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# Root Sets of Polynomials and Power Series with Finite Choices of Coefficients

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**Abstract** Given $H \subseteq \mathbb{C}$ two natural objects to study are the set of zeros of polynomials with coefficients in $H$,

$$\left\{z \in \mathbb{C} : \exists k > 0,\ \exists (a_n) \in H^{k+1}, \sum_{n=0}^{k} a_n z^n = 0\right\},$$

and the set of zeros of a power series with coefficients in $H$,

$$\left\{z \in \mathbb{C} : \exists (a_n) \in H^{\mathbb{N}}, \sum_{n=0}^{\infty} a_n z^n = 0\right\}.$$

In this paper, we consider the case where each element of $H$ has modulus 1. The main result of this paper states that for any $r \in (1/2, 1)$, if $H$ is $2\cos^{-1}(\frac{5-4|r|^2}{4})$-dense in $S^1$, then the set of zeros of polynomials with coefficients in $H$ is dense in $\{z \in \mathbb{C} : |z| \in [r, r^{-1}]\}$, and the set of zeros of power series with coefficients in $H$ contains the annulus $\{z \in \mathbb{C} : |z| \in [r, 1)\}$. These two statements demonstrate

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quantitatively how the set of polynomial zeros/power series zeros fill out the natural annulus containing them as $H$ becomes progressively more dense.

**Keywords** Root set · Littlewood polynomials · Unimodular polynomials

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## 1 Introduction

Let $H \subseteq \mathbb{C}$ be a finite set. Given such a $H$ we define the root set of polynomials with coefficients in $H$ to be:

$$R(H) := \left\{ z \in \mathbb{C} : \exists k > 0,\ \exists (a_n) \in H^{k+1}, \sum_{n=0}^{k} a_n z^n = 0 \right\}.$$

Similarly, we define the root set of power series with coefficients in $H$ to be

$$R^*(H) := \left\{ z \in \mathbb{C} : \exists (a_n) \in H^{\mathbb{N}}, \sum_{n=0}^{\infty} a_n z^n = 0 \right\}.$$

The study of the sets $R(H)$ and $R^*(H)$ can be dated back to Littlewood [6] who studied the case where $H = \{-1, 1\}$. Since then many related works have appeared, most notable amongst these are the number theoretic results of Beaucoup, Borwein, Boyd and Pinner [3], and Borwein, Erdélyi and Littmann [4], who studied the distribution of roots and multiple roots. Related work also appeared in Bousch [5], where it was shown that $R(\{-1, 1\})$ is dense in $\{z : |z|^4 \in [1/2, 2]\}$. In Shmerkin and Solomyak [8] some measure theoretic and topological properties of $R(\{-1, 0, 1\})$ are studied in detail.

In what follows, we will adopt the following notational conventions:

$$S^r := \left\{ z \in \mathbb{C} : |z| = r \right\},\ B(z, r) := \left\{ z' \in \mathbb{C} : |z' - z| < r \right\},$$

and given some interval $I$ in $\mathbb{R}$ let

$$A_I := \{ z \in \mathbb{C} : |z| \in I \}.$$

In this paper, we focus on the case where $H$ is a subset of the unit circle $S^1$. Under this assumption it is straightforward to show that

$$R(H) \subseteq A_{[1/2,2]} \text{ and } R^*(H) \subseteq A_{[1/2,1)}.$$

Intuitively, one might expect that if we allowed $H$ to become a progressively more dense subset of $S^1$, then $R(H)$ and $R^*(H)$ would begin to fill out their respective annuli. The main result of this paper shows that this intuition is correct.

Before stating this result we need to define a metric on $S^1$ to properly quantify the density of $H$. Given $e^{i\theta}, e^{i\theta'} \in S^1$ let $d(e^{i\theta}, e^{i\theta'}) = \min\{|\theta - \theta'|, |2\pi - (\theta - \theta')|\}$. This metric measures the interior angle of the sector of $S^1$ determined by the two radii $e^{i\theta}$ and $e^{i\theta'}$.

**Theorem 1.1** *Fix $r \in (1/2, 1)$. Suppose $H \subseteq S^1$ is $2\cos^{-1}(\frac{5-4r^2}{4})$-dense. Then $A_{[r,1)} \subseteq R^*(H)$ and $R(H)$ is dense in $A_{[r,r^{-1}]}$.*

The sets $R(H)$ and $R^*(H)$ are related by the following formula.

**Proposition 1.2** *Let $H \subseteq \mathbb{C}$ be any finite set, then the following relations hold:*

$$R(H) = \frac{1}{R(H)},$$

*and*

$$\overline{R(H)} \cap B(0, 1) = R^*(H) \cap B(0, 1).$$

In the statement of Proposition 1.2, $\overline{A}$ denotes the closure of a set $A$, and $\frac{1}{A}$ denotes the set $\{z \in \mathbb{C} : z^{-1} \in A\}$.

*Proof* Given $z \in \mathbb{C}$ suppose there is polynomial $P \in H[x]$ such that,

$$P(z) = \sum_{n=0}^{k} a_n z^n = 0.$$

We can construct another polynomial $Q \in H[x]$ such that $Q(1/z) = 0$. Just consider $Q(x) = x^k P(\frac{1}{x})$ with $k = \deg P$. Therefore, whenever $z \in R(H)$ we also have $1/z \in R(H)$. This proves our first relation.

Now we shall show that,

$$\overline{R(H)} \cap B(0, 1) = R^*(H) \cap B(0, 1).$$

Without loss of generality we can assume that $H \subseteq \{z : |z| \leq 1\}$. If $z^* \in \overline{R(H)} \cap B(0, 1)$ then we can find a sequence $(z_i) \in R(H)^{\mathbb{N}}$ with:

$$z_i \to z^*.$$

Moreover, since $z^* \in B(0, 1)$ there exists a positive number $M$ such that $1 < M < \frac{1}{|z^*|}$. Now let us consider any polynomial in $H[x]$

$$P(x) = \sum_{n=0}^{k} a_n x^n.$$

The following result holds for $|z_i| \leq M|z^*|$

$$\begin{aligned}|P(z^*) - P(z_i)| &\leq \sum_{n=0}^{k} |a_n||(z^*)^n - z_i^n| \\ &= \sum_{n=0}^{k} |a_n||z^* - z_i||(z^*)^{n-1} + (z^*)^{n-2}z_i + ... + z_i^{n-1}| \\ &\leq \sum_{n=0}^{k} |a_n||z^* - z_i|n(M|z^*|)^{n-1} \\ &\leq |z^* - z_i| \sum_{n=0}^{k} |a_n|n(M|z^*|)^{n-1}.\end{aligned}$$

Since $|a_n| \leq 1$ and $M|z^*| < 1$ the latter summation can be bounded uniformly with respect to $k$, namely:

$$\sum_{n=0}^{k} |a_n|n(M|z^*|)^{n-1} \leq C,$$

where $C > 0$ is a constant that only depends upon $z^*$.

Each $z_i$ is the root of some polynomial $P_i \in H[x]$, in which case by the above, for $i$ sufficiently large we have

$$|P_i(z^*)| = |P_i(z^*) - P_i(z_i)| \leq C|z^* - z_i|. \tag{1}$$

For the sequence $(P_i)$ there is either a uniform upper bound for the degrees of the $P_i$, or there exists a subsequence along which the degrees tend to infinity. In the first case, there must exist a polynomial $Q \in H[x]$ and a subsequence $(P_{i_j})$ such that $P_{i_j} = Q$ for all $i_j$. By (1) we must then have $Q(z^*) = 0$. Suppose deg $Q = L$, then

$$T(x) = Q(x) \sum_{n=0}^{\infty} x^{n(L+1)}$$

is a power series with digits in $H$. For this particular power series we clearly have $T(z^*) = 0$. Therefore, in the first case we have $z^* \in R^*(H)$. Now suppose there exists a subsequence $(P_{i_j})$ such that deg $P_{i_j} \to \infty$. Via a diagonalisation argument, one can assume without loss of generality that there exists a sequence $(a_n) \in H^{\mathbb{N}}$ and an increasing sequence of natural numbers $(l_n)$, such that for all $i_j \geq l_n$ the coefficient of the degree $n$ term of $P_{i_j}$ is $a_n$. In other words, as the $i_j$ become sufficiently large the lower order terms of the $P_{i_j}$'s start to coincide. It follows from (1) then that for this sequence $(a_n)$ we must have

$$\sum_{n=0}^{\infty} a_n (z^*)^n = 0.$$

Therefore, $z^* \in R^*(H)$ and $\overline{R(H)} \cap B(0,1) \subseteq R^*(H) \cap B(0,1)$.

Now suppose $z^* \in R^*(H) \cap B(0,1)$. Then there is a sequence $(a_n) \in H^{\mathbb{N}}$ such that

$$\sum_{n=0}^{\infty} a_n (z^*)^n = 0.$$

This series is absolutely and uniformly convergent in $B(0,c)$ for any $0 < c < 1$. Since $z^* \in B(0,1)$ it is contained in one of these sets for $c$ sufficiently close to 1. We see that the function:

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is holomorphic on the interior of the unit disc, and therefore, the roots of $P$ must form a discrete set. Since $z^*$ belongs to the root set of $P$ there must exist $r > 0$ such that

$$\{z \in \mathbb{C} : P(z) = 0\} \cap B(z^*, r) = \{z^*\}.$$

Now, suppose that $z^* \notin \overline{R(H)}$, then there exists a ball $B(z^*, r')$ such that $B(z^*, r') \subseteq B(z^*, r)$ and

$$R(H) \cap \overline{B(z^*, r')} = \emptyset.$$

We can then consider the following integral with $P_N(x) = \sum_{n=0}^{N} a_n x^n$

$$I_N = \int_{\partial B(z^*, r')} \frac{P_N'(x)}{P_N(x)} \text{dx}.$$

By our conditions on $r'$ we see that $P_N$ has no zeros in $\overline{B(z^*, r')}$ for all $N \in \mathbb{N}$. Therefore, by the argument principle (see [1, p. 152]) we must have $I_N \equiv 0$. One can also assume that $r'$ is sufficiently small so that $P_N$ converges to $P$ absolutely and uniformly. Therefore,

$$0 = \lim_{N \to \infty} I_N = \int_{\partial B(z^*, r')} \frac{P'(x)}{P(x)} \text{dx}.$$

However, it follows from another application of the argument principle, and the fact that $P(x)$ has a single zero in $\overline{B(z^*, r')}$ at $z^*$, that the above integral cannot be 0. This contradiction implies $z^* \in \overline{R(H)}$ and our proof is complete. □

It is natural to wonder whether there exists a set $H$ such that the sets $R(H)$ and $R^*(H)$ fill up their ambient annuli, that is $A_{[1/2,2]}$ and $A_{[1/2,1)}$ respectively. In fact such a $H$ cannot exist. For any $H \subseteq S^1$ there exists $z \in \mathbb{C}$ with modulus $1/2$ and $\delta > 0$, such that $R(H) \cap B(z, \delta) = \emptyset$ and $R^*(H) \cap B(z, \delta) = \emptyset$. This is because of the following simple reasoning. Since $H$ is a finite set there exists $z \in \mathbb{C}$ such that $|z| = 1/2$ and

$$|a_i + a_j z| > 1/2$$

for all $a_i, a_j \in H$. Equivalently

$$|a_i + a_j z| > \frac{|z|^2}{1 - |z|} \tag{2}$$

for all $a_i, a_j \in H$. By continuity, equation (2) holds under small perturbations of $z$. Therefore, there must exist $\delta > 0$, such that for all $z' \in B(z, \delta)$ we have

$$|a_i + a_j z'| > \frac{|z'|^2}{1 - |z'|}.$$

Since

$$\left| \sum_{n=2}^{k} a_n (z')^n \right| \leq \frac{|z'|^2}{1 - |z'|}$$

for all $(a_n) \in H^k$ and $k \in \mathbb{N}$, it follows that $z'$ cannot be the zero of a power series or a polynomial. Therefore we must have $R(H) \cap B(z, \delta) = \emptyset$ and $R^*(H) \cap B(z, \delta) = \emptyset$.

## 2 Proof of Theorem 1.1

We now turn our attention to proving Theorem 1.1. We start with the following technical proposition.

**Proposition 2.1** *Let $z \in A_{(1/2,1)}$. Suppose $H$ is $2\cos^{-1}(\frac{5-4|z|^2}{4})$-dense, then for any $z' \in \overline{B(0,2)}$ there exists $a \in H$ such that $z^{-1}(z' - a) \in \overline{B(0,2)}$.*

*Remark 2.2* It is useful to point out that the conclusion of this proposition is equivalent to:

$$\overline{B(0,2)} \subset \bigcup_{a \in H} a + z\overline{B(0,2)}.$$

*Proof* Let us start by fixing $z' \in \overline{B(0,2)}$. Consider the point $z'z^{-1}$. Clearly $z'z^{-1} \in \overline{B(0, 2|z|^{-1})}$. Let

$$S(z'z^{-1}, |z|^{-1}) := \{\omega \in \mathbb{C} : |\omega - z'z^{-1}| = |z|^{-1}\}$$

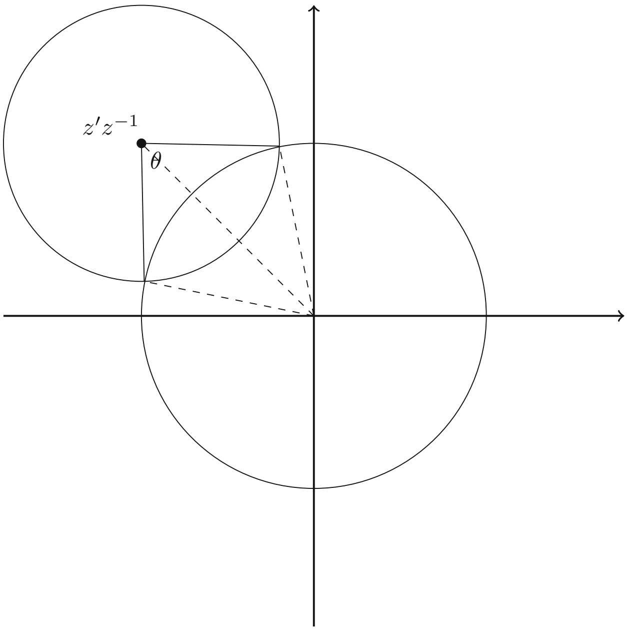


**Fig. 1** A diagram of $S^{|z|^{-1}}_{z'z^{-1}}$ intersecting $\overline{B(0,2)}$

be the circle centered at $z'z^{-1}$ with radius $|z|^{-1}$.

Since $z \in A_{(1/2,1)}$ we must have $S(z'z^{-1}, |z|^{-1}) \cap \overline{B(0,2)} \neq \emptyset$. In fact this intersection must contain an arc of $S(z'z^{-1}, |z|^{-1})$. This arc is parameterised by two radii of $S(z'z^{-1}, |z|^{-1})$ with interior angle $\theta$. See Fig. 1 for a diagram describing the intersection of $S(z'z^{-1}, |z|^{-1})$ with $\overline{B(0,2)}$. It is easy to see that the angle $\theta$ is minimised when $z'z^{-1}$ is as far from the origin as possible, i.e., when $z'$ has modulus 2. Employing elementary techniques from geometry we can see that the angle $\theta$ is at least twice the size of a particular angle of the triangle whose sides have length $|z|^{-1}$, 2, and $2|z|^{-1}$ (see Fig. 1). Therefore, we can use the well-known cosine rule from trigonometry to show that $\theta$ is always bounded below by

$$2\cos^{-1}\left(\frac{5-4|z|^2}{4}\right).$$

Since $H$ is $2\cos^{-1}(\frac{5-4|z|^2}{4})$-dense as a subset of $S^1$, there must exist $a \in H$ such that $z'z^{-1} - az^{-1}$ is contained in the arc of $S(z'z^{-1}, |z|^{-1})$ which intersects $\overline{B(0,2)}$. In particular, for this choice of $a$ we have $z^{-1}(z' - a) \in \overline{B(0,2)}$. □

Theorem 1.1 now follows almost immediately from Proposition 2.1.

*Proof of Theorem 1.1* By the relations given in Proposition 1.2 to prove Theorem 1.1 it is sufficient to only prove the statement relating to $R^*(H)$. Fix $r \in (1/2, 1)$ and let $H$ be a $2\cos^{-1}(\frac{5-4r^2}{4})$-dense subset of $S^1$. Note that $H$ is automatically $2\cos^{-1}(\frac{5-4|z|^2}{4})$-dense for any $z \in A_{[r,1)}$. So we can apply Proposition 2.1 for any $z \in A_{[r,1)}$. Let us now fix $z \in A_{[r,1)}$ and apply Proposition 2.1 when $z' = 0$. So there exists $a_0 \in H$ such that $x_0 := z^{-1}(-a_0) \in \overline{B(0,2)}$. Rearranging yields

$$0 = a_0 + x_0 z.$$

Applying Proposition 2.1 again with $x_0$ in the place of $z'$ yields $a_1$ and $x_1 := z^{-1}(x_0 - a_1)$, such that $x_1 \in \overline{B(0,2)}$ and

$$0 = a_0 + a_1 z + x_1 z^2. \tag{3}$$

One can then apply Proposition 2.1 with $z' = x_1$ . Repeating this procedure indefinitely yields a sequence $(a_n)$ and $(x_n)$ such that $x_{n+1} = z^{-1}(x_n - a_{n+1})$ for all $n \in \mathbb{N}$. The terms in $(x_n)$ remain in $\overline{B(0,2)}$. Therefore, we are able to repeatedly apply the substitution $x_{n+1} = z^{-1}(x_n - a_{n+1})$ in (3) and we obtain

$$0 = \sum_{n=0}^{\infty} a_n z^n.$$

Therefore, $z \in R^*(H)$. Since $z$ was arbitrary we have $A_{[r,1)} \subseteq R^*(H)$. □

The proof of Theorem 1.1 was based upon ideas from $\beta$-expansions. The argument given relied upon adapting methods from [2,7]. The proof can easily be adapted to show that under the hypothesis of the theorem, for every $z' \in \overline{B(0,2)}$ there exists $(a_n) \in H^{\mathbb{N}}$ such that $\sum_{n=0}^{\infty} a_n z^n = z'$.

## 3 Some Further Problems

There are some more challenging problems related to root sets $R(H)$, $R^*(H)$. We mentioned in the beginning of this paper that there exist various results of multiple roots [3,4]. We say that a $z \in \mathbb{C}$ is a multiple root of a holomorphic function $f$ of order $k$ if for all integers $i = 0, 1, 2, \ldots, k$

$$f^{(i)}(z) = 0.$$

Adopting the notation in this paper, we can define for any integer $k \geq 0$:

$$R_k(H) := \left\{ z \in \mathbb{C} : \exists k > 0,\ \exists (a_n) \in H^{k+1}, \right.$$
$$\left. P(w) = \sum_{n=0}^{k} a_n w^n,\ z \text{ is a } k\text{-th order root of } P(w) \right\}.$$

$$R_k^*(H) := \Bigg\{ z \in \mathbb{C} : \exists (a_n) \in H^{\mathbb{N}}, \\ P(w) = \sum_{n=0}^{\infty} a_n w^n,\ z \text{ is a } k\text{-th order root of } P(w) \Bigg\}.$$

Not so much has been studied about the above multiple root set, some partial results can be found in [8]. We can, for example, consider the following questions:

- Are $R_k(H)$, $R_k^*(H)$ dense in any non-degenerate annulus?
- What about the connectedness and path-connectness of $R_k(H)$, $R_k^*(H)$?
- What can we say about the boundary of $R_k(H)$, $R_k^*(H)$?

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