Comput. Methods Funct. Theory (2015) 15:655–687
DOI 10.1007/s40315-015-0139-6

Fourier–Mellin Transforms for Circular Domains

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Received: 18 January 2015 / Revised: 15 June 2015 / Accepted: 29 June 2015 /
Published online: 3 September 2015

Abstract Generalized Fourier–Mellin transforms for analytic functions defined in simply connected circular domains are derived. Circular domains are taken to be those with boundaries that are a finite union of circular arcs, including straight line edges. The results are an extension to circular domains of the generalized Fourier transforms for convex polygons (having only straight line edges) derived by Fokas and Kapaev (IMA J Appl Math 68:355–408, 2003). First, a new, elementary derivation of the latter result for polygons is given based on Cauchy's integral formula and a spectral representation of the Cauchy kernel. This rederivation extends in a natural way to the case of circular domains once an adapted spectral representation of the Cauchy kernel is established. Domains with boundaries that are a combination of circular arc and straight line edges can be treated similarly. The newly derived transforms are generalizations of the classical Fourier and Mellin transforms to general circular domains. It is shown by example how they can be used to solve boundary value problems for Laplace's equation in such domains. The notions of *spectral matrix* and *fundamental contour*, which arise naturally in the formulation, are also introduced.

Keywords Transform method · Circular domains · Fourier transform · Mellin transform

Mathematics Subject Classification 34K10 · 34B27 · 34M25

Communicated by Nikos Stylianopoulos.

Dedicated to Ed Saff, for lending his leadership and scholarship in equal measure.

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1 Introduction

The last two decades have seen the emergence of a set of mathematical ideas and techniques which have become known collectively as the *unified transform method*. It has been pioneered by Fokas and collaborators [16,18] and it is often called the *Fokas method*. For the most recent survey of this method the reader is referred to [17]. Part of this collective involves the reappraisal of classical boundary value problems for linear partial differential equations using the lessons learned from the study of nonlinear integrable systems. While the scope of the unified transform method is broad, the present article has been inspired by novel insights that this impactful method has brought to the area of complex analysis and function theory.

Specifically, in studying mixed boundary value problems for Laplace's equation in a convex polygon, Fokas and Kapaev [14] produced what appears to be a completely novel constructive method for their solution. The heart of the method, which developed ideas set out earlier by Fokas and Gel'fand [12], is the spectral analysis of one of the equations of a certain Lax pair (by means of Riemann–Hilbert techniques) and it leads eventually to the analysis of a so-called *global relation*. Interestingly, the approach also gives rise to a new integral representation for analytic functions in a convex polygon, one that generalizes the classical Fourier transform (naturally associated to a strip geometry) to more complicated domain types. Crowdy and Davis [8] have shown, in the context of a fluid dynamics application and by direct comparison with traditional Fourier methods how, even for the simple strip (or channel) geometry, the new approach of [14] already affords analytical advantages in practice.

The result of Fokas and Kapaev [14]—who derived it from the spectral analysis of a parameter-dependent ordinary differential equation and the use of Riemann–Hilbert methods—has been rederived from various other perspectives, including an approach via the fundamental differential form [13,15], and the consideration of fundamental solutions and Green's integral representation [24]. The key result of [14], however, is concisely stated as follows. If $f(z)$ is analytic in a bounded, simply connected, convex, N-sided polygon with sides $\{S_j | j = 1, \ldots, N\}$ then one can write

$$f(z) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{\mathcal{L}_j} \tilde{\rho}_j(k) \mathrm{e}^{\mathrm{i}kz} \, \mathrm{d}k, \tag{1}$$

where $\mathcal{L}_j$ are a special set of rays in the spectral k-plane (depending on the geometry of the polygon, and defined here in Sect. 2) and where the so-called *spectral functions* are defined as

$$\tilde{\rho}_j(k) = \int_{S_j} f(z') \mathrm{e}^{-\mathrm{i}kz'} \, \mathrm{d}z'. \tag{2}$$

These spectral functions satisfy the *global relation*

$$\sum_{j=1}^{N} \tilde{\rho}_j(k) = 0, \quad k \in \mathbb{C}. \tag{3}$$

In the context of boundary value problems for Laplace's equation for a harmonic function $q(x, y)$—taken to be the real part of some analytic function $f(z)$, say—these global relations can be used to determine the unknown boundary data for $f(z)$ and, hence, the solution for $q(x, y)$ inside the polygonal domain [14,16,18].

Ultimately, however, the statements (1)–(3) constitute a basic result in function theory, one that does not appear to have been reported elsewhere in the literature: it is a representation result for analytic functions in simply connected polygons. As such, this author was led to seek out an elementary derivation of (1)–(3) that does not require the language of Lax pairs, differential forms, distribution or spectral theory, or the mathematical technology of Riemann–Hilbert methods. Such a derivation, which we believe to be new, is presented in Sect. 2. It relies on nothing more than elementary properties of complex numbers, of the exponential function, and use of Cauchy's integral formula—all concepts available to a typical undergraduate after a first course in complex analysis.

Significantly the new derivation, which is geometric in nature, points the way to an important generalization: the formulation of the analogous transform methods for analytic functions defined in general *circular domains* (and, hence, to the study of boundary value problems for Laplace's equation in such domains). Where a polygon has N edges that are all straight lines (with zero curvature), we here define a circular domain to be one having N edges that each has constant, generally non-zero, curvature (including the possibility of straight line edges). This includes simple circular discs, of course, but also so-called "polycircular arc" domains which are the natural generalizations of N-sided polygons to regions with N circular arc edges. We derive the natural analogues of (1)–(3) for such domains. In Sects. 5–7 it is shown how these new transform pairs can be used to solve harmonic boundary value problems in a suite of examples which also allow us to outline some computational methods based on our approach.

Spence and Fokas [25] have presented, in the context of boundary value problems for Poisson's equation and several other partial differential equations, generalizations of the unified transform method to "boundary value problems in polar coordinates", including wedge regions. Our work is related to theirs, but our derivation here is more geometrical—not requiring any choice of coordinate system, or separation of variables—and the final statement of the results is different and applies to circular domains that are much more general than the wedge domains considered in [25]. Similarly, Fokas and Pinotsis [15] have developed the ideas of Fokas and Zyskin [13] and given a derivation of the transform pair of Fokas and Kapaev [14] by considering the Cauchy kernel, but the details of their derivation are different to that given here: it is our more geometrical viewpoint that provides the pathway to generalization to circular domains. Our treatment is, however, limited to consideration of analytic functions or, from the point of view of applications to boundary value problems, to Laplace's equation.

This paper treats only simply connected domains. However, the important generalization of the transform method to the case of multiply connected circular domains has been made in a companion paper [9].

2 A Transform Method for Convex Polygons

This section presents a new and elementary derivation of the transform method for convex polygons first introduced by Fokas and Kapaev [14] who used other techniques. In Sect. 3, it will be generalized to circular domains.

We start with a basic geometrical observation. If z' lies on some finite length slit on the real axis and z is in the upper half-plane—see Fig. 1—then, on geometrical grounds,

$$0 < \arg[z - z'] < \pi. \tag{4}$$

It follows trivially that

$$\int_0^\infty e^{ik(z-z')}\, dk = \left[\frac{e^{ik(z-z')}}{i(z-z')}\right]_0^\infty = \frac{1}{i(z'-z)} \tag{5}$$

or,

$$\frac{1}{z'-z} = i\int_0^\infty e^{ik(z-z')}\, dk, \quad 0 < \arg[z-z'] < \pi, \tag{6}$$

where it is easy to check that the contribution from the upper limit of integration vanishes for the particular choices of z' and z to which we have restricted consideration.

On the other hand, suppose z' lies on some other finite length slit now making angle χ with the positive real axis and suppose that z is in the half-plane shown in Fig. 2 (the half-plane "to the left" of the slit as one follows its tangent with uniform inclination angle χ). Now the affine transformation

$$z' \mapsto e^{-i\chi}(z'-\alpha), \quad z \mapsto e^{-i\chi}(z-\alpha), \tag{7}$$

for example, where the (unimportant) constant α is shown in Fig. 2, returns the slit to the real axis, and z to the upper half-plane, so

$$0 < \arg[e^{-i\chi}(z-\alpha) - e^{-i\chi}(z'-\alpha)] < \pi. \tag{8}$$

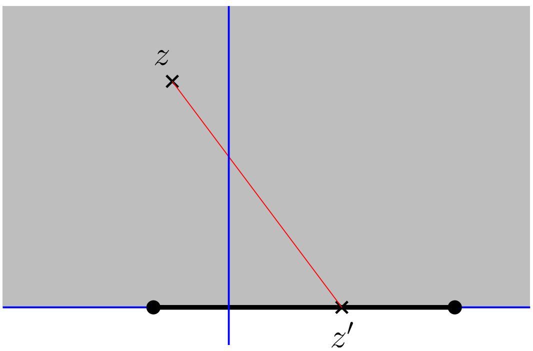


Fig. 1 Geometrical positioning of z and z' for the validity of (6)

(In Fig. 2 we have drawn the slit to intersect the imaginary axis, but this is not necessary and our arguments go through for any slit making angle χ to the real axis.) Hence, on use of (6) now with the substitutions (7), we can write

$$\frac{1}{\mathrm{e}^{-\mathrm{i}\chi}(z'-\alpha)-\mathrm{e}^{-\mathrm{i}\chi}(z-\alpha)} = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}k(\mathrm{e}^{-\mathrm{i}\chi}(z-\alpha)-\mathrm{e}^{-\mathrm{i}\chi}(z'-\alpha))}\,\mathrm{d}k, \tag{9}$$

or, on cancellation of α and rearrangement,

$$\frac{1}{z'-z} = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi}k(z-z')}\mathrm{e}^{-\mathrm{i}\chi}\,\mathrm{d}k. \tag{10}$$

This integral identity is valid uniformly for all z and z' having the geometrical positioning depicted in Fig. 2.

Now consider a bounded convex polygon P with N sides $\{S_j \mid j = 1, \ldots, N\}$. Figure 3 shows an example with $N = 3$. Geometrically the shaded polygon P can be viewed as the *intersection* of $N = 3$ half-plane regions of the kind just considered.

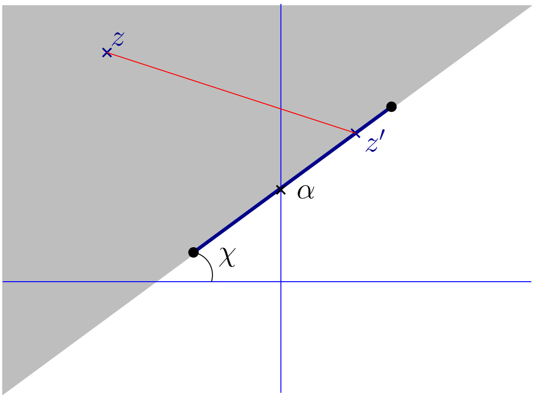


Fig. 2 Geometrical positioning of z and z' for the validity of (10)

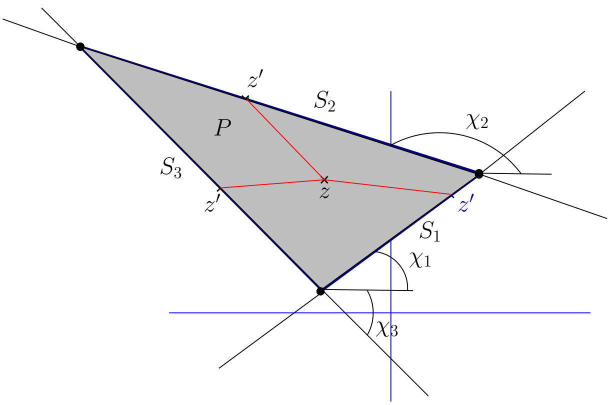


Fig. 3 A convex polygon P as an intersection of $N = 3$ half-planes with N angles $\{\chi_j \mid j = 1, 2, 3\}$. Formula (10) can be used in the Cauchy integral formula with $\chi = \chi_j$ when z' is on side S_j (for $j = 1, 2, 3$)

For a function $f(z)$ analytic in P, Cauchy's integral formula provides that for $z \in P$,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial P} \frac{f(z')\,\mathrm{d}z'}{z'-z}. \tag{11}$$

We can separate the boundary integral into a sum over the N sides:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \int_{S_j} f(z') \left\{ \frac{1}{(z'-z)} \right\} \mathrm{d}z'. \tag{12}$$

Now if side S_j has inclination angle χ_j then (10) can be used, with $\chi \mapsto \chi_j$, to re-express the Cauchy kernel uniformly for all $z \in P$ and for z' positioned on the respective sides:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \int_{S_j} f(z') \left\{ \mathrm{i} \int_0^{\infty} \mathrm{e}^{\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_j} k(z-z')} \mathrm{e}^{-\mathrm{i}\chi_j}\, \mathrm{d}k \right\} \mathrm{d}z'. \tag{13}$$

On reversing the order of integration we can write

$$f(z) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{\mathcal{L}} \rho_{jj}(k) \mathrm{e}^{-\mathrm{i}\chi_j} \mathrm{e}^{\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_j} kz}\, \mathrm{d}k, \tag{14}$$

where, for integers m, n between 1 and N, we define the *spectral matrix*

$$\rho_{mn}(k) \equiv \int_{S_n} f(z') \mathrm{e}^{-\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_m} kz'}\, \mathrm{d}z', \tag{15}$$

with $\mathcal{L} = [0, \infty)$ defined to be the *fundamental contour* for straight line edges. The transform pair for polygons can thus be stated as

$$\begin{cases} f(z) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{\mathcal{L}} \rho_{jj}(k) \mathrm{e}^{-\mathrm{i}\chi_j} \mathrm{e}^{\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_j} kz}\, \mathrm{d}k, \\ \rho_{jj}(k) = \int_{S_j} f(z') \mathrm{e}^{-\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_j} kz'}\, \mathrm{d}z', \quad j = 1, \ldots, N. \end{cases} \tag{16}$$

This is our final result.

2.1 The Global Relations

The elements of the spectral matrix, or "spectral functions", have important analytical structure. Observe that, for any $k \in \mathbb{C}$, and for any $m = 1, \ldots, N$,

$$\sum_{n=1}^{N} \rho_{mn}(k) = \sum_{n=1}^{N} \int_{S_n} f(z') \mathrm{e}^{-\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_m} kz'}\, \mathrm{d}z' = \int_{\partial P} f(z') \mathrm{e}^{-\mathrm{i}\mathrm{e}^{-\mathrm{i}\chi_m} kz'}\, \mathrm{d}z' = 0, \tag{17}$$

where we have used both the fact that $f(z')e^{-ie^{-i\chi_m}kz'}$ (for $m = 1, \ldots, N$) is analytic inside P, and Cauchy's theorem. There are N such global relations relating different elements of the spectral matrix, but each is an equivalent statement of the analyticity of $f(z)$ in the domain P.

While the global relations relate all elements of the spectral matrix, it is worth emphasizing that only the *diagonal* elements of the spectral matrix appear in the integral representation (14).

Transform pairs for unbounded polygons, such as semi-strips, can also be written down with minor modifications to the derivation above. The only difference is that the global relations are now valid in restricted sectors of the k-plane where the spectral functions are well defined [14].

2.2 Connection with Fokas and Kapaev [14]

Our statement (16) of the transform pair differs from (1)–(3) given by Fokas and Kapaev [14] in two ways. First, the notion of a spectral matrix was not introduced there. Second, the rays $\{\mathcal{L}_j | j = 1, \ldots, N\}$ in the spectral plane of [14] are absent. But it is easy to generate them.

By defining, for each $j = 1, \ldots, N$, the change of spectral variable given by

$$\lambda_j = e^{-i\chi_j}k, \quad j = 1, \ldots, N, \tag{18}$$

in both the spectral functions *and* the integral representation for $f(z)$, then, rather than a sum of N contributions over the single fundamental contour in the spectral plane, the transform pair (16) can be written as a sum of contributions from N (generally distinct) rays in a spectral plane:

$$f(z) = \frac{1}{2\pi}\sum_{j=1}^{N}\int_{\mathcal{L}_j}\tilde{\rho}_j(\lambda)e^{i\lambda z}\, d\lambda, \quad \tilde{\rho}_j(\lambda) \equiv \int_{S_j} f(z')e^{-i\lambda z'}\, dz', \tag{19}$$

where $\mathcal{L}_j = [0, \infty e^{-i\chi_j})$ is the ray defined by $\arg[\lambda] = -\chi_j$ (these coincide with the definitions given in [14]). Figure 4 shows the rays for the polygon shown in Fig. 3. With formulas (16) thus modified they are equivalent to (1) and (2).

The changes of variables (18) have the effect of reducing the N-by-N spectral matrix to just N distinct spectral functions. It is therefore tempting to argue that this obviates the need to introduce the spectral matrix. But we will see that this feature of reduction of the spectral matrix to just N spectral functions is *not* shared by more general domain types (notably, the circular domains considered later). Furthermore, it has been observed by previous authors studying the unified transform method (see [10,16], for example) that for simple polygonal domains, it is possible, using the global relations and their so-called "invariant properties", to express all spectral functions in terms of the given boundary data, using only algebraic manipulations [16]. This procedure of generating new spectral functions by transformations of the spectral argument often generates the other (off-diagonal) elements of the spectral matrix (15)

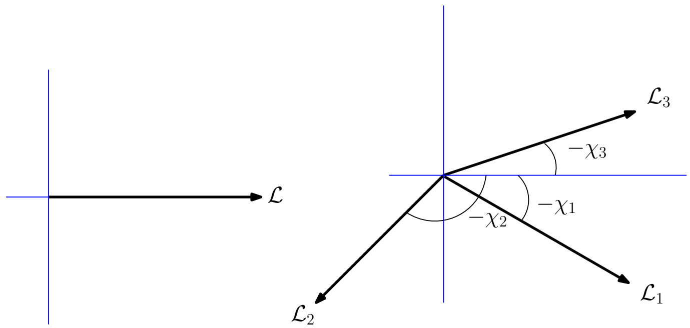


Fig. 4 The fundamental contour $\mathcal{L} = [0, \infty)$ for straight line edges (shown *left*). The spectral rays $\{\mathcal{L}_j \,|\, j = 1, 2, 3\}$ from Fokas and Kapaev [14] for the $N = 3$ polygon P shown in Fig. 3 (*right*). The latter rays are images of $\mathcal{L}$ under the N transformations (18)

that has arisen naturally in our derivation above. We would argue that the notion of a *spectral matrix* and the *fundamental contour* (for particular edge types) are very natural ones for these transform formulations. Both concepts will appear again in the generalization to circular domains given in Sect. 3.

2.3 Unbounded Polygonal Regions

For *unbounded* regions exterior to some bounded polygon (and, hence, containing the point at infinity), a little thought reveals that it is geometrically impossible to "cover" the entire region as a finite intersection of half-plane regions. The next best thing is to subdivide the domain into an atlas of polygonal subregions that *are* the intersection of half-planes having a representation of the kind introduced above. This realization has already been made, albeit from alternative perspectives, by Charalambopoulos, Dassios and Fokas [4] for the exterior of an equilateral triangle, for example, where an atlas of 6 convex subregions outside the triangle is used.

3 A Transform Method for Circular Domains

To find generalized transform schemes for circular domains we start by considering D as the simple unit disc; this basic geometrical unit will now replace the "half-plane" regions used for polygons. It should be clear that, to extend our approach to D, we must identify the particular spectral representation of the Cauchy kernel that is uniformly valid for z' on the domain boundary ∂D and for z inside D.

For values $|z| < 1$ in D, consider the integral

$$I \equiv \int_{L_1} \frac{1}{1 - \mathrm{e}^{2\pi \mathrm{i} k}} z^k \,\mathrm{d}k + \int_{L_2} z^k \,\mathrm{d}k + \int_{L_3} \frac{\mathrm{e}^{2\pi \mathrm{i} k}}{1 - \mathrm{e}^{2\pi \mathrm{i} k}} z^k \,\mathrm{d}k. \tag{20}$$

This integral is taken around what we will call the *fundamental contour* for circular arc edges shown in Fig. 5; it is the generalization of the fundamental contour for straight

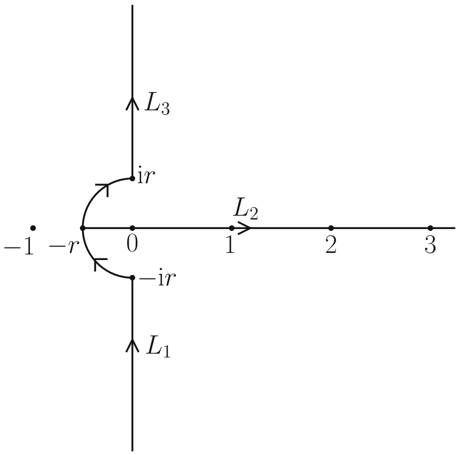


Fig. 5 The fundamental contour for circular arc edges with $0 < r < 1$

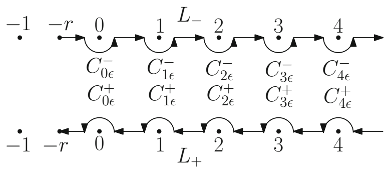


Fig. 6 Contours L_- and L_+ in the spectral k-plane used to establish (21)

line edges shown on the left in Fig. 4. To explain it, choose $0 < r < 1$. The contour L_1 is the union of the negative imaginary axis $(-\mathrm{i}\infty, -\mathrm{i}r]$ and the arc of the quarter circle $|k| = r$ in the third quadrant traversed in a clockwise sense; the contour L_2 is the real interval $[-r, \infty)$; the contour L_3 is the arc of the quarter circle $|k| = r$ in the second quadrant traversed in a clockwise sense together with the portion of the positive imaginary axis $[\mathrm{i}r, \mathrm{i}\infty)$. All integrals in (20) are non-singular and it is easy to verify directly that all integrands are exponentially decaying as $|k| \to \infty$ uniformly for all $|z| < 1$. (Readers already familiar with the method of Fokas and Kapaev [14] might recognize this fundamental contour as similar to that associated with an infinite semi-strip, and there is a good reason for this that is explained in Appendix A.)

For $|z| < 1$ it can be shown that

$$I = \int_{L_1} \frac{1}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} z^k \, \mathrm{d}k + \int_{L_2} z^k \, \mathrm{d}k + \int_{L_3} \frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} z^k \, \mathrm{d}k = \frac{1}{1 - z}. \tag{21}$$

To derive (21), note that the second integral in I can be written as

$$\begin{aligned}\int_{L_2} z^k \, \mathrm{d}k &= \mathcal{P} \int_{L_2} \left[\frac{1 - \mathrm{e}^{2\pi \mathrm{i}k}}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} \right] z^k \, \mathrm{d}k \\ &= \mathcal{P} \int_{L_2} \left[\frac{1}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} \right] z^k \, \mathrm{d}k - \mathcal{P} \int_{L_2} \left[\frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} \right] z^k \, \mathrm{d}k, \end{aligned} \tag{22}$$

where $\mathcal{P}$ denotes the principal value integral. But, with the contours L_- and L_+ shown in Fig. 6 for some $0 < \epsilon < r$,

$$\int_{L_-}\left[\frac{z^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]\mathrm{d}k = +\mathcal{P}\int_{L_2}\left[\frac{z^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]\mathrm{d}k + \sum_{n=0}^{\infty}\lim_{\epsilon\to 0}\int_{C_{n\epsilon}^-}\left[\frac{z^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]\mathrm{d}k$$
$$\int_{L_+}\left[\frac{\mathrm{e}^{2\pi \mathrm{i}k}z^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]\mathrm{d}k = -\mathcal{P}\int_{L_2}\left[\frac{\mathrm{e}^{2\pi \mathrm{i}k}z^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]\mathrm{d}k + \sum_{n=0}^{\infty}\lim_{\epsilon\to 0}\int_{C_{n\epsilon}^+}\left[\frac{\mathrm{e}^{2\pi \mathrm{i}k}z^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]\mathrm{d}k. \tag{23}$$

The contours L_- and L_+ are made up of the union of the radius-ϵ semi-circles $\{C_{n\epsilon}^{\pm}|n \geq 0\}$ centred at $k = n$ (for $n \geq 0$) traversed in an anti-clockwise sense together with the portions of the real k-axis between them. Hence, after an explicit computation of the integrals around $\{C_{n\epsilon}^{\pm}|n \geq 0\}$, and on substituting for the principal value integrals in (22) using (23), we can write

$$I = \int_{\mathcal{L}_-}\left[\frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]z^k\,\mathrm{d}k + \sum_{n=0}^{\infty}z^n + \int_{\mathcal{L}_+}\left[\frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\right]z^k\,\mathrm{d}k, \tag{24}$$

where $\mathcal{L}_- \equiv L_1 \cup L_-$ and $\mathcal{L}_+ \equiv L_+ \cup L_3$ are shown in Fig. 7. But the integral around $\mathcal{L}_-$ vanishes because its integrand is analytic in the fourth quadrant; similarly, the integral around $\mathcal{L}_+$ vanishes because its integrand is analytic in the first quadrant. Hence, for $|z| < 1$, we have

$$I = \sum_{n=0}^{\infty}z^n = \frac{1}{1-z}, \tag{25}$$

which establishes our result.

Several remarks are in order:

1. The residue calculus steps just given reveal that the integral (20) around the fundamental contour in Fig. 5 can be deformed to the modified integral

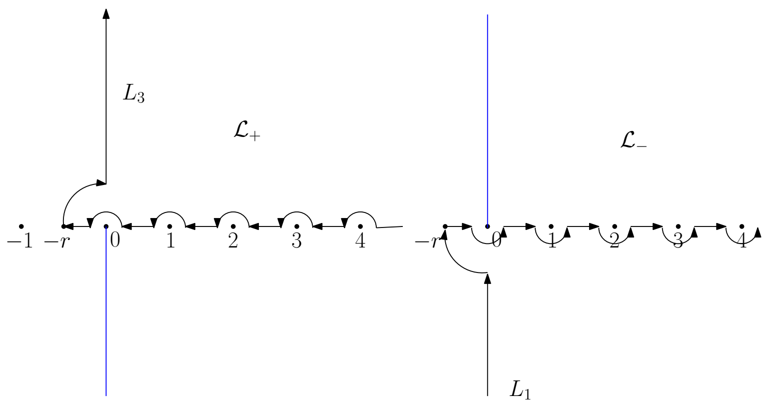
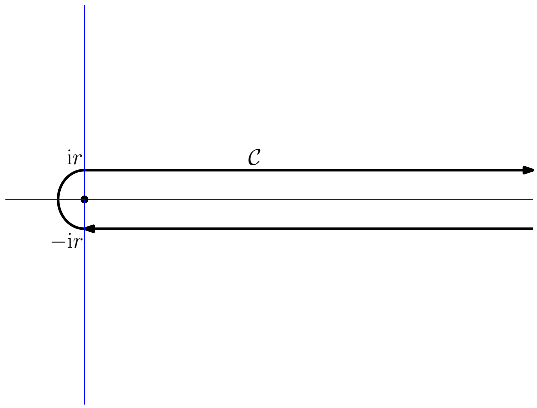


Fig. 7 The contours $\mathcal{L}_+$ and $\mathcal{L}_-$

Fig. 8 The contour $\mathcal{C}$



$$I = \oint_{\mathcal{C}} \frac{z^k}{1 - \mathrm{e}^{2\pi \mathrm{i} k}} \, \mathrm{d}k, \tag{26}$$

with the contour $\mathcal{C}$ encircling the real axis as shown in Fig. 8.

2. The reader might recognize the contour $\mathcal{C}$ as that appearing in the classical *Watson transformation* (see [25]) and, indeed, our construction of the spectral form (21) of the Cauchy kernel could alternatively have been made by invoking this transformation and giving a modified inversion formula based on (26). But we were led to our preferred form (21) in a quite different way. To see how, next we give the transform pair for the unit disc then, in Appendix A, show how to derive it independently from a logarithmic conformal mapping (a simple Schwarz–Christoffel mapping taking the disc to a semi-strip) coupled with the result for polygons of Sect. 2. Appendix A can be viewed as giving an algorithmic *construction* of the Watson contour $\mathcal{C}$ in Fig. 8 from the results of Sect. 2.
3. While the modified form (26) might, at first sight, appear to be a more concise statement of (20) we continue to use the latter. This is because (20) can be derived, after a logarithmic change of variable, from application of the transform method for a polygon (details are in Appendix A), hence the associated integration contours in Fig. 5 are, by the very nature of that construction, the ones along which exponential decay of the integrands is guaranteed. In (26), integration is along the deformed contour $\mathcal{C}$ that remains uniformly close to the infinite array of simple pole singularities along the positive real axis and, if high precision is desired, this can compromise accuracy of any numerical quadrature of the inverse transforms. Many such approximation theory questions will be properly addressed in future work.

3.1 Transform Pair for Interior of the Unit Disc

Let z' be a point on the unit circle and let $|z| < 1$ be a point inside the unit disc as depicted in Fig. 9 (this is the analogue of Fig. 1, but now for the unit disc instead of

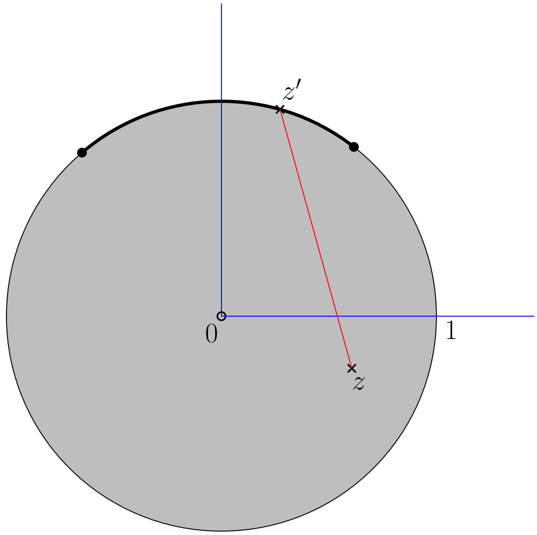


Fig. 9 Geometry of z' and z for derivation of (31)

the upper half-plane). Then $|z/z'| < 1$ uniformly and it follows on letting $z \mapsto z/z'$ in (21) that

$$\frac{z'}{z'-z} = \int_{L_1} \frac{1}{1-\mathrm{e}^{2\pi i k}} \frac{z^k}{z'^k}\,\mathrm{d}k + \int_{L_2} \frac{z^k}{z'^k}\,\mathrm{d}k + \int_{L_3} \frac{\mathrm{e}^{2\pi i k}}{1-\mathrm{e}^{2\pi i k}} \frac{z^k}{z'^k}\,\mathrm{d}k. \tag{27}$$

The Cauchy kernel is therefore given by the spectral representation

$$\frac{1}{z'-z} = \int_{L_1} \frac{1}{1-\mathrm{e}^{2\pi i k}} \frac{z^k}{z'^{k+1}}\,\mathrm{d}k + \int_{L_2} \frac{z^k}{z'^{k+1}}\,\mathrm{d}k + \int_{L_3} \frac{\mathrm{e}^{2\pi i k}}{1-\mathrm{e}^{2\pi i k}} \frac{z^k}{z'^{k+1}}\,\mathrm{d}k. \tag{28}$$

The Cauchy integral formula for a function $f(z)$ analytic in the unit disc is

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|z'|=1} \frac{f(z')}{z'-z}\,\mathrm{d}z'. \tag{29}$$

On substitution of (the uniformly valid) representation (28) for the Cauchy kernel we find

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|z'|=1} f(z') \left[\int_{L_1} \frac{1}{1-\mathrm{e}^{2\pi i k}} \frac{z^k}{z'^{k+1}}\,\mathrm{d}k + \int_{L_2} \frac{z^k}{z'^{k+1}}\,\mathrm{d}k \right.$$
$$\left. + \int_{L_3} \frac{\mathrm{e}^{2\pi i k}}{1-\mathrm{e}^{2\pi i k}} \frac{z^k}{z'^{k+1}}\,\mathrm{d}k \right] \mathrm{d}z'. \tag{30}$$

On swapping the order of integration we arrive at the transform pair

$$\begin{cases} f(z) = \frac{1}{2\pi \mathrm{i}}\left\{\int_{L_1} \frac{\rho(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}} z^k \,\mathrm{d}k + \int_{L_2} \rho(k) z^k \,\mathrm{d}k + \int_{L_3} \frac{\rho(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}} z^k \,\mathrm{d}k\right\}, \\ \rho(k) = \oint_{|z'|=1} \frac{f(z')}{z'^{k+1}} \,\mathrm{d}z'. \end{cases} \tag{31}$$

The global relation is

$$\rho(k) = 0, \quad k \in -\mathbb{N} \tag{32}$$

since, for this discrete set of k-values, the integrand of the integral defining $\rho(k)$ is analytic in the unit disc.

It is a simple and instructive exercise to verify the transform pair (31) for simple test analytic functions such as $f(z) = z^p$ for $p \in \mathbb{Z}^+$.

Equation (31) provides an alternative way to represent analytic functions in the disc. It is clear from our development that it is the natural generalization of the Fourier transform to the unit disc (or, more properly, the Fourier–Mellin transform—see the discussion in Appendix A and Appendix B, and the later comments at the end of Sect. 6). In complex analysis, it is more common to associate analytic functions in discs with Taylor series but, as we have seen, one can also write down the Fourier–Mellin transform pairs (31) too.

3.2 Transform Pair for Exterior of the Unit Disc

The transform pair for the *exterior* of the unit disc follows similarly. For z' on the unit circle with $|z| > 1$ then $|z'/z| < 1$ uniformly and setting $z \mapsto z'/z$ in (21) gives the identity

$$\frac{z}{z-z'} = \int_{L_1} \frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \left(\frac{z'}{z}\right)^k \mathrm{d}k + \int_{L_2} \left(\frac{z'}{z}\right)^k \mathrm{d}k + \int_{L_3} \frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \left(\frac{z'}{z}\right)^k \mathrm{d}k. \tag{33}$$

Hence we have the spectral representation of the Cauchy kernel given by

$$\frac{1}{z-z'} = \int_{L_1} \frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \frac{z'^k}{z^{k+1}} \,\mathrm{d}k + \int_{L_2} \frac{z'^k}{z^{k+1}} \,\mathrm{d}k + \int_{L_3} \frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \frac{z'^k}{z^{k+1}} \,\mathrm{d}k. \tag{34}$$

For $|z| > 1$ the Cauchy integral formula

$$f(z) = -\frac{1}{2\pi \mathrm{i}} \oint_{|z'|=1} \frac{f(z')}{z'-z} \,\mathrm{d}z' \tag{35}$$

holds for a function analytic outside the unit disc that decays like $1/z$ as $z \to \infty$. Use of (the uniformly valid) expression (34) in the Cauchy integral formula then produces the following transform pair for such a function:

$$\begin{cases} f(z) = \frac{1}{2\pi \mathrm{i}}\left\{\int_{L_1} \frac{\rho(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{\mathrm{d}k}{z^{k+1}} + \int_{L_2}\rho(k)\frac{\mathrm{d}k}{z^{k+1}} + \int_{L_3}\frac{\rho(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{\mathrm{d}k}{z^{k+1}}\right\}, \\ \rho(k) = \oint_{|z'|=1} f(z')z'^k\,\mathrm{d}z'. \end{cases} \tag{36}$$

The global relation is

$$\rho(k) = 0, \quad k \in -\mathbb{N} \tag{37}$$

since, for this discrete set of k-values, the integrand of the integral defining $\rho(k)$ is analytic outside the disc and is $\mathcal{O}(1/z^2)$ as $z \to \infty$.

For future use, we prefer to write (36) with the addition of two minus signs:

$$\begin{cases} f(z) = -\frac{1}{2\pi \mathrm{i}}\left\{\int_{L_1} \frac{\rho(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{\mathrm{d}k}{z^{k+1}} + \int_{L_2}\rho(k)\frac{\mathrm{d}k}{z^{k+1}} + \int_{L_3}\frac{\rho(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{\mathrm{d}k}{z^{k+1}}\right\}, \\ \rho(k) = -\oint_{|z'|=1} f(z')z'^k\,\mathrm{d}z', \end{cases} \tag{38}$$

which clearly does not affect the transform pair. The reason for this is to preserve a convention that the spectral function is the integral around the boundary traversed in such a way that the domain of interest is *on the left* as the boundary is traversed.

3.3 Transform Pair for Polycircular Domains

Just as a convex polygon was understood in Sect. 2 to be the intersection of N half-plane regions, a polycircular domain is now understood to be the intersection of N circular discs.

Consider now the convex simply connected region D bounded by the arcs $\{C_j | j = 1, \ldots, N\}$ of N circles $\{|z - \delta_j| = q_j | j = 1, \ldots, N\}$ as shown in Fig. 10 for $N = 3$. The Cauchy integral formula for a function $f(z)$ analytic in this region is

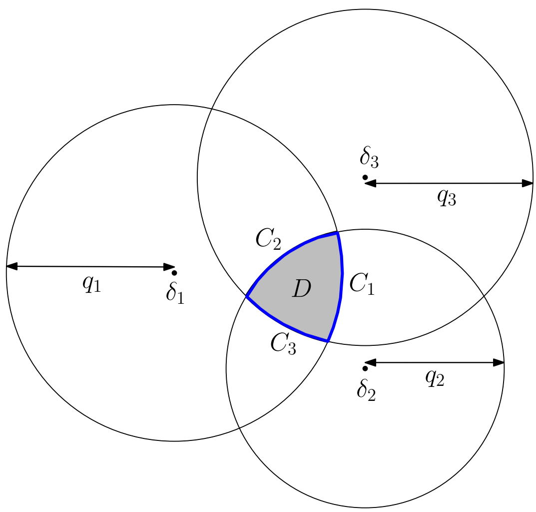


Fig. 10 A polycircular domain D with $N = 3$ sides denoted by C_1, C_2 and C_3

$$f(z)=\frac{1}{2\pi \mathrm{i}}\oint_{\partial D}\frac{f(z')}{z'-z}\,\mathrm{d}z'=\frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{N}\int_{C_j}\frac{f(z')}{z'-z}\,\mathrm{d}z', \tag{39}$$

where ∂D denotes the boundary of D and where, in the second equality, the integral around ∂D has been separated into the N separate integrals around the individual circular arcs $\{C_j | j=1,\ldots,N\}$.

Focussing on the portion of the integral along C_j for which it holds, uniformly for z in D, that $|z-\delta_j|<|z'-\delta_j|$ the Cauchy kernel for z' on C_j has the spectral representation

$$\begin{aligned}\frac{1}{z'-z}&=\frac{1}{(z'-\delta_j)-(z-\delta_j)}\\&=\int_{L_1}\frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{(z-\delta_j)^k}{(z'-\delta_j)^{k+1}}\,\mathrm{d}k+\int_{L_2}\frac{(z-\delta_j)^k}{(z'-\delta_j)^{k+1}}\,\mathrm{d}k\\&\quad+\int_{L_3}\frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{(z-\delta_j)^k}{(z'-\delta_j)^{k+1}}\,\mathrm{d}k,\end{aligned} \tag{40}$$

where we have set $z\mapsto(z-\delta_j)/(z'-\delta_j)$ in (21). It is important to write this in the rescaled form

$$\begin{aligned}\frac{1}{z'-z}&=\int_{L_1}\frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{1}{q_j}\left(\frac{z-\delta_j}{q_j}\right)^k\left[\frac{z'-\delta_j}{q_j}\right]^{-k-1}\mathrm{d}k\\&\quad+\int_{L_2}\frac{1}{q_j}\left(\frac{z-\delta_j}{q_j}\right)^k\left[\frac{z'-\delta_j}{q_j}\right]^{-k-1}\mathrm{d}k\\&\quad+\int_{L_3}\frac{\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\frac{1}{q_j}\left(\frac{z-\delta_j}{q_j}\right)^k\left[\frac{z'-\delta_j}{q_j}\right]^{-k-1}\mathrm{d}k.\end{aligned} \tag{41}$$

Now for $z\in D$ (see Fig. 10) comprising the intersection of the circular discs we can substitute (41) into the Cauchy integral formula (39) when z' sits on each of the separate boundary arcs $\{C_j | j=1,\ldots,N\}$ to find

$$\begin{aligned}f(z)=\frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{N}\Bigg\{&\int_{L_1}\frac{\rho_{jj}(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\left[\frac{z-\delta_j}{q_j}\right]^k\mathrm{d}k+\int_{L_2}\rho_{jj}(k)\left[\frac{z-\delta_j}{q_j}\right]^k\mathrm{d}k\\&+\int_{L_3}\frac{\rho_{jj}(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}}\left[\frac{z-\delta_j}{q_j}\right]^k\mathrm{d}k\Bigg\},\end{aligned} \tag{42}$$

where we have swapped the order of integration and introduced the N-by-N *spectral matrix*

$$\rho_{mn}(k)\equiv\frac{1}{q_m}\int_{C_n}\left[\frac{z'-\delta_m}{q_m}\right]^{-k-1}f(z')\,\mathrm{d}z'. \tag{43}$$

Global relations for this system are

$$\sum_{n=1}^{N} \rho_{mn}(k) = 0, \quad k \in -\mathbb{N} \tag{44}$$

for any $m = 1, 2, \dots, N$. There are N such global relations but each is an equivalent statement of the analyticity of $f(z)$ in the domain D.

It is important to remark that, unlike in the case of polygons, it is not clear that it is now possible to invoke any changes of variables in the spectral k-plane, akin to (18), that will reduce the N-by-N spectral matrix to just N spectral functions.

Variants of the above should be obvious. Suppose, on the other hand, that the polycircular domain D is that shown in Fig. 11: D is the intersection of the *exterior* of the circle $|z-\delta_1| = q_1$ and the *interior* of the circles $|z-\delta_2| = q_2$ and $|z-\delta_3| = q_3$. Let $\{C_j | j = 1, 2, 3\}$ again denote the boundary arcs making up the boundary ∂D as shown in Fig. 11 and such that the domain D remains to the left as ∂D is traversed. By arguments similar to those given above we deduce that the appropriate integral representation is

$$\begin{aligned} f(z) = -\frac{1}{2\pi \mathrm{i}} \Bigg\{ &\int_{L_1} \frac{\rho_{11}(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \left(\frac{q_1}{z-\delta_1}\right)^{k+1} \mathrm{d}k + \int_{L_2} \rho_{11}(k) \left(\frac{q_1}{z-\delta_1}\right)^{k+1} \mathrm{d}k \\ &+ \int_{L_3} \frac{\rho_{11}(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \left(\frac{q_1}{z-\delta_1}\right)^{k+1} \mathrm{d}k \Bigg\} \\ &+ \frac{1}{2\pi \mathrm{i}} \sum_{j=2}^{3} \Bigg\{ \int_{L_1} \frac{\rho_{jj}(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \left[\frac{z-\delta_j}{q_j}\right]^{k} \mathrm{d}k + \int_{L_2} \rho_{jj}(k) \left[\frac{z-\delta_j}{q_j}\right]^{k} \mathrm{d}k \\ &+ \int_{L_3} \frac{\rho_{jj}(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \left[\frac{z-\delta_j}{q_j}\right]^{k} \mathrm{d}k \Bigg\}, \end{aligned} \tag{45}$$

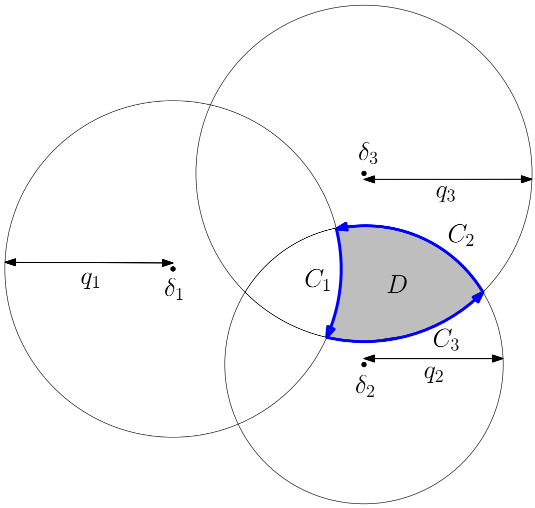


Fig. 11 A polycircular domain D with $N = 3$ (directed) sides denoted by C_1, C_2 and C_3. Note that C_1 is an arc of $|z-\delta_1| = q_1$ now traversed in a clockwise sense

where the matrix of spectral functions is defined for $n = 1, 2, 3$ by

$$\rho_{1n}(k) \equiv \frac{1}{q_1} \int_{C_n} \left(\frac{z' - \delta_1}{q_1} \right)^k f(z')\, \mathrm{d}z',$$

$$\rho_{mn}(k) \equiv \frac{1}{q_m} \int_{C_n} \left(\frac{z' - \delta_m}{q_m} \right)^{-k-1} f(z')\, \mathrm{d}z', \quad m = 2, 3. \tag{46}$$

Global relations are

$$\sum_{n=1}^{3} \rho_{mn}(k) = 0, \quad k \in -\mathbb{N}, \; m = 1, 2, 3. \tag{47}$$

3.4 Domains with Both Straight and Circular Edges

It should be clear that our approach generalizes to more general domains with boundaries comprising a *mixture* of circular arc and straight line edges. Correspondingly, the derivation of the relevant transform pairs involves a mixture of the results of Sects. 2 and 3. This is best illustrated by means of an explicit example (given in Sect. 6) from which the general construction can be discerned.

4 Applications to Boundary Value Problems

In the remainder of the paper we survey just a few of the many possible applications of this new transform perspective for circular domains, in particular to the solution of boundary value problems for Laplace's equation. We proceed under the assumption that solving for the functions in the spectral matrix will lead, on substitution of this spectral data into the integral representations for the relevant analytic function, to the solution of the stated boundary value problems. That this is true is not obvious, but it has been established for boundary value problems for Laplace equations in convex polygons by Fulton, Fokas and Xenophontos [21] and for general elliptic equations in arbitrary convex domains by Ashton [1].

Here we focus on boundary value problems where the effectiveness of the transform approach can be tested against other mathematical schemes. The following examples also give us the opportunity to introduce some numerical schemes we have devised for the analysis of the global relation.

5 The Problem of Dual Fourier Series

Even for the simple unit disc the new transform approach affords a reappraisal of some boundary value problems previously tackled by other means. A problem considered by Shepherd [20] is to find the set of real coefficients $\{A_n\}$ satisfying the conditions

$$\cos(m\theta) = \sum_{n \geq 0} A_n \cos n\theta, \quad 0 < \theta < \pi/2 \text{ (or } C_1),$$

$$-\sin(m\theta) = \sum_{n \geq 1} A_n \sin n\theta, \quad \pi/2 < \theta < \pi \text{ (or } C_2), \tag{48}$$

where m is some positive integer. This has become known as a problem of *dual Fourier series* [20,23,26]; indeed, Shepherd solved this problem, separately for even and odd m, by a sequence of ingenious manipulations of integral representations of the Fourier series coefficients; he also included an application to a problem in potential theory. But the same results follow in a more algorithmic way by an analysis of the global relations for this system as they arise in the new transform method described above.

First note that the problem can be reformulated as a mixed boundary value problem for a function $f(z)$ analytic in the unit z-disc. Let

$$f(z) = \sum_{n\geq 0} A_n z^n \tag{49}$$

be the Taylor expansion of such a function valid convergent for $|z| < 1$. Suppose it satisfies the mixed boundary conditions on the two "faces" of the unit circle $|z| = 1$ where $z = e^{i\theta}$:

$$\begin{aligned} \mathrm{Re}[f(z)] &= \cos m\theta, \quad -\pi/2 < \theta < \pi/2, \\ \mathrm{Im}[f(z)] &= -\sin m\theta, \quad \pi/2 < \theta < 3\pi/2. \end{aligned} \tag{50}$$

If such a function $f(z)$ can be found then the coefficients in (49) will satisfy (50) as z tends to the boundary.

Since $f(z)$ is analytic in the unit disc we have the following transform pair representation for it:

$$\begin{cases} f(z) = \frac{1}{2\pi i}\left\{\int_{L_1} \frac{\rho(k)}{1-e^{2\pi ik}} z^k\, dk + \int_{L_2} \rho(k) z^k\, dk + \int_{L_3} \frac{\rho(k)e^{2\pi ik}}{1-e^{2\pi ik}} z^k\, dk\right\}, \\ \rho(k) = \oint_{|z'|=1} \frac{f(z')}{z'^{k+1}}\, dz'. \end{cases} \tag{51}$$

The global relation is

$$\rho(k) = 0, \quad k \in -\mathbb{N}. \tag{52}$$

Equation (52) can be used to find the spectral function $\rho(k)$, hence $f(z)$, and, in turn, the coefficients $\{A_n\}$.

On C_1 we write

$$f(z) = \cos m\theta + i\left[a_0 + \sum_{n\geq 1} a_n e^{2n\theta i} + \overline{a_n} e^{-2n\theta i}\right], \tag{53}$$

where $a_0 \in \mathbb{R}$ and $\{a_n \in \mathbb{C} | n \geq 1\}$ are to be determined. Here we have decomposed the unknown imaginary part of $f(z)$ as a period-π Fourier series over the interval $[-\pi/2, \pi/2]$. Similarly, for z on C_2 we write

$$f(z) = \left[b_0 + \sum_{n \geq 1} b_n \mathrm{e}^{2n\theta \mathrm{i}} + \overline{b_n} \mathrm{e}^{-2n\theta \mathrm{i}} \right] - \mathrm{i} \sin m\theta, \tag{54}$$

where $b_0 \in \mathbb{R}$ and $\{b_n \in \mathbb{C} | n \geq 1\}$ are to be determined and, here, we have decomposed the unknown real part as a period-π Fourier series. On substitution of (53) and (54) into (52) we arrive at the following linear system for the unknown coefficients:

$$\begin{aligned} &a_0\, \mathcal{A}(0,k) + \sum_{n \geq 1} a_n \mathcal{A}(2n,k) + \overline{a_n} \mathcal{A}(-2n,k) \\ &\quad + b_0\, \mathcal{B}(0,k) + \sum_{n \geq 1} b_n \mathcal{B}(2n,k) + \overline{b_n} \mathcal{B}(-2n,k) = r(k), \quad k \in -\mathbb{N}, \\ &a_0\, \overline{\mathcal{A}(0,k)} + \sum_{n \geq 1} \overline{a_n}\, \overline{\mathcal{A}(2n,k)} + a_n\, \overline{\mathcal{A}(-2n,k)} \\ &\quad + b_0\, \overline{\mathcal{B}(0,k)} + \sum_{n \geq 1} \overline{b_n}\, \overline{\mathcal{B}(2n,k)} + b_n\, \overline{\mathcal{B}(-2n,k)} = \overline{r(k)}, \quad k \in -\mathbb{N}, \end{aligned} \tag{55}$$

where

$$\mathcal{A}(n,k) = \begin{cases} \frac{1}{n-k} \left[\mathrm{e}^{\mathrm{i}(n-k)\pi/2} - \mathrm{e}^{-\mathrm{i}(n-k)\pi/2} \right], & n \neq k, \\ \mathrm{i}\pi, & n = k, \end{cases} \tag{56}$$

$$\mathcal{B}(n,k) = \begin{cases} \frac{1}{\mathrm{i}(n-k)} \left[\mathrm{e}^{3\mathrm{i}(n-k)\pi/2} - \mathrm{e}^{\mathrm{i}(n-k)\pi/2} \right], & n \neq k, \\ \pi, & n = k, \end{cases} \tag{57}$$

and

$$r(k) = -\int_{-\pi/2}^{\pi/2} \cos m\theta\, \mathrm{e}^{-\mathrm{i}k\theta}\, \mathrm{d}\theta + \mathrm{i} \int_{\pi/2}^{3\pi/2} \sin m\theta\, \mathrm{e}^{-\mathrm{i}k\theta}\, \mathrm{d}\theta. \tag{58}$$

The system (55) can be solved numerically by truncating the infinite sums at some integer N—so that there is a total of $4N + 2$ real unknown coefficients—and then picking $N_k > 4N + 2$ values of k in (55) to produce an overdetermined system of N_k equations from which the unknown coefficients can be found by a least-squares algorithm. This procedure was carried out and the results checked against those derived by the alternative scheme given by Shepherd [20] (it can also be solved using a method based on conformal slit maps as used recently by the author [7] in a mixed boundary value problem of the same general kind).

While we have used Fourier series representations for the unknown boundary data above, another possibility (akin to Chebyshev-type expansions) is described in the example of Sect. 7 and can be applied in the above example too.

A strong argument in favour of the transform approach to this problem just demonstrated is that it is more algorithmic than the rather more ad hoc approach used by Shepherd [20]. Our approach also frames this problem as just another boundary value problem that is amenable to analysis using the same basic mathematical ideas underpinning the unified transform method.

6 Mixed Boundary Value Problem in a Semi-Disc

A second boundary value problem that can be used to test the method is to find a function $f(z)$ analytic in D defined to be the upper half unit semi-disc $|z| = 1, \mathrm{Im}[z] > 0$ as shown in Fig. 12. This is an interesting example in that D has both a circular arc boundary and a straight line boundary and it affords an opportunity to see how to combine the ideas given earlier in this paper.

Suppose the boundary conditions on an analytic function $f(z)$ in such a domain are given to be

$$\begin{aligned} &\mathrm{Re}[f(z)] = r(z, \overline{z}), \quad && \text{on } C, \\ &\mathrm{Im}[f(z)] = 0, && \text{on } L, \end{aligned} \tag{59}$$

where $r(z, \overline{z})$ is some given real-valued function.

The semi-disc can be understood as the intersection of the upper half-plane and the interior of the unit disc. Hence, by combining the ideas presented earlier for polygons regions (in Sect. 2) and for circular regions (in Sect. 3)—that is, by substituting the appropriate uniformly valid spectral representation of the Cauchy kernel having split the Cauchy integral formula into the two boundary portions—it is straightforward to derive the integral transform representation

$$\begin{aligned} f(z) = \frac{1}{2\pi \mathrm{i}} \Bigg\{ &\int_{L_1} \frac{\rho_{11}(k)}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} z^k \, \mathrm{d}k + \int_{L_2} \rho_{11}(k) z^k \, \mathrm{d}k \\ &+ \int_{L_3} \frac{\rho_{11}(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1 - \mathrm{e}^{2\pi \mathrm{i}k}} z^k \, \mathrm{d}k \Bigg\} + \frac{1}{2\pi} \int_0^\infty \rho_{22}(k) \mathrm{e}^{\mathrm{i}kz} \, \mathrm{d}k, \end{aligned} \tag{60}$$

where the spectral matrix has components

$$\begin{aligned} \rho_{11}(k) &= \int_C f(z') z'^{(-k-1)} \, \mathrm{d}z', \quad \rho_{22}(k) = \int_L f(z') \mathrm{e}^{-\mathrm{i}kz'} \, \mathrm{d}z', \\ \rho_{21}(k) &= \int_C f(z') \mathrm{e}^{-\mathrm{i}kz'} \, \mathrm{d}z', \quad \rho_{12}(k) = \int_L f(z') z'^{(-k-1)} \, \mathrm{d}z'. \end{aligned} \tag{61}$$

Two global relations can be expressed as

$$\rho_{11}(k) + \rho_{12}(k) = 0, \quad k \in -\mathbb{N} \tag{62}$$

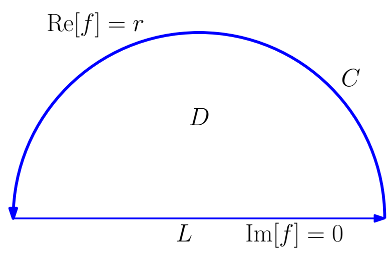


Fig. 12 Boundary value problem in D, the upper half semi-disc, with boundary comprising the circular portion C and straight line portion L

and

$$\rho_{21}(k) + \rho_{22}(k) = 0, \quad k \in \mathbb{C}. \tag{63}$$

These two conditions are not independent, but both turn out to be helpful in solving the problem.

First we will make use of (62) to find $f(z)$ on C. Some preliminary manipulations reveal that

$$\int_C \overline{f(z')} z'^{(-k-1)}\, \mathrm{d}z' = -\overline{\rho_{11}}(-k), \quad \int_L \overline{f(z')} z'^{(-k-1)}\, \mathrm{d}z' = \overline{\rho_{12}}(k). \tag{64}$$

The boundary condition on C is

$$f(z) + \overline{f(z)} = 2r(z, \bar{z}), \tag{65}$$

whence, on multiplying it by z^{-k-1} and integrating the resulting relation along C, we find

$$\rho_{11}(k) - \overline{\rho_{11}}(-k) = R_1(k), \quad \forall\, k \in \mathbb{C}, \tag{66}$$

where

$$R_1(k) \equiv \int_C \frac{2r}{z^{k+1}}\, \mathrm{d}z. \tag{67}$$

The boundary condition on L is

$$f(z) = \overline{f(z)}. \tag{68}$$

On multiplying this by z^{-k-1} and integrating the resulting relation along L we find

$$\rho_{12}(k) = \overline{\rho_{12}}(k), \quad \forall\, k \in \mathbb{C}. \tag{69}$$

The complex conjugate of (62) is

$$\overline{\rho_{11}}(k) + \overline{\rho_{12}}(k) = 0, \quad k \in -\mathbb{N}. \tag{70}$$

Now (69) implies

$$\overline{\rho_{11}}(k) + \rho_{12}(k) = 0, \quad k \in -\mathbb{N} \tag{71}$$

while (66), with $k \mapsto -k$, gives

$$\overline{\rho_{11}}(k) = \rho_{11}(-k) - R_1(-k) \quad \forall\, k \in \mathbb{C} \tag{72}$$

which can be used in (71) to produce

$$\rho_{11}(-k) - R_1(-k) + \rho_{12}(k) = 0, \quad k \in -\mathbb{N}. \tag{73}$$

Finally (62) can be used to eliminate $\rho_{12}(k)$ to give a relation depending only on $\rho_{11}(k)$:

$$\rho_{11}(-k) = R_1(-k) + \rho_{11}(k), \quad k \in -\mathbb{N}. \tag{74}$$

Equation (74) will be analysed to find $f(z)$ on C and, hence, $\rho_{11}(k)$.

We will represent $f(z)$ on C using a Fourier series on the interval $\theta \in [0, \pi]$ where $z = \mathrm{e}^{\mathrm{i}\theta}$. For $z \in C$ we therefore write

$$f(z) = r(z, \overline{z}) + \mathrm{i}\left[a_0 + \sum_{n\geq 1} a_n \mathrm{e}^{2\mathrm{i}n\theta} + \overline{a_n}\mathrm{e}^{-2\mathrm{i}n\theta}\right], \tag{75}$$

where the set of coefficients $\{a_n | n = 0, 1, 2, \ldots\}$ is to be found. On substitution of (75) into the definition of $\rho_{11}(k)$, and use of the definition of $R_1(k)$, we find

$$\rho_{11}(k) = \frac{R_1(k)}{2} - \int_0^{\pi} \mathrm{e}^{-\mathrm{i}k\theta}\left[a_0 + \sum_{n\geq 1} a_n \mathrm{e}^{2\mathrm{i}n\theta} + \overline{a_n}\mathrm{e}^{-2\mathrm{i}n\theta}\right]\mathrm{d}\theta. \tag{76}$$

On substitution into (74) we arrive at the system

$$\begin{aligned} &a_0\left[\int_0^{\pi}[-\mathrm{e}^{\mathrm{i}k\theta} + \mathrm{e}^{-\mathrm{i}k\theta}]\mathrm{d}\theta\right] + \sum_{n\geq 1} a_n\left[\int_0^{\pi}[-\mathrm{e}^{\mathrm{i}(k+2n)\theta} + \mathrm{e}^{\mathrm{i}(2n-k)\theta}]\mathrm{d}\theta\right] \\ &\quad + \sum_{n\geq 1}\overline{a_n}\left[\int_0^{\pi}[-\mathrm{e}^{\mathrm{i}(k-2n)\theta} + \mathrm{e}^{\mathrm{i}(-2n-k)\theta}]\mathrm{d}\theta\right] = \frac{1}{2}[R_1(k) + R_1(-k)], \quad k \in -\mathbb{N}. \end{aligned} \tag{77}$$

All the matrix elements in this linear system can be determined in closed form. As in Sect. 5, the system can be solved numerically by truncating the infinite sums and forming an overdetermined system by evaluating (77) at a sufficiently large number of choices of $k \in -\mathbb{N}$.

Once $\rho_{11}(k)$ has been found the global relation (62) gives $\rho_{12}(k)$, but the latter is not needed in the integral representation (60). But with $f(z)$ determined on C, the spectral function $\rho_{22}(k)$ needed in (60) can be found from the *second* global relation (63). This is because the boundary data on C are now known and can be used to compute $\rho_{21}(k)$ with (63) then yielding $\rho_{22}(k)$.

This boundary value problem can be solved using alternative methods based on the Schwarz reflection principle. Suppose we pick the data for the above boundary value problem to be

$$r(z, \overline{z}) = \frac{1}{2}[z + \overline{z}] = \cos\theta, \quad z = e^{i\theta}. \tag{78}$$

Then it is clear that the solution is

$$f(z) = z. \tag{79}$$

This, and other simple choices of $f(z)$, provides a check on the transform solution. It should be clear that the transform method will work; however, when the boundary condition on L is more complicated and arguments based on Schwarz reflection fail.

It is the appearance of both e^{ikz} (Fourier-type) and z^k (Mellin-type) in the integral representation (60) that prompts us to refer to the general transform methods of this paper as "Fourier–Mellin transforms for circular domains". We also show, in Appendix B, how to apply the very same method of Appendix A to derive the classical (complex) Mellin transform pair for an infinite wedge region. Together, Appendix A and Appendix B should convince the reader that the transform pairs we have derived above really *are* the natural extensions of the classical Fourier/Mellin transforms to general circular domains.

7 Mixed Boundary Value Problem in a Lens-Shaped Domain

To conclude, we solve a boundary value problem for $f(z)$ in the lens-shaped region D shown in Fig. 13 having two circular arc boundaries denoted by C_1 and C_2. The boundary conditions are taken to be

$$\begin{aligned} \mathrm{Re}[f(z)] &= r(z, \overline{z}), \quad \text{on } C_1, \\ \mathrm{Im}[f(z)] &= 0, \quad \text{on } C_2, \end{aligned} \tag{80}$$

where $r(z, \overline{z})$ is some given real-valued function.

The general transform pair for an N-sided circular arc polygon was given in (42). In the present example $N = 2$ with

$$q_1 = 1, \quad \delta_1 = 0, \quad q_2 = 1, \quad \delta_2 = 1. \tag{81}$$

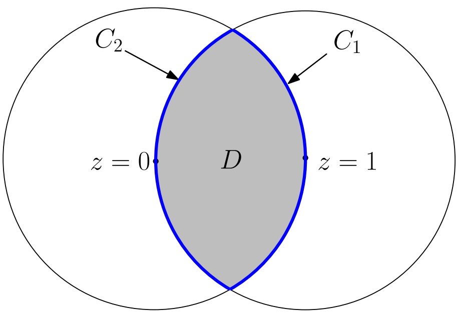


Fig. 13 The convex lens-shaped domain D formed by the intersection of the circular discs $|z| < 1$ and $|z - 1| < 1$. The boundary arc of D with $|z| = 1$ is denoted by C_1, the arc with $|z - 1| = 1$ is denoted by C_2

We can therefore write

$$f(z) = \frac{1}{2\pi \mathrm{i}} \left\{ \int_{L_1} \frac{\rho_{11}(k)}{1-\mathrm{e}^{2\pi \mathrm{i}k}} z^k \,\mathrm{d}k + \int_{L_2} \rho_{11}(k) z^k \,\mathrm{d}k + \int_{L_3} \frac{\rho_{11}(k)\mathrm{e}^{2\pi \mathrm{i}k}}{1-\mathrm{e}^{2\pi \mathrm{i}k}} z^k \,\mathrm{d}k \right.$$
$$+ \int_{L_1} \frac{\rho_{22}(k)[z-1]^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \,\mathrm{d}k + \int_{L_2} \rho_{22}(k)[z-1]^k \,\mathrm{d}k$$
$$\left. + \int_{L_3} \frac{\rho_{22}(k)\mathrm{e}^{2\pi \mathrm{i}k}[z-1]^k}{1-\mathrm{e}^{2\pi \mathrm{i}k}} \,\mathrm{d}k \right\}, \tag{82}$$

where

$$\rho_{11}(k) = \oint_{C_1} [z']^{-k-1} f(z')\,\mathrm{d}z', \quad \rho_{22}(k) = \oint_{C_2} [z'-1]^{-k-1} f(z')\,\mathrm{d}z'. \tag{83}$$

The other two elements of the spectral matrix are

$$\rho_{12}(k) = \oint_{C_2} [z']^{-k-1} f(z')\,\mathrm{d}z', \quad \rho_{21}(k) = \oint_{C_1} [z'-1]^{-k-1} f(z')\,\mathrm{d}z'. \tag{84}$$

The global relation we will analyse is

$$\rho_{11}(k) + \rho_{12}(k) = 0, \quad k \in -\mathbb{N}. \tag{85}$$

We expect singularities of $f(z)$ at the two points where C_1 and C_2 intersect. We are therefore motivated to introduce the representations of the boundary values given by

$$f(z) = \begin{cases} r(z,\overline{z}) + \mathrm{i}\left[a_0 + \sum_{n\ge 1} a_n \zeta^n + \frac{\overline{a_n}}{\zeta^n}\right], & \text{on } C_1, \\ b_0 + \sum_{n\ge 1} b_n \zeta^n + \frac{\overline{b_n}}{\zeta^n}, & \text{on } C_2, \end{cases} \tag{86}$$

where the coefficients $a_0, b_0 \in \mathbb{R}$, $\{a_n, b_n \in \mathbb{C} | n \ge 1\}$ are to be found; these expressions encode the known boundary conditions (80). On each boundary the complex variable ζ has a different identification with the variable z given by

$$z = \begin{cases} g(\zeta), & \text{for } z \text{ on } C_1, \\ 1 - g(\zeta), & \text{for } z \text{ on } C_2, \end{cases} \tag{87}$$

with

$$g(\zeta) = -\frac{\omega(\zeta,\alpha)\omega(\zeta,1/\alpha)}{\omega(\zeta,\overline{\alpha})\omega(\zeta,1/\overline{\alpha})}, \quad \omega(\zeta,\alpha) = (\zeta - \alpha), \tag{88}$$

where $\alpha = \mathrm{i}s$ for some $0 < s < 1$, and where $\zeta = \mathrm{e}^{\mathrm{i}\theta}$ for $\theta \in [0,\pi]$. For D shown in Fig. 13 we choose $s = 0.2679492$. $g(\zeta)$ has an interpretation as a conformal slit mapping [6] and, with the value of s just stated, (88) maps the unit ζ-disc to the unbounded region exterior to a circular arc slit on the unit z-circle for $-\pi/3 < \arg[z] < \pi/3$. The upper half semi-circle in the ζ plane, i.e., $\zeta = e^{i\theta}$ for $\theta \in [0, \pi]$ maps in a one-to-one fashion onto the slit $-\pi/3 < \arg[z] < \pi/3$. The relevant s value is found using Newton's method to ensure that $g(1) = e^{-i\pi/3}$. It is helpful to think of $g(\zeta)$ in (88) as a generalization of the classical Joukowski map to one taking the unit disc to the exterior of a circular arc segment (rather than a straight line segment). We anticipate that $g(\zeta)$ will find great utility in the general numerical implementation of our method.

On substitution of (86) into (85), we find

$$\begin{aligned} &a_0 A(k,0) + \sum_{n\geq 1}\{a_n A(k,n) + \overline{a_n} A(k,-n)\} \\ &\quad + b_0 B(k,0) + \sum_{n\geq 1}\{b_n B(k,n) + \overline{b_n} B(k,-n)\} = R(k), \quad k \in -\mathbb{N}, \end{aligned} \tag{89}$$

where

$$A(k,n) \equiv i\int_0^{\pi} \frac{\zeta^n g'(\zeta)}{g(\zeta)^{k+1}}\, i\zeta\, d\theta, \quad B(k,n) \equiv -\int_0^{\pi} \frac{\zeta^n g'(\zeta)}{(1-g(\zeta))^{k+1}}\, i\zeta\, d\theta, \tag{90}$$

and

$$R(k) \equiv -\int_0^{\pi} \frac{\zeta^n g'(\zeta)}{g(\zeta)^{k+1}}\, i\zeta\, d\theta. \tag{91}$$

This system can be solved by truncating the infinite sums and choosing sufficiently many values of $k \in -\mathbb{N}$ to form an overdetermined system solvable by a least-squares procedure for the unknown coefficients.

The effectiveness of this scheme is tested by making particular choices of data $r(z, \overline{z})$ for which analytical solutions are known via alternative conformal mapping methods [6]. A composition of the sequence of conformal maps

$$\begin{aligned} &\zeta \mapsto \eta = \frac{1-\zeta}{1+\zeta}; \quad \eta \mapsto \xi = e^{i\pi/4}\eta; \\ &\xi \mapsto \chi = \xi^{4/3}; \quad \chi \mapsto z = \frac{1}{2} - \frac{i\sqrt{3}}{2}\left[\frac{1-\chi}{1+\chi}\right] \end{aligned} \tag{92}$$

transplants the upper half unit semi-disc in the ζ-plane to the lens-shape domain D in the z-plane. The exponent $4/3$ in the third of the sequence of maps is a consequence of the fact, following from spherical trigonometry arguments, that the angle of intersection between C_1 and C_2 at $z = 1/2 + i\sqrt{3}/2$ is $2\pi/3$. Suppose the inverse of the composite conformal mapping (92) is denoted by $h(z)$ so that

$$\zeta = h(z). \tag{93}$$

Suppose now that we pick, say,

$$f(z) = h(z)^2 = \zeta^2. \tag{94}$$

This function is analytic in D, clearly real on the real ζ axis (and hence on C_2), and it has real part

$$r(z, \overline{z}) = \frac{1}{2}\left[h^2(z) + \frac{1}{h^2(z)}\right] \tag{95}$$

on the upper half unit ζ-disc (and, hence, on C_1). Formula (95) was therefore used as data in the transform solution scheme described above and the resulting solutions checked against $f(z)$ as given in (94). Other choices of $f(z)$ were also tested. The transform method was found to accurately retrieve all such test functions $f(z)$.

As a remark, our use of the function $g(\zeta)$ here is closely related to the use of Chebyshev polynomial expansions although, as already mentioned, not of the standard kind because they have been tailored to a circular arc slit (rather than a straight line interval). The expansions, however, also include square root singularities at the ends of the arcs so they are *not* just expansions in polynomials. As already mentioned, the many approximation theory challenges arising from the numerical implementation of the new transform approach for circular domains presented here await much further investigation (see [16,19,21,22] for a survey of progress in this respect on polygonal domains).

8 Discussion

This paper has presented several new ideas and results.

First, we have given a new derivation of the representation of analytic functions in convex N-sided simply connected polygons due to Fokas and Kapaev [14]. The derivation is elementary, relying on use of Cauchy's integral formula and a suitable spectral representation of the Cauchy kernel obtained from simple properties of the exponential function. The notion of a *spectral matrix* and the idea of a *fundamental contour* for straight line edges arise naturally in this approach. It was also shown that, by an N-fold change of spectral variables, the N-by-N spectral matrix reduces to the same set of N spectral functions considered by Fokas and Kapaev [14].

Then, by replacing the spectral representation of the Cauchy kernel by the one relevant to circular discs (rather than half-planes), the very same sequence of constructive steps starting from Cauchy's integral formula is used to derive the natural extensions of the transforms to general circular domains, both bounded and unbounded, understood geometrically to be intersections of N circular discs.

More general circular domains with boundaries comprising a mixture of straight lines and circular arcs are treated too (cf. Sect. 6). Geometrically these are intersections of half-planes and circular discs and a hybrid construction clearly leads to the relevant transform pairs.

The reduction of the spectral matrix to a set of N spectral functions appears to be special to the case of polygons and does not extend (at least, in any obvious way) to domains with circular boundaries of non-zero curvature. We have argued that the notions of a spectral matrix, and the fundamental contour, are natural.

Several simple illustrative examples have been given of the scope of the new method in the context of examples where the approach can be checked by alternative means. It is clear that the approach has great flexibility and possible future applications. Already the method has been extended by the author to the case of multiply connected circular domains; full details can be found in [9] where some important physical applications of the method are also discussed.

We also touched on several new approximation theory questions, including the optimal choice of integration contour when performing the transform inversion by numerical quadrature, and the best ways to represent unknown data on circular arc boundaries. Such matters are the subject of ongoing investigation.

Other questions arise: are there special boundary value problems in circular domains, perhaps with sufficient geometrical symmetries, where the analysis of the global relations leads to previously unrecognized explicit solutions using just algebra (as has been the case when the unified transform method has been applied to elliptic PDEs in simple polygons [10,16])? Can the new transform formulation here lead to fast and accurate new general numerical methods for the solution of harmonic boundary value problems in general circular domains (evidence now exists that this is possible for polygons [11,16,19])? A natural advantage of transform methods—as we have seen by our very construction where the singular Cauchy kernel has been replaced by a non-singular spectral integral (cf. (6) and (28))—is that they often lead to the possibility of alternative numerical formulations that do not involve *singular* integral equations (as do most standard boundary integral methods).

We have deliberately steered away from the very mathematical ideas that gave rise to the discovery of (1)–(3) in the first place, i.e., the spectral analysis of Lax pair equations, differential forms, etc. [14]. This was done in the conviction that (1)–(3) is a basic result in analytic function theory and should be derivable as such. But the formulation in terms of the spectral analysis of differential forms is the gateway to generalization to other partial differential equations [16,18,25] and it is now of interest to determine how one might reproduce the new transform pairs for circular domains found herein starting from, for example, the spectral analysis of an appropriate differential form [13]. This, in turn, might point the way to generalization of Fokas' unified transform method to other partial differential equations (e.g., the modified Helmholtz equation, or Helmholtz equation) in the more general circular domains of the kind considered here.

Crowdy and Fokas [5] have shown how boundary value problems for the biharmonic equation in polygonal domains in two dimensions can be formulated by appropriate extension of the ideas of Fokas and Kapaev [14] who restricted to harmonic fields. Similarly, the present extended formulation is useful for the solution of biharmonic boundary value problems in general circular domains. There are currently very few alternative techniques to solve biharmonic problems in such domains. Work in this direction is in progress.

The extension of all these ideas to boundary value problems for Laplace's equation in three dimensions—that is, regions interior and exterior to spheres—is an important open problem. In this vein it is worth pointing out that Ashton [2,3] has recently extended the unified transform approach of Fokas to various elliptic boundary value problems defined in convex polyhedra.

Acknowledgments The author is grateful for many useful discussions, over the years, with Thanasis Fokas (whose work has inspired the investigations herein), Euan Spence and Elena Luca. This work is supported by an Established Career Fellowship from the Engineering and Physical Sciences Research Council in the UK. Funds from the Department of Mathematics at Imperial College to support the ACCA-UK network are also acknowledged.

Data accessibility This paper contributes a new theoretical methodology and does not therefore contain any reportable data.

Appendix A: Alternative Derivation of (31)

Here we show how to rederive the transform pair for the unit disc by combining a simple conformal mapping and the result for a general polygon.

Introduce the (Schwarz–Christoffel) conformal mapping

$$\eta = \log z, \quad z = e^{\eta} \tag{96}$$

and take the branch cut of the logarithm to be along the negative z-axis with the choice $-\pi < \arg[z] < \pi$ (say). The map then transplants the unit disc to the left semi-strip in the η-plane (Fig. 14):

$$-\infty < \mathrm{Re}[\eta] < 0, \quad -\pi < \mathrm{Im}[\eta] < \pi. \tag{97}$$

Now let

$$F(\eta) \equiv f(z(\eta)), \tag{98}$$

where $f(z)$ is analytic in the unit disc. Since $F(\eta)$ inherits the property of analyticity, now in the left semi-strip, the transform expression for a function analytic in this polygon, and vanishing as $|\eta| \to \infty$, is

$$F(\eta) = \frac{1}{2\pi}\int_{\tilde{L}_1} e^{ik\eta}\rho_1(k)\,dk + \frac{1}{2\pi}\int_{\tilde{L}_2} e^{ik\eta}\rho_2(k)\,dk + \frac{1}{2\pi}\int_{\tilde{L}_3} e^{ik\eta}\rho_3(k)\,dk, \tag{99}$$

where $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}_3$ are rays from the origin with arguments $0, -\pi/2$ and $-\pi$. The spectral functions are

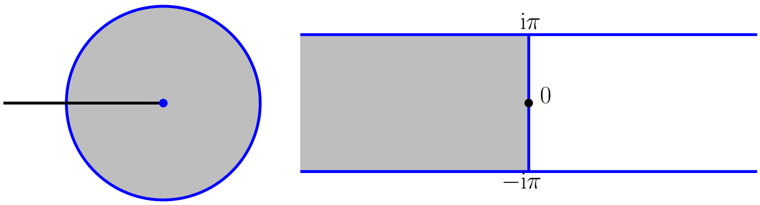


Fig. 14 The logarithmic transformation (96) takes the unit z disc to the left semi-strip in an η-plane

$$\rho_1(k) = \int_{-\mathrm{i}\pi-\infty}^{-\mathrm{i}\pi} F(\eta)\mathrm{e}^{-\mathrm{i}k\eta}\,\mathrm{d}\eta, \quad \rho_2(k) = \int_{-\mathrm{i}\pi}^{\mathrm{i}\pi} F(\eta)\mathrm{e}^{-\mathrm{i}k\eta}\,\mathrm{d}\eta,$$
$$\rho_3(k) = \int_{\mathrm{i}\pi}^{\mathrm{i}\pi-\infty} F(\eta)\mathrm{e}^{-\mathrm{i}k\eta}\,\mathrm{d}\eta. \tag{100}$$

We will return later to the restriction to functions vanishing as $|\eta| \to \infty$.

Since $f(z)$ is single-valued in the z-disc, we need to enforce the constraint

$$F(\eta) = F(\eta - 2\pi\mathrm{i}). \tag{101}$$

On multiplying condition (101) by $\mathrm{e}^{-\mathrm{i}k\eta}$ and integrating along the upper side of the semi-strip, we find

$$\int_{\mathrm{i}\pi}^{\mathrm{i}\pi-\infty} \mathrm{e}^{-\mathrm{i}k\eta} F(\eta)\,\mathrm{d}\eta = \int_{\mathrm{i}\pi}^{\mathrm{i}\pi-\infty} \mathrm{e}^{-\mathrm{i}k\eta} F(\eta - 2\pi\mathrm{i})\,\mathrm{d}\eta = \mathrm{e}^{2\pi k} \int_{-\mathrm{i}\pi}^{-\mathrm{i}\pi-\infty} \mathrm{e}^{-\mathrm{i}k\zeta} F(\zeta)\,\mathrm{d}\zeta, \tag{102}$$

where we have used the change of variable $\zeta = \eta - 2\pi\mathrm{i}$. We conclude that

$$\rho_3(k) = -\mathrm{e}^{2\pi k}\rho_1(k). \tag{103}$$

The global relation for the representation of $F(\eta)$ in the left semi-strip is

$$\rho_1(k) + \rho_2(k) + \rho_3(k) = 0, \quad \mathrm{Im}[k] \geq 0. \tag{104}$$

Use of (103) in (104) leads to

$$\rho_1(k) = \frac{\rho_2(k)}{\mathrm{e}^{2\pi k} - 1}, \tag{105}$$

which is valid on $\tilde{L}_1$ and $\tilde{L}_3$. Relation (99) then gives

$$F(\eta) = \frac{1}{2\pi}\int_{\tilde{L}_1} \mathrm{e}^{\mathrm{i}k\eta}\frac{\rho_2(k)}{\mathrm{e}^{2\pi k} - 1}\,\mathrm{d}k + \frac{1}{2\pi}\int_{\tilde{L}_2} \mathrm{e}^{\mathrm{i}k\eta}\rho_2(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\tilde{L}_3} \mathrm{e}^{\mathrm{i}k\eta}\frac{\mathrm{e}^{2\pi k}\rho_2(k)}{\mathrm{e}^{2\pi k} - 1}\,\mathrm{d}k. \tag{106}$$

On redefining the spectral variable to be

$$\lambda = \mathrm{i}k, \tag{107}$$

and on use of (96), the transform pair for the unit disc becomes

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_0^{\mathrm{i}\infty} \frac{\rho(\lambda)\mathrm{e}^{2\pi\mathrm{i}\lambda}}{1 - \mathrm{e}^{2\pi\mathrm{i}\lambda}} z^\lambda\,\mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\int_0^{\infty} \rho(\lambda) z^\lambda\,\mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{0} \frac{\rho(\lambda)}{1 - \mathrm{e}^{2\pi\mathrm{i}\lambda}} z^\lambda\,\mathrm{d}\lambda,$$
$$\rho(\lambda) = \oint_{|z'|=1} f(z')[z']^{-\lambda-1}\,\mathrm{d}z', \tag{108}$$

and the spectral rays $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}_3$ are rotated by angle $\pi/2$ to be almost those defining the fundamental contour in Fig. 5.

The global relation can be deduced from (105): since $\rho_1(k)$ must be analytic in the upper half k-plane (105) implies that

$$\rho_2(k) = 0, \quad k \in \mathrm{i}\mathbb{N}, \tag{109}$$

or, equivalently,

$$\rho(\lambda) = 0, \quad \lambda \in -\mathbb{N}. \tag{110}$$

The only difference now between what has just been derived and the formulae (31) is the integration contour. But this is because in (31) we allow for $f(z)$ to be non-zero at $z = 0$ while the derivation of (108) assumes *a priori* that $F(\eta) \to 0$ as $|\eta| \to \infty$. The radius-r circular arc deformations of the above contours $\tilde{L}_j$ to L_j for $j = 1, 2, 3$ as shown in Fig. 5 are included to add in this extra constant term contribution from the residue at $k = 0$.

Appendix B: The Classical (Complex) Mellin Transform Pair

It is very instructive to see how the classical (complex) Mellin transform pair manifests itself in our much more general transform approach. The reader will then see how our generalized transforms for circular domains are really just generalizations of the Mellin transform.

Let $f(z)$ be analytic in a wedge in a complex z-plane (see Fig. 15) defined by

$$-\theta < \arg[z] < 0 \tag{111}$$

for some $0 < \theta < 2\pi$. We now seek a transform pair for a function $f(z)$ which is analytic in this wedge.

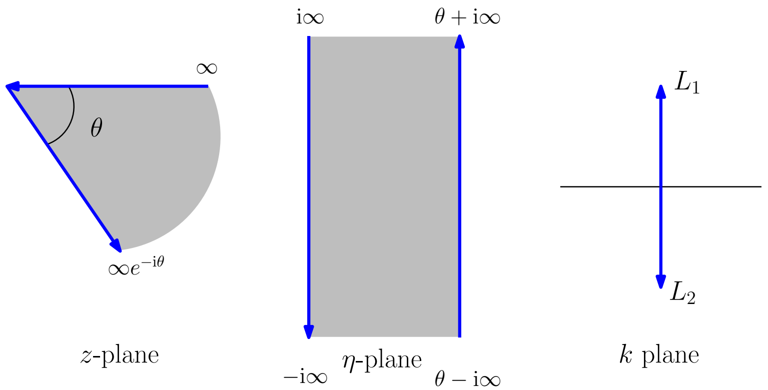


Fig. 15 The logarithmic transformation from a wedge to a vertical strip

In Appendix A, at the end, we had to perform the change of spectral variable (107). To avoid this, instead we modify the original logarithmic change of variable (96) to

$$\eta = \mathrm{i}\log z, \quad z = \mathrm{e}^{-\mathrm{i}\eta}. \tag{112}$$

Under this transformation the wedge in the z-plane is transplanted to a vertical strip in the η plane given by

$$0 < \mathrm{Re}[\eta] < \theta. \tag{113}$$

Let

$$F(\eta) \equiv f(z(\eta)). \tag{114}$$

Since $f(z)$ is analytic in the z-wedge, and $z(\eta)$ is analytic in the vertical η-strip, then $F(\eta)$ is analytic in the vertical η-strip. Hence, from the transform representation of functions analytic in polygons we can write

$$F(\eta) = \frac{1}{2\pi}\left[\int_0^{\mathrm{i}\infty} \rho_1(k)\mathrm{e}^{\mathrm{i}k\eta}\,\mathrm{d}k + \int_0^{-\mathrm{i}\infty} \rho_2(k)\mathrm{e}^{\mathrm{i}k\eta}\,\mathrm{d}k\right], \tag{115}$$

where

$$\rho_1(k) = \int_{\mathrm{i}\infty}^{-\mathrm{i}\infty} F(\eta)\mathrm{e}^{-\mathrm{i}k\eta}\,\mathrm{d}\eta = -\int_0^{\infty} \mathrm{i}f(z)z^{k-1}\,\mathrm{d}z \tag{116}$$

and

$$\rho_2(k) = \int_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty} F(z)\mathrm{e}^{-\mathrm{i}k\eta}\,\mathrm{d}\eta = \int_0^{\infty\mathrm{e}^{-\mathrm{i}\theta}} \mathrm{i}f(z)z^{k-1}\,\mathrm{d}z. \tag{117}$$

But the global relation says that

$$\rho_1(k) + \rho_2(k) = 0, \quad k \in \mathrm{i}\mathbb{R}. \tag{118}$$

Hence we can write (115) as

$$F(\eta) = \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \rho_1(k)\mathrm{e}^{\mathrm{i}k\eta}\,\mathrm{d}k. \tag{119}$$

If we now define

$$\Phi(k) = \mathrm{i}\rho_1(k) \tag{120}$$

then (116) and (119) imply

$$\begin{cases} \Phi(k) = \int_0^\infty f(z) z^{k-1}\, \mathrm{d}z, \\ f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Phi(k) z^{-k}\, \mathrm{d}k. \end{cases} \tag{121}$$

This is the classical complex *Mellin transform pair* with the inverse transform taken along the classical Bromwich contour running parallel along the imaginary k-axis. Since the new transforms for circular domains in (42) are built on the results of Appendix A, which themselves are patently "Mellin transforms for a semi-strip", it is natural to call our generalized transform pairs "Fourier–Mellin transforms for circular domains". In the latter, the Bromwich contour is replaced by our fundamental contour in Fig. 5.

Fokas and Kapaev [14] studied some boundary value problems for Laplace's equation in a wedge, but without first making the logarithmic transformation (112). The scheme above can be a convenient alternative to that approach, one that connects the approach to more classical Mellin transform techniques.

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