Ann. Math. Québec (2016) 40:435–452
DOI 10.1007/s40316-016-0057-7

p-Adic and analytic properties of period integrals and values of *L*-functions

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Received: 15 February 2015 / Accepted: 1 January 2016 / Published online: 17 February 2016

Abstract The Ichino-Ikeda conjecture is an identity that relates a ratio of special values of automorphic L-functions to a ratio of period integrals. Both sides of this identity are expected to satisfy certain equidistribution properties when the data vary, and indeed it has been possible to transfer such properties from one side of the identity to the other in cases where the identity is known. The present article studies parallels between complex-analytic and p-adic equidistribution properties and relates the latter to questions about Galois cohomology.

Keywords Galois representation · Shimura variety · Special values of L-functions

Résumé La conjecture d'Ichino-Ikeda est une identité qui met en relation un rapport entre valeurs spéciales de fonctions L automorphes et un rapport entre périodes d'intégrales. On attend à ce que les deux côtés de cette identité vérifient certaines propriétés d'équidistribution lorsqu'on fait varier les données ; en effet, dans certains cas on a réussi à transférer de telles propriétés d'un côté de l'identité à l'autre. Cet article développe des parallèles entre les propriétés d'équidistribution dans le cadre complexe-analytique et dans le cadre p-adique; dans la situation p-adique ces propriétés sont liées à des questions sur la cohomologie galoisienne.

Mathematics Subject Classification Primary 11F80; Secondary 11F70 · 11G18 · 11F67

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement No. 290766 (AAMOT). The author was partially supported by NSF grant DMS-1404769.

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1 Introduction

Let F be a CM field of degree $2d$ over $\mathbb{Q}$, $F^+ \subset F$ its maximal totally real subfield, $c \in Gal(F/F^+)$ complex conjugation. Let Π and Π' be everywhere tempered automorphic representations of $GL(n)_F$ and $GL(n-1)_F$, respectively. We assume

$$\Pi^\vee \xrightarrow{\sim} \Pi^c; \quad \Pi^{\prime,\vee} \xrightarrow{\sim} \Pi^{\prime,c}$$

and both Π_∞ and Π'_∞ are cohomological. Then Π and Π' descend to L-packets $\Phi(\Pi, V)$ and $\Phi(\Pi', V')$ of automorphic representations of unitary groups $U(V)$ and $U(V')$ whenever V and V' are hermitian spaces over F of dimension n and $n-1$, respectively. These packets may be empty for certain pairs (V, V'), depending on local conditions, but the descent is always non-trivial to the quasi-split unitary groups. (For this and similar claims, see [22] and, more generally, [25] and [21].) We assume $\Phi(\Pi, V)$ and $\Phi(\Pi', V')$ are packets of cuspidal automorphic representations. The packets are stable provided the original representations Π and Π' are cuspidal; this will usually be assumed as well. In general, for each place v of F^+ there are local packets $\Phi_v(\Pi, V)$ and $\Phi_v(\Pi', V')$ such that

$$\Phi(\Pi, V) \subset \prod_v \Phi_v(\Pi, V); \quad \Phi(\Pi', V') \subset \prod_v \Phi_v(\Pi', V')$$

with equality if the packets are stable.

The Ichino-Ikeda conjecture, in the version for unitary groups due to N. Harris, expresses the central value $L(\frac{1}{2}, \Pi \times \Pi')$ of the Rankin-Selberg tensor product in terms of period integrals of forms in $\Phi(\Pi, V) \times \Phi(\Pi', V')$. Fix a pair (V, V'); we assume V' embeds as a non-degenerate hermitian subspace of V, so that $G' = U(V')$ is naturally a subgroup of $G = U(V)$. Let $\mathcal{A}_0(V)$ and $\mathcal{A}_0(V')$ denote the spaces of cusp forms on $G = U(V)$ and $G' = U(V')$, respectively. Define

$$L_{can}(f, f') = \int_{G'(F^+)\backslash G'(\mathbf{A})} f(g')f'(g')dg', \quad f \in \mathcal{A}_0(V), \ f' \in \mathcal{A}_0(V')$$

where dg' is Tamagawa measure. Fix $(\pi, \pi') \in \Phi(\Pi, V) \times \Phi(\Pi', V')$, and let $f \in \pi$, $f' \in \pi'$ be factorizable functions: $f = \otimes'_v f_v$, $f' = \otimes'_v f'_v$, relative to isomorphisms $\pi \xrightarrow{\sim} \otimes'_v \pi_v$, $\pi' \xrightarrow{\sim} \otimes'_v \pi'_v$. The precise formula is

$$\frac{|L_{can}(f, f')|^2}{||f||_2^2 ||f'||_2^2} = 2^{-\beta} C_0 \Delta^S_{U(n)} \prod_{v \in S} Z_v(f_v, f'_v) P^S\left(\frac{1}{2}, \pi, \pi'\right) \tag{1.1}$$

where $L_{can}(\bullet, \bullet')$ is the period integral, $|| \bullet ||_2$ is the L_2 norm (on $G(F^+)\backslash G(\mathbf{A})$ or $G'(F^+)\backslash G'(\mathbf{A})$, relative to Tamagawa measure), and the ratio of partial L-functions

$$P^S(s, \pi, \pi') = \frac{L^S(s, \Pi \times \Pi')}{L^S(s + \frac{1}{2}, \pi, Ad) L^S(s + \frac{1}{2}, \pi', Ad)}$$

is the main term. Here

- β is an integer determined by the size of the L-packet $\Phi(\Pi, V) \times \Phi(\Pi', V')$;
- S is a finite set of places of F^+ including all archimedean places and primes where either π_v or π'_v is ramified;
- the superscript ^{S} refers to partial L-functions: the denominator (respectively the numerator) is an L-function over F^+ (respectively over F) with factors at places in S (respectively at places of F dividing S) removed.

- C_0 is an elementary constant (a quotient of volumes which can be taken in $\mathbb{Q}^{\times}$);
- $\Delta^S_{U(n)} = \alpha\pi^m, \alpha \in \overline{\mathbb{Q}}, m \in \mathbb{Z}$ (a product of special values of partial abelian L-functions, also interpreted as the L-function of the *Gross motive* attached to G),
- each $Z_v(f_v, f'_v) = \frac{I(f_v, f'_v)}{|f_v|^2_v |f'_v|^2_v}$ is a normalized local zeta integral, where $I(f_v, f'_v)$ is the unnormalized local zeta integral defined as follows:

$$I(f_v, f'_v) = \int_{G'(F_v^+)} c(f_v)(g'_v)c(f'_v)(g'_v)dg'_v \tag{1.2}$$

with

$$c(f_v)(g) = \langle \pi_v(g_v)f_v, f_v\rangle_v; \quad c(f'_v)(g'_v) = \langle \pi'_v(g'_v)f'_v, f'_v\rangle_v. \tag{1.3}$$

Here the pairings in (1.3) are invariant hermitian forms on π_v and π'_v, respectively, with the property that the L^2 pairings on π and π', respectively, factor

$$\langle f_1, f_2\rangle_2 = Q(\pi)\prod_v \langle f_{1,v}, f_{2,v}\rangle_v; \ \langle f'_1, f'_2\rangle_2 = Q(\pi')\prod_v \langle f'_{1,v}, f'_{2,v}\rangle_v \tag{1.4}$$

whenever $f_i = \otimes'_v f_{i,v}$, $f'_j = \otimes'_v f'_{j,v}$, $i, j = 1, 2$, with respect to the factorizations of π and π' chosen above, and with constants $Q(\pi)$ and $Q(\pi')$ to be specified in Sect. 3. Moreover, the denominators $|f_v|^2_v$, $|f'_v|^2_v$ of the normalized local zeta integrals are defined by

$$|f_v|^2_v = \langle f_v, f_v\rangle_v; \quad |f'_v|^2_v = \langle f'_v, f'_v\rangle_v. \tag{1.5}$$

- $L(*, *, Ad)$ is the Langlands L-function attached to the adjoint representation of the L-group of G or G'.

See [36,37] for the most complete results on this conjecture to date.

The local Gan-Gross-Prasad conjectures for unitary groups, proved (at finite places) by Beuzart-Plessis, implies that the left-hand side of (1.1) vanishes for local reasons except for at most one pair $V' \subset V$ and one (π, π'). More precisely, for each $v \in S$, there is a unique pair of *local* hermitian spaces $V'_v \subset V_v$ and a unique $\pi_v \times \pi'_v \in \Phi_v(\Pi, V_v) \times \Phi_v(\Pi', V'_v)$ such that

$$Hom_{U(V'_v)}(\pi_v \times \pi'_v, \mathbb{C}) \neq 0. \tag{1.6}$$

We assume the pair (Π, Π') is *coherent* in the sense that there are global hermitian spaces V and V' with the given localizations. We assume we have chosen V, V' and consider a *family* of pairs (π, π') compatible with this choice, and $(f, f') \in \pi \times \pi'$. Here we use the word "family" alternatively as it is understood in the setting of analytic number theory, (see for example [31,32]) or in the sense of p-adic analytic families. Then (1.1) translates (complex or p-adic) analytic properties of the periods into analogous properties of the special values, and vice versa.

The present paper contains no new results; its purpose is rather to review a number of standard conjectures on these properties in the light of the formula (1.1). We mainly focus on identifying questions about the p-adic behavior of special values, especially central values, that are suggested by the Ichino-Ikeda conjecture. Some of these questions are analogous to expectations about analytic properties of special values that appear to be widely believed by specialists but that to my knowledge have not been formulated in print. Section 2 collects a few of these expectations; these are primarily meant to motivate the speculations in the later sections, but it may be useful to have them all in one place. Section 3 formulates an automorphic version of Deligne's conjecture on critical values of L-functions. This allows us to replace (1.1) by an expression relating algebraic numbers, whose integral properties are studied in the last two sections.

It is a pleasure to dedicate this paper to Glenn Stevens, who narrowly escaped being my roommate when we were graduate students at Harvard. I thank the organizers of the Glenn Stevens conference for inviting me to contribute to this volume and Akshay Venkatesh, Peter Sarnak, Haruzo Hida, Eric Urban, Chris Skinner, and the referee for the first draft, for helping me to make the questions and expectations approximately sensible; any remaining nonsense is mine alone. I especially thank Ariane Mézard for refusing to allow me to cancel my lecture at the Fontaine conference in 2010, after I learned that my intended topic had become obsolete, and in this way forcing me to think about the topic of the present note.

2 Complex-analytic variation

Let (V, V'), (π, π') be as in the previous section. Imitating the terminology of Gan and Ichino, we say the representation $\Pi \times \Pi'$ of $GL(n) \times GL(n-1)$ is the *last name* of the representation $\pi \times \pi'$ of $G \times G'$. The pair (V, V'), together with the finite set S of places and, for each $v \in S$, the unique $\pi_v \times \pi'_v$—taken up to inertial equivalence—satisfying (1.6), comprise the *first name* of $\pi \times \pi'$. Here if v is non-archimedean, we say two representations are inertially equivalent if they belong to the same Bernstein component; this means, roughly, that the semisimplifications of the associated local Langlands parameters are equivalent on inertia. If v is archimedean, π_v and π'_v are in the discrete series and inertial equivalence is taken to mean isomorphism. In particular, two representations can have the same first names but different last names; a given last name designates an L-packet and the first names distinguish the different members of the L-packet (some items in the first name are redundant). If $T \subset S$ then one obtains the first name at T by ignoring the data at $v \in S \backslash T$.

The hypotheses imply that the completed L-functions (including archimedean terms) satisfy a functional equation

$$L(s, \Pi \times \Pi') = \varepsilon(s, \Pi \times \Pi') L(1-s, \Pi \times \Pi') \tag{2.1}$$

where the sign $\varepsilon(\frac{1}{2}, \Pi \times \Pi') = \pm 1$. The coherence assumption implies that the sign is in fact $+1$. This property is determined by the *first name* of $\Pi \times \Pi'$. Here is a vague expectation, inspired by a well-known conjecture of Goldfeld on twists of elliptic curves.

Expectation 2.1 (a) *Let $T \subsetneq S$ be a proper subset. Let $\mathcal{F} = (V' \subset V; \{\pi_v \times \pi'_v\})$ be a first name satisfying the coherence assumption, $\mathcal{F}_T$ the restriction of $\mathcal{F}$ to T. Then (in a sense to be made more precise in Expectation 2.4 below) almost all $\Pi \times \Pi'$ with first name $\mathcal{F}' \supset \mathcal{F}_T$ satisfying the coherence assumption have the property that*

$$L\left(\frac{1}{2}, \Pi \times \Pi'\right) \neq 0. \tag{2.2}$$

(b) *Moreover, for fixed Π with first name $(V, \{\pi_v\}, v \in S)$, almost all Π' with first name $(V', \{\pi'_v\}, v \in S)$ have the property (2.2), and likewise if the roles of V and V' are reversed.*

(c) *Point (b) remains true when Π' runs over cohomological automorphic representations of $GL(r)$ for any $r \leq n-1$.*

Remark 2.2 I repeat that in equidistribution questions like this one, the first name can be specified only up to inertial equivalence.

Remark 2.3 One can ask for a weaker version of points (b) and (c) of Expectation 2.1: for given Π there is at least one Π' satisfying (2.2). When $r = n - 1$ a number of results of this kind are known, for example [9,17]. For $r < n - 1$ it is considered extremely difficult even to find one Π'.

The vagueness is in the expression "almost all." The indicated collection $C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})$ of $\Pi \times \Pi'$ with first name specified at $S\backslash T$ is an example of a family; it is infinite because no restriction has been made at $S\backslash T$, except the coherence assumption. Analytic number theorists define height functions $h(\Pi \times \Pi') = \prod_{v\in S\backslash T} h_v(\Pi_v \times \Pi'_v)$ on families of this kind. The local height h_v measures the size of the infinitesimal character if v is archimedean and the Artin conductor of $\Pi_v \times \Pi'_v$ if v is non-archimedean. See for example the definition of the Iwaniec-Sarnak analytic conductor in [35], 2.12.2, or the discussion on p. 728 of [23]; however, we will be loose about the exponent in order to allow for qualitative statements. The first important property is

$$\left|\left\{\Pi \times \Pi' \subset C\left(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T}\right), h(\Pi \times \Pi') < N\right\}\right| < \infty, \quad \forall N > 0. \tag{2.3}$$

Denote by $C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})_{<N}$ the set on the left-hand side of (2.3), and let $C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})^{\neq 0}_{<N}$ the subset of $\Pi \times \Pi'$ such that $L(\frac{1}{2}, \Pi \times \Pi') \neq 0$. In view of (2.3), Expectation 2.1 can be made quantitative:

Expectation 2.4 *Under the hypotheses of Expectation 2.1, one has*

$$\lim_{N\to\infty} \frac{|C\left(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T}\right)^{\neq 0}_{<N}|}{|C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})_{<N}|} = 1. \tag{2.4}$$

Assuming the validity of Conjecture (1.1), Expectation 2.4 can be seen as a statement about the equidistribution of the spectrum of $G \times G'$ relative to the diagonal cycle $G'(F^+)\backslash G'(\mathbf{A}) \subset G'(F^+)\backslash G'(\mathbf{A}) \times G(F^+)\backslash G(\mathbf{A})$. A different kind of equidistribution result concerns the growth of the central value $L(\frac{1}{2}, \Pi \times \Pi')$ as $\Pi \times \Pi'$ varies over $C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})^{\neq 0}_{<N}$ (which is effectively the same as variation over $C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})_{<N}$ under Expectation 2.4). We assume the height $h(\Pi \times \Pi')$ factors as $h(\Pi)h(\Pi')$, with each factor satisfying the analogue of (2.3). We assume $h(\Pi \times \Pi')$ has been normalized in such a way that

$$\left|L\left(\frac{1}{2}, \Pi \times \Pi'\right)\right| <<_{C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})} h(\Pi \times \Pi')^{\frac{1}{2}} \tag{2.5}$$

is the convexity bound on central values, obtained by any means necessary (the Rankin-Selberg integral, or the Langlands-Shahidi method), where the implied constant depends on the family $C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})$. Then the relevant generalized Lindelöf Hypothesis would be

Conjecture 2.5 *For any $\varepsilon > 0$ one has*

$$\left|L\left(\frac{1}{2}, \Pi \times \Pi'\right)\right| <<_{C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T},\varepsilon)} h(\Pi \times \Pi')^{\varepsilon}$$

where the implied constant depends on the family and on ε.

Subconvexity in this setting would be any result of the form

$$\left|L\left(\frac{1}{2}, \Pi \times \Pi'\right)\right| <<_{C(\Pi^{S\backslash T} \times \Pi'^{,S\backslash T})} h(\Pi \times \Pi')^{\frac{1}{2}-\varepsilon_0} \tag{2.6}$$

for some fixed $\varepsilon_0 > 0$.

I learned the following expectation from Venkatesh; I don't know whether or not it has a name:

Expectation 2.6

$$|L(1,\pi,Ad)| \asymp_{C(\Pi^{S\setminus T})} h(\Pi)^{\varepsilon}; \quad |L(1,\pi',Ad)| \asymp_{C(\Pi'^{,S\setminus T})} h(\Pi')^{\varepsilon} \tag{2.7}$$

Note that $L(s,\pi,Ad) = L(s,\Pi,As^{\pm})$ is an Asai L-function attached to Π, with sign depending on the parity of n. Thus (2.6) is actually a bound for L-functions of $GL(n)$ and $GL(n-1)$. The upper bounds

$$|L(1,\pi,Ad)| << _{C(\Pi^{S\setminus T})} h(\Pi)^{\varepsilon}; \quad |L(1,\pi',Ad)| << _{C(\Pi'^{,S\setminus T})} h(\Pi')^{\varepsilon}$$

thus probably follow (as Venkatesh has explained) from Theorem 2 of Li in [23]; however, I have not checked that the Asai L-functions satisfy all the properties required for that proof. The lower bounds are also expected; see [26], where they are proved in some cases assuming the non-existence of Siegel zeroes.

Subconvexity is an equidistribution statement for the absolute values of the periods $L_{can}(f,f')$ in (1.1), as f and f' vary over norm 1 vectors. The size of the periods depends both on the ratio of L-values and on the absolute values of the local zeta integrals, and in principle one can separate the effect of varying f and f' over vectors in a fixed $\pi \times \pi'$ from the effect of varying $\Pi \times \Pi'$ in a family.

In particular, if you are willing to grant Expectation 2.6 and the Lindelöf Hypothesis 2.5, then the Ichino-Ikeda conjecture implies that the growth of the periods is purely local:

$$|L_{can}(f,f')|^2 = O\left(\left|\prod_{v\in S} Z_v(f_v,f'_v)\right|\right) \tag{2.8}$$

when f and f' vary over unit vectors in π and π'.

This is mentioned only in order to motivate the p-adic analogue. The (generalized) Lindelöf Hypothesis is widely believed because it is an immediate consequence of the (generalized) Riemann Hypothesis. However, any evidence in favor of the Lindelöf Hypothesis is considered precious. The article [35] of Venkatesh obtains bounds on periods, generally much weaker than 2.8, and uses them to prove subconvexity and other equidistribution applications in a number of settings.

3 Rationality properties

Under our hypotheses, each representation $\pi \in \Phi(\Pi, V)$ (resp. $\pi' \in \Phi(\Pi', V')$) contributes non-trivially to the cohomology of an automorphic vector bundle (coherent sheaf) on the Shimura variety attached to G (resp. G'). More precisely, the Shimura varieties are attached to the groups G^+, $G'^{,+}$ of similitudes of V and V', respectively, with rational similitude factor, and one needs to extend π and π' to automorphic representations $G\pi$ and $G\pi'$ of G^+ and $G'^{,+}$. This has to be done in a way that is compatible with the structure of the automorphic vector bundles and the central characters of $G\pi$ and $G\pi'$ have to be chosen so that the period integral L_{can} extends. Details can be found in [15], especially in Sect. 5.1, where it is explained that the period invariants introduced below have only a mild dependence on these supplementary choices (see below). See [15] as well for all claims (and disclaimers[1]) regarding the rational structures on π and π' defined by coherent cohomology.

Suppose now we have chosen f and f' as in (1.1) to be vectors in π and π', respectively, that are rational over a number field L relative to the rational structure defined by coherent cohomology. (We recall that the identification of an automorphic form with a coherent cohomology class depends on a notion of trivialization of automorphic vector bundles at CM points, and the theory in [15] is worked out relative to a fixed conjugacy class of CM points. We gloss over this point in what follows.) Let $P(\pi, \pi')$ and $P(\pi^\vee, \pi'^{,\vee})$ (resp. $Q(\pi)$, $Q(\pi')$) be the Gross-Prasad periods (resp. normalized Petersson norms) defined in Definition 5.15 [resp. formula (3.13)] of [15]. These are elements of $E(\pi) \otimes E(\pi') \otimes \mathbb{C}$, where $E(\pi)$ and $E(\pi')$ are number fields that play the role of coefficient fields of π and π' and associated motives $M(\pi)$ and $M(\pi')$.[2] Thus if f and f' are defined over $L \supset E(\pi) \cdot E(\pi')$ as above, then $P(\pi, \pi')^{-1} L_{can}(f, f)'$, $P(\pi^\vee, \pi'^{,\vee})^{-1} \overline{L_{can}(f, f')}$, $Q(\pi)^{-1} ||f, f||^2$, and $Q(\pi')^{-1} ||f', f'||^2$, all belong to L. Moreover, these quotients all transform in an appropriate way under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, so that the expression on the left hand side of (1.1) is an $E(\pi) \otimes E(\pi')$-multiple of

$$\mathcal{P}(\pi, \pi') := \frac{P(\pi, \pi') P(\pi^\vee, \pi'^{,\vee})}{Q(\pi) Q(\pi')}. \tag{3.1}$$

(There is a misprint in formula (5.10) of [15]; the factor $P(\pi^\vee, \pi'^{,\vee})$ was omitted.)

The constants $Q(\pi)$ and $Q(\pi')$ were already seen in (1.4).

The following conjecture is a consequence of the Tate conjecture, applied to an appropriate theory of motives:

Conjecture 3.1 (a) *Up to $E(\pi)^\times$-multiples, the invariant $Q(\pi)$ depends only on Π_f and the archimedean part of the first name of π. (Likewise for $Q(\pi')$.)*

(b) *Up to $E(\pi) \cdot E(\pi')$-multiples, the invariant $P(\pi, \pi')$ depends only on $\Pi_f \times \Pi'_f$ and the archimedean part of the first names of π and π'.*

In particular, the invariants do not depend on the inner forms of G and G' at finite places, nor on the first names of π and π' at finite places. It follows that the set of invariants of the form $Q(\pi)$ attached to a given Π has 2^{dn} members, corresponding to the number of elements in the union of the local discrete series L-packets at the archimedean places of F^+; however, when n is even, up to half of these invariants may not be realized globally because of a global sign obstruction. (In that case Yoshida has shown how to construct alternative invariants by using quadratic base change.)

Conjecture 3.1 is a consequence of Conjecture 3.3.10 of [19], a much more general conjecture on the multiplicative relations among the various $Q(\pi)$, which is also implied by the Tate conjecture. Conjecture 3.1 for holomorphic discrete series has been established in most cases by using relations of the $Q(\pi)$ to critical values of L-functions (for example, see Corollary 3.5.12 of [13]). As hinted above, the actual Petersson norms of arithmetically normalized forms depend on the central character of the extension of π to a representation $G\pi$ of the similitude group G^+. This annoying feature of the theory must be taken into account in formulating conjectures carefully, as in Conjecture 3.3.10 of [19], but we will disregard it and assume our $G\pi$ and $G\pi'$ have been chosen consistently.

[1] For example, the assumption that the Gross-Prasad period invariants are defined correctly depends in general on the hypothesis that local archimedean pairings extend continuously to the Fréchet completions of moderate growth.

[2] More precisely, $Q(\pi) \in E(\pi) \otimes \mathbb{C}$ and $Q(\pi') \in E(\pi') \otimes \mathbb{C}$. We realize $Q(\pi)$ as an element of $E(\pi) \otimes E(\pi') \otimes \mathbb{C}$ by embedding $E(\pi) \xrightarrow{\sim} E(\pi) \otimes 1 \subset E(\pi) \otimes E(\pi')$, and similarly for $Q(\pi')$.

The main terms on the right-hand side of (1.1) are critical values of motivic L-functions, and are therefore conjecturally algebraic multiples of the period invariants defined by Deligne in [4]. We write $M(\Pi)$ and $M(\Pi')$ for the hypothetical motives over F, of rank n and $n-1$, respectively, over their respective coefficient fields $E(\Pi)$ and $E(\Pi')$, attached to Π and Π'. Here one expects $E(\Pi)$ to be the field of definition of the finite part of the automorphic representation Π, and likewise for $E(\Pi')$, but this is all hypothetical in general. With the usual normalization, $M(\Pi)$ and $M(\Pi')$ are of weights $n-1$ and $n-2$, respectively. We write $c(\frac{1}{2}, \Pi \times \Pi')$ for the Deligne period attached to the special value $L(\frac{1}{2}, \Pi \times \Pi')$, which equals $L(n-1, M(\Pi) \otimes M(\Pi'))$, a critical value of the motivic L-function (viewed by restriction of scalars to $\mathbb{Q}$). Similarly, we write $c(1, \Pi, Ad)$ and $c(1, \Pi', Ad)$ for the Deligne periods attached to the special values $L(1, \pi, Ad)$ and $L(1, \pi', Ad)$; this is an abuse of notation because the L-functions are attached to the adjoint representations of the L-groups of G and G', and not of $GL(n)$, $GL(n-1)$; however, the L-functions do only depend on Π and Π'. Deligne's conjecture is

Conjecture 3.2 (Deligne) *Let $E(\Pi)$, $E(\Pi')$, and $E(\Pi, \Pi')$ denote the coefficient fields of the restrictions of scalars from F to $\mathbb{Q}$ of $M(\Pi)$, $M(\Pi')$, and $M(\Pi) \otimes M(\Pi')$, respectively. Then as elements of $E(\Pi, \Pi') \otimes \mathbb{C}$, resp. $E(\Pi) \otimes \mathbb{C}$, resp. $E(\Pi') \otimes \mathbb{C}$, we have*

$$L\left(\frac{1}{2}, \Pi \times \Pi'\right) \sim c(\frac{1}{2}, \Pi \times \Pi')$$
$$L(1, \pi, Ad) \sim c(1, \Pi, Ad)$$
$$L(1, \pi', Ad) \sim c(1, \Pi', Ad)$$

where the notation $\sim$ means, respectively, up to scalar multiples in $E(\Pi, \Pi')$, $E(\Pi)$, and $E(\Pi')$.

This conjecture is predicated on a valid theory of motives for automorphic representations. To verify the conjecture in the automorphic setting, one invokes the following principle:

Principle 3.3 *The Deligne periods $c(\frac{1}{2}, \Pi \times \Pi')$, $c(1, \Pi, Ad)$, and $c(1, \Pi', Ad)$ can all be expressed, up to factors in the relevant coefficient fields, as products of periods of the form $Q(\pi)$ and $Q(\pi')$ and of periods of abelian motives, as π and π' vary over the packets $\Phi(\Pi, V)$ and $\Phi(\Pi', V')$ for various hermitian spaces V and V'.*

In fact, the multiplicative relations among the periods $Q(\pi)$ for varying π imply that the Deligne periods can be written in many ways as products of these normalized Petersson norms. Assuming a reasonable theory of motives, the article [16] gives precise conjectural versions of Principle 3.3 when F is an imaginary quadratic field. More general (hypothetical) factorizations are worked out in Section 4 of [11], and versions of these relations are proved in many cases in [11,12], and [24]

A precise conjectural relation among the period invariants appearing on the left-hand and right-hand sides of (1.1) is given in Conjecture 5.16 of [15] (with the left-hand side correctly defined as in (3.1) above). In what follows, we will want to view algebraic numbers such as $P(\pi, \pi')^{-1} L_{can}(f, f)$ as elements of p-adic fields (or integer rings), and to relate p-adic properties of these algebraic numbers to p-adic properties of the algebraic parts of the special values of L-functions on the right-hand side of (1.1). One difficulty is that the period invariants in the numerators and denominators of the two sides do not correspond. For example, one can often interpret $L_{can}(\bullet, \bullet)$ as a cohomological cup product (cf. [17]). In these cases the Gross-Prasad periods $P(\pi, \pi')$ and $P(\pi^\vee, \pi'^{,\vee})$ can be normalized to equal 1, but the critical value $L(\frac{1}{2}, \Pi \times \Pi')$ is conjecturally never an algebraic number.

4 p-adic analytic variation

The theme of this section is that the structure of the Ichino-Ikeda conjecture appears to be tailor-made for interpolation in p-adic families. The presence of the adjoint L-function in the denominator is consistent with the appearance of (elements of) congruence modules in Hida's definitions of p-adic L-functions in Hida families. Moreover, the presence of the adjoint L-function on the right-hand side of 1.1 is the natural counterpart of the presence of the norm on the denominator of the right-hand side. However, both sides of the conjecture need to be replaced by algebraic numbers before we can consider their algebraic variation.

In what follows we assume all the Deligne periods $c(\frac{1}{2}, \Pi \times \Pi')$, $c(1, \Pi, Ad)$, $c(1, \Pi', Ad)$, as well as $Q(\pi)$ and $Q(\pi')$, to be normalized up to p-adic units. This is at least reasonable when the motives $M(\Pi)$ and $M(\Pi')$ are *ordinary* at primes dividing p. In the ordinary case we also assume the invariants can be normalized consistently, up to p-adic units in an ordinary (Hida) family containing an integral ordinary vector in π. We assume the relations postulated in Principle 3.3 are valid up to p-adic units.

With our normalized periods, we let

$$\tilde{L}^S\left(\frac{1}{2}, \Pi \times \Pi'\right) = \frac{L^S\left(\frac{1}{2}, \Pi \times \Pi'\right)}{c\left(\frac{1}{2}, \Pi \times \Pi'\right)}$$

and define $\tilde{L}^S(1, \pi, Ad)$, $\tilde{L}^S(1, \pi', Ad)$, $\tilde{Q}(\pi)$, and $\tilde{Q}(\pi')$ analogously. These are all algebraic numbers. It is well-known that $\Delta^S_{U(n)}$ is an algebraic multiple of $(2\pi i)^{a(U(n))}$ for a certain integer $a(U(n))$; we define $\tilde{\Delta}^S_{U(n)} = (2\pi i)^{-a(U(n))}\Delta^S_{U(n)}$. We also define $\tilde{L}_{can}(f, f')$ by dividing $L_{can}(f, f')$ by a an appropriately normalized $P(\pi, \pi')$, so that (1.1) becomes an identity of algebraic numbers when the quantities are replaced by their versions with $\tilde{}$.

Remark 4.1 The computations in [16] of the Deligne periods in terms of the invariants $Q(\pi)$ make use of hermitian pairings on the cohomological realizations of the motives, and the multiplicative relations among the various $Q(\pi)$ are based on identifying the automorphic realizations of the motives with exterior powers of one another. As long as $p > n \geq 2$, it should be possible to make sense of these identifications p-integrally. Here one must bear in mind that the relations between the natural integral normalizations of the $Q(\pi)$, for example, as the group G varies among inner forms, reflect subtle properties of Galois cohomology (see [30] for results when $n = 2$), and a precise conjecture will need to take these subtleties into account.

In this section we always assume $p > 2$.

4.1 The adjoint L-functions

We begin by considering the denominator of the left-hand side. The value at $s = 1$ of $L(s, \pi, Ad)$ is critical and is therefore conjecturally an algebraic multiple of the corresponding Deligne period. (Automorphic versions of this conjecture are proved in many cases in [11,12].) I would like to say that the Bloch-Kato conjecture expresses the quotient of $L(1, \pi, Ad)$ by the Deligne period in terms of orders of Galois cohomology groups. Obviously matters can't be so simple, because the Deligne period is only defined up to a scalar factor in the coefficient field, and there are additional complications when the coefficient field is not rational.

We take as a model the results of [5,6] for elliptic and Hilbert modular forms. Let τ be a cohomological automorphic representation of $GL(2)_{F^+}$ attached to a Hilbert modular form, all of whose weights have the same parity. Let $S(\tau)$ be the set of finite primes at which τ is ramified. Let $\rho_\tau : Gal(\overline{\mathbb{Q}}/F^+) \to GL(2, \mathcal{O})$ denote the p-adic Galois representation attached to τ, where $\mathcal{O}$ is the p-adic integer ring generated by the Fourier coefficients of a newform in τ; let k be the residue field of $\mathcal{O}$, and let $\bar{\rho}_\tau : Gal(\overline{\mathbb{Q}}/F^+) \to GL(2, k)$ be the residual representation of ρ_τ. Let $Ad^0(\rho_\tau)$ denote the adjoint action of $Gal(\overline{\mathbb{Q}}/F^+)$ on the trace zero subspace of $End(\rho_\tau)$. We define $\Lambda(s, Ad^0(\rho_\tau))$ to be the completed L-function of $Ad^0(\rho_\tau)$ (including Γ-factors). Here is a rough version of Dimitrov's statement; as far as I know the hypotheses have not been relaxed. A precise statement can be found in [6].

Theorem 4.2 (Dimitrov) *Let $p \notin S(\tau)$ be a prime unramified in F^+. Assume*

(a) *p is sufficiently large relative to the archimedean component τ_∞ of τ.*
(b) *the image of $\bar{\rho}_\tau$ is sufficiently large (condition ($\mathbf{LI}_{Ind(\rho)}$) in [6]).*

There is an ideal $Tam(Ad^0(\rho_\tau)) \subset \mathcal{O}$, depending only on the local factors of τ at primes in $S(\tau)$, and a normalized period $c(1, \tau, Ad^0) \in \mathbb{C}^\times$, such that

$$Tam(Ad^0(\rho_\tau))Fitt\left(H^1_f(F^+, Ad^0(\rho_\tau)) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right) = \iota_p\left(\frac{\Lambda(1, Ad^0(\rho_\tau))}{c^*(1, \tau, Ad^0)}\right)\mathcal{O}. \quad (4.1)$$

Here ι_p is a chosen embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$, H^1_f is the Bloch-Kato Selmer group, $Fitt$ denotes the Fitting ideal, and $c^*(1, \tau, Ad)$ is a version of the Deligne period $c(1, \tau, Ad)$ introduced above, modified to account for the Γ-factors.

Now assume π is an automorphic representation of the unitary group G, defined over a number field $E = E(\pi)$, which we view as a subfield of $\mathbb{C}$. Fix a prime w of E dividing p and let $\mathcal{O}_{E,w} \subset E$ be the ring of integers, localized at w; let $\mathcal{O}$ be the w-adic completion of $\mathcal{O}_{E,w}$. Let $S = S(\pi)$ denote the set of places of F^+ at which either π or the group G is ramified. The generalization of the right hand side of (4.1) is the ideal generated by the p-adic realization of $\tilde{L}^S(1, \pi, Ad)$, which is one of the factors in the denominator of the right-hand side of 1.1. The left-hand side of (4.1) should similarly be related to the corresponding factor of the denominator of the left-hand side of 1.1, namely the factor $Q(\pi)^{-1}||f||_2^2 = Q(\pi)^{-1} < f, f >$, where $Q(\pi)$ is the p-integral invariant introduced above and f is an integral generator of the module of coherent cohomological forms in π of some optimal level.

One doesn't really know how to optimize the level in general, but it is reasonable to interpret the quotient $Q(\pi)^{-1} < f, f >$ as a generator of a *congruence ideal*. More precisely, let $\mathbb{T}_\pi$ denote the localized Hecke algebra, that appears in the Taylor-Wiles method and its generalizations, specifically the generalization to coherent cohomology in [14]. This is a finite $\mathcal{O}$-algebra, acting on a (localized) $\mathcal{O}$-module M_π of coherent cohomology on the Shimura variety attached to G. We let $M_{\pi,w} \subset M_\pi$ be the $\mathcal{O}_{E,w}$-module of E-rational forms contained in M_π (in other words, the coherent cohomology of a model over $\mathcal{O}_{E,w}$). The localization has the effect that

$$M_\pi \otimes \mathbb{Q}_p \xrightarrow{\sim} \oplus_{\pi' \equiv \pi} V(\pi') \quad (4.2)$$

where π' runs through automorphic representations of G contributing to the same coherent cohomology space as π, the relation $\equiv$ denotes congruence (of the corresponding Galois representations) modulo the maximal ideal of $\mathcal{O}$, and $V(\pi')$ is the subspace of (rational) coherent cohomology cut out by the Hecke eigenvalues of π'. Let $\lambda_\pi : \mathbb{T}_\pi \to \mathcal{O}$ be the character of the Hecke algebra on forms in $V(\pi)$, $I_\pi = \ker \lambda_\pi$, $J_\pi = Ann_{\mathbb{T}_\pi}(J_\pi) \subset \mathbb{T}_\pi$ and $C(\pi) = \lambda_\pi(J_\pi)$ the congruence ideal.

In order to continue, it is reasonable to make the following hypotheses:

Hypothesis 4.3 *Suppose $p \notin S$ and $G(F_v^+)$ is quasi-split for every finite place $v \in S$. Then*

(i) *There is an invariant complex-valued inner product $<,>$ on $M_{\pi,w}$ with respect to which the action of $\mathbb{T}_\pi$ is hermitian, and $Q(\pi)$ can be taken to be the free $\mathcal{O}_{E,w}$-submodule of $\mathbb{C}$ generated by $< f, g >$ where f belongs to an $\mathcal{O}_{E,w}$-basis of $M_{\pi,w} \cap V(\pi)$ and g runs over $M_{\pi,w}$.*
(ii) *$C(\pi) = Q(\pi)^{-1} < f, f > \mathcal{O}$ if f belongs to an $\mathcal{O}$-basis of $M_\pi \cap V(\pi)$.*
(iii) *$C(\pi) = Fitt(H^1_f(F^+, Ad^0(\rho_\tau)) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$.*

If we don't assume that $G(F_v^+)$ is quasi-split at ramified primes, then the module $M_{\pi,w}$ may be missing *level-lowering congruences*—there may be a congruence between the p-adic Galois representation attached to the base change Π of π and that attached to some Π_1 that is less ramified than Π at places dividing S, but there may be local obstructions at places in S to descending Π_1 to G. In that case, the Petersson norm of an integral generator of the π-isotypic component of M_π may not be sufficiently divisible by p; see [30] for a well-known example. (I thank Chris Skinner for reminding me of this possibility.) Point (iii) of Hypothesis 4.3 is the sort of thing that can be derived from the Taylor-Wiles method as a consequence of an identity of the form $R_\pi \xrightarrow{\sim} T_\pi$, where R_π is the deformation ring of the Galois representation attached to π. When the Taylor-Wiles method applies, it implies that T_π is a complete intersection and that M_π is free over T_π; this implies Point (i). When $n > 2$, one only knows how to prove $R_\pi \xrightarrow{\sim} T_\pi$ theorems in minimal level; in the setting of coherent cohomology, this is proved in [14], for the moment under fairly restrictive hypotheses. Nevertheless, it seems fair to admit this for heuristic purposes.

Equation (4.1) also incorporates the Γ-factors as well as local Tamagawa factors. Now we draw the lesson for the L-function $L(s, \pi, Ad) = L(s, Ad(\rho_{p,\pi}))$, where $\rho_{p,\pi}$ is the p-adic representation of $Gal(\overline{\mathbb{Q}}/F)$ attached to Π and where $Ad(\rho_{p,\pi})$ is the p-adic representation of $Gal(\overline{\mathbb{Q}}/F^+)$ (note that F has been replaced by F^+) on the Lie algebra of $End(\rho_{p,\pi})$. Assuming Dimitrov's theorem generalizes to automorphic representations of unitary groups for which Hypothesis 4.3 is valid, it is reasonable to expect the following version of the Bloch-Kato conjecture for the adjoint motive attached to Π:

Expectation 4.4 *Let $\pi \in \Phi(\Pi, V)$. Let $S(\pi)$ be the set of finite places of F at which π is ramified. Let $p \notin S(\pi)$ be a prime unramified in F. Assume*

(a) *p is sufficiently large relative to the archimedean component π_∞ of π.*
(b) *the residual image of $\rho_{p,\pi}$ is sufficiently large.*

There is an ideal $Tam(Ad(\rho_{p,\pi})) \subset \mathcal{O}$, depending only on the local factors of π at primes in $S(\pi)$, and a normalized period $c^*(1, \pi, Ad) \in \mathbb{C}^\times$, such that

$$Tam(Ad(\rho_{p,\pi}))Fitt\left(H^1_f(F^+, Ad(\rho_{p,\pi})) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right) = \iota_p\left(\frac{\Lambda(1, Ad(\rho_{p,\pi}))}{c^*(1, \pi, Ad)}\right)\mathcal{O}. \tag{4.3}$$

As in the statement of Theorem 4.2, $c^*(1, \pi, Ad)$ is a modified version of the hypothetical $c(1, \pi, Ad)$ introduced above, modified to account for the Γ-factors. Write $\tilde{\Lambda}(1, Ad(\rho_{p,\pi})) = \frac{\Lambda(1, Ad(\rho_{p,\pi}))}{c^*(1,\pi,Ad)}$. Combining Expectation 4.4 with Hypothesis 4.3, we obtain the following expression for the denominator of the Ichino-Ikeda conjecture.

$$Tam(Ad(\rho_{p,\pi}))\tilde{Q}(\pi)\mathcal{O} = \tilde{\Lambda}(1, Ad(\rho_{p,\pi}))\mathcal{O}. \tag{4.4}$$

Here we are assuming $\tilde{Q}(\pi)$ generates the congruence ideal $C(\pi)$. The next question is the following:

Question 4.5 (1) Can one actually expect an identity of algebraic numbers:

$$Tam(Ad(\rho_{p,\pi}))\tilde{Q}(\pi) = \tilde{\Lambda}(1, Ad(\rho_{p,\pi}))? \tag{4.5}$$

(2) As π varies in a p-adic family, do the expressions (4.5), suitably p-stabilized, p-adically interpolate into an identity of analytic functions of π?

By "p-stabilized" I mean, here and below, that the local factors at p and at ∞ have been modified according to the recipe provided by Coates in [2] for L-functions of motives that are ordinary at p, or more generally by Perrin-Riou and others. The content of Question 4.5 (2) corresponds to the conjecture first formulated by Greenberg in [10] as a general framework for Hida's construction of p-adic L-functions for ordinary families. In view of Expectation 4.4, Question 4.5 (2) is a version of the Main Conjecture of Iwasawa theory, at least for families satisfying Panchishkin's condition on p-adic valuations of Frobenius eigenvalues [28], and was again formulated in [10]. The factorization of the left-hand side has the following meaning: if the p-adic L-function on the right hand side of (1) is meant to measure all congruences between π and other automorphic forms, the term Tam on the left accounts for level-raising congruences at S, while the term $\tilde{Q}(\pi)$ accounts for congruences without changing the level. As mentioned above, the formulation has to be modified if level-lowering is impossible on the group G for local reasons.

For $n = 2$, a version of this question was answered in the affirmative in an unpublished paper [34] of Eric Urban (in many cases); π is allowed to vary in a Hida family, and the L-function is twisted by cyclotomic Dirichlet characters.

4.2 Local zeta integrals

Let v be a finite prime in S, not dividing p. Write $G_v = G(F_v^+)$, $G'_v = G(F_v^+)$. Although it has not been proved, it seems to be generally believed that, for any $f_v \in \pi_v$, $f'_v \in \pi'_v$, rational over $E(\Pi_v, \Pi'_v)$, there is a polynomial $P(f_v, f'_v, T) \in E(\Pi, \Pi')[T]$ such that

$$Z_v(f_v, f'_v) = P\left(f_v, f'_v, q_v^{\frac{1}{2}+s}\right)|_{s=\frac{1}{2}} \cdot \frac{L(\frac{1}{2}, \pi_v \times \pi'_v)}{L(1, \pi_v, Ad)L(1, \pi'_v, Ad)}. \tag{4.6}$$

Moreover, I would expect it to be possible to find f_v and f'_v, rational over $E(\Pi_v, \Pi'_v)$, such that $P(f_v, f'_v, T) = 1$, as in the usual adelic theory of zeta functions.

Question 4.6 Suppose π_v and π'_v have p-integral models and admit no non-trivial p-adic deformations. Can one find integral vectors f_v, f'_v such that $P_v(f_v, f'_v, q_v^{\frac{1}{2}+s})|_{s=\frac{1}{2}} = 1$?

The meaning of the terms in the first sentence is left deliberately vague. Integral models are understood in the sense of [20] and p-adic deformations are understood in the sense of [8].

I expect the answer to 4.6 to be affirmative, and I expect π_v and π'_v to have p-integral models when π and π' are coherent cohomological of level prime to p, or cohomological of any level. If π_v and π'_v have (infinitesimal) p-adic deformations, there's no reason to assume there are integral f_v and f'_v that trivialize the variable part of the local zeta integral. In other words, as f_v and f'_v run over integral vectors, the minimal p-adic valuation $def(\pi_v, \pi'_v)$ of $P_v(f_v, f'_v, q_v^{\frac{1}{2}+s})|_{s=\frac{1}{2}}$ may well be positive.

Suppose S contains no prime dividing p. Define the local defect $def_{loc}(\pi, \pi')$ of (π, π') to be the sum over $v \in S$, of $def(\pi_v, \pi'_v)$. Write val_p for the p-adic valuation.

Question 4.7 Suppose G_v and G'_v are quasi-split at all finite primes $v \in S$, and S contains no prime dividing p. Is $val_p(|\tilde{L}_{can}(f, f')|^2) \geq def_{loc}(\pi, \pi')$ for all $(f, f') \in \pi \times \pi'$?

This question is meant to express the following idea – that all local congruences between $\otimes_{v\in S}\pi_v \times \pi'_v$ and other representations of $\prod_{v\in S} G_v$ are realized in the module of p-integral automorphic forms on $G \times G'$. I pose the question not because I believe the answer is necessarily affirmative but in order to focus attention on the relevant phenomena. It may be that the answer is positive provided the images of the corresponding Galois representations are sufficiently large, so that the Taylor-Wiles method applies. Or it may become positive if one adds enough auxiliary primes to S. Alternatively, it is quite likely that the p-adic valuations of local zeta integrals are reflected in the denominators as well as the numerators of the period side of the Ichino-Ikeda formula, and there may be no simple way to separate their contributions. I would especially expect this to be the case if one drops the quasi-split assumption.

If $\pi \times \pi'$ is ramified at p, the p-adic local Langlands correspondence comes into play, and I don't know what to expect. Section 5 represents a first attempt to address this question.

4.3 Rankin-Selberg L-functions

We continue to work in the setting of the Ichino-Ikeda conjecture; in particular, the pair (π, π') is assumed to have a base change to a *coherent* pair (Π, Π') and satisfies the local Gan-Gross-Prasad constraints 1.6. In particular, the global sign of the functional equation is always $+1$.

Assuming we can relate the denominators of the two sides of (1.1) as in (4.4), or even (4.5), applied to Π' as well as to Π, we derive the consequences for the numerator. The term $Tam(Ad(\rho_{p,\pi}))$ is a product of factors at primes in $S(\pi)$ (including archimedean primes), so it is appropriate to define

$$Z^{Tam,S}(\pi, \pi'; f, f') = Tam(Ad(\rho_{p,\pi}))^{-1} \prod_{v\in S} Z_v(f_v, f'_v);$$

as before, this definition depends on the chosen factorizations $\pi \xrightarrow{\sim} \otimes'_v \pi_v$, $\pi' \xrightarrow{\sim} \otimes'_v \pi'_v$. The Ichino-Ikeda formula then implies, under (4.4), and assuming $p > 2$,

$$C_0 \tilde{\Delta}^S_{U(n)} \tilde{Z}^{Tam,S}(\pi, \pi'; f, f') \tilde{L}^S\left(\frac{1}{2}, \Pi \times \Pi'\right) \mathcal{O} = |\tilde{L}_{can}(f, f')|^2 \mathcal{O}. \tag{4.7}$$

(An adjustment is needed to account for the presence of Γ-factors in (4.4). This is incorporated into the archimedean factors of $\tilde{Z}^{Tam,S}$. We need not dwell on the details.)

Again, we can ask about more precise normalizations.

Question 4.8 Suppose G_v and G'_v are quasi-split at all finite primes $v \in S$, and S contains no prime dividing p.

(1) Can one actually expect an identity of algebraic numbers if the periods are appropriately chosen:

$$C_0 \tilde{\Delta}^S_{U(n)} \tilde{Z}^{Tam,S}(\pi, \pi'; f, f') \tilde{L}^S\left(\frac{1}{2}, \Pi \times \Pi'\right) = |\tilde{L}_{can}(f, f')|^2? \tag{4.8}$$

(2) As π varies in a p-adic family, with the local components of f and f' fixed at primes in S, do the expressions (4.8), suitably p-stabilized, p-adically interpolate into an identity of analytic functions of $\pi \times \pi'$ (and the p-stabilizations)?

(3) In particular, does the central value of the p-adic L-function factorize as the product of two p-adic analytic functions, corresponding to $\tilde{L}_{can}(f, f')$ and its complex conjugate?

These questions are subject to the same caveats as those that followed Question 4.7; one doesn't really know how the local zeta integrals reflect the arithmetic of the motives attached to Π and Π', even under the quasi-split hypothesis.

Factorizations of p-adic L-functions as in question (3) have been obtained in a number of situations in low dimension; see for example [3,18,27,33]. Whenever there is a factorization of the central value of the complex L-function – references [33] and [27] use factorizations due to Waldspurger—there is a clear route to constructing a p-adic factorization as well. The parameters in the hypothetical factorizable p-adic L-function are sometimes called "anticyclotomic" variables. The following question has nothing to do with the Ichino-Ikeda conjecture.

Question 4.9 Does the factorization of the central value of the standard L-function of a unitary group in [19] admit a p-adic interpolation?

Returning to the situation of Question 4.8, we arrive at a p-adic analogue of Question 2.1. Because the role of the local zeta integrals is unclear, we are not even confident enough to formulate an Expectation.

Question 4.10 Suppose G_v and G'_v are quasi-split at all finite primes $v \in S$, and S contains no prime dividing p. In what follows, we assume (2) and (3) of Question 4.8 have affirmative answers.

(1) Suppose each of π and π' varies in a connected ordinary family, or more generally, in a family that satisfies the Panchishkin condition of [28]. When can one assume the p-adic L-function $L_p(s_{cent}, \pi \times \pi')$ to be divisible by p; i.e., to have non-trivial Iwasawa μ-invariant? Here s_{cent} is the central value, corresponding to $s = \frac{1}{2}$ in the Ichino-Ikeda conjecture.
(2) Let f and f' be modular forms varying over families as in (1); in other words, we suppose f and f' are functions of the p-adic parameters of π and π' (so that one has picked out a vector f from each π in the family). When can one assume the normalized p-adic pairing $\tilde{L}_{can}(f, f')$ to be divisible by p?
(3) Are the answers to questions (1) and (2) determined entirely by local considerations, and by the residual p-adic Galois representations attached to π and π'?

Question 4.10 has to do with non-vanishing modulo p of normalized special values of L-functions. It would be nice to be able to say that the hypothetical anticyclotomic p-adic L-function has trivial μ-invariant as a function of an (ordinary or Panchishkin) family. Alternatively, one can always divide by a power of p and replace $L_p(s_{cent}, \pi \times \pi')$ by an element of an appropriate Hida-type Hecke algebra that is not divisible by p. One would then like to say that the result is the correctly normalized p-adic L-function; there is no rule that allows us to determine when a p-adic interpolation of normalized special values of complex L-functions is correctly normalized, because there is no universal rule for choosing integral periods. At the same time, one would like to say that the linear forms on the space of p-integral $f \in \pi$ spanned by the $\tilde{L}_{can}(f, f')$, with f' p-integral in varying π', span the lattice of integral linear forms. But the local defect $def_{loc}(\pi, \pi')$ discussed in the previous section, even corrected by dividing by the Tamagawa factors as in (4.7), may contribute extraneous powers of p.

If one assumes the appropriate Iwasawa-Greenberg Main Conjecture, one would expect the μ-invariant of $L_p(s_{cent}, \pi \times \pi')$ to be reflected in the corresponding Selmer group. The residual Galois representations attached to π and π' are constant on the p-adic families, and may contribute a non-trivial μ-invariant that is likewise constant in the family. Thus we weaken the conditions.

Question 4.11 (1) Under the hypotheses of Question 4.8, suppose the p-adic L-functions $L_p(s_{cent}, \pi \times \pi')$ exist and are normalized correctly. Fix π and let π' vary among ALL (ordinary or Panchishkin) p-adic families with fixed first name, as in Question 2.1. Is it possible that all the $L_p(s_{cent}, \pi \times \pi')$ are divisible by p?
(2) Same as (1), without the restriction on the first name.

Remark 4.12 We have neglected to address another aspect of the Ichino-Ikeda conjecture, namely the role of the factors $C_0\tilde{\Delta}^S_{U(n)}$. The function $\Delta_{U(n)}$ is a product of special values of abelian L-functions of F^+, and its p-adic valuation is therefore related (by the Main Conjecture for totally real fields, proved by Wiles) to orders of ideal class groups of F. This factor is independent of the choice of the (coherent) pair (Π, Π') and it is inconceivable that it is relevant to all the Selmer groups that occur. I therefore assume that the presence of the factors $C_0\tilde{\Delta}^S_{U(n)}$ should be explained by a choice of normalization of measures and pairings, and with the correct normalizations these factors will not affect the answers to Questions 4.10 and 4.11.

Remark 4.13 In unpublished work, Eric Urban has shown how to attach a p-adic distribution on $U(n-1)$ to the restriction of an automorphic representation of $U(n)$, and thus to define a p-adic version of the pairing L^{can}. It is an open problem to determine whether or not the ordinary projection of this distribution is non-trivial.

5 p-adic continuity of invariant linear forms

In this section we assume V (and therefore V') totally definite. Thus the representations Π_∞ and Π'_∞ are always *finite-dimensional*; this will be important in what follows. We fix Π and Π' such that $L(\frac{1}{2}, \Pi \times \Pi') \neq 0$ and the (unique) pair π, π' in $\Phi(\Pi, V) \times \Phi(\Pi', V')$ such that $L_{can} : \pi \times \pi' \rightarrow \mathbb{C}$ does not vanish identically. In what follows the superscript $^\vee$ designates the contragredient; $\pi^\vee$, and $\pi'^{,\vee}$ are the complex conjugates of π and π', respectively. Let $f, f', f^\vee$, and $f'^{,\vee}$ vary among vectors in $\pi, \pi', \pi^\vee$, and $\pi'^{,\vee}$ respectively, of the form

$$f = f_\infty \otimes f_p \otimes f^{p,\infty}; \quad f' = f'_\infty \otimes f'_p \otimes f'^{,p,\infty} \tag{5.1}$$

with

$$f_\infty \in \pi_\infty = \otimes_{v|\infty}\pi_v; \quad f_p \in \pi_p = \otimes_{v|p}\pi_v$$

and with similar definitions and factorizations for $f', f^\vee$, and $f'^{,\vee}$. We fix the factors $f^{p,\infty}$, $f'^{,p,\infty}$, $f^{\vee,p,\infty}$, $f^{\vee,\prime,p,\infty}$ for the remainder of the paper, with the property that

Hypothesis 5.1 $L_{can}(f, f') \cdot L_{can}(f^\vee, f'^{,\vee}) \neq 0$, *for some* $f, f', f^\vee, f'^{,\vee}$ *satisfying* (5.1).

A more complete version of the Ichino-Ikeda conjecture lets $f, f', f^\vee$, and $f'^{,\vee}$ vary freely:

$$\frac{L_{can}(f, f')L_{can}(f^\vee, f'^{,\vee})}{\langle f, f^\vee\rangle_2\langle f', f'^{,\vee}\rangle_2} = 2^{-\beta}C_0\Delta^S_{U(n)}\prod_{v\in S} Z_v(f_v, f'_v, f^\vee_v, f'^{,\vee}_v)P^S\left(\frac{1}{2}, \pi, \pi'\right) \tag{5.2}$$

Here each $Z_v(f_v, f'_v, f^\vee_v, f'^{,\vee}_v) = \frac{I(f_v, f'_v, f^\vee_v, f'^{,\vee}_v)}{\langle f_v, f^\vee\rangle_v\langle f'_v, f'^{,\vee}\rangle_v}$ is a normalized local zeta integral, where $I(f_v, f'_v, f^\vee_v, f'^{,\vee}_v)$ is the unnormalized local zeta integral defined as in (1.2), with $c(f_v)$ and $c(f'_v)$ replaced by the matrix coefficients

$$c(f_v, f_v^\vee)(g) = \langle \pi_v(g_v) f_v, f_v^\vee \rangle_v; c(f_v', f'^{,\vee})(g_v') = \langle \pi_v'(g_v') f_v', f_v'^{,\vee} \rangle_v, \tag{5.3}$$

and it is assumed that

$$c(f_v, f_v^\vee)(1) \cdot c(f_v', f_v'^{,\vee})(1) \neq 0$$

so that the normalization is well-defined.

With these assumptions, L_{can} restricts to pairings

$$\pi_{p,\infty} \otimes \pi_{p,\infty}' \to \mathbb{C}; \quad \pi_{p,\infty}^\vee \otimes \pi_{p,\infty}'^{,\vee} \to \mathbb{C} \tag{5.4}$$

where $\pi_{p,\infty} = \pi_p \otimes \pi_\infty$ and similarly for the other three. Note that π_∞ and π_∞' and their contragredients are finite-dimensional algebraic representations. We may assume the four spaces on the left-hand side of (5.4) are defined over the CM field $E = E(\Pi, \Pi')$ (as algebraic representations of G and G' in the case of π_∞ and π_∞') and choose constants $P(\pi, \pi')$, such that $\tilde{L}_{can} := P(\pi, \pi')^{-1} L_{can} : \pi \otimes \pi' \to \mathbb{C}$ takes values in E; for the map $L_{can} : \pi^\vee \otimes \pi'^{,\vee} \to \mathbb{C}$ we can take $P(\pi^\vee, \pi'^{,\vee})$ to be the complex conjugate of $P(\pi, \pi')$. Since V and V' are totally definite, the constants $Q(\pi)$ and $Q(\pi')$ can be taken to be in E, and in fact we may normalize them both to equal 1.

Write $f_{p,\infty} = f_\infty \otimes f_p \in \pi_{p,\infty}$, $f_{p,\infty}' = f_\infty' \otimes f_p' \in \pi_{p,\infty}'$ and likewise for the other two factors, and define the unnormalized local zeta integral

$$I_{p,\infty}(f_{p,\infty}, f_{p,\infty}', f_{p,\infty}^\vee, f_{p,\infty}'^{,\vee}) = \prod_{v|p} I_v(f_v, f_v', f_v^\vee, f_v'^{,\vee}) \times \prod_{v|\infty} I_v(f_v, f_v', f_v^\vee, f_v'^{,\vee}),$$

with notation as in (1.2). Applying the Ichino-Ikeda formula, and recalling that we have set $Q(\pi) = Q(\pi') = 1$ in (1.4), we find

$$L_{can}(f, f') L_{can}(f^\vee, f'^{,\vee}) = I_{p,\infty}(f_{p,\infty}, f_{p,\infty}', f_{p,\infty}^\vee, f_{p,\infty}'^{,\vee}) \times C \tag{5.5}$$

where

$$C = 2^{-\beta} C_0 \Delta_{U(n)}^S \prod_{v \in S'} Z_v(f_v, f_v') P^S\left(\frac{1}{2}, \pi, \pi'\right)$$

is a constant for the purposes of the present section. Here S' is the subset of S consisting of places not dividing p or ∞. Moreover, we may assume (possibly after modifying the local pairing at archimedean primes, and compensating elsewhere) that $C \in E$. Certainly the left-hand side of (5.5) is algebraic, and with this normalization we now know that $I_{p,\infty}(f_{p,\infty}, f_{p,\infty}', f_{p,\infty}^\vee, f_{p,\infty}'^{,\vee})$ is algebraic.

Write $G_{p,\infty} \times G_p'$ for $\prod_{v|p} G_v \times G_v' = (R_{F^+ \otimes \mathbb{Q}_p/\mathbb{Q}_p} G \times G')(\mathbb{Q}_p)$. Choose an embedding $\iota_p : E \hookrightarrow K$ where K is a finite extension of $\mathbb{Q}_p$, and use ι_p to view $\pi_\infty \otimes \pi_\infty'$ as a K-valued algebraic representation of $G \times G'$, and thus of $G_{p,\infty} \times G_p'$. We may then view $\pi_{p,\infty} \otimes \pi_{p,\infty}'$ as an irreducible *locally algebraic representation* of $G_p \times G_{p,\infty}'$ with values in the non-archimedean local field K. The set $\mathcal{C}(\sigma)$ of admissible Banach space completions of a locally algebraic representation σ of a reductive group G over $\mathbb{Q}_p$ has been studied in connection with the p-adic local Langlands program, notably in [1,7,29]. The set of such completions is well understood for $G = GL(2, \mathbb{Q}_p)$ but not in other cases. Nevertheless, we may assume for the sake of argument that it has the structure of a p-adic analytic space of some sort.

Question 5.2 For which admissible Banach completions $C_\gamma(\pi_{p,\infty} \otimes \pi_{p,\infty}') \in \mathcal{C}(\pi_{p,\infty} \otimes \pi_{p,\infty}')$ does L_{can} extend continuously?

We fix $f^{\vee}_{p,\infty}$ and $f'^{,\vee}_{p,\infty}$.

Principle 5.3 *Assuming the linear form*

$$L(f^{\vee}_{p,\infty}, f'^{,\vee}_{p,\infty}) : \pi_{p,\infty} \otimes \pi'_{p,\infty} \to K, \;\; f_{p,\infty} \times f'_{p,\infty} \mapsto \langle f_{p,\infty}, f^{\vee}_{p,\infty} \rangle \langle f'_{p,\infty}, f'^{,\vee}_{p,\infty} \rangle \tag{5.6}$$

extends continuously to the completion $C_{\gamma}(\pi_{p,\infty} \otimes \pi'_{p,\infty})$, L_{can} extends continuously to $C_{\gamma}(\pi_{p,\infty} \otimes \pi'_{p,\infty})$ if and only if the linear form

$$f_{p,\infty} \times f'_{p,\infty} \mapsto I_{p,\infty}(f_{p,\infty}, f'_{p,\infty}, f^{\vee}_{p,\infty}, f'^{,\vee}_{p,\infty})$$

extends continuously.

This is clearly a consequence of the Expression 5.5. Nevertheless, there is no reason a priori to assume (5.6) extends continuously when $f^{\vee}_{p,\infty}$ and $f'^{,\vee}_{p,\infty}$ are locally algebraic vectors. The admissible Banach space completion $C_{\gamma}(\pi_{p,\infty} \otimes \pi'_{p,\infty})$ has a dual (more than one, actually) that has no reason to be admissible, nor to contain the locally algebraic vectors in the contragredient of $\pi_{p,\infty} \otimes \pi'_{p,\infty}$. The statement of the principle is meant to suggest a vague analogy with (2.8) in the complex analytic theory. It is also meant to be a possible starting point for a study of Banach space completions of locally algebraic representations, and their reductions modulo p, by means of restrictions (possibly derived) to large subgroups. Some results in the case where $G = GL(2, \mathbb{Q}_p)$ and G' is a non-split maximal torus have been obtained by Morra in his thesis; but the general pattern is by no means clear.

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