup_{0 \neq u \in V} \frac{\int_M |\Delta u|^2 \, dv_g}{\int_{\partial M} u^2 \, ds_g}, \tag{4.13}$$

where $H^2_N(M) := \{u \in H^2(M) \, : \, \partial_\nu u = 0 \;\; on \; \partial M\}$.

To prove this theorem, we first need to state a counterpart of Theorem 4.5 for the eigenvalue problem (1.5). Although the argument is classic and standard, for the sake of completeness, we state the theorem and we include a brief discussion on its proof. Consider

$$Z_1 := \left\{ v \in C^\infty(M) : \Delta^2 v = 0, \;\; \text{in } M \text{ and } \partial_\nu v = 0 \;\; \text{on } \partial M \right\},$$

and let $Z_1/\mathbb{R}$ be the subspace of Z_1 orthogonal to the constants with respect to the following inner product

$$(f, g)_{\mathcal{W}_1} = \int_{\partial M} f g \, ds_g. \tag{4.14}$$

We denote by $\mathcal{V}_1/\mathbb{R}$ the completion of $Z_1/\mathbb{R}$ with respect to inner product (4.9) and by $\mathcal{W}_1$ the completion of Z_1 with respect to inner product (4.14). The Hilbert space $\mathcal{V}_1/\mathbb{R}$ is a subspace of $H^2_N(M)/\mathbb{R}$ and $\mathcal{V}_1 \subset H^2_N(M)$ is

$$\mathcal{V}_1 = \mathcal{V}_1/\mathbb{R} \oplus \mathbb{R}.$$

Since, for every $f \in \mathcal{W}_1$, we have

$$(f, f)_{\mathcal{W}_1} = \|f\|_{L^2(\partial M)} \leq \xi_2^{-1} \|\Delta f\|_{L^2(M)},$$

the embedding $i : \mathcal{V}_1/\mathbb{R} \to \mathcal{W}_1$ is continuous. Then by the compactness of the trace embedding, $H^{1/2}(\partial M) \hookrightarrow L^2(\partial M)$, we conclude that the embedding is compact. Let

$$L : \mathcal{V}_1 \to \mathcal{V}_1',$$
$$f \mapsto (f, \cdot),$$

where $\mathcal{V}_1'$ is the dual space, and let

$$i_1 : \mathcal{W}_1 \to \mathcal{V}_1'$$
$$f \mapsto (f, g)_{\mathcal{W}_1}.$$

The linear map L is an isomorphism. Thus, the linear operator $K = \pi \circ L^{-1} \circ i_1 \circ i$: $\mathcal{V}_1/\mathbb{R} \to \mathcal{V}_1/\mathbb{R}$ is a positive compact self-adjoint operator with strictly positive eigenvalues. Here $\pi : \mathcal{V}_1 \to \mathcal{V}_1/\mathbb{R}$ is the orthogonal projection onto $\mathcal{V}_1/\mathbb{R}$. The inverse of eigenvalues of K give the positive eigenvalues of (1.5), and the eigenfunctions are the same. We summarize this discussion in the following theorem.

Theorem 4.8 *Let M be a manifold with C^2 boundary. Then the spectrum of eigenvalue problem (1.5) consists of a countable set of non-negative eigenvalues*

$$0 = \xi_1 < \xi_2 \leq \cdots \leq \xi_k \leq \cdots$$

with finite multiplicities, and the corresponding eigenfunctions $\{\psi_k\}$ form a complete orthogonal system for $\mathcal{V}_1$.

Now, the proof of Theorem 4.7 is similar to that of Theorem 4.6. We leave the details of the proof to the interested reader.

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