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ORIGINAL RESEARCH

σ-Approximately module amenable Banach algebras

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Abstract In this paper, we define the notion of sigma-approximate module amenability of Banach algebras and give some properties about this notion. Also for Banach $\mathfrak{A}$-bimodule $\mathcal{A}$, and $\mathcal{J}$, the closed ideal of $\mathcal{A}$ generated by elements of form $(\alpha \cdot a)b - a(b \cdot \alpha)$, $(a, b \in \mathcal{A}, \quad \alpha \in \mathfrak{A})$, we considered some corollaries about $\widehat{\sigma}$-approximate amenability of $\frac{\mathcal{A}}{\mathcal{J}}$ as a Banach $\mathcal{A}$-bimodule, where $\widehat{\sigma} : \frac{\mathcal{A}}{\mathcal{J}} \to \frac{\mathcal{A}}{\mathcal{J}}$ by $\widehat{\sigma}(a + \mathcal{J}) = \sigma(a) + \mathcal{J}$ has a dense range.

Keywords σ-Approximate module amenable · Banach algebras · Banach module · σ-Module derivation

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Introduction

The concept of module amenability for Banach algebras was introduced by Amini [1]. Let $\mathfrak{A}$ and $\mathcal{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha \cdot (ab) = (\alpha \cdot a)b \quad \text{and} \quad (ab) \cdot \alpha = a(b \cdot \alpha),$$

for all $a, b \in \mathcal{A}$, $\alpha \in \mathfrak{A}$. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatibility of actions:

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x \quad \text{and} \quad a \cdot (x \cdot \alpha) = (a \cdot x) \cdot \alpha,$$

for all $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$, $x \in \mathcal{X}$, and the same for the other side actions. Then, we say that $\mathcal{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. If moreover, $\alpha \cdot x = x \cdot \alpha$, $(\alpha \in \mathfrak{A}, x \in \mathcal{X})$, then $\mathcal{X}$ is called a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule. Note that, when $\mathcal{A}$ acts on itself by algebra multiplication from both sides, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule because $\mathcal{A}$ does not satisfy $a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b$, $(\alpha \in \mathfrak{A}, \quad a, b \in \mathcal{A})$ [1].

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-bimodule and acts on itself by algebra multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. Also, if $\mathcal{A}$ is a commutative Banach algebra, then it is a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule.

Now suppose that $\mathcal{X}$ be an $\mathcal{A}$-$\mathfrak{A}$-bimodule, then a continuous map $T : \mathcal{A} \to \mathcal{X}$ is called an $\mathfrak{A}$-bimodule map, if $T(a \pm b) = T(a) \pm T(b)$ and $T(\alpha \cdot a) = \alpha \cdot T(a)$ and $T(a \cdot \alpha) = T(a) \cdot \alpha$, for each $\alpha \in \mathfrak{A}$, $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$. The space of all $\mathfrak{A}$-bimodule maps $T : \mathcal{A} \to \mathcal{X}$ such that $T(ab) = T(a)T(b)$, $(a, b \in \mathcal{A})$, is denoted by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{X})$. Also we denote $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A})$, by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. A bounded $\mathfrak{A}$-bimodule map $D : \mathcal{A} \to \mathcal{X}$ is called a module derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad (a, b \in \mathcal{A}).$$

D is not necessary linear, but its boundedness implies its norm continuity, because it preserves subtraction. When $\mathcal{X}$ is commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule, each $x \in \mathcal{X}$ defines a module derivation

$$\delta_x(a) = a \cdot x - x \cdot a, \quad (a \in \mathcal{A}),$$

which is called an inner module derivations.

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. A σ-module derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$

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is a bounded $\mathfrak{A}$-bimodule map $D:\mathcal{A}\longrightarrow\mathcal{X}$ satisfying

$$D(ab)=\sigma(a)\cdot D(b)+D(a)\cdot\sigma(b),\quad (a,b\in\mathcal{A}).$$

When $\mathcal{X}$ is commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule, each $x\in\mathcal{X}$ defines a σ-module derivation

$$\delta_x^\sigma:\mathcal{A}\longrightarrow\mathcal{X},\quad \delta_x^\sigma(a)=\sigma(a)\cdot x-x\cdot\sigma(a),\quad (a\in\mathcal{A}),$$

which is called a σ-inner module derivation.

σ-Approximate module amenability

We start this section with definition of sigma-approximate module amenability, then we consider some hereditary properties of this concept.

Definition 1 Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\sigma\in Hom_{\mathfrak{A}}(\mathcal{A})$. We say that $\mathcal{A}$ is a σ-approximately module amenable (σ-(AMA)), if for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule, $\mathcal{X}$, every σ-module derivation $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ is σ-approximately inner, i.e, there is a net $(x_i)_{i\in\mathfrak{I}}\in\mathcal{X}^*$ such that $D(a)=\lim_i\delta_{x_i}^\sigma(a)=\lim_i\sigma(a)x_i-x_i\sigma(a)$, $(a\in\mathcal{A})$. Also we say that $\mathcal{A}$ is a σ-approximately module contractible ($\sigma-(AMC)$), if for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule, $\mathcal{X}$, every σ-module derivation $D:\mathcal{A}\longrightarrow\mathcal{X}$ is σ-approximately inner.

The two following results is the σ-approximate version of [1, Proposition 2.1] and [5], respectively.

Proposition 2 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\sigma\in Hom_{\mathfrak{A}}(\mathcal{A})$. Suppose that $\mathfrak{A}$ has a bounded approximate identity and $\mathcal{A}$ is σ-approximately amenable. Then $\mathcal{A}$ is σ-(AMA).*

Proof Let $\mathcal{X}$ be a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule and $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ be a σ-module derivation. By [1, Proposition 2.1], D is a σ-derivation, i.e, D is $\mathbb{C}$-linear. Now since $\mathcal{A}$ is σ-approximately amenable, $\mathcal{A}$ is σ-(AMA). □

Proposition 3 *Let $\mathcal{A}$ be an essential left Banach $\mathfrak{A}$-bimodule and $\sigma\in Hom_{\mathfrak{A}}(\mathcal{A})$. If $\mathcal{A}$ is σ-approximately amenable, then $\mathcal{A}$ is σ-(AMA).*

Proof Let $\mathcal{X}$ be a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule and $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ be a σ-module derivation. Since $\mathcal{A}$ is an essential left Banach $\mathfrak{A}$-bimodule, D is $\mathbb{C}$-linear [5]. Now since $\mathcal{A}$ is σ-approximately amenable, D is σ-approximately inner and thus $\mathcal{A}$ is σ-(AMA). □

Proposition 4 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\sigma\in Hom_{\mathfrak{A}}(\mathcal{A})$. If $\mathcal{A}$ is σ-(AMA), then $\mathcal{A}$ is $(\lambda\circ\sigma,\mu\circ\sigma)$-(AMA), for each $\lambda,\mu\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.*

Proof Let $\mathcal{X}$ be a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule and $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ be a $(\lambda\circ\sigma,\mu\circ\sigma)$-module derivation. Then $\mathcal{X}$ is an $\mathcal{A}$-module derivation with the following module actions:

$$a*x=\lambda(a)\cdot x\quad\text{and}\quad x*a=x\cdot\mu(a),\quad (a\in\mathcal{A},\ x\in\mathcal{X}).$$

It is easy to see that $\mathcal{X}$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule with this product. We have

$$\begin{aligned}D(ab)&=(\lambda\circ\sigma)(a)\cdot D(b)+D(a)\cdot(\mu\circ\sigma)(b)\\&=\sigma(a)*D(b)+D(a)*\sigma(b),\quad (a,b\in\mathcal{A}).\end{aligned}$$

Thus, D is a σ-module derivation. So there exists a net $(x_i)\in\mathcal{X}^*$ such that $D(a)=\lim_i\delta_{x_i}^\sigma(a)$, $(a\in\mathcal{A})$. So we have

$$\begin{aligned}D(a)&=\lim_i(\sigma(a)*x_i-x_i*\sigma(a))\\&=\lim_i((\lambda\circ\sigma)(a)\cdot x_i-x_i\cdot(\mu\circ\sigma)(a)),\quad (a\in\mathcal{A}).\end{aligned}$$

Which shows that D is $(\lambda\circ\sigma,\mu\circ\sigma)$-approximately inner. Thus, $\mathcal{A}$ is $(\lambda\circ\sigma,\mu\circ\sigma)$-(AMA). □

Corollary 5 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule. If $\mathcal{A}$ is (AMA), then $\mathcal{A}$ is (λ,μ)-(AMA), for each $\lambda,\mu\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.*

Proposition 6 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\sigma\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Suppose that σ is an idempotent epimorphism and $\mathcal{A}$ is σ-(AMA). Then, $\mathcal{A}$ is (AMA).*

Proof Let $\mathcal{X}$ be a commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule and $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ be a module derivation. So $\widetilde{D}=D\circ\sigma$ is a σ-module derivation, because, for each $a,b\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ we have

$$\begin{aligned}\widetilde{D}(ab)&=D\circ\sigma(ab)=D(\sigma(a)\sigma(b))\\&=\sigma(a)(D\circ\sigma)(b)+(D\circ\sigma)(a)\sigma(b),\end{aligned}$$

and

$$\widetilde{D}(\alpha a)=D(\sigma(\alpha a))=D(\alpha\sigma(a))=\alpha D(\sigma(a)).$$

Since $\mathcal{A}$ is σ-(AMA), there exists a net $(x_i)_{i\in\mathfrak{I}}\in\mathcal{X}^*$ such that $\widetilde{D}(a)=\lim_i(\sigma(a)x_i-x_i\sigma(a))$, $(a\in\mathcal{A})$. Now for each $b\in\mathcal{A}$, there exists $a\in\mathcal{A}$ such that $b=\sigma(a)$. Therefore,

$$\begin{aligned}D(b)=D(\sigma(a))=\widetilde{D}(a)&=\lim_i(\sigma(a)x_i-x_i\sigma(a))\\&=\lim_i(bx_i-x_ib),\quad (b\in\mathcal{A}).\end{aligned}$$

So D is approximately inner and $\mathcal{A}$ is (AMA). □

Proposition 7 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-bimodules and $\sigma\in Hom_{\mathfrak{A}}(\mathcal{A})$ and $\tau\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$. Suppose that $\varphi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$ be a surjective map such that $\varphi\circ\sigma=\tau\circ\varphi$. If $\mathcal{A}$ is σ-(AMA), then $\mathcal{B}$ is τ-(AMA).*

Proof Let $\mathcal{X}$ be a commutative Banach $\mathcal{B}$-$\mathfrak{A}$-bimodule and $\mathcal{D}:\mathcal{B}\rightarrow\mathcal{X}^*$ be a τ-module derivation. Then, $(\mathcal{X},*)$ can be considered as a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule by the following module actions:

$$a * x = \varphi(a) \cdot x \quad \text{and} \quad x * a = x \cdot \varphi(a), \quad (a \in \mathcal{A},\, x \in X).$$

Therefore, $\widetilde{D} = D \circ \varphi : \mathcal{A} \to (\mathcal{X}^*, *)$ is a σ-module derivation, because

$$\begin{aligned}\widetilde{D}(ab) &= D(\varphi(a)\varphi(b)) \\ &= D(\varphi(a))\tau(\varphi(b)) + \tau(\varphi(a))D(\varphi(b)) \\ &= \widetilde{D}(a)\varphi(\sigma(b)) + \varphi(\sigma(a))\widetilde{D}(b) \\ &= \widetilde{D}(a) * \sigma(b) + \sigma(a) * \widetilde{D}(b), \quad (a, b \in \mathcal{A}).\end{aligned}$$

Since $\mathcal{A}$ is σ-(AMA), there exists a net $(x_i)_{i \in \mathfrak{I}} \in \mathcal{X}^*$ such that $\widetilde{D} = \lim_i \delta^{\sigma}_{x_i}$. So we have

$$\begin{aligned}\widetilde{D}(a) &= \lim_i \sigma(a) * x_i - x_i * \sigma(a) \\ &= \lim_\alpha \varphi(\sigma(a)) \cdot x_i - x_i \cdot \varphi(\sigma(a)) \\ &= \lim_\alpha \tau(\varphi(a)) \cdot x_i - x_i \cdot \tau(\varphi(a)), \quad (a \in \mathcal{A}).\end{aligned}$$

Since φ is a surjective map, so $D(b) = \lim_i \tau(b) \cdot x_i - x_i \cdot \tau(b)$, $(b \in \mathcal{B})$. Hence, $\mathcal{B}$ is τ-(AMA). □

Proposition 8 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach $\mathfrak{A}$-modules and $\varphi \in Hom_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$ be a surjective map. If $\mathcal{A}$ is (AMA), then $\mathcal{B}$ is σ-(AMA), for each $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$.*

Proof Let $\mathcal{X}$ be a Banach $\mathcal{B}$-$\mathfrak{A}$-bimodule and $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$. Then $(\mathcal{X}, *)$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule with the following module actions:

$$a * x = \sigma(\varphi(a)) \cdot x \quad \text{and} \quad x * a = x \cdot \sigma(\varphi(a)), \quad (a \in \mathcal{A}, x \in \mathcal{X}).$$

Now, let $D : \mathcal{B} \to \mathcal{X}^*$ be a σ-module derivation. So $\widetilde{D} = D \circ \varphi : \mathcal{A} \to (\mathcal{X}^*, *)$ is a module derivation, because for each $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$, we have

$$\widetilde{D}(\alpha a) = D(\varphi(\alpha a)) = D(\alpha\varphi(a)) = \alpha D(\varphi(a)),$$

and

$$\begin{aligned}\widetilde{D}(ab) &= D(\varphi(ab)) \\ &= D(\varphi(a))\sigma(\varphi(b)) + \sigma(\varphi(a))D(\varphi(b)) \\ &= \widetilde{D}(a) * b + a * \widetilde{D}(b).\end{aligned}$$

So there exists a net $(x_i)_{i \in \mathfrak{I}} \in X^*$ such that $\widetilde{D} = \lim_i \delta_{x_i}$ and we have

$$\begin{aligned}\widetilde{D}(a) &= \lim_i \delta_{x_i}(a) \\ &= \lim_i (a * x_i - x_i * a) \\ &= \lim_i \sigma(\varphi(a)) \cdot x_i - x_i \cdot \sigma(\varphi(a)), \quad (a \in \mathcal{A}).\end{aligned}$$

Since φ is surjective, for each $b \in \mathcal{B}$, there exists $a \in \mathcal{A}$, such that $b = \varphi(a)$. So for each $b \in \mathcal{B}$ we have

$$\begin{aligned}D(b) = D(\varphi(a)) = \widetilde{D}(a) &= \lim_i \sigma(\varphi(a)) \cdot x_i - x_i \cdot \sigma(\varphi(a)) \\ &= \lim_i \sigma(b) \cdot x_i - x_i \cdot \sigma(b).\end{aligned}$$

Which shows that D is σ-approximately inner. Thus, $\mathcal{B}$ is σ-(AMA). □

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with compatible actions and $\mathcal{J}$ be the closed ideal of $\mathcal{A}$ generated by elements of form $(\alpha \cdot a)b - a(b \cdot \alpha)$, for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Then, the quotient Banach algebra $\frac{\mathcal{A}}{\mathcal{J}}$ is Banach $\mathcal{A}$-bimodule with compatible actions [2]. The following Lemma is proved in [3].

Lemma 9 Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. Suppose that $\sigma \in Hom_{\mathfrak{A}}(\mathcal{A})$ such that $\sigma(\mathcal{J}) \subseteq \mathcal{J}$. Then $\widehat{\sigma} : \frac{\mathcal{A}}{\mathcal{J}} \to \frac{\mathcal{A}}{\mathcal{J}}$ by $\widehat{\sigma}(a + \mathcal{J}) = \sigma(a) + \mathcal{J}$ is $\mathbb{C}$-linear.

Proposition 10 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. Let σ be as in above lemma. If $\frac{\mathcal{A}}{\mathcal{J}}$ is $\widehat{\sigma}$-approximately amenable, then $\mathcal{A}$ is σ-(AMA).*

Proof Let $\mathcal{X}$ be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. It is easy to see that $\mathcal{J} \cdot \mathcal{X} = \mathcal{X} \cdot \mathcal{J} = 0$. So $\mathcal{X}$ is a Banach $\frac{\mathcal{A}}{\mathcal{J}}$-bimodule with the following module actions;

$$(a + \mathcal{J}) \cdot x = ax \quad \text{and} \quad x \cdot (a + \mathcal{J}) = xa, \quad (a \in \mathcal{A}, x \in \mathcal{X}).$$

Suppose that $D : \mathcal{A} \to \mathcal{X}^*$ be a σ-module derivation. Define $\widehat{D} : \frac{\mathcal{A}}{\mathcal{J}} \to \mathcal{X}^*$ by $\widehat{D}(a + \mathcal{J}) = D(a)$, $(a \in \mathcal{A})$. $\widehat{D}$ is well defined [3, Proposition 2.6] and it is $\mathbb{C}$-linear [1, Proposition 2.1]. Also, it is easy to see that $\widehat{D}(ab + \mathcal{J}) = \widehat{D}(a + \mathcal{J})\ \widehat{\sigma}(b + \mathcal{J}) + \widehat{\sigma}(a + \mathcal{J})\widehat{D}(b + \mathcal{J})$. Moreover according to the above Lemma, $\widehat{\sigma}$ is $\mathbb{C}$-linear. Therefore, $\widehat{D}$ is $\widehat{\sigma}$-derivation. Thus, there exists a net $(x_i)_{i \in \mathfrak{I}} \in X^*$ such that $\widehat{D} = \lim_i \delta^{\widehat{\sigma}}_{x_i}$ and we have

$$\begin{aligned}D(a) = \widehat{D}(a + \mathcal{J}) &= \lim_i (\widehat{\sigma}(a) \cdot x_i - x_i \cdot \widehat{\sigma}(a)) \\ &= \lim_i (\sigma(a) + \mathcal{J}) \cdot x_i - x_i \cdot (\sigma(a) + \mathcal{J}) \\ &= \lim_i \sigma(a)x_i - x_i\sigma(a), \quad (a \in \mathcal{A}).\end{aligned}$$

Which shows that D is σ-approximately inner and therefore $\mathcal{A}$ is σ-(AMA). □

In [4], section 3, we stated some properties of σ-approximate contractibility when $\mathcal{A}$ has an identity and considered some corollaries when $\sigma(\mathcal{A})$ is dense in $\mathcal{A}$. In proof of the following proposition we use those results. Recall that, the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$, where f is a continuous linear functional on $\mathfrak{A}$.

Proposition 11 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with trivial left actions and σ be as in above lemma. Suppose that $\mathcal{A}$ is σ-(AMA). If $\frac{\mathcal{A}}{\mathcal{J}}$ has an identity and $\overline{\widehat{\sigma}(\frac{\mathcal{A}}{\mathcal{J}})} = \frac{\mathcal{A}}{\mathcal{J}}$, then $\frac{\mathcal{A}}{\mathcal{J}}$ is $\widehat{\sigma}$-approximately amenable.*

Proof By [4, Corollary 3.3.], we can assume that $\mathcal{X}$ is a σ-unital Banach $\frac{\mathcal{A}}{\mathcal{J}}$-bimodule. Let $e + \mathcal{J}$ be the identity in $\frac{\mathcal{A}}{\mathcal{J}}$. So $\widehat{\sigma}(e + \mathcal{J})$ is a unit for $\widehat{\sigma}(\frac{\mathcal{A}}{\mathcal{J}})$. Thus by density of $\widehat{\sigma}(\frac{\mathcal{A}}{\mathcal{J}})$ in $\frac{\mathcal{A}}{\mathcal{J}}$, we see that $\widehat{\sigma}(e + \mathcal{J}) = e + \mathcal{J}$. Now let $\widehat{D} : \frac{\mathcal{A}}{\mathcal{J}} \to \mathcal{X}^*$ be a $\widehat{\sigma}$-derivation. By [4, Lemma 3.7], $\widehat{D}(e + \mathcal{J}) = 0$. Now similar to [3, Proposition 2.7], we can see $\mathcal{X}$ as a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule and $D = \widehat{D} \circ \pi : \mathcal{A} \to \mathcal{X}^*$ is a σ-module derivation, where $\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{J}}$ is the natural $\mathfrak{A}$-module map. Since $\mathcal{A}$ is σ-(AMA), there exists a net $(x_i)_{i\in\mathfrak{I}} \in X^*$ such that $D = \lim_i \delta^{\widehat{\sigma}}_{x_i}$ and we have,

$$\widehat{D}(a + \mathcal{J}) = \lim_i (\sigma(a) + \mathcal{J}) \cdot x_i - x_i \cdot (\sigma(a) + \mathcal{J})$$
$$= \lim_i \widehat{\sigma}(a + \mathcal{J}) \cdot x_i - x_i \cdot \widehat{\sigma}(a + \mathcal{J}), \quad (a \in \mathcal{A}).$$

Which means that $\widehat{D}$ is $\widehat{\sigma}$-approximately inner and $\widehat{\sigma}$-approximately amenable. □

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then, $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$, is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions. Then, $\mathcal{A}$ is a Banach algebra and a Banach $\mathfrak{A}^\#$-bimodule with the following actions:

$$(\alpha, \lambda)a = \alpha a + \lambda a \quad \text{and} \quad a(\alpha, \lambda) = a\alpha + a\lambda,$$
$$(\alpha \in \mathfrak{A}, \lambda \in \mathbb{C}, \mathfrak{a} \in \mathcal{A}).$$

Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions and let $\mathcal{A}^\# = \mathcal{A} \oplus \mathfrak{A}^\#$. Then $(\mathcal{A}^\#, \cdot)$ is a Banach algebra, where the multiplication $\cdot$ is defined by $(a, \mathfrak{u}) \cdot (\mathfrak{b}, \mathfrak{v}) = (\mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{v} + \mathfrak{u}\mathfrak{b}, \mathfrak{u}\mathfrak{v})$, $(a, b \in \mathcal{A}, \mathfrak{u}, \mathfrak{v} \in \mathfrak{A})$. Also $\mathcal{A}^\#$ is a Banach $\mathfrak{A}^\#$-bimodule with the following module actions:

$$(a, \mathfrak{u}) \cdot \mathfrak{v} = (a \cdot \mathfrak{v}, \mathfrak{u}\mathfrak{v}) \quad \text{and} \quad \mathfrak{v} \cdot (a, \mathfrak{u}) = (\mathfrak{v} \cdot a, \mathfrak{v}\mathfrak{u})$$
$$(a \in \mathcal{A}, \mathfrak{u}, \mathfrak{v} \in \mathfrak{A}^\#).$$

So $\mathcal{A}^\#$ is a unital Banach $\mathfrak{A}^\#$-bimodule with compatible actions.

A similar result of [5, Theorem 3.1], for approximate module amenability, is as follows:

Proposition 12 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule, $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Then $\widehat{\sigma}(a, \mathfrak{u}) = \sigma(\mathfrak{a}) \oplus \mathfrak{u}$, $(a \in \mathcal{A}, \mathfrak{u} \in \mathfrak{A}^\#)$ is in $\mathrm{Hom}^\#_{\mathfrak{A}}(\mathcal{A}^\#)$ and the following are equivalent;*

(i) $\mathcal{A}$ is σ-(AMA) as an $\mathfrak{A}^\#$-bimodule.
(ii) $\mathcal{A}^\#$ is $\widehat{\sigma}$-(AMA) as an $\mathfrak{A}^\#$-bimodule.

Proof It is easy to see that $\widehat{\sigma} \in \mathrm{Hom}^\#_{\mathfrak{A}}(\mathcal{A}^\#)$.

$i \Rightarrow 2$. Let $\mathcal{X}$ be a commutative Banach $\mathcal{A}^\#-\mathfrak{A}^\#$-bimodule and $\widehat{D} : \mathcal{A}^\# \to \mathcal{X}^*$ be a $\widehat{\sigma}$-module derivation. By [4, Lemma 3.1], $\widehat{D}(1) = 0$. So $D = \widehat{D}\,|_{\mathcal{A}}: \mathcal{A} \to \mathcal{X}^*$ is a σ-module derivation. Thus by the hypothesis, there exists a net $(x_i)_{i\in\mathfrak{I}} \in X^*$ such that $D = \lim_i \delta^{\sigma}_{x_i}$. Note that $\mathcal{X}$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}^\#$-module and $\widehat{D}(a, 0) = \lim_i \widehat{\sigma}(a, 0)x_i - x_i\widehat{\sigma}(a, 0)$, $(a \in \mathcal{A})$. Also we have

$$\widehat{D}(a, \mathfrak{u}) = \widehat{D}((a, 0) + (0, \mathfrak{u})) = \widehat{D}(a, 0) + \widehat{D}(0, \mathfrak{u})$$
$$= \widehat{D}(a, 0), \quad ((a, \mathfrak{u}) \in \mathcal{A}^\#).$$

Thus, $\widehat{D}$ is $\widehat{\sigma}$-approximately inner and therefore $\mathcal{A}^\#$ is $\widehat{\sigma}$-(AMA).

$ii \Rightarrow i$. Let $\mathcal{X}$ be a commutative Banach $\mathcal{A}$-$\mathfrak{A}^\#$-bimodule and $D : \mathcal{A} \to \mathcal{X}^*$ be a σ-module derivation. Define $\widehat{D} : \mathcal{A}^\# \to \mathcal{X}^*$ by $\widehat{D}(a, \mathfrak{u}) = D(a)$, $((a, \mathfrak{u}) \in \mathcal{A})$. Thus $\widehat{D}$ is $\widehat{\sigma}$-$\mathfrak{A}^\#$-module derivation, because,

$$\begin{aligned}\widehat{D}((a, \mathfrak{u})(b, \mathfrak{v})) &= \widehat{D}((ab + a\mathfrak{v} + \mathfrak{u}b), \mathfrak{u}\mathfrak{v})\\
&= D(ab + a\mathfrak{v} + \mathfrak{u}b)\\
&= D(ab) + +D(a)\mathfrak{v} + \mathfrak{u}D(b)\\
&= \sigma(a)D(b) + D(a)\sigma(b) + D(a)\mathfrak{v} + \mathfrak{u}D(b)\\
&= (\sigma(a) + \mathfrak{u})D(b) + D(a)(\sigma(b) + \mathfrak{v})\\
&= \widehat{\sigma}(a, \mathfrak{u})D(b) + D(a)\widehat{\sigma}(b, \mathfrak{v})\\
&= \widehat{\sigma}(a, \mathfrak{u})\widehat{D}(b, \mathfrak{v}) + \widehat{D}(a, \mathfrak{u})\widehat{\sigma}(b, \mathfrak{v}),\\
&\quad (a, b \in \mathcal{A}, \mathfrak{u}, \mathfrak{v} \in \mathfrak{A}^\#).\end{aligned}$$

and by (ii) is a module $\widehat{D}$-approximately inner. Therefore, D is module approximately inner. So $\mathcal{A}$ is σ-(AMA). □

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