Quasi-invariance of Gaussian measures for the periodic Benjamin-Ono-BBM equation

Giuseppe Genovese[1] · Renato Lucà[2,3] · Nikolay Tzvetkov[4]

Abstract

The BBM equation is a Hamiltonian PDE which revealed to be a very interesting test-model to study the transformation property of Gaussian measures along the flow, after Tzvetkov (Sigma 3:e28-35). In this paper we study the BBM equation with critical dispersion (which is a Benjamin-Ono type model). We prove that the image of the Gaussian measures supported on fractional Sobolev spaces of increasing regularity are absolutely continuous, but we cannot identify the density, for which new ideas are needed.

1 Introduction

We study the equation of the Benjamin-Ono type

$$\partial_t u + \partial_t |D_x| u + \partial_x u + \partial_x (u^2) = 0, \quad u(0, x) = u_0(x), \tag{1.1}$$

✉ Giuseppe Genovese
  giuseppe.genovese@math.uzh.ch

  Renato Lucà
  rluca@bcamath.org

  Nikolay Tzvetkov
  nikolay.tzvetkov@u-cergy.fr

[1] University of Zurich Institute of Mathematics Winterthurerstrasse, 190-8057 Zurich, Switzerland

[2] BCAM - Basque Center for Applied Mathematics, 48009 Bilbao, Spain

[3] Ikerbasque, Basque Foundation for Science, 48011 Bilbao, Spain

[4] Department of Mathematics (AGM), University of Cergy-Pontoise, 2, av. Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France

posed on the one dimensional flat torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, where

$$|D_x|u(x) := \sum_{n \neq 0} |n|\hat{u}(n)e^{inx} \,.$$

In (1.1) u is real valued and we shall consider only zero average solutions (note that the average is preserved under the evolution). The global existence and uniqueness of solutions for (1.1) has been obtained in [17] for data in H^s with $s > \frac{1}{2}$. We denote by Φ_t the associated flow and write $u(t) := \Phi_t u$. It can be proved that the quantity

$$\mathcal{E}[u] := \left(\|u\|_{L^2}^2 + 4\pi \|u\|_{H^{\frac{1}{2}}}^2 \right)^{1/2} \tag{1.2}$$

is conserved by the evolution of (1.1); see Lemma 2.4 in [27]. In particular, the norm $H^{\frac{1}{2}}$ remains bounded in time.

We are interested in the transformation property along Φ_t of the Gaussian measures defined as follows. Let $\{h_n\}_{n \in \mathbb{N}}$, $\{l_n\}_{n \in \mathbb{N}}$ be two independent sequences of independent $\mathcal{N}(0, 1)$ random variables. Set

$$g_n := \begin{cases} \frac{1}{\sqrt{2}}(h_n + il_n) & n \in \mathbb{N} \\ \frac{1}{\sqrt{2}}(h_n - il_n) & -n \in \mathbb{N} \end{cases} \,.$$

We denote by γ_s the Gaussian measure on H^{s-} induced by the random Fourier series

$$\varphi_s(x) = \sum_{n \neq 0} \frac{g_n}{|n|^{s+1/2}} e^{inx} \tag{1.3}$$

and E_s the associated expectation value. We also define

$$\tilde{\gamma}_s(A) := E_s[1_{A \cap \{\mathcal{E}[u] \leqslant R\}}] \,. \tag{1.4}$$

Our main result is the quasi-invariance of $\tilde{\gamma}_s$ along Φ_t. We extend previous achievements of [10, 27] for $\beta > 1$ (in the notation of [10]) to the minimal dispersion case $\beta = 1$.

We recall that a measure μ on a space X is quasi-invariant with respect to a map $\Phi : X \to X$ if its image under Φ is absolutely continuous with respect to μ.

Theorem 1.1 *Let $s > 1$ and $R > 0$. Then the measures $\tilde{\gamma}_s$ are quasi-invariant along the flow of* (1.1). *For any $t \in \mathbb{R}$ there is $p = p(t, R) > 1$ such that the densities of the transported measures lie in $L^p(\tilde{\gamma}_s)$.*

Remark 1.2 The cut-off $1_{\{\mathcal{E}[u] \leqslant R\}}$ used to define the measure $\tilde{\gamma}_s$ allows to prove not only the quasi-invariance of the transported measure, but the fact that it belongs to $L^p(\tilde{\gamma}_s)$ for some $p > 1$, at any time. On the other hand, if we are only interested to the quasi-invariance of the reference measure we can upgrade the result to the limit

case $R \to \infty$, namely proving the quasi-invariance of the transport of the measure γ_s under Φ_t, for all $t \in \mathbb{R}$. For details we refer to Sect. 3.2 of [7].

The transport of Gaussian measures under given transformations is a classical subject of probability theory, starting from the classical works of Cameron-Martin [3] for shifts and Girsanov [13] for non-anticipative maps (*i.e.* adapted). The anticipative (or non-adapted) case is more difficult to deal with and a crucial role is played by the generator of the transformation. Kuo [16] established a Jacobi formula which generalises the Girsanov formula in case of maps with trace-class generators and Ramer [25] extended it to Hilbert-Schmidt generators (for a comparison of the Girsanov and Ramer change of variable formula see [30]). Further developments have been achieved in the context of Malliavin calculus, see for instance [4, 5, 28], essentially establishing Jacobi formulas for Gaussian measures in functional spaces for more general classes of maps.

In [27] the third author introduced a new method to prove quasi-invariance of Gaussian measures along the flow of dispersive PDEs. This paper triggered a renewed interest in the subject from the viewpoint of dispersive PDEs, which translates into studying the evolution of random initial data (such as Brownian motion and related processes). For recent developments on the topic, see [2, 6–12, 14, 19–24, 26], although this list might be not exhaustive.

The technique of [27] permits to treat flow maps whose differential is not in the Hilbert-Schmidt class, thus improving on the classical results. However it is only used to prove absolute continuity of the transported Gaussian measure without providing an explicit approximation of the density of the infinite dimensional change of coordinates induced by the flow, which is an important open question for many Hamiltonian PDEs and related models. Recent progresses in this direction have been made in [18], [6], [10], where the techniques developed allowed to get an exact formula for the density.

Identifying the density is one major difference between the present work and [10]. Indeed whereas in [10] we could prove the strong convergence of a sequence of approximating densities, here the minimal amount of dispersion in (1.1) prevents us to employ the same method and the question remains open. This challenging problem presents similarly also for the DNSL gauge group (see discussion below). Developing a robust technique to identify the density of the transported measures in these kind of problems would represent the completion of the programme started in [27]. Another important difference is that here we work without the exponential cut-off on the H^s norm, which was central in our previous work [10], and this introduces a number of technical difficulties. In particular we have to use finer probabilistic estimates.

The proof relies on the study of the derivative at time zero of the $H^{s+\frac{1}{2}}$ Sobolev norm

$$F[u] = \frac{d}{dt} \|\Phi_t u\|^2_{H^{s+\frac{1}{2}}} \bigg|_{t=0} \tag{1.5}$$

which plays the key role in the method introduced by the third author in [27]. The crucial point of the present work is that this object is not trivial to bound in any sense w.r.t. $\tilde{\gamma}_s$. Indeed we prove that it behaves as a sub-exponential random variable, which gives precisely the endpoint estimate on the $L^p(\tilde{\gamma}_s)$ norm in order to apply the

argument of [27]. We faced the same kind of difficulties studying the gauge group associated to the derivative nonlinear Schrödinger equation, and indeed our strategy unrolls similarly as in [9]. The analogies between the DNSL gauge and the Eq. (1.1) are interesting. Despite their simplicity, both models are critical for the method of [27] in the sense mentioned above: the term (1.5) is a sub-exponential random variable w.r.t. the Gaussian measure and, more importantly, it cannot be bounded within its support in terms of Sobolev norms (another notable example is the nonlinear wave equation studied in [14, 21], where the renormalisation needed in higher dimension complicates things further). Intuitively, this is due to the minimal amount of dispersion in the models, which for Eq. (1.1) can be clearly seen by comparison with the usual BBM equation with dispersion parameter strictly greater than one (denoted by γ in [27], β in [10]).

We conclude discussing the assumption $s > 1$. This assumption is only used in Lemma 4.2, where is necessary in order for $\|\partial_x u\|_{L^\infty}$ to be finite γ_s-almost surely. On the other hand, the other probabilistic arguments that we used only requires $s > 1/2$. It would be certainly interesting trying to relax $s > 1$, however this would require new ideas in order to avoid the use of Lemma 4.2.

The rest of the paper is organised as follows. Section 2 contains the necessary deterministic estimates for of the $H^{s+\frac{1}{2}}$ norm at $t = 0$ (F defined above). In Sect. 3 we prove the convergence in $L^2(\tilde{\gamma}_s)$ of suitable truncations of F. In Sect. 4 we show that F, as a random variable w.r.t. $\tilde{\gamma}_s$, has an exponential tail. Finally in Sect. 5 we ultimate the proof of the main result.

Notations Given a function $f : \mathbb{T} \to \mathbb{R}$ with zero average, we define its Sobolev norm H^s as

$$\|f\|_{H^s}^2 := \sum_{n=1}^{\infty} |n|^{2s} |\hat{f}(n)|^2 \, .$$

Note that with this definition the norms L^2 and H^0 differs by a factor $2\sqrt{\pi}$. This is why this factor appears in (1.2). A ball of radius R and centred in zero in the H^s topology is denoted by $B^s(R)$. We drop the superscript for $s = 0$ (ball of L^2). We write $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}$. We write $X \lesssim Y$ if there is a constant $c > 0$ such that $X \leqslant cY$ and $X \simeq Y$ if $Y \lesssim X \lesssim Y$. We underscore the dependency of c on the additional parameter a writing $X \lesssim_a Y$. C, c always denote constants that often vary from line to line within a calculation. We denote by P_N the orthogonal projection on Fourier modes $\leqslant N$, namely

$$P_N(u) = \sum_{|n| \leqslant N} \hat{u}(n) e^{inx} \, ,$$

where $\hat{u}(n)$ is the n-th Fourier coefficient of $u \in L^2$. Also, we denote the Littlewood-Paley projector by $\Delta_0 := P_1$, $\Delta_j := P_{2^j} - P_{2^{j-1}}$, $j \in \mathbb{N}$. We use the standard notation $[A, B] := AB - BA$ to denote the commutator of the operators A, B. We will use the following well-known tail bounds for sequences of independent centred Gaussian random variables $X_1, \ldots, X_d$ (see for instance [29]):

$$P\left(\left|\sum_{i=1}^{d}|X_i| - E[\sum_{i=1}^{d}|X_i|]\right| \geqslant \lambda\right) \leqslant C\exp\left(-c\frac{\lambda^2}{d}\right) \qquad (1.6)$$

and the Bernstein inequality

$$P\left(\left|\sum_{i=1}^{d}|X_i|^2 - E[\sum_{i=1}^{d}|X_i|^2]\right| \geqslant \lambda\right) \leqslant C\exp\left(-c\min\left(\lambda, \frac{\lambda^2}{d}\right)\right). \qquad (1.7)$$

2 Smoothing estimates

We will work with the truncated equation

$$\partial_t u + \partial_t |D_x| u + \partial_x u + \partial_x P_N((P_N u)^2) = 0, \quad u(0, x) = u_0(x). \qquad (2.1)$$

We denote by Φ_t^N the associated flow and $\Phi_t^{N=\infty} = \Phi_t$. The flow Φ_t^N is well defined, since the global well-posedness of (2.1) is clear by the fact that at fixed N the nonlinear part of the evolution regards only the Fourier modes in $[-N, N]$. One has a local existence time which depends on the L^2 norm on the initial datum (this clearly fails for $N = \infty$), and then one can globalize the solutions obtained using the Cauchy theorem taking advantage of the invariance of (1.2) under the flow Φ_t^N; see Lemma 2.4 in [27].

The crucial quantity we deal with is

$$F_N(t, u) := \frac{d}{dt}\|P_N\Phi_t^N u\|_{H^{s+\frac{1}{2}}}^2. \qquad (2.2)$$

We will abbreviate $F_N = F_N(0, u)$.

In this section all the integrals are taken over $x \in \mathbb{T}$. We always omit the dx to simplify the notations.

Proposition 2.1 *We have*

$$F_N(t, u) = F_{1,N}(t, u) + F_{2,N}(t, u) + F_{3,N}(t, u), \qquad (2.3)$$

where

$$F_{1,N}(t, u) := \frac{2}{\pi}\int_{\mathbb{T}}\left||D|^s P_N u(t)\right|^2 \partial_x P_N u(t), \qquad (2.4)$$

$$F_{2,N}(t, u) := -\frac{4}{\pi}\int_{\mathbb{T}}(|D_x|^s P_N u(t))\left[|D_x|^s, P_N u(t)\right]\partial_x P_N u(t), \qquad (2.5)$$

$$F_{3,N}(t, u) := \frac{2}{\pi}\int_{\mathbb{T}}(|D_x|^s P_N u(t))\frac{\partial_x |D_x|^s}{1 + |D_x|}((P_N u(t))^2). \qquad (2.6)$$

Proof From (2.1) we have

$$\partial_t |D_x|^\sigma P_N u(t) = \frac{|D_x|^\sigma}{1 + |D_x|} \left(-\partial_x P_N u(t) - \partial_x P_N ((P_N u(t))^2) \right). \qquad (2.7)$$

Since

$$\int_{\mathbb{T}} \left(\frac{|D_x|^\sigma}{1 + |D_x|} (\partial_x P_N u(t)) \right) |D_x|^\sigma P_N u(t)$$

$$= \int_{\mathbb{T}} \left(\partial_x \left(\frac{|D_x|^\sigma}{\sqrt{1 + |D_x|}} P_N u(t) \right) \right) \frac{|D_x|^\sigma}{\sqrt{1 + |D_x|}} P_N u(t)$$

$$= \frac{1}{2} \int_{\mathbb{T}} \partial_x \left(\left(\frac{|D_x|^\sigma}{\sqrt{1 + |D_x|}} P_N u(t) \right)^2 \right) = 0, \qquad (2.8)$$

pairing (2.7) in L^2 with $|D_x|^\sigma P_N u(t)$ we can compute

$$\frac{d}{dt} \|P_N u(t)\|_{H^\sigma}^2 = -\frac{2}{\pi} \int_{\mathbb{T}} \left(|D_x|^\sigma P_N u(t) \right) \frac{|D_x|^\sigma}{1 + |D_x|} \partial_x \left((P_N u(t))^2 \right); \qquad (2.9)$$

note that on the r.h.s. we can write $\partial_x ((P_N u)^2)$ in place of $P_N \partial_x ((P_N u)^2)$ by orthogonality.

Choosing $\sigma = s + 1/2$ into (2.9) we get

$$\frac{d}{dt} \|P_N u(t)\|_{H^{s+1/2}}^2 = -\frac{2}{\pi} \int_{\mathbb{T}} \left(|D_x|^{s+1/2} P_N u(t) \right) \frac{|D_x|^{s+1/2}}{1 + |D_x|} \partial_x ((P_N u(t))^2).$$

This implies

$$\frac{d}{dt} \|P_N u(t)\|_{H^{s+1/2}}^2 = -\frac{2}{\pi} \int_{\mathbb{T}} \left(\frac{|D_x|}{1 + |D_x|} |D_x|^s P_N u(t) \right) |D_x|^s \partial_x ((P_N u(t))^2).$$

Thus, writing

$$\frac{|D_x|}{1 + |D_x|} = 1 - \frac{1}{1 + |D_x|}$$

we arrive to

$$\frac{d}{dt} \|P_N u(t)\|_{H^{s+1/2}}^2 = I_1(t) + I_2(t),$$

where

$$I_1(t) = -\frac{2}{\pi} \int_{\mathbb{T}} (|D_x|^s P_N u(t)) |D_x|^s \partial_x ((P_N u(t))^2)$$

$$= -\frac{4}{\pi} \int_{\mathbb{T}} (|D_x|^s P_N u(t)) |D_x|^s ((\partial_x (P_N u(t))) P_N u(t))$$

and

$$I_2(t) = \frac{2}{\pi} \int_{\mathbb{T}} \left(\frac{|D_x|^s}{1+|D_x|} P_N u(t) \right) |D_x|^s \left(\partial_x (P_N u(t))^2 \right)$$
$$= \frac{2}{\pi} \int_{\mathbb{T}} (|D_x|^s P_N u(t)) \frac{\partial_x |D_x|^s}{1+|D_x|} \left((P_N u(t))^2 \right).$$

Using

$$|D_x|^s \left((\partial_x P_N u(t))(P_N u(t)) \right)$$
$$= \left(|D_x|^s \partial_x P_N u(t) \right) P_N u(t) + \left[|D_x|^s, P_N u(t) \right] \partial_x P_N u(t)$$

we can rewrite I_1 as

$$I_1(t) = -\frac{4}{\pi} \int_{\mathbb{T}} (|D_x|^s P_N u(t)) \, (|D_x|^s \partial_x P_N u(t))(P_N u(t))$$
$$- \frac{4}{\pi} \int_{\mathbb{T}} (|D_x|^s P_N u(t)) \left[|D_x|^s, P_N u(t) \right] \partial_x P_N u(t). \qquad (2.10)$$

Integrating by parts we can rewrite the first term on the r.h.s. of (2.10) as

$$\frac{2}{\pi} \int_{\mathbb{T}} \left| |D_x|^s P_N u(t) \right|^2 \partial_x P_N u(t). \qquad (2.11)$$

Thus we arrive to

$$I_1(t) \quad = \frac{2}{\pi} \int_{\mathbb{T}} \left| |D_x|^s P_N u(t) \right|^2 \partial_x P_N u(t)$$
$$- \frac{4}{\pi} \int_{\mathbb{T}} (|D_x|^s P_N u(t)) \left[|D_x|^s, P_N u(t) \right] \partial_x P_N u(t). \qquad (2.12)$$

Since $I_1(t) = F_{1,N}(t, u) + F_{2,N}(t, u)$ and $I_2(t) = F_{3,N}(t, u)$ the proof is concluded. $\qquad \square$

Proposition 2.2 *Let $s \geqslant 0$. The solutions of* (2.1) *satisfy for all $t \in \mathbb{R}$:*

$$\left| \frac{d}{dt} \| P_N u(t) \|_{H^{s+1/2}}^2 \right| \lesssim \| P_N u(t) \|_{H^s}^2 \| \partial_x P_N u(t) \|_{L^\infty}. \qquad (2.13)$$

Proof By (2.3) it suffices to show that

$$|F_{1,N}(t, u)| + |F_{2,N}(t, u)| + |F_{3,N}(t, u)| \lesssim \| P_N u(t) \|_{H^s}^2 \| \partial_x P_N u(u) \|_{L^\infty}. \quad (2.14)$$

This is immediate in the case of $F_{1,N}(t, u)$, by Hölder's inequality.

For $F_{2,N}(t,u)$ we use the following commutator estimate from [15], valid for f periodic with zero average:

$$\left\| \left[|D_x|^s, f \right] g \right\|_{L^2} \lesssim \|\partial_x f\|_{L^\infty} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty} .$$

Since $P_N u(t)$ has zero average we obtain

$$\left\| \left[|D_x|^s, P_N u(t) \right] \partial_x P_N u(t) \right\|_{L^2} \lesssim \|\partial_x P_N u(t)\|_{L^\infty} \|\partial_x P_N u(t)\|_{H^{s-1}}$$
$$+ \|P_N u(t)\|_{H^s} \|\partial_x P_N u(t)\|_{L^\infty}$$
$$\lesssim \|P_N u(t)\|_{H^s} \|\partial_x P_N u(t)\|_{L^\infty},$$

whence

$$|F_{2,N}(t,u)| \simeq \left| \int_{\mathbb{T}} (|D_x|^s P_N u(t)) \left[|D_x|^s, P_N u(t) \right] \partial_x P_N u(t) \right|$$
$$\lesssim \|P_N u(t)\|_{H^s}^2 \|\partial_x P_N u(t)\|_{L^\infty} .$$

The contribution of $F_{3,N}(t,u)$ is even smaller. Indeed, since $\frac{\partial_x}{1+|D_x|}$ is bounded on L^2 we have

$$|F_{3,N}(t,u)| \simeq \left| \int_{\mathbb{T}} (|D_x|^s P_N u) \frac{\partial_x |D_x|^s}{1+|D_x|} \left((P_N u)^2 \right) \right|$$
$$\lesssim \|P_N u(t)\|_{H^s} \|(P_N u(t))^2\|_{H^s} \lesssim \|P_N u(t)\|_{H^s}^2 \|P_N u(t)\|_{L^\infty} \quad (2.15)$$

where we used $\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{L^\infty} + \|g\|_{H^s} \|f\|_{L^\infty}$ with $f = g = P_N u(t)$ in the last bound. Then (2.14) for $F_{3,N}(t,u)$ follows since $P_N u(u)$ has zero average, thus $\|P_N u(t)\|_{L^\infty} \lesssim \|\partial_x P_N u(t)\|_{L^\infty}$. $\qquad\qquad\square$

3 Second moment estimates

The goal of this section is to prove the $L^2(\gamma_s)$ convergence of the term F_N (recall once again that we are using the simplified notation $F_N = F_N(0,u)$, namely we mean that the time derivative in (2.2) is evaluated at $t = 0$). This is a crucial result in our paper, as it allows us to exploit the random cancellations in F_N to get bounds on this quantity which appear prohibitive to achieve deterministically.

Proposition 3.1 *For all $N > M$ it holds*

$$\|F_N - F_M\|_{L^2(\gamma_s)} \lesssim \frac{1}{M^{\frac{2s-1}{4}}}, \quad s \in \left(\frac{1}{2}, \frac{3}{2} \right] \quad (3.1)$$

$$\|F_N - F_M\|_{L^2(\gamma_s)} \lesssim \frac{1}{\sqrt{M}}, \quad s > \frac{3}{2} . \quad (3.2)$$

The above result concerns the full range $s > \frac{1}{2}$, even though in the rest of the paper only the case $s > 1$ will be studied. Considering $s > \frac{1}{2}$ here could be however useful in a future attempt to extend also the main results of this paper to $s > \frac{1}{2}$.

We start by a simple result on the decay of discrete convolutions. The way we use these bounds is explained in Remark 3.4.

Lemma 3.2 *Let $M \in \mathbb{N}$. Let $x, y > 0$ with $x + y > 1$. Let $p, q \geqslant 1$ such that $\max(\frac{1}{p}, \frac{1}{q}) < x$ and $\max(\frac{p-1}{p}, \frac{q-1}{q}) < y$. Then there is $c = c(p, q, x, y) > 0$ such that*

$$\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y} \leqslant \frac{c}{\langle m \rangle^r}, \quad r := \min\left(x - \frac{1}{p}, \frac{1}{q} - (1 - y)\right) \tag{3.3}$$

$$\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y} \leqslant \frac{c 1_{\{|m| \geqslant \frac{2M}{3}\}}}{\langle m \rangle^{x - \frac{1}{p}}} + \frac{c}{\langle M \rangle^{x - \frac{1}{q}} \langle m \rangle^{\frac{1}{q} - (1 - y)}}. \tag{3.4}$$

Remark 3.3 *If $x, y > 1$, taking $p = \infty$ and $q = 1$ we recover the well-known convolution estimate for powers.*

Proof We can assume $m \neq 0$, otherwise the statement is immediate. We have

$$\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y} \leqslant 1_{\{|m| \geqslant \frac{2M}{3}\}} \sum_{\{|n| \geqslant \frac{|m|}{2}\}} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y}$$

$$+ \sum_{\{|m - n| \geqslant \frac{|m|}{2}\} \cap \{|n| \geqslant M\}} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y}. \tag{3.5}$$

Note that the second summand is sufficient to bound the l.h.s. if $M > \frac{3|m|}{2}$, otherwise the first term is needed.

We estimate separately the two summands by the Hölder inequality. We have

$$\sum_{\{|n| \geqslant \frac{|m|}{2}\}} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y} \leqslant \left(\sum_{\{|n| \geqslant \frac{|m|}{2}\}} \frac{1}{\langle n \rangle^{xp}} \right)^{\frac{1}{p}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^{\frac{yp}{p-1}}} \right)^{\frac{p-1}{p}} \leqslant \frac{c_1(p, y)}{|m|^{x - \frac{1}{p}}}$$

Similarly

$$\sum_{\{|n| \geqslant M\} \cap \{|m - n| \geqslant \frac{|m|}{2}\}} \frac{1}{\langle n \rangle^x \langle m - n \rangle^y} \leqslant \left(\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^{xq}} \right)^{\frac{1}{q}} \left(\sum_{\{|n| \geqslant \frac{|m|}{2}\}} \frac{1}{\langle n \rangle^{\frac{yq}{q-1}}} \right)^{\frac{q-1}{q}}$$

$$\leqslant \frac{c_1(q, x)}{|m|^{\frac{1}{q} - (1 - y)} \langle M \rangle^{x - \frac{1}{q}}}.$$

So we obtained (3.4), and (3.3) also follows taking $M = 0$. □

Remark 3.4 The following particular cases of Lemma 3.2 will be useful in the sequel.

i) For $\frac{1}{2} < s < \frac{3}{2}$ we set

$$x - \frac{1}{p} = \frac{1}{q} - (1 - y) = s - \frac{1}{2} \tag{3.6}$$

and get

$$\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^{2s-1} \langle m - n \rangle} \leqslant \frac{c}{\langle m \rangle^{s - \frac{1}{2}}}, \tag{3.7}$$

$$\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^{2s-1} \langle m - n \rangle} \leqslant \frac{c}{\langle m \rangle^{s - \frac{1}{2}}} \left(1_{\{|m| \gtrsim M\}} + \frac{1}{\langle M \rangle^{s - \frac{1}{2}}} \right). \tag{3.8}$$

ii) For $s \geqslant \frac{3}{2}$ we set $p = q = 1$, which gives

$$\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^{2s-1} \langle m - n \rangle} \leqslant \frac{c}{\langle m \rangle}, \tag{3.9}$$

$$\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^{2s-1} \langle m - n \rangle} \leqslant \frac{c}{\langle m \rangle} \left(1_{\{|m| \gtrsim M\}} + \frac{1}{\langle M \rangle} \right). \tag{3.10}$$

iii) For $s > \frac{1}{2}$ by the same choice (3.6) we get

$$\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^s \langle m - n \rangle^s} \leqslant \frac{c}{\langle m \rangle^{s - \frac{1}{2}}}, \tag{3.11}$$

$$\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^s \langle m - n \rangle^s} \leqslant \frac{c}{\langle m \rangle^{s - \frac{1}{2}}} \left(1_{\{|m| \gtrsim M\}} + \frac{1}{\langle M \rangle^{s - \frac{1}{2}}} \right). \tag{3.12}$$

iv) For $s > \frac{1}{2}$ we set $\frac{1}{p} = \varepsilon, q = 1$, where $\varepsilon > 0$ may be chosen arbitrarily small, and get

$$\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^{2s+1} \langle m - n \rangle} \leqslant \frac{c}{\langle m \rangle}, \tag{3.13}$$

$$\sum_{|n| \geqslant M} \frac{1}{\langle n \rangle^{2s+1} \langle m - n \rangle} \leqslant \frac{c}{\langle m \rangle} \left(1_{\{|m| \gtrsim M\}} + \frac{1}{\langle M \rangle^{2s}} \right). \tag{3.14}$$

Thanks to Proposition 2.1, we can split the proof of Proposition 3.1 into three steps, one statement for each $F_{i,N}$, $i = 1, 2, 3$ (recall (2.4), (2.5) and (2.6)). Again, we are using the shorten notation $F_{i,N} = F_{i,N}(0, u)$.

We feel the need to warn the reader about the next somewhat lengthy computations. In particular the proof of subsequent Lemma 3.6 is a long enumeration of cases, each

of which reduces to a term already estimated in the proof of Lemma 3.5. The proofs of Lemmas 3.5 and 3.7 are similar, but independent one from each other.

In what follows we use crucially the Wick formula for expectation values of multilinear forms of Gaussian random variables in the following form. Let $\ell \in \mathbb{N}$ and S_ℓ be the symmetric group on $\{1, \ldots, \ell\}$, whose elements are denoted by σ. Recalling that $\hat{u}(-n) = \overline{\hat{u}(n)}$

$$E_s\left[\prod_{j=1}^{\ell} \hat{u}(n_j)\hat{u}(-m_j)\right] = E_s\left[\prod_{j=1}^{\ell} \hat{u}(n_j)\overline{\hat{u}(m_j)}\right] = \sum_{\sigma \in S_\ell} \prod_{j=1}^{\ell} \frac{\delta_{m_j, n_{\sigma(j)}}}{|n_j|^{2s+1}}, \quad (3.15)$$

where E_s is the expectation w.r.t. γ_s. In the following we shall use $\ell = 3$ and often refer to the elements of S_3 as *contractions* (of indices).

The following set will appear in the next three proofs. Given a vector $a \in \mathbb{Z}^3$ we denote a_j its components and we define

$$A_{N,M} = \{a_1, a_2, a_3 \neq 0, \; a_1 + a_2 + a_3 = 0, \; N \geqslant \max_j(|a_j|) > M\}. \quad (3.16)$$

Note that if $a \in A_{N,M}$ we must have

$$a_j \neq -a_{j'} \quad \text{for all} \quad j, j' \in \{1, 2, 3\}. \quad (3.17)$$

This is because of the restrictions $a_j \neq 0$, $a_1 + a_2 + a_3 = 0$.

Lemma 3.5 *For all $N > M$ it holds*

$$\|F_{1,N} - F_{1,M}\|_{L^2(\gamma_s)} \lesssim \frac{1}{M^{\frac{s}{2}-\frac{1}{4}}}, \quad s \in \left(\frac{1}{2}, \frac{3}{2}\right] \quad (3.18)$$

$$\|F_{1,N} - F_{1,M}\|_{L^2(\gamma_s)} \lesssim \frac{1}{\sqrt{M}}, \quad s > \frac{3}{2}. \quad (3.19)$$

Proof We have (recall (2.4))

$$F_{1,N} - F_{1,M} = \frac{2i}{\pi} \sum_{n \in A_{N,M}} |n_1|^s |n_2|^s n_3 \, \hat{u}(n_1)\hat{u}(n_2)\hat{u}(n_3).$$

and taking the modulus squared (recall $\hat{u}(-m_j) = \overline{\hat{u}(m_j)}$)

$$|F_{1,N} - F_{1,M}|^2 = \frac{4}{\pi^2} \sum_{(n,m) \in A_{N,M}^2} |n_1|^s |n_2|^s n_3 |m_1|^s |m_2|^s m_3 \prod_{j=1}^{3} \hat{u}(n_j)\hat{u}(-m_j).$$

$$(3.20)$$

When taking the expected value of (3.20) w.r.t. γ_s we use the Wick formula (3.15) with $\ell = 3$. This gives

$$
\|F_{1,N} - F_{1,M}\|^2_{L^2(\gamma_s)} = \frac{4}{\pi^2} \sum_{\sigma \in S_3} \sum_{n \in A_{N,M}} \frac{|n_1|^s |n_{\sigma(1)}|^s |n_2|^s |n_{\sigma(2)}|^s n_3 n_{\sigma(3)}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}
$$

(3.21)

It is easy to see that the contractions $\sigma = (1, 2, 3)$ and $\sigma = (2, 1, 3)$ give the same contributions. Also, the remaining contractions give all the same contributions. Thus we may reduce to the cases $\sigma = (1, 2, 3)$ and (say) $\sigma = (1, 3, 2)$.

The contribution relative to $\sigma = (1, 2, 3)$ is

$$
\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s} |n_2|^{2s} n_3^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s} |n_2|^{2s} |n_3|^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} .
$$

(3.22)

We write

$$
\text{r.h.s. of (3.22)} \lesssim \sum_{\substack{|n_j| \leqslant N \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{1}{\langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle^{2s-1}}
$$

$$
\leqslant \sum_{\substack{|n_2|,|n_3| \leqslant N \\ \max(|n_2|,|n_3|) \gtrsim M}} \frac{1}{\langle n_2 \rangle \langle n_2 - n_3 \rangle \langle n_3 \rangle^{2s-1}}
$$

$$
\lesssim \sum_{|n_2|>M} \frac{1}{\langle n_2 \rangle} \sum_{n_3 \in \mathbb{Z}} \frac{1}{\langle n_2 - n_3 \rangle \langle n_3 \rangle^{2s-1}}
$$

(3.23)

$$
+ \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n_2 \rangle} \sum_{|n_3|>M} \frac{1}{\langle n_2 - n_3 \rangle \langle n_3 \rangle^{2s-1}} .
$$

(3.24)

In the second inequality we used the symmetry of the r.h.s. under $m_3 \leftrightarrow -m_3$ and that $n_1 + n_2 + n_3 = 0$ and $\max_j(|n_j|) > M$ imply $\max(|n_2|, |n_3|) \gtrsim M$.

For $s \in (\frac{1}{2}, \frac{3}{2}]$ the inner sums in (3.23) and (3.24) can be estimated respectively by (3.7) and (3.8)

$$
(3.23) \leqslant \sum_{|n_2|>M} \frac{1}{\langle n_2 \rangle} \frac{1}{\langle n_2 \rangle^{s-\frac{1}{2}}} = \sum_{|n_2|>M} \frac{1}{\langle n_2 \rangle^{s+\frac{1}{2}}} \lesssim \frac{1}{M^{s-\frac{1}{2}}},
$$

(3.25)

$$
(3.24) \leqslant \sum_{|n_2| \gtrsim M} \frac{1}{\langle n_2 \rangle^{s+\frac{1}{2}}} + \frac{1}{M^{s-\frac{1}{2}}} \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n_2 \rangle^{s+\frac{1}{2}}} \lesssim \frac{1}{M^{s-\frac{1}{2}}}.
$$

(3.26)

For $s > \frac{3}{2}$ we estimate the inner sums of (3.23) and (3.24) using the inequalities (3.9) and (3.10) and obtain

$$(3.23) \leqslant \sum_{|n_2| > M} \frac{1}{\langle n_2 \rangle^2} \lesssim \frac{1}{M} \,, \tag{3.27}$$

$$(3.24) \leqslant \sum_{|n_2| \gtrsim M} \frac{1}{\langle n_2 \rangle^2} + \frac{1}{M} \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n_2 \rangle^2} \lesssim \frac{1}{M} \,. \tag{3.28}$$

The contribution relative to $\sigma = (1, 3, 2)$ is

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^s n_2 |n_3|^s n_3}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^{s+1} |n_3|^{s+1}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \tag{3.29}$$

Again we can write

$$\text{r.h.s. of (3.29)} \lesssim \sum_{\substack{|n_j| \leqslant N \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{1}{\langle n_1 \rangle \langle n_2 \rangle^s \langle n_3 \rangle^s}$$

$$= \sum_{\substack{|n_2|, |n_3| \leqslant N \\ \max(|n_1|, |n_2|) > M}} \frac{1}{\langle n_1 \rangle \langle n_2 \rangle^s \langle n_1 - n_2 \rangle^s}$$

$$\leqslant \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_1 \rangle} \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n_2 \rangle^s \langle n_1 - n_2 \rangle^s} \tag{3.30}$$

$$+ \sum_{n_1 \in \mathbb{Z}} \frac{1}{\langle n_1 \rangle} \sum_{|n_2| \gtrsim M} \frac{1}{\langle n_2 \rangle^s \langle n_1 - n_2 \rangle^s} \,. \tag{3.31}$$

The inner sums in (3.30) and (3.31) are estimated respectively by (3.11) and (3.12) and we obtain

$$(3.30) \leqslant \sum_{|n_1| > M} \frac{1}{\langle n_1 \rangle^{s + \frac{1}{2}}} \lesssim \frac{1}{M^{s - \frac{1}{2}}} \,, \tag{3.32}$$

$$(3.31) \leqslant \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_1 \rangle^{s + \frac{1}{2}}} + \frac{1}{M^{s - \frac{1}{2}}} \sum_{n_1 \in \mathbb{Z}} \frac{1}{\langle n_1 \rangle^{s + \frac{1}{2}}} \lesssim \frac{1}{M^{s - \frac{1}{2}}} \,. \tag{3.33}$$

$\square$

Lemma 3.6 *For all $N > M$ it holds*

$$\|F_{2,N} - F_{2,M}\|_{L^2(\gamma_s)} \lesssim \frac{1}{M^{\frac{s}{2}-\frac{1}{4}}}, \qquad s \in \left(\frac{1}{2}, \frac{3}{2}\right] \tag{3.34}$$

$$\|F_{2,N} - F_{2,M}\|_{L^2(\gamma_s)} \lesssim \frac{1}{\sqrt{M}}, \qquad s > \frac{3}{2}. \tag{3.35}$$

Proof In fact, we will reduce to a sum of contributions which are the same as the ones handled in the previous lemma.

Recalling that

$$\left[|D_x|^s, P_N u(t)\right]\partial_x P_N u(t) = |D_x|^s(P_N u(t)\partial_x P_N u(t)) - P_N u(t)|D_x^s|\partial_x P_N u(t)$$

and proceeding as in the proof of Lemma 3.5 we obtain

$$F_{2,N} - F_{2,M} = \frac{4i}{\pi} \sum_{n \in A_{N,M}} |n_1|^s n_2(|n_2 + n_3|^s - |n_2|^s)\, \hat{u}(n_1)\hat{u}(n_2)\hat{u}(n_3).$$

Taking the modulus squared

$$\begin{aligned}
&|F_{2,N} - F_{2,M}|^2 \\
&= \frac{16}{\pi^2} \sum_{(n,m) \in A_{N,M}^2} |n_1|^s n_2(|n_2 + n_3|^s - |n_2|^s)|m_1|^s m_2(|m_2 + m_3|^s - |m_2|^s) \\
&\quad \prod_{j=1}^{3} \hat{u}(n_j)\hat{u}(-m_j)
\end{aligned} \tag{3.36}$$

and using the Wick formula (3.15) with $\ell = 3$ we arrive to

$$\begin{aligned}
&\|F_{2,N} - F_{2,M}\|_{L^2(\gamma_s)}^2 \\
&= \frac{16}{\pi^2} \sum_{\sigma \in S_3} \sum_{n \in A_{N,M}} \frac{|n_1|^s |n_{\sigma(1)}|^s n_2 n_{\sigma(2)}(|n_2 + n_3|^s - |n_2|^s)(|n_{\sigma(2)} + n_{\sigma(3)}|^s - |n_{\sigma(2)}|^s)}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}}.
\end{aligned} \tag{3.37}$$

Before evaluating all the contributions relative to the different σ, we recall an useful inequality to handle the difference

$$\left||a + b|^s - |a|^s\right|.$$

We distinguish two cases, namely

$$|a| \leqslant 2|b|, \quad |a| > 2|b|.$$

In the first case we have immediately

$$\left| |a + b|^s - |a|^s \right| \lesssim |b|^s, \qquad \text{for } |a| \leqslant 2|b|. \tag{3.38}$$

In the second case we use, for $s > 0$, the Taylor expansion (converging for $|x| < 1$)

$$(1 + x)^s = 1 + \sum_{k \geqslant 1} \frac{(s)_k}{k!} x^k,$$

where $(s)_k$ is defined by

$$(s)_0 = 1, \quad (s)_k := \prod_{j=0}^{k-1} (s - j), \quad k \geqslant 1.$$

Letting $x := \frac{|b|}{|a|} < \frac{1}{2}$ and using $|(s)_k| \leqslant k!$ we can bound

$$\left| |a + b|^s - |a|^s \right| = |a|^s \left| \sum_{k \geqslant 1} \left(\frac{|b|}{|a|} \right)^k \right| \lesssim |a|^{s-1} |b|, \qquad \text{for } |a| > 2|b|. \tag{3.39}$$

Now we are ready to estimate (3.37).

• Permutation $\sigma = (1, 2, 3)$. We need to handle

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M}} \frac{|n_1|^{2s} n_2^2 (|n_2 + n_3|^s - |n_2|^s)^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}.$$

If $|n_2| \leqslant 2|n_3|$ we can use (3.38) to bound $\left| |n_2 + n_3|^s - |n_2|^s \right| \lesssim |n_3|^s$ so that

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M \\ |n_2| \leqslant 2|n_3|}} \frac{|n_1|^{2s} n_2^2 |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^2 |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}.$$

This is done as (3.22) (exchanging $n_2 \leftrightarrow n_3$).

If $|n_2| > 2|n_3|$ we use (3.39) to bound $\left| |n_2 + n_3|^s - |n_2|^s \right| \lesssim |n_2|^{s-1} |n_3|$ and we reduce to estimate

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M \\ |n_2| > 2|n_3|}} \frac{|n_1|^{2s} |n_2|^2 |n_2|^{2s-2} |n_3|^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^{2s} |n_3|^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}.$$

We can again proceed as we have done for (3.22), getting the same decay rate.

- Permutation $\sigma = (1, 3, 2)$. We need to handle

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s} n_2 n_3 (|n_2+n_3|^s - |n_2|^s)(|n_2+n_3|^s - |n_3|^s)}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}} . \tag{3.40}$$

We have three possibilities:

(A) $|n_2| \leqslant 2|n_3|$ and $|n_3| \leqslant 2|n_2|$,
(B) $|n_3| > 2|n_2|$,
(C) $|n_2| > 2|n_3|$.

In the case (A) we use (3.38) to bound $\left||n_2+n_3|^s - |n_2|^s\right| \lesssim |n_3|^s$ and $\left||n_2+n_3|^s - |n_3|^s\right| \lesssim |n_2|^s$. Thus we reduce to estimate

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_3| \leqslant 2|n_2|}} \frac{|n_1|^{2s} n_2 n_3 |n_3|^s |n_2|^s}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s}|n_2|^{s+1}|n_3|^{s+1}}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}} .$$

This is done as (3.29).

If we are in the case (B) we have in particular $|n_2| < \frac{1}{2}|n_3| \leqslant 2|n_3|$, so we can use (3.38) to bound the difference $\left||n_2+n_3|^s - |n_2|^s\right| \lesssim |n_3|^s$ and (3.39) to bound the difference $\left||n_2+n_3|^s - |n_3|^s\right| \lesssim |n_3|^{s-1}|n_2|$. Thus we need to estimate

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_3|>2|n_2|}} \frac{|n_1|^{2s} n_2 n_3 |n_3|^s |n_3|^{s-1}|n_2|}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}} \leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s}|n_2|^2 |n_3|^{2s}}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}}$$

and this is done as (3.22). The case (C) is the same as (B) exchanging $n_2 \leftrightarrow n_3$.

- Permutation $\sigma = (2, 1, 3)$. We need to handle

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^s |n_2|^s n_2 n_1 (|n_2+n_3|^s - |n_2|^s)(|n_1+n_3|^s - |n_1|^s)}{|n_1|^{2s+1}|n_2|^{2s+1}|n_3|^{2s+1}} .$$

We distinguish

(A) $|n_2| \leqslant 2|n_3|, |n_1| \leqslant 2|n_3|$
(B) $|n_2| \leqslant 2|n_3|, |n_1| > 2|n_3|$
(C) $|n_2| > 2|n_3|, |n_1| \leqslant 2|n_3|$;   (same as (B) switching $n_2 \leftrightarrow n_1$)
(D) $|n_2| > 2|n_3|, |n_1| > 2|n_3|$.

In the case (A) we use (3.38) to bound $\left| |n_2 + n_3|^s - |n_2|^s \right| \lesssim |n_3|^s$ and $\left| |n_1 + n_3|^s - |n_1|^s \right| \lesssim |n_3|^s$. Thus we need to estimate

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_1| \leqslant 2|n_3|}} \frac{|n_1|^s |n_2|^s n_2 n_1 |n_3|^s |n_3|^s}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{s+1} |n_2|^{s+1} |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}},$$

that is done as (3.29).

In the case (B) we use (3.38) to bound $\left| |n_2 + n_3|^s - |n_2|^s \right| \lesssim |n_3|^s$ and (3.39) to bound $\left| |n_1 + n_3|^s - |n_1|^s \right| \lesssim |n_1|^{s-1}|n_3|$, so that

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_1|>2|n_3|}} \frac{|n_1|^s |n_2|^s n_2 n_1 |n_3|^s |n_1|^{s-1} |n_3|}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s} |n_2|^{s+1} |n_3|^{s+1}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}},$$

that is estimated as (3.29). The case (C) is the same as (B) using the symmetry $n_2 \leftrightarrow n_1$.

In the case (D) we use (3.39) to bound $\left| |n_2 + n_3|^s - |n_2|^s \right| \lesssim |n_2|^{s-1}|n_3|$ and $\left| |n_1 + n_3|^s - |n_1|^s \right| \lesssim |n_1|^{s-1}|n_3|$, so that

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2|>2|n_3| \text{ and } |n_1|>2|n_3|}} \frac{|n_1|^s |n_2|^s n_2 n_1 |n_2|^{s-1} |n_3| |n_1|^{s-1} |n_3|}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s} |n_2|^{2s} |n_3|^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}},$$

that is estimated as (3.22).

• Permutation $\sigma = (2, 3, 1)$. We need to handle

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^s |n_2|^s n_2 n_3 (|n_2 + n_3|^s - |n_2|^s)(|n_3 + n_1|^s - |n_3|^s)}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} .$$

We distinguish

(A) $|n_2| \leqslant 2|n_3|$, $|n_3| \leqslant 2|n_1|$;
(B) $|n_2| \leqslant 2|n_3|$, $|n_3| > 2|n_1|$;
(C) $|n_2| > 2|n_3|$, $|n_3| \leqslant 2|n_1|$;
(D) $|n_2| > 2|n_3|$, $|n_3| > 2|n_1|$.

In the case (A) we use (3.38) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_3|^s$ and $\left||n_3 + n_1|^s - |n_3|^s\right| \lesssim |n_1|^s$ so that

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_3| \leqslant 2|n_1|}} \frac{|n_1|^s |n_2|^s n_2 n_3 |n_3|^s |n_1|^s}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^{s+1} |n_3|^{s+1}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} ,$$

that is estimated as (3.29).

In the case (B) we use (3.38) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_3|^s$ and (3.39) to bound $\left||n_3 + n_1|^s - |n_3|^s\right| \lesssim |n_3|^{s-1}|n_1|$, thus

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_3| > 2|n_1|}} \frac{|n_1|^s |n_2|^s n_2 n_3 |n_3|^s |n_3|^{s-1} |n_1|}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{s+1} |n_2|^{s+1} |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} ,$$

that is again estimated as (3.29).

In the case (C) we use (3.39) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_2|^{s-1}|n_3|$ and (3.38) to bound $\left||n_3 + n_1|^s - |n_3|^s\right| \lesssim |n_1|^s$, thus

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M \\ |n_2| > 2|n_3| \text{ and } |n_3| \leqslant 2|n_1|}} \frac{|n_1|^s |n_2|^s n_2 n_3 |n_2|^{s-1} |n_3| |n_1|^s}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^{2s} |n_3|^2}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \, ,$$

that is estimated as (3.22).

In the case (D) we use (3.39) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_2|^{s-1}|n_3|$ and $\left||n_3 + n_1|^s - |n_3|^s\right| \lesssim |n_3|^{s-1}|n_1|$. We arrive to

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M \\ |n_2| > 2|n_3| \text{ and } |n_3| > 2|n_1|}} \frac{|n_1|^s |n_2|^s n_2 n_3 |n_2|^{s-1} |n_3| |n_3|^{s-1} |n_1|}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{s+1} |n_2|^{2s} |n_3|^{s+1}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \, ,$$

that is again estimated as (3.29).

• Permutation $\sigma = (3, 2, 1)$. We need to handle

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^s |n_3|^s n_2^2 (|n_2 + n_3|^s - |n_2|^s)(|n_2 + n_1|^s - |n_2|^s)}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

We distinguish

(A) $|n_2| \leqslant 2|n_3|, \ |n_2| \leqslant 2|n_1|$;

(B) $|n_2| \leqslant 2|n_3|, \ |n_2| > 2|n_1|$;

(C) $|n_2| > 2|n_3|, \ |n_2| \leqslant 2|n_1|$;   (same as (B) switching $n_1 \leftrightarrow n_3$)

(D) $|n_2| > 2|n_3|, \ |n_2| > 2|n_1|$.

In the case (A) we use (3.38) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_3|^s$ and $\left||n_2 + n_1|^s - |n_2|^s\right| \lesssim |n_1|^s$. We arrive to

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_2| \leqslant 2|n_1|}} \frac{|n_1|^s |n_3|^s n_2^2 |n_3|^s |n_1|^s}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{2s} |n_2|^2 |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} ,$$

that is estimated as (3.22).

In the case (B) we use (3.38) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_3|^s$ and (3.39) to bound $\left||n_2 + n_1|^s - |n_2|^s\right| \lesssim |n_2|^{s-1}|n_1|$. We arrive to

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2| \leqslant 2|n_3| \text{ and } |n_2|>2|n_1|}} \frac{|n_1|^s |n_3|^s n_2^2 |n_3|^s |n_2|^{s-1} |n_1|}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{s+1} |n_2|^{s+1} |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} ,$$

that is estimated as (3.29).

The case (C) is the same as (B) exchanging $n_1 \leftrightarrow n_3$.

In the case (D) we use (3.39) to bound $\left||n_2 + n_3|^s - |n_2|^s\right| \lesssim |n_2|^{s-1}|n_3|$ and $\left||n_2 + n_1|^s - |n_2|^s\right| \lesssim |n_2|^{s-1}|n_1|$. We arrive to

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M \\ |n_2|>2|n_3| \text{ and } |n_2|>2|n_1|}} \frac{|n_1|^s |n_3|^s n_2^2 |n_2|^{s-1} |n_3| |n_2|^{s-1} |n_1|}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1+n_2+n_3=0 \\ \max_j(|n_j|)>M}} \frac{|n_1|^{s+1} |n_2|^{2s} |n_3|^{s+1}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} ,$$

that is estimated as (3.29).

● Permutation $\sigma = (3, 1, 2)$. We need to handle

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M}} \frac{|n_1|^s |n_3|^s n_2 n_1 (|n_2 + n_3|^s - |n_2|^s)(|n_1 + n_2|^s - |n_1|^s)}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

We note that renaming the indeces (n_1, n_3, n_2) with (n_2, n_1, n_3) this reduces to

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j (|n_j|) > M}} \frac{|n_2|^s |n_1|^s n_3 n_2 (|n_3 + n_1|^s - |n_3|^s)(|n_2 + n_3|^s - |n_2|^s)}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}},$$

that is the contribution of the permutation $\sigma = (2, 3, 1)$. This completes the proof. □

Lemma 3.7 *Let* $s > \frac{1}{2}$. *We have for all* $N > M$

$$\|F_{3,N} - F_{3,M}\|_{L^2(\gamma_s)} \lesssim \frac{1}{\sqrt{M}}. \tag{3.41}$$

Proof We have

$$F_{3,N} - F_{3,M} = \frac{2i}{\pi} \sum_{n \in A_{N,M}} |n_1|^s \frac{(n_2 + n_3)|n_2 + n_3|^s}{1 + |n_2 + n_3|} u(n_1)u(n_2)u(n_3).$$

Taking the modulus squared

$$|F_{3,N} - F_{3,M}|^2 = \frac{4}{\pi^2} \sum_{(n,m) \in A_{N,M}^2} |n_1|^s \frac{(n_2 + n_3)|n_2 + n_3|^s}{1 + |n_2 + n_3|} |m_1|^s$$

$$\frac{(m_2 + m_3)|m_2 + m_3|^s}{1 + |m_2 + m_3|} \prod_{j=1}^{3} u(n_j)u(-m_j) \tag{3.42}$$

and using the Wick formula (3.15) with $\ell = 3$ we arrive to

$$\|F_{3,N} - F_{3,M}\|_{L^2(\gamma_s)}^2$$

$$= \frac{4}{\pi^2} \sum_{\sigma \in S_3} \sum_{n \in A_{N,M}} \frac{|n_1|^s |n_{\sigma(1)}|^s \frac{(n_2+n_3)|n_2+n_3|^s}{1+|n_2+n_3|} \frac{(n_{\sigma(2)}+n_{\sigma(3)})|n_{\sigma(2)}+n_{\sigma(3)}|^s}{1+|n_{\sigma(2)}+n_{\sigma(3)}|}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\leqslant \frac{4}{\pi^2} \sum_{\sigma \in S_3} \sum_{n \in A_{N,M}} \frac{|n_1|^s |n_{\sigma(1)}|^s |n_2 + n_3|^s |n_{\sigma(2)} + n_{\sigma(3)}|^s}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}. \tag{3.43}$$

It is easy to see that the contractions $\sigma = (1, 2, 3)$ and $\sigma = (1, 3, 2)$ give the same contributions. Also, the remaining contractions gives all the same contributions. Thus we may reduce to the cases (say) $\sigma = (1, 2, 3)$ and $\sigma = (2, 1, 3)$.

The contribution relative to $\sigma = (1, 2, 3)$ is

$$
\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_2 + n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}
$$

$$
\lesssim_s \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_2|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}
$$

$$
+ \sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^{2s} |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} \tag{3.44}
$$

By the symmetry $n_2 \leftrightarrow n_3$ it suffices to handle the first term on the r.h.s., for which we have

$$
\lesssim_s \sum_{\substack{|n_1|, |n_2| \leqslant N \\ |n_1| \gtrsim M}} \frac{1}{\langle n_1 \rangle^{2s+1} \langle n_2 \rangle \langle n_2 - n_1 \rangle} + \sum_{\substack{|n_1|, |n_2| \leqslant N \\ |n_2| \gtrsim M}} \frac{1}{\langle n_1 \rangle^{2s+1} \langle n_2 \rangle \langle n_2 - n_1 \rangle}
$$

$$
\lesssim_s \sum_{|n_2| \gtrsim M} \frac{1}{\langle n_2 \rangle} \sum_{n_1 \in \mathbb{Z}} \frac{1}{\langle n_1 \rangle^{2s+1} \langle n_2 - n_1 \rangle} \tag{3.45}
$$

$$
+ \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n_2 \rangle} \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_1 \rangle^{2s+1} \langle n_2 - n_1 \rangle} . \tag{3.46}
$$

The inner sums of (3.45) and (3.46) are estimated by (3.13) and (3.14). We have

$$
(3.45) \leqslant \sum_{|n_2| > M} \frac{1}{\langle n_2 \rangle^2} \lesssim \frac{1}{M} , \tag{3.47}
$$

$$
(3.46) \lesssim \sum_{|n_2| > M} \frac{1}{\langle n_2 \rangle^2} \lesssim \frac{1}{M} . \tag{3.48}
$$

The contribution relative to $\sigma = (2, 1, 3)$ is

$$
\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^s |n_2|^s |n_2 + n_3|^s |n_1 + n_3|^s}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}} . \tag{3.49}
$$

Using

$$|n_2 + n_3|^s |n_1 + n_3|^s \lesssim_s (|n_2|^s + |n_3|^s)(|n_1|^s + |n_3|^s)$$

and exchanging the indices, we can reduce (3.49) to a sum of terms of the form

$$\sum_{\substack{|n_j| \leqslant N, n_j \neq 0 \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{|n_1|^s |n_2|^s |n_3|^{2s}}{|n_1|^{2s+1} |n_2|^{2s+1} |n_3|^{2s+1}}$$

$$\lesssim \sum_{\substack{|n_j| \leqslant N \\ n_1 + n_2 + n_3 = 0 \\ \max_j(|n_j|) > M}} \frac{1}{\langle n_1 \rangle \langle n_2 \rangle^{s+1} \langle n_3 \rangle^{s+1}}$$

$$\leqslant \sum_{\max(|n_1|, |n_2|) \gtrsim M} \frac{1}{\langle n_1 \rangle \langle n_2 \rangle^{s+1} \langle n_1 - n_2 \rangle^{s+1}}$$

$$\leqslant \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_1 \rangle} \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n_2 \rangle^{s+1} \langle n_1 - n_2 \rangle^{s+1}} \tag{3.50}$$

$$+ \sum_{n_1 \in \mathbb{Z}} \frac{1}{\langle n_1 \rangle} \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_2 \rangle^{s+1} \langle n_1 - n_2 \rangle^{s+1}} . \tag{3.51}$$

Again (3.50) and (3.51) can be estimated by using (3.11) and (3.12). We have

$$(3.50) \leqslant \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_1 \rangle^{s+\frac{3}{2}}} \lesssim \frac{1}{M^{s+\frac{1}{2}}} , \tag{3.52}$$

$$(3.51) \leqslant \sum_{|n_1| \gtrsim M} \frac{1}{\langle n_1 \rangle^{s+\frac{3}{2}}} + \frac{1}{M^{s+\frac{1}{2}}} \sum_{n_1 \in \mathbb{Z}} \frac{1}{\langle n_1 \rangle^{s+\frac{3}{2}}} \lesssim \frac{1}{M^{s+\frac{1}{2}}} . \tag{3.53}$$

$\square$

4 Tail estimates

The goal of this section is to prove the following proposition, that is the key quantitative estimate in the study of the quasi-invariance of $\tilde{\gamma}_s$.

Proposition 4.1 *Let $s > 1$. For all $N \in \mathbb{N} \cup \{\infty\}$ it holds*

$$\|F_N\|_{L^p(\tilde{\gamma}_s)} \lesssim C(R)p . \tag{4.1}$$

We state and prove immediately two useful tail bounds in view of Proposition 2.2.

Lemma 4.2 *Let $s > 1$ and $\kappa > 0$. There is $c(R) > 0$ such that for all $N \in \mathbb{N} \cup \{\infty\}$ we have*

$$\tilde{\gamma}_s(\|P_N \partial_x u\|_{L^\infty} \geqslant t^\kappa) \lesssim e^{-c(R)t^{2s\kappa}}. \tag{4.2}$$

Proof First we prove the statement for $s > 3/2$. We bound

$$\|P_N \partial_x u\|_{L^\infty} \leqslant \sum_{j \in \mathbb{N}} 2^j \sup_{x \in \mathbb{T}} |\Delta_j P_N u| \leqslant \sum_{j \in \mathbb{N}} \sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)| \tag{4.3}$$

and estimate the deviation probability for the r.h.s.

Let j_t the largest element of $\mathbb{N}$ such that

$$2^j < \frac{t^\kappa}{R} \quad \text{for} \quad j < j_t; \tag{4.4}$$

we set $j_t = 0$ if (4.4) is never satisfied. We split

$$\sum_{j \in \mathbb{N}} \sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)| \leqslant \sum_{0 \leqslant j < j_t} \sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)| + \sum_{j \geqslant j_t} \sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)|.$$

The first summand is easily evaluated. Indeed

$$\sum_{0 \leqslant j < j_t} \sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)| \leqslant 2^{\frac{3j_t}{2}} \|\Delta_j u\|_{L^2} \lesssim 2^{j_t} \|\Delta_j u\|_{\dot{H}^{\frac{1}{2}}} \leqslant 2^{j_t} R \leqslant t^\kappa, \tag{4.5}$$

where we first used the Cauchy–Schwartz and then the Bernstein inequality. Since the above inequality holds $\tilde{\gamma}_s$-a.s. we have

$$\tilde{\gamma}_s\left(\sum_{0 \leqslant j < j_t} \sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)| \geqslant t^\kappa \right) = 0. \tag{4.6}$$

To estimate the contribution for $j \geqslant j_t$ we introduce a sequence $\{\sigma_j\}_{j \geqslant j_t}$ defined as

$$\sigma_j := c_0(j + 1 - j_t)^{-2}, \tag{4.7}$$

where $c_0 > 0$ is sufficiently small in such a way that $\sum_{j \geqslant j_t} \sigma_j \leqslant 1$. Then we bound

$$\tilde{\gamma}_s\left(\sum_{|n| \simeq 2^j} 2^j |(P_N u)(n)| \geqslant t^\kappa \right) \leqslant \sum_{j \geqslant j_t} \tilde{\gamma}_s\left(\sum_{|n| \simeq 2^j} |(P_N u)(n)| \geqslant 2^{-j} \sigma_j t^\kappa \right). \tag{4.8}$$

As the γ_s-expectation of $\sum_{|n| \simeq 2^j} |(P_N u)(n)|$ is bounded by $C2^{j\left(\frac{1}{2}-s\right)}$ and

$$C2^{j\left(\frac{1}{2}-s\right)} \leqslant \frac{1}{2} 2^{-j} \sigma_j t^\kappa \quad \text{for } s > 3/2 \text{ and } t > C_s, \tag{4.9}$$

where C_s is a sufficiently large constant (only depending on s)[1], we have as consequence of inequality (1.6) that

$$\tilde{\gamma}_s\left(\sum_{|n|\simeq 2^j}|(P_N u)(n)| \geqslant 2^{-j}\sigma_j t^\kappa\right) \leqslant C\exp\left(-c\frac{2^{-2j}\sigma_j^2 t^{2\kappa}}{\sum_{|n|\simeq 2^j}n^{-2s-1}}\right)$$

$$\leqslant C\exp\left(-ct^{2\kappa}\sigma_j^2 2^{2j(s-1)}\right); \qquad (4.10)$$

Thus

$$\text{r.h.s. of (4.8)} \lesssim \sum_{j\geqslant j_t} e^{-c\sigma_j^2 2^{j(2s-2)}t^{2\kappa}} \lesssim e^{-c2^{j_t(2s-2)}t^{2\kappa}} \lesssim e^{-c\frac{t^{2s\kappa}}{R^{2s-2}}}, \qquad (4.11)$$

that concludes the proof when $s > 3/2$ (note that the second inequality is in fact justified as long as $s > 1$). To handle the case $1 < s \leqslant 3/2$ we note that the above argument still works as long as we restrict to frequencies j such that[2]

$$C2^{j\left(\frac{1}{2}-s\right)} < \frac{1}{2}2^{-j(1+\varepsilon)}t^\kappa < \frac{1}{2}2^{-j}\sigma_j t^\kappa; \qquad (4.12)$$

where $\varepsilon > 0$ will be later chosen sufficiently small; in fact such that $\varepsilon < s - \frac{1}{2}$ (we could then have chosen the sequence σ_j so that $2^{-j\varepsilon} < \sigma_j$). Thus, denoting with j_t^* the largest integer such that

$$C2^{j_t^*\left(\frac{1}{2}-s\right)} < \frac{1}{2}2^{-j_t^*(1+\varepsilon)}t^\kappa \qquad (4.13)$$

holds, we need to handle the frequencies $j > j_t^*$ (we set $j_t^* = 0$ if (4.13) is never satisfied). Namely, it suffices to show

$$\tilde{\gamma}_s(\|(\mathrm{Id} - P_{2^{j_t^*}})P_N\partial_x u\|_{L^\infty} \geqslant t^\kappa) \lesssim e^{-ct^{2s\kappa}}.$$

Note that for $s > 1/2$ we have $j_t^* \gg_t j_t$ (in fact we gain a power of t working with j_t^* in place of j_t). We have by definition of j_t^* that

$$2^{j_t^*\left(\frac{1}{2}-s\right)} \gtrsim 2^{-j_t^*(1+\varepsilon)}t^\kappa, \qquad (4.14)$$

namely

$$2^{j_t^*} \gtrsim t^{\frac{\kappa}{\frac{3}{2}+\varepsilon-s}}. \qquad (4.15)$$

---

[1] For instance we can quantify C_s as follows $C_s^\kappa = \sup_j \frac{2C}{c_0}2^{j\left(\frac{3}{2}-s\right)}(j+1)^2$, noting that this is finite for $s > 3/2$. The restriction $t > C_s$ is harmless as the statement (4.2) is trivial for $t < C_s$.

[2] Note that the second inequality is true for j sufficiently large. This is again harmless since small frequencies are handled using (4.4), (4.5).

Now we bound as above

$$\tilde{\gamma}_s(\|(\mathrm{Id} - P_{2^{j_t^*}})P_N\partial_x u\|_{L^\infty} \leqslant \sum_{j > j_t^*} \tilde{\gamma}_s(\|\Delta_j P_N\partial_x u\|_{L^\infty} > \sigma_j t^\kappa). \qquad (4.16)$$

Then we use that for all $\varepsilon' > 0$ we have

$$\gamma_s(\|\Delta_j P_N\partial_x u\|_{L^\infty} > \sigma_j t^\kappa) \lesssim C \exp\left(-ct^{2\kappa}\sigma_j^2 2^{2j(s-1-\varepsilon')}\right). \qquad (4.17)$$

This is true since $\|\Delta_j P_N\partial_x u\|_{L^2(\gamma_s)} \simeq C2^{j(1-s)}$. Using this fact we can show that for all $q > 2$ we have $\|\Delta_j P_N\partial_x u\|_{L^q(\gamma_s)} \simeq \sqrt{q}2^{j(1-s)}$. Using this and the Minkowski's integral inequality we can prove that for all $p < \infty$ we have

$$\|\|\Delta_j P_N\partial_x u\|_{L^p_x}\|_{L^q(\gamma_s)} \simeq \sqrt{q}2^{j(1-s)}, \quad \text{for all } q > p.$$

By this estimate for the moments the following tail bound follows

$$\gamma_s(\|\Delta_j P_N\partial_x u\|_{L^p} > \sigma_j t^\kappa) \lesssim C \exp\left(-ct^{2\kappa}\sigma_j^2 2^{2j(s-1)}\right).$$

From this inequality and $\|\Delta_j P_N\partial_x u\|_{L^\infty} \lesssim 2^{j/p}\|\Delta_j P_N\partial_x u\|_{L^p}$ we get (4.17). Thus

$$\text{r.h.s. of (4.16)} \lesssim \sum_{j > j_t} e^{-c\sigma_j^2 2^{j(2s-2-2\varepsilon')}t^{2\kappa}} \lesssim e^{-c2^{j_t^*(2s-2-2\varepsilon')}t^{2\kappa}}$$

$$\lesssim e^{-ct^{\frac{\kappa}{3/2+\varepsilon-s}(2s-2-2\varepsilon')}t^{2\kappa}} \lesssim e^{-ct^{2s\kappa}}, \qquad (4.18)$$

where: in the second inequality we used $2s - 2 - 2\varepsilon' > 0$ (which is true for all $s > 1$ taking ε' sufficiently small. In the third inequality we used (4.15). The last inequality holds as long as $s > \frac{1}{2} + \varepsilon$ and $\varepsilon' \ll \varepsilon$. Since we can take $\varepsilon, \varepsilon'$ arbitrarily small, the proof is complete (we recall that we always have a restriction $s > 1$ coming from the second inequality in (4.11)). $\square$

Lemma 4.3 *Let $\kappa > 0$ and*

$$a := \frac{4\kappa s}{2s - 1}, \qquad b := \frac{2}{2s - 1} \qquad (4.19)$$

Then there are $C, c > 0$, such that the bound

$$\tilde{\gamma}_s(\|P_N u\|_{H^s} \geqslant t^\kappa) \leqslant C \exp\left(-c\frac{t^a}{R^b}\right), \qquad (4.20)$$

holds for all $N \in \mathbb{N}$ and $t \gtrsim (\ln N)^{\frac{2}{\kappa}}$.

Proof Let

$$X_N^{(s)} := \sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j P_N u \|_{L^2} .$$

We prove the statement for $X_N^{(s)}$ in place of $\| P_N u \|_{H^s}$. This is sufficient since $\| P_N u \|_{H^s} \leqslant X_N^{(s)}$. Let j_t the largest element of $\mathbb{N} \cup \{\infty\}$ such that

$$2^{j(s-\frac{1}{2})} < \frac{t^\kappa}{2R} \quad \text{for} \quad j < j_t . \tag{4.21}$$

We split

$$X_N^{(s)} = \sum_{0 \leqslant j < j_t} X_{j,N}^{(s)} + \sum_{j \geqslant j_t} X_{j,N}^{(s)} . \tag{4.22}$$

The first summand is easily evaluated. Indeed since $s > \frac{1}{2}$ we have that

$$\sum_{0 \leqslant j < j_t} X_{j,N}^{(s)} \leqslant \sum_{0 \leqslant j < j_t} 2^{j(s-\frac{1}{2})} \| \Delta_j u \|_{\dot{H}^{\frac{1}{2}}} < \frac{t^\kappa}{R} R = t^\kappa , \quad \text{(we used (4.21))},$$

holds $\tilde{\gamma}_s$-a.s., therefore

$$\tilde{\gamma}_s \left(\sum_{0 \leqslant j < j_t} X_{j,N}^{(s)} \geqslant t^\kappa \right) = 0 . \tag{4.23}$$

Let σ_j as in (4.7). We have

$$\begin{aligned}
\tilde{\gamma}_s(X_{j,N}^{(s)} \geqslant \sigma_j t^\kappa) &= \tilde{\gamma}_s(2^{js} \| \Delta_j P_N u \|_{L^2} \geqslant \sigma_j t^\kappa) \\
&= \tilde{\gamma}_s(\| \Delta_j P_N u \|_{L^2}^2 \geqslant 2^{-2js} \sigma_j^2 t^{2\kappa}) .
\end{aligned} \tag{4.24}$$

We note that

$$E_s[\| \Delta_j P_N u \|_{L^2}^2] = \sum_{\substack{n \simeq 2^j \\ |n| \leqslant N}} E_s[|\hat{u}(n)|^2] \simeq \sum_{\substack{n \simeq 2^j \\ |n| \leqslant N}} \frac{1}{n^{2s+1}} \simeq 2^{-2js} 1_{\{j \leqslant \log_2 N\}} . \tag{4.25}$$

Therefore

$$\begin{aligned}
\tilde{\gamma}_s(\| \Delta_j P_N u \|_{L^2}^2 \geqslant 2^{-2js} \sigma_j^2 t^{2\kappa}) &\lesssim \tilde{\gamma}_s(\| \Delta_j P_N u \|_{L^2}^2 \\
&\geqslant E_s[\| \Delta_j P_N u \|_{L^2}^2] + 2^{-2js} \sigma_j^2 t^{2\kappa}) \\
&\lesssim \tilde{\gamma}_s(|\| \Delta_j P_N u \|_{L^2}^2 \\
&\quad -E_s[\| \Delta_j P_N u \|_{L^2}^2]| \geqslant 2^{-2js} \sigma_j^2 t^{2\kappa}) \tag{4.26}
\end{aligned}$$

This last term can be bounded by the Bernstein inequality (1.7). We conclude

$$\tilde{\gamma}_s(X_{j,N}^{(s)} \geqslant \sigma_j t^\kappa) \leqslant C \exp\left(-c \min\left(\sigma_j^2 t^{2\kappa} 2^j, \sigma_j^4 t^{4\kappa} 2^j\right)\right). \qquad (4.27)$$

Therefore for any fixed $j \geqslant j_t$ we have

$$\tilde{\gamma}_s(X_{j,N}^{(s)} \geqslant \sigma_j t^\kappa) \leqslant 2 e^{-\sigma_j^2 t^{2\kappa} 2^j} \qquad (4.28)$$

provided

$$t \gtrsim j^{\frac{2}{\kappa}}.$$

The estimate extends to all $j \geqslant j_t$ for

$$t \gtrsim \max_{j_t \leqslant j \lesssim \ln N} \gtrsim (\ln N)^{\frac{2}{\kappa}}.$$

Note that by definition of j_t (see (4.21)) we have

$$2^{j_t} \gtrsim \left(\frac{t^\kappa}{2R}\right)^{\frac{1}{s-\frac{1}{2}}}. \qquad (4.29)$$

Thus, using (4.28)-(4.29) we can estimate

$$\begin{aligned}
\tilde{\gamma}_s\left(\sum_{j \geqslant j_t} X_{j,N}^{(s)} \geqslant t^\kappa\right) &\leqslant \sum_{j \geqslant j_t} \tilde{\gamma}_s\left(X_{j,N}^s \geqslant \sigma_j t^\kappa\right) \\
&\lesssim \sum_{j \geqslant j_t} e^{-\sigma_j^2 t^{2\kappa} 2^j} \\
&\lesssim e^{-t^{2\kappa} 2^{j_t}} \lesssim \exp\left(-c \frac{t^{4\kappa s/(2s-1)}}{R^{2/(2s-1))}}\right)
\end{aligned}$$

for some absolute constants $C, c > 0$.   □

Remark 4.4 By the same proofs one can show the statements of Lemma 4.2 and Lemma 4.3 for all the measures $\tilde{\gamma}_{s,M}(A) = E_s[1_{\{\mathcal{E}[P_M u] \leqslant R\} \cap A}]$ with $M \geqslant N$ (we only dealt with $M = \infty$).

The small deviations of F_N are evaluated as follows. First of all we note that by Proposition 3.1 there is $C > 0$ such that for any $M \geqslant N \in \mathbb{N}$

$$\|F_N - F_M\|_{L^2(\gamma_s)} \leqslant \frac{C}{N^\upsilon}, \quad \upsilon := \min(\frac{1}{2}, \frac{2s-1}{4}). \qquad (4.30)$$

Since F_N is a trilinear form of Gaussian random variables, we see that (4.30) implies by hyper-contractivity that for all $p > 2$ there is $C > 0$ (possibly different from above) for which

$$\|F_N - F_M\|_{L^p(\gamma_s)} \leqslant \frac{Cp^{\frac{3}{2}}}{N^\upsilon} . \tag{4.31}$$

From (4.31) we obtain the following result in the standard way

Proposition 4.5 *Let $s > 1$ and $N \in \mathbb{N}$. There are $C, c > 0$ such that*

$$\gamma_s\left(|F_N - F| \geqslant t\right) \leqslant Ce^{-ct^{\frac{2}{3}}N^{\frac{2\upsilon}{3}}} . \tag{4.32}$$

Proposition 4.6 *Let $s > 1$ and $t \leqslant N^{2\upsilon}$. There are $c, C > 0$ such that*

$$\tilde{\gamma}_s(|F_N| \geqslant t) \leqslant Ce^{-ct} . \tag{4.33}$$

Proof Let us set $T := \lfloor t^{\frac{1}{2\upsilon}} \rfloor$ and notice that $T \leqslant N$. By the union bound

$$\tilde{\gamma}_s(|F_N| \geqslant t) \leqslant \gamma_s(|F_N - F_T| \geqslant t/2) + \tilde{\gamma}_s(|F_T| \geqslant t/2) . \tag{4.34}$$

By Proposition 4.5 we have

$$\gamma_s(|F_N - F_T| \geqslant t) \leqslant Ce^{-ct^{\frac{2}{3}}T^{\frac{2\upsilon}{3}}} \leqslant Ce^{-ct} . \tag{4.35}$$

On the other hand since $t \geqslant T^{2\upsilon}$ the estimate of Proposition 4.7 applies to the second summand of (4.34). This concludes the proof. $\qquad \square$

Now we complete the proof of Proposition 4.1, combining Proposition 4.7 and the following result describing larger deviations of F_N.

Proposition 4.7 *Let $s > 1$ and $t \geqslant N^{2\upsilon}$. There are $c(R), C > 0$ such that*

$$\tilde{\gamma}_s(|F_N| \geqslant t) \leqslant Ce^{-c(R)t} . \tag{4.36}$$

Proof By Proposition 2.2

$$\tilde{\gamma}_s(|F_N| \geqslant t) \leqslant \tilde{\gamma}_s\left(\|P_N u\|_{H^s}^2 \|\partial_x P_N u\|_{L^\infty} \geqslant t\right) .$$

Using Lemma 4.2 and Lemma 4.3 we get for $\hat{\kappa} \in (0, \frac{1}{2})$

$$\begin{aligned}
\tilde{\gamma}_s\left(\left(\|P_N u\|_{H^s}^2\right)^2 \|\partial_x P_N u\|_{L^\infty} \geqslant t\right) &\leqslant \tilde{\gamma}_s\left(\|P_N u\|_{H^s}^2 \geqslant t^{\hat{\kappa}}\right) \\
&\quad + \tilde{\gamma}_s\left(\|\partial_x P_N u\|_{L^\infty} \geqslant t^{1-2\hat{\kappa}}\right) \\
&\leqslant C\exp(-c(R)t^{\hat{a}}) , \tag{4.37}
\end{aligned}$$

with

$$\hat{a} := \min\left(\frac{4s}{2s-1}\hat{\kappa}\,,\ \frac{4s}{3-1}(1-2\hat{\kappa})\right). \tag{4.38}$$

Now we optimize

$$\hat{\kappa} := \frac{2s-1}{4s}, \qquad (\text{note } \hat{\kappa} \in (0, 1/2) \text{ for } s > 1/2); \tag{4.39}$$

in particular $\hat{a} = 1$ and the proof is concluded. □

5 Quasi-invariant measures

In this section we complete the proof of Theorem 1.1 by the method of [27].

Let us first introduce the set

$$E_N := \text{span}_{\mathbb{R}}\{(\cos(nx), \sin(nx))\,,\quad |n| \leqslant N,\quad n \neq 0\}.$$

Note $\dim E_N = 2N$. We denote by $E_N^{\perp}$ the orthogonal complement of E_N in the topology of $L^2(\mathbb{T})$. Letting $\gamma_{s,N}^{\perp}$ the measure induced on $E_N^{\perp}$ by the map

$$\varphi_s(\omega, x) = \sum_{|n|>N} \frac{g_n(\omega)}{|n|^{s+1/2}} e^{inx}, \tag{5.1}$$

the measure γ_s factorises over $E_N \times E_N^{\perp}$ as

$$\gamma_s(du) := \frac{1}{Z_N} e^{-\frac{1}{2}\|P_N u\|_{H^{s+\gamma/2}}^2} L_N(d P_N u)\, \gamma_{s,N}^{\perp}(d P_{>N} u), \tag{5.2}$$

where L_N is the Lebesgue measure induced on E_N by the isomorphism between $\mathbb{R}^{2N}$ and E_N and Z_N is a renormalisation factor. This factorisation is useful since we know by [27, Lemma 4.2] that the Lebesgue measure L_N is invariant under $\Phi_t^N P_N = P_N \Phi_t^N$.

The first important step toward the proof of the quasi-invariance of $\tilde{\gamma}_s$ is the following

Proposition 5.1 *Let $N \in \mathbb{N}$, $s > 1$ and $R > 0$. There exists $C(R)$ such that*

$$\frac{d}{dt}\left(\tilde{\gamma}_s(\Phi_t^N(A))\right)^{\frac{1}{p}} \leqslant C(R)p, \tag{5.3}$$

for all measurable set A and for all $p \geqslant 1$.

Proof Using the definition (1.4), the factorisation (5.2) and Proposition 4.1 of [27], we have for all measurable A

$$
\begin{aligned}
\tilde{\gamma}_s \circ \Phi_t^N(A) &= \int_{\Phi_t^N(A)} \gamma_s(du) 1_{\{\mathcal{E}[u] \leqslant R\}} \\
&= \int_A L_N(dP_N u) \gamma_{s,N}^\perp (dP_{>N} u) 1_{\{\mathcal{E}[u] \leqslant R\}} \exp\left(-\frac{1}{2} \|P_N \Phi_t^N u\|_{H^{s+\frac{1}{2}}}^2\right) \\
&= \int_A \tilde{\gamma}_s(du) \exp\left(\frac{1}{2}\|P_N u\|_{H^{s+\frac{1}{2}}}^2 - \frac{1}{2}\|P_N \Phi_t^N u\|_{H^{s+\frac{1}{2}}}^2\right) \quad (5.4)
\end{aligned}
$$

where we used that the Jacobian determinant is unitary (see [27, Lemma 4.2]) and in the second identity we used $\mathcal{E}[\Phi_t^N u] = \mathcal{E}[u]$; see Lemma 2.4 in [27]. Since

$$
t \in (\mathbb{R}, +) \to \Phi_t^N
$$

is a one parameter group of transformations, we can easily check that

$$
\frac{d}{dt}\left(\tilde{\gamma}_s \circ \Phi_t^N(A')\right)\bigg|_{t=\bar{t}} = \frac{d}{dt}\left(\tilde{\gamma}_s \circ \Phi_t^N(\Phi_{\bar{t}}^N A')\right)\bigg|_{t=0} \quad (5.5)
$$

for all measurable A'. Using (5.5) and (5.4) under the choice $A = \Phi_{\bar{t}}^N A'$, we arrive to

$$
\begin{aligned}
&\frac{d}{dt}\left(\tilde{\gamma}_s \circ \Phi_t^N(A)\right)\bigg|_{t=\bar{t}} \\
&= \frac{d}{dt}\int_{\Phi_{\bar{t}}^N(A)} \exp\left(\frac{1}{2}\|P_N u\|_{H^{s+\frac{1}{2}}}^2 - \frac{1}{2}\|P_N \Phi_t^N u\|_{H^{s+\frac{1}{2}}}^2\right) \tilde{\gamma}_s(du)\bigg|_{t=0} \\
&= -\frac{1}{2}\int_{\Phi_{\bar{t}}^N(A)} \tilde{\gamma}_s(du) \frac{d}{dt}\|P_N \Phi_t^N u\|_{H^{s+\frac{1}{2}}}^2\bigg|_{t=0} \\
&= -\frac{1}{2}\int_{\Phi_{\bar{t}}^N(A)} \tilde{\gamma}_s(du) F_N. \quad (5.6)
\end{aligned}
$$

By the Hölder inequality and Proposition 4.1, we get

$$
\left|\int_{\Phi_{\bar{t}}^N(A)} \tilde{\gamma}_s(du) F_N\right| \leqslant C(R)\, p\, (\tilde{\gamma}_s(\Phi_{\bar{t}}^N(A)))^{1-\frac{1}{p}}. \quad (5.7)
$$

Thus we conclude that there is $C(R)$ such that

$$
\frac{d}{dt}\left(\tilde{\gamma}_s \circ \Phi_t^N(A)\right) \leqslant C(R)\, p\, (\tilde{\gamma}_s(\Phi_t^N(A)))^{1-\frac{1}{p}}. \quad (5.8)
$$

From (5.8) we get (5.3). $\qquad\square$

We are now able to control quantitatively the growth in time of $\tilde{\gamma}_s(\Phi_t(A))$.

Proposition 5.2 *Let $s > 1$ and $R > 0$. There exists $C(R) > 1$ such that*

$$\tilde{\gamma}_s(\Phi_t(A)) \lesssim \tilde{\gamma}_s(A)^{(C(R)^{-|t|})}, \qquad \forall t \in \mathbb{R}. \tag{5.9}$$

Proof We will prove that

$$\tilde{\gamma}_s(\Phi_t^N(A)) \lesssim \tilde{\gamma}_s(A)^{(C(R)^{-|t|})}, \qquad \forall t \in \mathbb{R}, \tag{5.10}$$

with a constant $C(R)$ which is independent on $N \in \mathbb{N}$. We can then promote this inequality to the case $N = \infty$, namely to (5.9), proceeding as in Sect. 8 of [27].

To prove (5.10) we rewrite (5.8) as

$$\frac{d}{dt}\left(\left(\tilde{\gamma}_s \circ \Phi_t^N(A)\right)^{\frac{1}{p}}\right) \leqslant C(R)$$

and integrating this over $[0, t]$ we get

$$(\tilde{\gamma}_s \circ \Phi_t^N)(A) \leqslant (C(R)|t| + \tilde{\gamma}_s(A)^{\frac{1}{p}})^p$$

It will be useful to rewrite the r.h.s. as

$$
\begin{aligned}
(|t| + \tilde{\gamma}_s(A)^{\frac{1}{p}})^p &= \tilde{\gamma}_s(A)\left(1 + \frac{C(R)|t|}{\tilde{\gamma}_s(A)^{\frac{1}{p}}}\right)^p \\
&= \tilde{\gamma}_s(A)e^{p\log\left(1 + C(R)|t|\tilde{\gamma}_s(A)^{-\frac{1}{p}}\right)}.
\end{aligned}
\tag{5.11}
$$

Now we can pick

$$p = p(A) = \log\frac{1}{\tilde{\gamma}_s(A)} \quad \text{in such a way that} \quad \tilde{\gamma}_s(A)^{-\frac{1}{p}} = e. \tag{5.12}$$

Thus

$$(\tilde{\gamma}_s \circ \Phi_t^N)(A) \leqslant \tilde{\gamma}_s(A)e^{p\log(1 + C(R)e|t|)} \leqslant \tilde{\gamma}_s(A)e^{p\,C(R)e|t|}. \tag{5.13}$$

Then we claim that for $|t|$ small enough

$$e^{p\,C(R)e|t|} \leqslant \tilde{\gamma}_s(A)^{-1/2}. \tag{5.14}$$

To have that, it must be

$$p\,C(R)e|t| \leqslant \frac{1}{2}\log\frac{1}{\tilde{\gamma}_s(A)} = \frac{p}{2} \tag{5.15}$$

which is true for $|t| \leqslant \frac{1}{2eC(R)}$. Plugging (5.14) into (5.13) we arrive to

$$(\tilde{\gamma}_s \circ \Phi_t^N)(A) \leqslant \tilde{\gamma}_s(A)^{1/2}, \qquad |t| \leqslant \frac{1}{2eC(R)} \, . \qquad (5.16)$$

Then, it is easy to see that the desired bound (5.9) follows by iteration of the estimate (5.16) (the constants $C(R)$ in (5.9)-(5.16) differs by an irrelevant factor). For details we refer to the proof of Lemma 3.3 in [9]. □

By the previous result, the flow Φ_t maps zero measure sets into zero measure sets, for all $t \in \mathbb{R}$. Therefore, for $s > 1$, we have proved that $\tilde{\gamma}_s \circ \Phi_t$ is absolutely continuous w.r.t. $\tilde{\gamma}_s$ (and so w.r.t γ_s) with a density $f_s(t, u) \in L^1(\tilde{\gamma}_s)$. We finish with a more precise evaluation of the integrability of f_s.

Proposition 5.3 *Let $s > 1$. There exists $p = p(t, R) > 1$ such that $f_s(t, u) \in L^p(\tilde{\gamma}_s)$.*

Proof Once we have proved the inequality (5.3), the proof of the statement proceeds exactly as that of [9, Proposition 3.4] (the flow $\mathscr{G}$ must be replaced by Φ). From the proof we see that $p = (1 - e^{-|t| \ln C(R)})^{-1}$. □

References

1. Bourgain, J.: Periodic nonlinear Schrödinger equation and invariant measures. Commun. Math. Phys. **166**, 1–26 (1994)
2. Burq, N., Thomann, L.: *Almost sure scattering for the one dimensional nonlinear Schrödinger equation*, arXiv:2012.13571 (2020)
3. Cameron, R., Martin, W.: Transformations of Wiener Integrals under translations. Annals Math. **45**, 386–396 (1944)
4. Cruzeiro, A.B.: Equations différentielles ordinaires: non-explosion et measures quasi invariantes. J. Funct. Anal. **54**, 193–205 (1983)
5. Cruzeiro, A.B.: Equations différentielles sur l'espace de Wiener et formules de Cameron-Martin non linéaires. J. Funct. Anal. **54**, 206–227 (1983)
6. Debussche, A., Tsustumi, Y.: Quasi-Invariance of Gaussian measures transported by the Cubic NLS with third-order dispersion on $\mathbb{T}$, J. Funct. Anal. 281, (2021)

7. Forlano, J., Seong, K.: *Transport of Gaussian measures under the flow of one-dimensional fractional nonlinear Schrödinger equations*, arXiv:2102.13398 [math.AP] (2021)
8. Forlano, J., Trenberth, W.: On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equation, Ann. Inst. Henri Poincare. Anal. Non Lineaire **36**, 1987–2025 (2019)
9. Genovese, G., Lucá, R., Tzvetkov, N.: Quasi-invariance of low regularity Gaussian measures under the gauge map of the periodic derivative NLS, arXiv:2008.10001v1 (2020)
10. Genovese, G., Lucà, R., Tzvetkov, N.: Transport of Gaussian measures with exponential cut-off for Hamiltonian PDEs, arXiv:2103.04408 (2021)
11. Genovese, G., Lucà, R., Valeri, D.: Gibbs measures associated to the integrals of motion of the periodic DNLS. Sel. Math. New Ser. **22**(3), 1663–1702 (2016)
12. Genovese, G., Lucà, R., Valeri, D.: Invariant measures for the periodic derivative nonlinear Schrödinger equation. Mathematische Annalen **374**(3–4), 1075–1138 (2019)
13. Girsanov, I.: On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. Theory Prob Appl. **5**, 285–301 (1960)
14. Gunaratnam, T. S., Oh, T., Tzvetkov, N., Weber, H.: *Quasi-invariant Gaussian measures for the nonlinear wave equation in three dimensions*, arXiv:1808.03158 (2018)
15. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure App. Math. **41**, 891–907 (1988)
16. Kuo, H.-H.: Integration theory on infinite-dimensional manifolds. Trans. Amer. Math. Soc. **159**, 57–78 (1971)
17. Mammeri, Y.: Long time bounds for the periodic Benjamin-Ono-BBM equation. Nonlinear Anal. **71**(10), 5010–5021 (2009)
18. Nahmod, A., Rey-Bellet, L., Sheffield, S., Staffilani, G.: Absolute continuity of Brownian bridges under certain gauge transformations. Math. Res. Lett. **18**, 875–887 (2011)
19. Oh, T., Seong, K.: *Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation in negative Sobolev spaces*, arXiv:2012.06732 [math.AP] (2020)
20. Oh, T., Tzvetkov, N.: Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. Probab. Theory Relat. Fields **169**, 1121–1168 (2017)
21. Oh, T., Tzvetkov, N.: Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation. JEMS **22**, 1785–1826 (2020)
22. Oh, T., Sosoe, P., Tzvetkov, N.: An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation, J. Ec. Polytech. Math. **5**, 793–841 (2018)
23. T. Oh, Y. Tsutsumi and N. Tzvetkov, *Quasi-invariant Gaussian measures for the cubic nonlinear Schrödinger equation with third-order dispersion*, C. R. Acad. Sci. Paris, Ser. I, 357 (2019), 366-381
24. Planchon, F., Tzvetkov, N., Visciglia, N.: Transport of Gaussian measures by the flow of the nonlinear Schrödinger equation. Math. Ann. **378**, 389–423 (2020)
25. Ramer, R.: On nonlinear transformations of Gaussian measures. J. Funct. Anal. **15**, 166–187 (1974)
26. Sosoe, P., Trenberth, W.J., Xiao, T.: Quasi-invariance of fractional Gaussian fields by nonlinear wave equation with polynomial nonlinearity. Differ. Integral Equ. **33**, 393–430 (2020)
27. Tzvetkov, N.: Quasi-invariant Gaussian measures for one dimensional Hamiltonian PDEs. Forum Math Sigma **3**(e28), 35 (2015)
28. Üstünel, A.S., Zakai, M.: Transformation of measure on Wiener space. Springer, Berlin (2013)
29. Vershynin, R.: High-dimensional probability. Cambridge University Press, UK (2018)
30. Zakai, M., Zeitouni, O.: When does the ramer formula look like the girsanov formula? Ann. Probab. **20**(3), 1436–1440 (1992)