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A New Proof of the Lattice Structure of Many-to-Many Pairwise-Stable Matchings

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Abstract This paper studies, under substitutable and cardinal monotone preferences, the lattice structure of the set $S(P)$ of many-to-many pairwise-stable matchings. It proves that the selection matchings are increasing functions on $S(P)$. This fact, along with a few auxiliary results, is then used to prove that $S(P)$ is a distributive lattice. The contribution of this paper is not new, but the alternative proof is interesting as it avoids the use of abstract lattice theory.

Keywords Pairwise-stable matching · Lattice · Cardinal monotonicity

Mathematics Subject Classification 06D10 · 91B68

1 Introduction

In many-to-many two-sided matching markets, definition of stability is a subtle issue; *pairwise-stability*, *setwise-stability,* and *corewise-stability* are common concepts of stability,[1] standard definitions of which imply that "any setwise-stable matching is pairwise-stable and must be in the core."[2]

Under *strict*, *substitutable,* and *cardinal monotone* preferences, Konish and Ünver [9] observed, through an example [9, p. 65, Example 2], that the *core* may be

[1] See Sotomayor [14, p. 56] and Echenique and Oviedo[3, p. 245] for precise definitions. Setwise-stability (group-stability) was first studied by Roth [13].

[2] See Sotomayor [14, p. 58, line 3].

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empty, which answers the open question from Sotomayor [14, p. 58, line 10] "*whether or not many-to-many core matchings always exist is apparently still an open question*"; Blair [2] showed, also through an example, that the set of pairwise-stable matchings and the core might be *disjoint* [2, p. 622, Example 2.6]. Both examples illustrate, though from different perspectives, that the set of *setwise-stable* matchings might be *empty*, which was also proved by Sotomayor [14, p. 60, Example 1].

Roth [11, p. 55, Theorem 2] and Blair [2, p. 623, Theorem 3.2] both proved the existence of many-to-many pairwise-stable matchings. Therefore, currently, only the set of pairwise-stable matchings $S(P)$ may be our study object.

Given two different many-to-many matchings μ_1 and μ_2, let each firm select its most preferred subset of employees from those that assigned to it at μ_1 and μ_2. The selections result in a many-to-many matching, which we denote $\lambda(\mu_1, \mu_2)$ and call *selection matching made by firms*. We may define, similarly, the *selection matching made by workers*, $v(\mu_1, \mu_2)$.

Alkan [1] proved, under *strict*, *substitutable*, and *cardinal monotone revealed* preferences, that the selection matchings of pairwise-stable matchings are themselves pairwise-stable,[3] and that the set of pairwise-stable matchings is a lattice [1, p. 743, Proposition 8].

The current paper proves that the selection matchings are "increasing" functions on the set $S(P)$. This fact, along with a few auxiliary results, is then used to prove that $S(P)$ is a distributive lattice. Although the contribution of this paper is not new, the alternative proof is interesting as it avoids the use of abstract lattice theory.

Hatfield and Milgrom [5] called "*law of aggregate demand*" to "cardinal monotonicity," under which they studied many-to-one *core* [5, p. 917, left column, line 33; right column, line 3] matchings with contracts.[4] But in many-to-many matching markets, the core and the set of pairwise-stable matchings might be disjoint; furthermore, the core might be empty, as we discussed in the second paragraph. So their results cannot be generalized to the current setting.

Hatfield and Kominers [6, 7] studied properties of *setwise-stable* matchings [6, p. 183, the last paragraph; p. 184, definition 4; 7, p.7, definition 2] in a *one-sided* matching game and a many-to-many matching game with contracts, respectively, which they claimed subsume many-to-many matching [6, p. 176, line 5; 7, p. 1, lines

[3] Stability of the selection matchings was called "*consensus property*" by Roth [12, p. 381, line 3], who claimed, under substitutability, in a many-to-one matching model, that the consensus property holds for firms [12, p. 385, Theorem 7]. Blair [2] pointed out that Roth's claim might not hold in his setting [2, p. 624, lines 43–45]. Echenique and Oviedo [3] extended Roth's claim to a many-to-many matching model, under the assumption that each worker $w \in W$ is chosen by one of the subsets of firms, $\mu(w)$ or $\mu'(w)$, that assigned to him or her at two different pairwise-stable matchings μ and μ' [3, p. 253, proposition 9.10]. Li [10, p. 390, example 3.1] justified that Roth's consensus property for firms is not assured under substitutability and analyzed that the selection matching will assign a worker to his or her least preferred assignment or the empty set $\emptyset$.

[4] Note that, they proved most of their results by "*revealed preference*", which was not an assumption of their setting and was not defined in their text. See Appendix in [5, p. 930, left column, line 7 in the proof of Theorem 1; p. 931, left column, lines 5 and 10; p. 932, left column, line 1 in the proof of Theorem 8; p. 933, right column, line 37].

1 to 2 of the Abstract].[5] But in many-to-many matching markets, the set of setwise-stable matchings is a moot point. So the current setting must be different substantially from theirs.

Blair [2] showed that the set of many-to-many pairwise-stable matchings is a lattice under an "appropriate" partial order.[6] However, Blair's lattice was not distributive [2, p. 627, Example 5.2], and the selection matchings might not be pairwise-stable [2, p. 624, line 43].

One distinction between the current paper and Alkan [1] lies in the conditions on preferences necessary for the main results: the current paper works with the *complete* preference orderings, whereas Alkan [1] worked with the *incomplete revealed* preference orderings of agents. Another, crucial, distinction is that Alkan proved the distributivity by a result in abstract lattice theory [1, p. 744, lines 25–30], which is an outside fact of his setting [1, p. 739, lines 22–23], whereas the current paper does it without outside fact.

Echenique and Oviedo [3] acknowledged and analyzed the failure of distributivity in their lattice [3, p. 254, lines 28–30].

The paper is organized as follows. In Sect. 2, we present the preliminary notations and definitions. In Sect. 3, we study the properties of the selection matchings and in Sect. 4 we prove the distributivity.

2 Preliminaries

There are two disjoint sets of agents: the set of n firms $F = \{f_1, \cdots, f_n\}$, and the set of m workers $W = \{w_1, \cdots, w_m\}$. Firms can hire groups of workers, and workers can be employed by sets of firms.

Each firm $f \in F$ has a complete, transitive, and strict preference list P_f over all the subsets of W; each worker $w \in W$ has a complete, transitive, and strict preference list P_w over all the subsets of F. Let $P = \{P_{f_1}, \cdots, P_{f_n}, P_{w_1}, \cdots, P_{w_m}\}$ denote a *Preference profile*.

Given a preference profile P, for each firm $f \in F$ and any two sets of workers $S, S' \subseteq W$, we write SP_fS' to mean f prefers S to S', and SR_fS' to mean f prefers S at least as well as S'; analogously, for each worker $w \in W$ and any two sets of firms $T, T' \subseteq F$, we write TP_wT' and TR_wT'. The alternatives, preferred by agent k to the empty set $\emptyset$, are called *acceptable* to k, otherwise are *unacceptable* to k.

A matching specifies which workers are employed by each firm, which firms employ each worker, and which firms and workers are unmatched. Formally, we give the following definition of a matching.

Definition 2.1 A *matching* μ is a function from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that for every worker $w \in W$, and for every firm $f \in F$:

[5] Gale and Shapley [4, p. 12, Example 3] provided an example that has no pairwise-stable matching in a one-sided matching problem with strict preferences. Since substitutability and law of aggregate demand in many-to-many matching markets both degenerated into strictness in one-to-one matching model, many-to-many pairwise-stable matching might not exist in one-sided many-to-many matching markets.

[6] This partial order then is called "Blair partial order" in the literature of matching theory.

(1) $\mu(w) \subseteq F$;
(2) $\mu(f) \subseteq W$;
(3) $w \in \mu(f) \Leftrightarrow f \in \mu(w)$.

We say an agent $k \in F \cup W$ is *matched* if $\mu(k) \neq \emptyset$, otherwise he is *unmatched*.

Given a preference profile P and a set S of workers, each firm f can determine which subset of S it would most prefer to hire; we call this *$f's$ choice* from S, and denote it by $C_f(S)$, viz., $C_f(S) \subseteq S$ and $C_f(S) R_f S'$ for all $S' \subseteq S$. Similarly, we define each worker *$w's$ choice*, $C_w(T)$, from a set T of firms.

We say a matching μ is *individually rational* if $C_k(\mu(k)) = \mu(k)$ for all agents k, μ is *blocked* by an agent k if $C_k(\mu(k)) \neq \mu(k)$ and by a worker–firm pair (w,f) if $w \notin \mu(f)$, but $w \in C_f(\mu(f) \cup \{w\})$ and $f \in C_w(\mu(w) \cup \{f\})$.

Definition 2.2 A matching is *pairwise-stable* if it is not blocked by any agent or any worker–firm pair.

It is well known that pairwise-stable matching always exists when agents' preferences satisfy substitutability, which was introduced by Kelso and Crawford [8], and which we state formally as below.

Definition 2.3 An agent $k's$ preference list P_k satisfies *substitutability* if, for any subset S of the opposite set that contains agent i, $i \in C_k(S)$ then $i \in C_k(S' \cup \{i\})$ for all $S' \subseteq S$.

Remark 2.4 If agent $k's$ preference list P_k is strict and substitutable, then $C_k(S_1 \cup S_2) = C_k(C_k(S_1) \cup S_2)$, where S_1 and S_2 are subsets of the opposite set.[7]

Alkan [1] introduced cardinal monotonicity, under which he proved the pairwise-stability of the selection matchings, and which we state as follows.

Definition 2.5 An agent $k's$ preference list P_k satisfies *cardinal monotonicity* if, for all subsets S of the opposite set and all $S' \subseteq S$, $|C_k(S')| \leqslant |C_k(S)|$.[8]

3 Monotonicity

Given a preference profile P, let $M(P)$ denote the set of many-to-many matchings, and let $S(P)$ denote the set of many-to-many pairwise-stable matchings. For any two matchings $\mu_1, \mu_2 \in M(P)$, let each firm select its most preferred subset of workers from those that assigned to it at μ_1 and μ_2, then the selections produce a many-to-many matching.

Formally, the *selection matching made by firms of μ_1 and μ_2*, $\lambda(\mu_1, \mu_2)$, is defined by $\lambda(\mu_1, \mu_2)(f) = C_f(\mu_1(f) \cup \mu_2(f))$ for all firms $f \in F$, and $\lambda(\mu_1, \mu_2)(w) = \{f | w \in$

[7] *Proof* By the definition of substitutability, $C_k(S_1 \cup S_2) \cap S_1 \subseteq C_k(S_1)$. Since $C_k(S_1 \cup S_2) \cap S_2 \subseteq S_2$, $(C_k(S_1 \cup S_2) \cap S_1) \cup (C_k(S_1 \cup S_2) \cap S_2) = C_k(S_1 \cup S_2) \cap (S_1 \cup S_2) = C_k(S_1 \cup S_2)$. So $C_k(S_1 \cup S_2) \subseteq C_k(S_1) \cup S_2 \subseteq S_1 \cup S_2$. By the definition of the choice function $C_k(\bullet)$, $C_k(S_1 \cup S_2) R_k C_k(C_k(S_1) \cup S_2)$ and $C_k(C_k(S_1) \cup S_2) R_k C_k(S_1 \cup S_2)$. Strictness gives $C_k(S_1 \cup S_2) = C_k(C_k(S_1) \cup S_2)$.

[8] $|S|$ denotes the number of agents in S.

$C_f(\mu_1(f) \cup \mu_2(f))\}$ for all workers $w \in W$. Similarly, the *selection matching made by workers of* μ_1 *and* μ_2, $v(\mu_1, \mu_2)$, is defined by $v(\mu_1, \mu_2)(w) = C_w(\mu_1(w) \cup \mu_2(w))$ for all workers $w \in W$, and $v(\mu_1, \mu_2)(f) = \{w | f \in C_w(\mu_1(w) \cup \mu_2(w))\}$ for all firms $f \in F$. So λ and v are binary operations on the set $M(P)$.

Let $\mu_1, \mu_2 \in M(P)$, define $\mu_1 \succeq_F^B \mu_2$ iff $C_f(\mu_1(f) \cup \mu_2(f)) = \mu_1(f)$ for all firms $f \in F$, and $\mu_1 \succeq_W^B \mu_2$ iff $C_w(\mu_1(w) \cup \mu_2(w)) = \mu_1(w)$ for all workers $w \in W$. Then for every matching $\mu \in M(P)$, $\mu \succeq_F^B \mu$ and $\mu \succeq_W^B \mu$. So we have two partial orders, $\succeq_F^B$ and $\succeq_W^B$, on the set $M(P)$, which include the stable ones.

Fix a matching $\mu \in M(P)$, let $\lambda_\mu(\mu') = \lambda(\mu, \mu')$ and $v_\mu(\mu') = v(\mu, \mu')$ for all $\mu' \in M(P)$, then λ_μ and v_μ are functions on $M(P)$. We investigate, on the partial sets $(M(P), \succeq_F^B)$ and $(M(P), \succeq_W^B)$, the monotonicity properties of λ_μ and v_μ, which are sufficient for the distributivity proved in the next section, and make our proof substantially different than that of Alkan [1].

Proposition 3.1 *When agents have substitutable preferences, let* $\mu, \mu_1, \mu_2 \in M(P)$, *if* $\mu_1 \succeq_F^B \mu_2$ *then* $\lambda_\mu(\mu_1) \succeq_F^B \lambda_\mu(\mu_2)$.

Proof Suppose $\mu_1 \succeq_F^B \mu_2$, then $C_f(\mu_1(f) \cup \mu_2(f)) = \mu_1(f)$ for all firms $f \in F$ by the definition of $\succeq_F^B$. By the definition of the selection matching made by firms, for all $f \in F$,

$$\begin{aligned}
& C_f(\lambda(\mu, \mu_1)(f) \cup \lambda(\mu, \mu_2)(f)) \\
&= C_f(C_f(\mu(f) \cup \mu_1(f)) \cup C_f(\mu(f) \cup \mu_2(f))) \\
&= C_f(\mu(f) \cup \mu_1(f) \cup \mu_2(f)) \\
&= C_f(\mu(f) \cup C_f(\mu_1(f) \cup \mu_2(f))) \\
&= C_f(\mu(f) \cup \mu_1(f)) \\
&= \lambda(\mu, \mu_1)(f),
\end{aligned}$$

where the first and the last equations hold by the definition of λ, the fourth equation holds by the assumption and all the others by substitutability (see Remark 2.4). Thus $\lambda_\mu(\mu_1) \succeq_F^B \lambda_\mu(\mu_2)$. □

Because workers and firms play a symmetric role in our model, we have directly the following proposition.

Proposition 3.2 *When agents have substitutable preferences, let* $\mu, \mu_1, \mu_2 \in M(P)$, *if* $\mu_1 \succeq_W^B \mu_2$ *then* $v_\mu(\mu_1) \succeq_W^B v_\mu(\mu_2)$.

Propositions 3.1 and 3.2 exhibit that λ_μ and v_μ are "increasing" functions on $(M(P), \succeq_F^B)$ and $(M(P), \succeq_W^B)$, respectively. The following lemma comes from Theorem 4.5 of Blair [2].

Lemma 3.3 *When agents have substitutable preferences, if* $\mu_1, \mu_2 \in S(P)$ *then* $\mu_1 \succeq_F^B \mu_2 \Leftrightarrow \mu_2 \succeq_W^B \mu_1$.

Lemma 3.3 sets a bridge between the two partial orders $\succeq_F^B$ and $\succeq_W^B$, which hence enlarges the domain of both functions, as we will prove below.

The following lemma comes from Proposition 8 of Alkan [1, p. 743].

Lemma 3.4 *When agents have substitutable and cardinal monotone preferences, the set $S(P)$ is closed under λ and v.*

Examples in Li [10, p. 390, Example 3.1] and Alkan [1, p. 745] showed that cardinal monotonicity is a necessary condition for the pairwise-stability of the selection matchings.

Proposition 3.5 *When agents have substitutable and cardinal monotone preferences, let $\mu, \mu_1, \mu_2 \in S(P)$, if $\mu_1 \succeq_F^B \mu_2$ then $v_\mu(\mu_1) \succeq_F^B v_\mu(\mu_2)$.*

Proof Suppose $\mu_1 \succeq_F^B \mu_2$, then Lemma 3.3 implies $\mu_2 \succeq_W^B \mu_1$, so $v_\mu(\mu_2) \succeq_W^B v_\mu(\mu_1)$ by Proposition 3.2. Since μ, μ_1, and μ_2 are pairwise-stable, Lemma 3.4 gives both $v_\mu(\mu_2)$ and $v_\mu(\mu_1)$ are pairwise-stable matchings, hence Lemma 3.3 implies $v_\mu(\mu_1) \succeq_F^B v_\mu(\mu_2)$. □

The following proposition holds immediately by symmetry.

Proposition 3.6 *When agents have substitutable and cardinal monotone preferences, let $\mu, \mu_1, \mu_2 \in S(P)$, if $\mu_1 \succeq_W^B \mu_2$ then $\lambda_\mu(\mu_1) \succeq_W^B \lambda_\mu(\mu_2)$.*

4 Distributivity

We will prove the distributivity in this section. The monotonicity properties proved in the previous section along with the properties proved below are the bases of our proof and make the proof substantially different than that of Alkan [1].

Proposition 4.1 *When agents have substitutable preferences, let $\mu_1, \mu_2 \in S(P)$, then the following equations hold for all firms $f \in F$:*

$$C_f(v(\mu_1, \mu_2)(f) \cup \mu_i(f)) = \mu_i(f), \quad i = 1, 2.$$

Proof We prove the equation for $i = 1$.

Suppose not then there exists a firm $f \in F$ such that $C_f(v(\mu_1, \mu_2)(f) \cup \mu_1(f)) \neq \mu_1(f)$. Since μ_1 is pairwise-stable, $C_f(v(\mu_1, \mu_2)(f) \cup \mu_1(f)) P_f \mu_1(f) R_f \emptyset$, there exists a worker $w \in W$ such that $w \in C_f(v(\mu_1, \mu_2)(f) \cup \mu_1(f))$ and $w \notin \mu_1(f)$; so $w \in v(\mu_1, \mu_2)(f)$, and so $w \in \mu_2(f)$ ($f \in \mu_2(w)$ by the definition of a matching) by the definition of v, and $w \in C_f(\mu_1(f) \cup \{w\})$ by substitutability. Because μ_1 is pairwise-stable and $w \notin \mu_1(f), f \notin C_w(\mu_1(w) \cup \{f\})$. Since $f \in \mu_2(w)$ by the above analysis, $f \notin C_w(\mu_1(w) \cup \mu_2(w))$ by substitutability, viz., $f \notin v(\mu_1, \mu_2)(w)$, which contradicts the above analysis that $f \in v(\mu_1, \mu_2)(w)$. □

Because firms and workers play a symmetric role in our model, we have the following result.

Proposition 4.2 *When agents have substitutable preferences, let $\mu_1, \mu_2 \in S(P)$, then the following equations hold for all workers $w \in W$*

$$C_w(\lambda(\mu_1, \mu_2)(w) \cup \mu_i(w)) = \mu_i(w), \quad i = 1, 2.$$

Lemmas 4.3 and 4.4 below come from Propositions 8 and 7 of Alkan [1], respectively.

Lemma 4.3 *When agents have substitutable and cardinal monotone preferences,* $(S(P), \lambda, v, \succeq_F^B)$ *and* $(S(P), v, \lambda, \succeq_W^B)$ *are lattice.*

Lemma 4.4 *When agents have substitutable and cardinal monotone preferences, the property of complementarity holds on* $S(P)$ *, viz., let* $\mu, \mu' \in S(P)$ *, then (i)* $\lambda(\mu, \mu')(k) \cap v(\mu, \mu')(k) = \mu(k) \cap \mu'(k)$ *and (ii)* $\lambda(\mu, \mu')(k) \cup v(\mu, \mu')(k) = \mu(k) \cup \mu'(k)$ *for all agents* $k \in F \cup W$.

The following proposition exhibits that λ is distributive over v on $S(P)$, and propositions 3.1, 4.1, and the complementarity are the foundation of the proof.

Proposition 4.5 *When agents have substitutable and cardinal monotone preferences, let* $\mu_1, \mu_2, \mu_3 \in S(P)$*, then*

$$\lambda(\mu_1, v(\mu_2, \mu_3)) = v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3)).$$

Proof Proposition 4.1 in conjunction with the definition of $\succeq_F^B$ gives $\mu_2 \succeq_F^B v(\mu_2, \mu_3)$ and $\mu_3 \succeq_F^B v(\mu_2, \mu_3)$, then Proposition 3.1 implies $\lambda(\mu_1, \mu_2) \succeq_F^B \lambda(\mu_1, v(\mu_2, \mu_3))$ and $\lambda(\mu_1, \mu_3) \succeq_F^B \lambda(\mu_1, v(\mu_2, \mu_3))$. Thus, $\lambda(\mu_1, v(\mu_2, \mu_3))$ is a lower bound of both $\lambda(\mu_1, \mu_2)$ and $\lambda(\mu_1, \mu_3)$. Since v is the meet operation of partial order $\succeq_F^B$ (Lemma 4.3), $v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3)) \succeq_F^B \lambda(\mu_1, v(\mu_2, \mu_3))$. Then by the definition of $\succeq_F^B$, for all firms $f \in F$,

$$C_f(v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) \cup \ \lambda(\mu_1, v(\mu_2, \mu_3))(f)) = v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f). \tag{4.1}$$

Suppose there is a firm $f \in F$ such that

$$v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) \neq \lambda(\mu_1, \ v(\mu_2, \mu_3))(f)$$

. Since $\lambda(\mu_1, v(\mu_2, \mu_3))$ is pairwise-stable by Lemma 3.4 and agents have strict preferences, Eq. (4.1) gives

$$v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) P_f \lambda(\mu_1, v(\mu_2, \mu_3))(f) R_f \emptyset.$$

So there exists a worker $w \in v(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f)$, but $w \notin \lambda(\mu_1, v(\mu_2, \mu_3))(f)$. Because agents have substitutable preferences, Eq. (4.1) implies

$$w \in C_f(\lambda(\mu_1, v(\mu_2, \mu_3))(f) \cup \{w\}).$$

Since $\lambda(\mu_1, v(\mu_2, \mu_3))(f) = C_f(\mu_1(f) \cup v(\mu_2, \mu_3)(f))$, $w \notin \mu_1(f) \cup v(\mu_2, \mu_3)(f)$ by substitutability. Then

$$w \notin \mu_1(f) \text{ and } w \notin \nu(\mu_2, \mu_3)(f). \tag{4.2}$$

Since $w \in \mu_1(f) \cup \mu_2(f) \cup \mu_3(f)$, $w \in \mu_2(f) \cup \mu_3(f)$. Then $w \in \lambda(\mu_2, \mu_3)(f)$ and $w \notin \mu_2(f) \cap \mu_3(f)$ by complementarity. So $w \notin \mu_1(f) \cup \mu_2(f)$ or $w \notin \mu_1(f) \cup \mu_3(f)$. Since $\lambda(\mu_1, \mu_2)(f) \subseteq \mu_1(f) \cup \mu_2(f)$ and $\lambda(\mu_1, \mu_3)(f) \subseteq \mu_1(f) \cup \mu_3(f)$, $w \notin \lambda(\mu_1, \mu_2)(f)$ or $w \notin \lambda(\mu_1, \mu_3)(f)$. So $w \notin \lambda(\mu_1, \mu_2)(f) \cap \lambda(\mu_1, \mu_3)(f)$ and consequently $w \notin \lambda(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) \cap \nu(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f)$ by complementarity. Since $w \in \nu(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f)$ by the assumption, again by complementarity $w \notin \lambda(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f)$. Since $\lambda(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) = C_f(\mu_1(f) \cup \mu_2(f) \cup \mu_3(f))$ by the definition of $\succeq_F^B$ and by substitutability, then $w \notin C_f(\mu_1(f) \cup \mu_2(f) \cup \mu_3(f))$.

By the assumption, $w \in \nu(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) \subseteq \lambda(\mu_1, \mu_2)(f) \cup \lambda(\mu_1, \mu_3)(f)$, and $w \notin \nu(\mu_2, \mu_3)(f)$ and $w \in \mu_2(f) \cup \mu_3(f)$ (see Eq. (4.2)) give $w \in \lambda(\mu_2, \mu_3)(f)$ by complementarity, thus $w \in \lambda(\mu_1, \mu_2)(f) \cap \lambda(\mu_2, \mu_3)(f)$ or $w \in \lambda(\mu_1, \mu_3)(f) \cap \lambda(\mu_2, \mu_3)(f)$. Therefore, $w \in \lambda(\lambda(\mu_1, \mu_2), \lambda(\mu_2, \mu_3))(f)$ or $w \in \lambda(\lambda(\mu_1, \mu_3), \lambda(\mu_2, \mu_3))(f)$ by complementarity. But $\lambda(\lambda(\mu_1, \mu_2), \lambda(\mu_2, \mu_3))(f) = C_f(\mu_1(f) \cup \mu_2(f) \cup \mu_3(f))$ and $\lambda(\lambda(\mu_1, \mu_3), \lambda(\mu_2, \mu_3))(f) = C_f(\mu_1(f) \cup \mu_2(f) \cup \mu_3(f))$ by substitutability. Therefore, $w \in C_f(\mu_1(f) \cup \mu_2(f) \cup \mu_3(f))$ which contradicts the above conclusion that $w \notin C_f(\mu_1(f) \cup \mu_2(f) \cup \mu_3(f))$.

Thus, $\nu(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))(f) = \lambda(\mu_1, \nu(\mu_2, \mu_3))(f)$ for all firms $f \in F$, viz., $\lambda(\mu_1, \nu(\mu_2, \mu_3)) = \nu(\lambda(\mu_1, \mu_2), \lambda(\mu_1, \mu_3))$. □

Because workers and firms play a symmetric role, Proposition 3.2, combined with the definition of $\succeq_W^B$ and Proposition 4.2, yields the following result.

Proposition 4.6 *When agents have substitutable and cardinal monotone preferences, let* $\mu_1, \mu_2, \mu_3 \in S(P)$, *then*

$$\nu(\mu_1, \lambda(\mu_2, \mu_3)) = \lambda(\nu(\mu_1, \mu_2), \nu(\mu_1, \mu_3)).$$

Example 5.2 of Blair [2] shows that, when agents do not have cardinal monotone preferences, the set of pairwise-stable matchings might not be a distributive lattice.

5 Conclusion

Lattice is a fundamental concept of optimization theory; it provides a concrete path to the optimal objectives. In many-to-many two-sided matching markets, the interests of agents on the same side of the market can be simultaneously maximized [4, p. 55, theorem 2]. Alkan [1] studied the lattice structure of many-to-many pairwise-stable matchings; he proved the distributivity by an abstract lattice theory which is outside his setting.

The current paper provides an alternative proof of the distributive lattice of many-to-many pairwise-stable matchings, which improves the study of the lattice structure in two-sided matching markets. The generality of our analysis is not only theoretically interesting but may lead to new insights on some open problems, such as the necessary and sufficient conditions for existence of a setwise-stable matching and a corewise-stable matching. These topics are left for future research.

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References

[1] Alkan, A.: A class of multipartner matching markets with a strong lattice structure. Econ. Theor. **19**(4), 737–746 (2002)
[2] Blair, C.: The lattice structure of the set of pairwise-stable matchings with multiple partners. Math. Oper. Res. **13**(4), 619–628 (1988)
[3] Echenique, F., Oviedo, J.: A theory of stability in many-to-many matching markets. Theor. Econ. **1**(2), 233–273 (2006)
[4] Gale, D., Shapley, L.S.: College admissions and the stability of marriage. Am. Math. Mon. **69**(1), 9–15 (1962)
[5] Hatfield, J.W., Milgrom, P.R.: Matching with contracts. Am. Econ. Rev. **95**(4), 913–935 (2005)
[6] Hatfield, J.W., Kominers, S.D.: Matching in networks with bilateral contracts. Am. Econ. J: Microeconomics **4**(1), 176–208 (2012)
[7] Hatfield, J.W., Kominers, S.D.: Contract design and stability on many-to-many matching. Harvard Business School, Mineo (2012)
[8] Kelso, A.S., Crawford, V.P.: Job matching, coalition formation, and gross substitutes. Econometrica **50**(6), 1483–1504 (1982)
[9] Konish, H., Ünver, M.U.: Credible group stability in many-to-many matching problems. J. Econ. Theor. **129**(1), 57–80 (2006)
[10] Li, J.: A note on Roth's consensus property of many-to-one matching. Math. Oper. Res. **38**(2), 389–392 (2013)
[11] Roth, A.E.: Stability and polarization of interests in job matching. Econometrica **52**(1), 47–57 (1984)
[12] Roth, A.E.: Conflict and coincidence of interest in job matching: some new results and open questions. Math. Oper. Res. **10**(3), 379–389 (1985)
[13] Roth, A.E.: The college admissions problem is not equivalent to the marriage problem. J. Econ. Theory **36**(2), 277–288 (1985)
[14] Sotomayor, M.: Three remarks on the many-to-many pairwise-stable matching problem. Math. Soc. Sci. **38**(1), 55–70 (1999)