Stoch PDE: Anal Comp (2014) 2:27–53
DOI 10.1007/s40072-013-0024-0

Stochastic variational inequalities associated with elasto-plastic torsion

Jong Uhn Kim

Received: 29 April 2013 / Accepted: 16 December 2013 / Published online: 3 January 2014

Abstract This paper is concerned with an initial value problem for a stochastic variational inequality associated with elasto-plastic torsion. Our goal is to establish the existence and uniqueness of a solution. The stochastic problem is reduced to essentially a deterministic problem, which is not covered by existing results on evolution variational inequalities. We propose a definition of a solution in the same spirit as for weak solutions of partial differential equations, and derive some basic consequences of our definition. Based on these results, we can prove the existence and uniqueness of a solution to the stochastic problem.

Keywords Stochastic noise · Brownian motions · Variational inequality · Elasto-plastic torsion · Subdifferential

Mathematics Subject Classification 35Q72 · 35R60 · 35R70 · 58E99 · 60H15

1 Introduction

The goal of this paper is to study a stochastic evolution variational inequality of the form

$$\frac{\partial u}{\partial t} - \Delta u + \partial I_{\mathfrak{K}}(u) \ni \frac{\partial M}{\partial t}, \tag{1.1}$$

J. U. Kim (✉)
Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA
e-mail: kim@math.vt.edu

Here $\partial I_{\mathfrak{K}}(\cdot)$ denotes the subdifferential of the indicator function $I_{\mathfrak{K}}(\cdot)$, where $\mathfrak{K}$ is a closed convex subset of a certain function space. In this paper, we will only consider $\mathfrak{K}$ which is exclusively associated with elasto-plastic torsion. The right-hand side of (1.1) represents a random noise where $M = M(t)$ is a certain Hilbert space valued continuous martingale.

The genesis of stochastic variational inequalities is the celebrated Skorohod problem [15]; see also [10]. For generalization to higher space dimensions, see [5,16]. This subject has evolved in various directions. The above (1.1) is one such version. When $\mathfrak{K}$ is the set of nonnegative functions, initial boundary value problems were discussed in [7–9,12]. In particular, [12] has inspired extensive research on (1.1) and related problems in general abstract setting; see [1–3,14,17], and references therein. In general abstract setting, a typical assumption on the convex set $\mathfrak{K}$ is that its interior is not empty. This assumption seems necessary to obtain suitable regularity so that a solution of (1.1) can be defined in an appropriate sense. However, this assumption excludes some important applications. For example, if $\mathfrak{K}$ is the set of nonnegative functions in the basic function class $H_0^1(G)$, then $\mathfrak{K}$ has empty interior with respect to $H_0^1(G)$-norm for every space dimension $d = 1, 2, \ldots$. When $\mathfrak{K}$ is associated with elasto-plastic torsion, it is given by

$$\mathfrak{K} = \left\{ v \in H_0^1(G) \middle| |\nabla v(x)| \leq 1, \qquad \text{for almost all } x \right\} \tag{1.2}$$

Obviously, this has also empty interior with respect to $H_0^1(G)$-norm. At present, there seems to be no result on the Cauchy problem for (1.1) when $\mathfrak{K}$ is defined by (1.2). In this paper, we will address this problem exclusively for $\mathfrak{K}$ defined by (1.2).

Let the initial condition be given by

$$u(0) = u_0 \in \mathfrak{K} \tag{1.3}$$

We are seeking for a stochastic process $u = u(t)$ defined on the time interval $[0, T]$ such that $u(t) \in \mathfrak{K}$, for almost all t, and (1.1) and (1.3) are satisfied in an appropriate sense with probability one. Following the general strategy in [12], we reduce (1.1) to an essentially deterministic problem, and devote a good portion of this work to the deterministic problem. More precisely, we will work with the following form of a deterministic problem:

$$\frac{\partial}{\partial t}\big(u - M\big) - \Delta u \in -\,\partial I_{\mathfrak{K}}(u) \tag{1.4}$$

If the right-hand side of (1.1) is replaced by $f \in L^2(0, T; H^{-1}(G))$, the existence and uniqueness of a solution was established in a general setting which covers the case of (1.2); see [4,11], where a solution is defined in terms of a variational inequality. In principle, we will adapt such formulation for the definition of a solution. However, we need substantial modification when the right-hand side of (1.1) is not an ordinary function with respect to the time variable. We note that $M(t)$ is only Hölder continuous in time variable, and is not of bounded variation. Also, some key technical devices used in [4,11] for the regularity and uniqueness of a solution do not seem to be adaptable to our case.

On the other hand, our definition of a solution requires substantial regularity of M with respect to space variables. More precisely, we will assume $M \in C([0, T]; C_0^1(\overline{G}))$. In Sect. 3, we will present the definition of a solution of the deterministic problem, and establish the uniqueness of a solution. At present, the existence of a solution according to our definition is not known. This is due to lack of basic estimates under our assumptions on $M(t)$. But, for the stochastic problem discussed in Sect. 4, this difficulty can be overcome by means of stochastic integrals, and we are able to establish the existence of a solution with probability one, which leads to the solution as a stochastic process. We will also show that this stochastic process with state-space $\mathfrak{K}$ is a Markov process, and that it has an invariant measure on $\mathfrak{K}$.

2 Notation and technical preliminaries

Throughout this paper, G is a bounded open subset of $\mathbb{R}^d$ with smooth boundary ∂G, and

$$C_0^1(\overline{G}) = \left\{h \middle| \quad h \in C(\overline{G}), \quad \nabla h \in C(\overline{G})^d, \quad h(x) = 0, \quad \text{for } x \in \partial G\right\}$$

which is a Banach space with the norm

$$\left\|h\right\| = \sup_{x\in\overline{G}} \left|\nabla h(x)\right|.$$

The imbedding $C_0^1(\overline{G}) \to H_0^1(G)$ is dense.

As above, the set $\mathfrak{K}$ is defined by

$$\mathfrak{K} = \left\{h \in H_0^1(G) \middle| \quad |\nabla h(x)| \le 1, \quad \text{for almost all } x\right\}$$

When $\mathfrak{X}$ is a Banach space and J is an interval in $\mathbb{R}$,

$$C_r(J; \mathfrak{X}) = \left\{f \middle| \quad f = f(t) \text{ is } \mathfrak{X}\text{-valued right-continuous with respect to } t \in J\right\}$$

When $\mathfrak{X}$ is a topological space, $\mathcal{B}(\mathfrak{X})$ denotes the set of all Borel subsets of $\mathfrak{X}$. When $\mathfrak{X}$ is a compact metric space, $\mathcal{M}(\mathfrak{X})$ is the space of Radon measures, and $\mathcal{M}(\mathfrak{X})^d$ is the set of all $\xi = (\xi_1, \ldots, \xi_d), \quad \xi_j \in \mathcal{M}(\mathfrak{X}), \quad j = 1, \ldots, d.$

Lemma 2.1 *Let $g \in \mathfrak{K}$. Then, for any $\epsilon > 0$, there is $g_\epsilon = g_\epsilon(x)$ such that*

$$g_\epsilon \in C_c^\infty(G), \qquad \left|\nabla g_\epsilon(x)\right| \le 1 + \epsilon, \qquad \forall x \in G \tag{2.1}$$

and

$$\left\|g - g_\epsilon\right\|_{H_0^1(G)} < \epsilon \tag{2.2}$$

Here $C_c^\infty(G)$ denotes the set of all functions in $C^\infty(\mathbb{R}^d)$ with compact support in G.

Proof We need a special partition of unity for G. Let

$$\overline{G} \subset \bigcup_{j=1}^{n+m} B_{r_j}(z_j), \quad z_j \in G$$

where $B_{r_j}(z_j)$ is an open ball in $\mathbb{R}^d$ with center $z_j \in G$ and radius $r_j > 0$ with the following properties.

(i) for $j = 1, \ldots, n$, $B_{r_j}(z_j) \cap \partial G$ is not empty, and if $z \in B_{r_j}(z_j) \cap \partial G$, then $(1-\lambda)z_j + \lambda z \in G$, for all $0 \le \lambda < 1$.
(ii) for $j = n+1, \ldots, n+m$, $\overline{B_{r_j}(z_j)} \subset G$.

Let $\{\alpha_j\}_{j=1}^{n+m}$ be a partition of unity subbordinate to $\{B_{r_j}(z_j)\}_{j=1}^{n+m}$ such that

$$\alpha_j \in C_c^\infty(B_{r_j}(z_j)), \qquad 0 \le \alpha_j(x) \le 1, \qquad \forall j, \quad \forall x$$

and

$$\sum_{j=1}^{n+m} \alpha_j(x) = 1, \quad \text{for all } x \in \overline{G}$$

We then define a mapping $\psi_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\psi_\lambda(x) = \sum_{j=1}^{n} \alpha_j(x)\big(z_j + \lambda(x - z_j)\big) + \sum_{j=n+1}^{n+m} \alpha_j(x)x$$

Then, for $x \in \overline{G}$,

$$\psi_\lambda(x) = x + \sum_{j=1}^{n} \alpha_j(x)(\lambda - 1)(x - z_j)$$

and thus,

$$\begin{aligned} \big|\psi_\lambda(x) - x\big| &\le C|\lambda - 1|, \qquad \forall x \in G, \quad \forall\lambda \\ \big\|D\psi_\lambda(x) - I\big\| &\le C|\lambda - 1|, \qquad \forall x \in G, \quad \forall\lambda \end{aligned}$$

for some constant $C > 0$, where I is the $d \times d$ identity matrix and $D\psi_\lambda$ is the derivative matrix of ψ_λ. Let $g \in H_0^1(G)$ be given such that $|\nabla g(x)| \le 1$, for almost all $x \in G$. We can extend g by

$$\tilde{g}(x) = \begin{cases} g(x), & \text{for } x \in G \\ 0, & \text{for } x \in \mathbb{R}^d \setminus G \end{cases}$$

We find that

$$\mathrm{supp}\,\left(\tilde{g}\circ\psi_\lambda\right)\subset G$$

for all $\lambda > 1$ with sufficiently small $|\lambda - 1|$.

For $\delta > 0$, let $\zeta_\delta \in C_c^\infty(\mathbb{R}^d)$ be a mollifier such that

$$\zeta_\delta(x)\geq 0,\quad \forall x,\qquad \left\|\zeta_\delta\right\|_{L^1(\mathbb{R}^d)}=1,\qquad \mathrm{supp}\ \zeta_\delta\subset B_\delta(0)$$

Now let $\epsilon > 0$ be given. Then, we can choose $\lambda > 1$ and $\delta > 0$ such that

$$g_\epsilon \overset{\mathrm{def}}{=} \left(\tilde{g}\circ\psi_\lambda\right)*\zeta_\delta$$

satisfies the properties (2.1) and (2.2). □

Lemma 2.2 *Let $f\in C([0,L];H_0^1(G))$ such that $f(t)\in\mathfrak{K}$, for all $t\in[0,L]$. Then, for each $\gamma>0$, there is $f_\gamma\in C([0,L];C_0^1(\overline{G}))$ such that*

$$\left\|f-f_\gamma\right\|_{C([0,L];H_0^1(G))}<\gamma \tag{2.3}$$

and

$$\left|\nabla f_\gamma(t,x)\right|\leq 1+\gamma,\qquad \forall(t,x)\in[0,L]\times\overline{G} \tag{2.4}$$

Proof Fix any $\gamma > 0$. There is $\delta > 0$ such that

$$\left\|f(t_1)-f(t_2)\right\|_{H_0^1(G)}<\frac{\gamma}{2},\qquad \text{for all } t_1,t_2\in[0,L] \text{ such that } |t_1-t_2|<\delta$$

Let $\left\{J_k\right\}_{k=1}^N$ be a family of open intervals such that the length of J_k is δ, its midpoint z_k belongs to $[0, L]$, and

$$[0,L]\subset\bigcup_{k=1}^N J_k$$

We then choose $\beta_k\in C_c^\infty(J_k)$ such that

$$0\leq\beta_k(t)\leq 1,\qquad \forall t,\qquad \sum_{k=1}^N\beta_k(t)=1,\qquad \forall t\in[0,L] \tag{2.5}$$

and define

$$h(t)=\sum_{k=1}^N\beta_k(t)f(z_k)$$

By virtue of (2.5),

$$\|f - h\|_{C([0,L];H_0^1(G))} < \frac{\gamma}{2}$$

We apply Lemma 2.1 to each $f(z_k)$ and obtain $f_{k,\gamma} \in C_c^\infty(G)$ such that

$$\|f(z_k) - f_{k,\gamma}\|_{H_0^1(G)} < \frac{\gamma}{2}$$

and

$$\left|\nabla f_{k,\gamma}(x)\right| \le 1 + \gamma, \qquad \forall x \in \overline{G}$$

Let

$$f_\gamma(t) = \sum_{k=1}^{N} \beta_k(t) f_{k,\gamma}$$

This f_γ satisfies (2.3) and (2.4). □

For convenience of notation, we write

$$C_t^* \stackrel{\text{def}}{=} C\big([0,t]; C_0^1(\overline{G})\big)^* = \text{ the dual of } C\big([0,t]; C_0^1(\overline{G})\big)$$

Suppose that $\xi \in \mathcal{M}\big([0,T] \times \overline{G}\big)^d$. For any $t \in (0,T]$, $\nabla \cdot \xi \in C_t^*$ is defined by

$$\begin{aligned} - < \nabla \cdot \xi, h >_{C_t^*, C_t} &= \int_{[0,t]\times\overline{G}} \nabla h(s,x) \cdot d\xi(s,x) \\ &= \sum_{j=1}^{d} \int_{[0,t]\times\overline{G}} \frac{\partial h}{\partial x_j}(s,x) d\xi_j(s,x), \qquad \xi = \big(\xi_1, \ldots, \xi_d\big), \end{aligned}$$

for all $h \in C\big([0,t]; C_0^1(\overline{G})\big)$, where $<, \ >_{C_t^*, C_t}$ denotes the duality pairing between $C\big([0,t]; C_0^1(\overline{G})\big)$ and its dual C_t^*.

Next let $\epsilon > 0$ and $\rho_\epsilon \in C_c^\infty(\mathbb{R})$ be a mollifier such that

$$\rho_\epsilon(t) \ge 0, \qquad \forall t, \qquad \text{supp } \rho_\epsilon \subset \big(-\epsilon, 0\big), \qquad \|\rho_\epsilon\|_{L^1(\mathbb{R})} = 1 \tag{2.6}$$

Fix any $t \in (0,T)$, and let $\epsilon \in (0, T-t)$. The convolution $\nabla \cdot \xi * \rho_\epsilon \in C_t^*$ is defined by

$$- < \nabla \cdot \xi * \rho_\epsilon, h >_{C_t^*, C_t} = \int_0^t \int_{(s,s+\epsilon)\times\overline{G}} \nabla h(s,x) \rho_\epsilon(s-z) \cdot d\xi(z,x)\, ds,$$

for all $h \in C\big([0,t]; C_0^1(\overline{G})\big)$,

Lemma 2.3 *Let $\xi \in \mathcal{M}\big([0,T] \times \overline{G}\big)^d$. Suppose that its variation satisfies*

$$\|\xi\|\big(\{0\} \times \overline{G}\big) = 0.$$

Fix any $t \in (0,T)$. For $\epsilon \in (0, T-t)$, it holds that

$$\|\nabla \cdot \xi * \rho_\epsilon\|_{C_t^*} \leq \|\nabla \cdot \xi\|_{C_t^*} + \theta(\epsilon) \tag{2.7}$$

where $\theta(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof Choose any $\psi = \psi(s,x) \in C\left([0,t]; C_0^1(\overline{G})\right)$ such that

$$\|\psi\|_{C([0,t];C_0^1(\overline{G}))} = \||\nabla\psi|\|_{C([0,t]\times\overline{G})} \leq 1$$

It is convenient to introduce

$$\hat{\psi}(s,x) = \begin{cases} \psi(s,x), & \text{for } 0 \leq s \leq t \\ \psi(t,x), & \text{for } s > t \\ \psi(0,x), & \text{for } s < 0 \end{cases}$$

Then,

$$\|\hat{\psi}\|_{C(\mathbb{R};C_0^1(\overline{G}))} \leq 1$$

and

$$\begin{aligned} - < \nabla \cdot \xi * \rho_\epsilon, \psi >_{C_t^*, C_t} &= \int_0^t \int_{(s,s+\epsilon)\times\overline{G}} \nabla\psi(s,x)\rho_\epsilon(s-z) \cdot d\xi(z,x)\, ds \\ &= \int_{[0,t]\times\overline{G}} \left(\int_{z-\epsilon}^{z} \nabla\hat{\psi}(s,x)\rho_\epsilon(s-z)\, ds\right) \cdot d\xi(z,x) \\ &\quad - \int_{[0,\epsilon]\times\overline{G}} \left(\int_{z-\epsilon}^{0} \nabla\hat{\psi}(s,x)\rho_\epsilon(s-z) ds\right) \cdot d\xi(z,x) \\ &\quad + \int_{(t,t+\epsilon]\times\overline{G}} \left(\int_{z-\epsilon}^{t} \nabla\hat{\psi}(s,x)\rho_\epsilon(s-z) ds\right) \cdot d\xi(z,x) \end{aligned} \tag{2.8}$$

Let

$$h_1(z,x) = \int_{z-\epsilon}^{z} \hat{\psi}(s,x)\rho_\epsilon(s-z)\,ds$$

$$h_2(z,x) = \int_{z-\epsilon}^{0} \hat{\psi}(s,x)\rho_\epsilon(s-z)\,ds$$

$$h_3(z,x) = \int_{z-\epsilon}^{t} \hat{\psi}(s,x)\rho_\epsilon(s-z)\,ds$$

Then, $h_j \in C([0,T]; C_0^1(\overline{G}))$ with

$$\|h_j\|_{C([0,T];C_0^1(\overline{G}))} \le 1, \qquad j = 1,2,3.$$

Thus,

$$\left| \int_{[0,t]\times\overline{G}} \left(\int_{z-\epsilon}^{z} \nabla\hat{\psi}(s,x)\rho_\epsilon(s-z)ds \right) \cdot d\xi(z,x) \right| = \left| < \nabla\cdot\xi, h_1 >_{C_t^*, C_t} \right| \le \|\nabla\cdot\xi\|_{C_t^*}$$

$$\left| \int_{[0,\epsilon]\times\overline{G}} \left(\int_{z-\epsilon}^{0} \nabla\hat{\psi}(s,x)\rho_\epsilon(s-z)ds \right) \cdot d\xi(z,x) \right| \le \|\xi\|\big([0,\epsilon]\times\overline{G}\big)$$

$$\left| \int_{(t,t+\epsilon]\times\overline{G}} \left(\int_{z-\epsilon}^{t} \nabla\hat{\psi}(s,x)\rho_\epsilon(s-z)ds \right) \cdot d\xi(z,x) \right| \le \|\xi\|\big((t,t+\epsilon]\times\overline{G}\big)$$

where $\|\xi\|$ denotes the variation of ξ. It follows from (2.8) that

$$\|\nabla\cdot\xi * \rho_\epsilon\|_{C_t^*} \le \|\nabla\cdot\xi\|_{C_t^*} + \|\xi\|\big([0,\epsilon]\times\overline{G}\big) + \|\xi\|\big((t,t+\epsilon]\times\overline{G}\big)$$

Since $\|\xi\|\big(\{0\}\times\overline{G}\big) = 0$, we have

$$\|\xi\|\big([0,\epsilon]\times\overline{G}\big) + \|\xi\|\big((t,t+\epsilon]\times\overline{G}\big) \to 0, \qquad \text{as } \epsilon \to 0$$

which yields (2.7). □

Lemma 2.4 *If $v \in C_r([0,T); L^2(G))$ and $v(t) \in \mathfrak{K}$, for almost all $t \in [0,T)$, then $v(t) \in \mathfrak{K}$, for each $t \in [0,T)$.*

Proof This follows from the fact that $\mathfrak{K}$ is a closed subset of $L^2(G)$. □

Lemma 2.5 *$\mathfrak{K}$ is is a convex compact metric space with the metric induced by the $L^2(G)$-norm.*

3 Deterministic problem

Throughout this section, we assume

$$\begin{cases} M = M(t,x) \in C([0,T]; C_0^1(\overline{G})), \\ M(0,x) = 0, \qquad \forall x \in \overline{G} \end{cases} \tag{3.1}$$

$$0 < T < \infty \quad \text{and} \quad u_0 \in \mathfrak{K} \quad \text{are given.} \tag{3.2}$$

We rewrite (1.1) as

$$\frac{\partial}{\partial t}\big(u - M\big) - \Delta u \in -\partial I_{\mathfrak{K}}(u) \tag{3.3}$$

and the initial condition is given by

$$u(0) = u_0 \in \mathfrak{K} \tag{3.4}$$

For our definition of a solution, we need some preliminary observation. Suppose that we have the following ideal situation.

$$u \in L^\infty(0,T; H_0^1(G)), \qquad u(t) \in \mathfrak{K}, \qquad \text{for almost all } t \tag{3.5}$$

and

$$\frac{\partial}{\partial t}\big(u - M\big) \in L^2(0,T; H^{-1}(G)) \tag{3.6}$$

We then set

$$F = \frac{\partial}{\partial t}\big(u - M\big) - \Delta u$$

so that $F \in L^2(0,T; H^{-1}(G))$ and (3.3) is satisfied if

$$< F, v - u >_{L^2(0,T;H^{-1}(G)),L^2(0,T;H_0^1(G))} \geq 0 \tag{3.7}$$

holds for all $v \in L^2(0,T; H_0^1(G))$ such that $v(t) \in \mathfrak{K}$, for almost all t. We can simply take the condition (3.7) as a part of the definition of a solution. Without (3.6), the condition (3.7) does not make sense. So we adopt the basic spirit in the definition of weak solutions of partial differential equations. Assuming conditions (3.5) and (3.6), we derive a necessary consequence of (3.7). If this necessary consequence can be expressed under the conditions weaker than (3.6), we replace (3.7) by this consequence of (3.7) as a part of definition of a solution. This is the motivation behind the following definition.

Definition 3.1 u is a said to be a solution of (3.3) and (3.4) on the interval $[0,T]$ if

(i) $u \in L^\infty(0,T; H_0^1(G))$ and $u(t) \in \mathfrak{K}$ for almost all $t \in [0,T]$.

(ii) There is $\xi \in \mathcal{M}\big([0,T] \times \overline{G}\big)^d$ such that $\|\xi\|\big(\{0\} \times \overline{G}\big) = 0$, and

$$\frac{\partial}{\partial t}\big(u - M\big) - \Delta u - \nabla \cdot \xi = 0 \tag{3.8}$$

in the sense of distributions over $(0,T) \times G$, and

$$\int_0^s < \frac{\partial v}{\partial t}, v - u + M > dt - \int_0^s < \Delta u, v - u + M > dt$$
$$+ \int_{[0,s]\times\overline{G}} \nabla M \cdot d\xi \geq \frac{1}{2}\|v(s) - u(s) + M(s)\|^2_{L^2(G)} - \frac{1}{2}\|v(0) - u_0\|^2_{L^2(G)} \tag{3.9}$$

for almost all $s \in [0,T)$, for each v such that

$$\frac{\partial v}{\partial t} \in L^2(0,T; H^{-1}(G)), \qquad v(t) \in \mathfrak{K}, \qquad \text{for almost all } t \in [0,T]$$

Here $< \cdot, \cdot >$ denotes the duality pairing between $H^1_0(G)$ and $H^{-1}(G)$.

We note that $\nabla \cdot \xi$ plays the role of $-\partial I_{\mathfrak{K}}(u)$.

Lemma 3.2 *The conditions* (i)*, (3.8) and (3.9) imply the initial condition (3.4) and the regularity:*

$$u \in C_r([0,T); W^{s,p}(G)), \qquad \textit{for all} \qquad 0 \leq s < 1, 1 \leq p < \infty$$

and $u(t) \in \mathfrak{K}$, *for each* $t \in [0,T)$.

Proof For each $t \in (0,T]$, define $\xi_t \in \mathcal{M}\big(\overline{G}\big)^d$ by

$$\xi_t(F) = \xi\big([0,t] \times F\big), \qquad \text{for each } F \in \mathcal{B}(\overline{G})$$

where ξ is the measure in (3.8) and (3.9). Then, it follows from (3.8) that

$$\frac{\partial}{\partial t}\big(u - M - \nabla \cdot \xi_t\big) = \Delta u$$

holds in the sense of distributions over $(0,T) \times G$. Thus, for each $\phi \in C^\infty_c(G)$,

$$\frac{\partial}{\partial t}\left(< u, \phi > - < M, \phi > + \int_G \nabla\phi \cdot d\xi_t\right) =< u, \Delta\phi >$$

in the sense of distributions over $(0,T)$, where $<,\ >$ stands for the inner product in $L^2(G)$. It follows that $< u, \phi > \in C_r([0,T))$. Choose any $t^* \in [0,T)$. There is a sequence $\{t_n\} \downarrow t^*$ such that $u(t_n) \in H_0^1(G)$ and $\left\| |\nabla u(t_n)| \right\|_{L^\infty(G)} \leq 1$, for all n. Since $u(t_n) \to u(t^*)$ in the sense of distributions over G, we find that $u(t^*) \in H_0^1(G)$ and $\left\| |\nabla u(t^*)| \right\|_{L^\infty(G)} \leq 1$. Thus, $u(t) \in \mathfrak{K}$, for all $t \in [0, T)$. Next choose an arbitrary sequence $\{t_n\} \downarrow t^* \in [0, T)$. Since $W^{1,\infty}(G)$ is compactly embedded into $W^{s,p}(G)$ for each $0 \leq s < 1$ and $1 \leq p < \infty$, there is a subsequence still denoted by $\{t_n\}$ such that

$$u(t_n) \to u(t^*) \qquad \text{strongly in } W^{s,p}(G)$$

Hence, $u \in C_r([0, T); W^{s,p}(G))$, for all $0 \leq s < 1$ and $1 \leq p < \infty$.

Next choose v in (3.9) such that $v(t) = u_0$, for all $t \in [0, T]$. It follows that

$$\|u_0 - u(s) + M(s)\|_{L^2(G)} \to 0, \qquad \text{as } s \downarrow 0.$$

which yields (3.4). □

Lemma 3.3 *Let ρ_ϵ be a mollifier satisfying (2.6). Suppose that u is a solution with corresponding $\xi \in \mathcal{M}\big([0, T] \times \overline{G}\big)^d$. Then, for each $t \in (0, T)$, and $0 < \epsilon < T - t$, there is $h = h(t, \epsilon, \xi) \in L^2(0, t; L^2(G)^d)$ such that*

$$< \nabla \cdot \xi * \rho_\epsilon, \phi >_{C_t^*, C_t} = < \nabla \cdot h, \phi >_{L^2(0,t;H^{-1}(G)), L^2(0,t;H_0^1(G))}, \quad \forall \phi \in C([0, t]; C_0^1(\overline{G}))$$

Proof It follows from (3.8) that

$$\frac{\partial}{\partial t}\big(u - M\big) * \rho_\epsilon - \Delta u * \rho_\epsilon - \nabla \cdot \xi * \rho_\epsilon = 0 \tag{3.10}$$

in the sense of distributions over $(0, T - \epsilon) \times G$, and hence,

$$\nabla \cdot \xi * \rho_\epsilon \in C([0, t]; H^{-1}(G))$$

Let

$$\mathfrak{X} = \left\{ \nabla\phi \middle| \phi \in C([0, t]; C_0^1(\overline{G})) \right\}$$

Then, $\mathfrak{X}$ is a subspace of $L^2(0, t; L^2(G)^d)$. For each $\mathfrak{x} = \nabla\phi \in \mathfrak{X}$, define

$$\Lambda(\mathfrak{x}) = < \nabla \cdot \xi * \rho_\epsilon, \phi >_{C_t^*, C_t} = < \nabla \cdot \xi * \rho_\epsilon, \phi >_{L^2(0,t;H^{-1}(G)), L^2(0,t;H_0^1(G))}$$

Then,

$$\left|\Lambda(\mathfrak{x})\right| \leq \left\|\nabla \cdot \xi * \rho_\epsilon\right\|_{L^2(0,t;H^{-1}(G))} \left\|\nabla\phi\right\|_{L^2(0,t;L^2(G)^d)}$$

and hence, Λ is a continuous linear functional on $\mathfrak{X}$. It can be extended to a continuous linear functional on $L^2(0,t;L^2(\mathbb{R})^d)$ by the Hahn-Banach theorem. Thus, there is $h = h(t,\epsilon,\xi) \in L^2(0,t;L^2(G)^d)$ such that

$$\begin{aligned}< \nabla\cdot\xi * \rho_\epsilon, \phi >_{C_t^*,C_t} &= - < h, \nabla\phi >_{L^2(0,t;L^2(G)^d)} \\ &= < \nabla\cdot h, \phi >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))}\end{aligned}$$

for all $\phi \in C([0,t];C_0^1(\overline{G}))$. □

Lemma 3.4 *Suppose that u is a solution with corresponding $\xi \in \mathcal{M}\big([0,T]\times\overline{G}\big)^d$. Then, for each $t \in (0,T)$,*

$$\big\|\nabla\cdot\xi\|_{C_t^*} \le \lim_{\epsilon\to 0} < -\nabla\cdot\xi * \rho_\epsilon, u * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \tag{3.11}$$

where ρ_ϵ is the same as above.

Proof It follows from (3.10) that for each $0 < \epsilon < T - t$,

$$\begin{aligned}&\int_0^s < \frac{\partial}{\partial t}v, v - \big(u - M\big) * \rho_\epsilon > dt - \int_0^s < \Delta u * \rho_\epsilon, v - \big(u - M\big) * \rho_\epsilon > dt \\ &- \int_0^s < \nabla\cdot\xi * \rho_\epsilon, v - u * \rho_\epsilon > dt - \int_0^s < \nabla\cdot\xi * \rho_\epsilon, M * \rho_\epsilon > dt \\ &= \frac{1}{2}\big\|v(s) - \big(u * \rho_\epsilon\big)(s) + \big(M * \rho_\epsilon\big)(s)\|^2_{L^2(G)} - \frac{1}{2}\big\|v(0) - \big(u * \rho_\epsilon\big)(0) \\ &\quad + \big(M * \rho_\epsilon\big)(0)\|^2_{L^2(G)}\end{aligned} \tag{3.12}$$

for all $s \in [0,t]$, and all $v \in C^1([0,t];C_0^1(\overline{G}))$. Here $<,\ >$ is the duality pairing between $H_0^1(G)$ and $H^{-1}(G)$.

By passing $\epsilon \to 0$ in (2.8) with ψ replaced by v and M, respectively,

$$\lim_{\epsilon\to 0} < \nabla\cdot\xi * \rho_\epsilon, v >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} = < \nabla\cdot\xi, v >_{C_t^*,C_t}$$

and

$$\lim_{\epsilon\to 0} < \nabla\cdot\xi * \rho_\epsilon, M >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} = < \nabla\cdot\xi, M >_{C_t^*,C_t} .$$

By means of (2.7), we see that

$$\begin{aligned}&\big| < \nabla\cdot\xi * \rho_\epsilon, M - M * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))}\big| \\ &\quad \le \big\|\nabla\cdot\xi * \rho_\epsilon\big\|_{C_t^*}\big\|M - M * \rho_\epsilon\big\|_{C([0,t];C_0^1(\overline{G}))} \to 0, \qquad \text{as } \epsilon \to 0\end{aligned}$$

It follows that

$$\lim_{\epsilon\to 0} < \nabla\cdot\xi * \rho_\epsilon, M * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} = < \nabla\cdot\xi, M >_{C_t^*,C_t}.$$

Now it is easy to see that

$$\lim_{\epsilon\to 0} < \nabla\cdot\xi * \rho_\epsilon, u * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \qquad \text{must exist}$$

because the limits of all other terms of (3.12) exist as $\epsilon \to 0$. Next by choosing $v \in C^1([0,t]; C_0^1(\overline{G}))$ such that $\left|\nabla v(s,x)\right| \le 1$, for all $(s,x) \in [0,t]\times\overline{G}$, and by comparing with the inequality (3.9), we conclude that

$$< -\nabla\cdot\xi, v >_{C_t^*,C_t} \le \lim_{\epsilon\to 0} < -\nabla\cdot\xi * \rho_\epsilon, u * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))}$$

For each $v \in C([0,t]; C_0^1(\overline{G}))$ such that $\left|\nabla v(s,x)\right| \le 1$, for all $(s,x) \in [0,t]\times\overline{G}$, there is a sequence $\{v_n\}$ in $C^1([0,T]; C_0^1(\overline{G}))$ such that $\left|\nabla v_n(s,x)\right| \le 1$, for all $(s,x) \in [0,t]\times\overline{G}$, and $v_n \to v$ in $C([0,t]; C_0^1(\overline{G}))$. Thus, (3.11) follows. □

Lemma 3.5 *Let u be a solution with corresponding $\xi \in \mathcal{M}\big([0,T]\times\overline{G}\big)^d$. Let v be another function such that*

$$v \in L^\infty(0,T; H_0^1(G)), \qquad v(t) \in \mathfrak{K}, \qquad \textit{for almost all } t \in (0,T)$$

It holds that

$$\lim_{\epsilon\to 0} < -\nabla\xi * \rho_\epsilon, u * \rho_\epsilon - v * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \ge 0 \quad (3.13)$$

for each $t \in (0,T)$.

Proof Fix any $\gamma > 0$, and $0 < t < T$. By Lemma 2.3, there is $0 < \epsilon_0 < \gamma$ such that $\epsilon_0 < T - t$, and

$$\left\|\nabla\cdot\xi * \rho_\epsilon\right\|_{C_t^*} \le \left\|\nabla\cdot\xi\right\|_{C_t^*} + \gamma, \qquad \text{for all } 0 < \epsilon \le \epsilon_0.$$

By virtue of Lemmas 2.2 and 3.3, there is $w_{\epsilon,\gamma}$ such that

$$\begin{aligned}
& w_{\epsilon,\gamma} \in C([0,t]; C_0^1(\overline{G})) \\
& \left|< \nabla\cdot\xi * \rho_\epsilon, v * \rho_\epsilon - w_{\epsilon,\gamma} >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))}\right| < \gamma \\
& \left\|w_{\epsilon,\gamma}\right\|_{C([0,t];C_0^1(\overline{G}))} \le 1 + \gamma
\end{aligned}$$

Hence, for each $0 < \epsilon \leq \epsilon_0$,

$$
\begin{aligned}
&< \nabla \cdot \xi * \rho_\epsilon, v * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \\
&=< \nabla \cdot \xi * \rho_\epsilon, v * \rho_\epsilon - w_{\epsilon,\gamma} >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \\
&\quad + < \nabla \cdot \xi * \rho_\epsilon, w_{\epsilon,\gamma} >_{C_t^*,C_t} \\
&\geq -\gamma - (1+\gamma)\left\|\nabla \cdot \xi * \rho_\epsilon\right\|_{C_t^*} \\
&\geq -\gamma - (1+\gamma)\left(\left\|\nabla \cdot \xi\right\|_{C_t^*} + \gamma\right)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\varliminf_{\epsilon\to 0} < -\nabla\xi * \rho_\epsilon, u * \rho_\epsilon - v * \rho_\epsilon >_{L^2(0,t;H^{-1}(G),L^2(0,t;H_0^1(G))} \\
&\quad \geq \varliminf_{\epsilon\to 0} < \nabla \cdot \xi * \rho_\epsilon, v * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \\
&\qquad + \varliminf_{\epsilon\to 0} < -\nabla \cdot \xi * \rho_\epsilon, u * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} \\
&\quad \geq \varliminf_{\epsilon\to 0} < \nabla \cdot \xi * \rho_\epsilon, v * \rho_\epsilon >_{L^2(0,t;H^{-1}(G)),L^2(0,t;H_0^1(G))} + \left\|\nabla \cdot \xi\right\|_{C_t^*} \\
&\quad \geq -\gamma - \gamma\left\|\nabla \cdot \xi\right\|_{C_t^*} - \gamma(1+\gamma)
\end{aligned}
$$

This yields (3.13). □

Theorem 3.6 *According to* Definition *3.1, there is at most one solution of (3.3) and (3.4).*

Proof Let u_1 and u_2 be solutions with corresponding measures ξ_1 and ξ_2, respectively. Let us set

$$w = u_1 - u_2$$

Then, it holds that

$$\frac{\partial w}{\partial t} - \Delta w - \nabla \cdot \xi_1 + \nabla \cdot \xi_2 = 0$$

in the sense of distributions over $(0, T) \times G$. Let ρ_ϵ be a mollifier satisfying (2.6). For $0 < \epsilon < T$, it holds that

$$\frac{\partial}{\partial t} w * \rho_\epsilon - \Delta w * \rho_\epsilon - \nabla \cdot \xi_1 * \rho_\epsilon + \nabla \cdot \xi_2 * \rho_\epsilon = 0$$

on the interval $[0, T - \epsilon)$. Choose any $t_* \in (0, T)$. By virtue of Lemma 3.5,

$$\varliminf_{\epsilon\to 0} < -\nabla\xi_1 * \rho_\epsilon, u_1 * \rho_\epsilon - u_2 * \rho_\epsilon >_{L^2(0,t_*;H^{-1}(G)),L^2(0,t_*;H_0^1(G))} \geq 0 \quad (3.14)$$

and

$$\underline{\lim}_{\epsilon\to 0} < \nabla\xi_2 * \rho_\epsilon, u_1 * \rho_\epsilon - u_2 * \rho_\epsilon >_{L^2(0,t_*;H^{-1}(G)),L^2(0,t_*;H_0^1(G))} \geq 0 \quad (3.15)$$

Since $w \in C_r([0,T); L^2(G)) \cap L^\infty(0,T; H_0^1(G))$, it follows from (3.14) and (3.15) that

$$\left\| w(t_*)\right\|^2_{L^2(G)} + 2\int_0^{t_*} \left\| w(t)\right\|^2_{H_0^1(G)} dt \leq 0 \quad (3.16)$$

which yields $w \equiv 0$. □

Theorem 3.7 *Let u be a solution with corresponding $\xi \in \mathcal{M}\big([0,T]\times \overline{G}\big)^d$. Suppose that $\mathcal{O}$ is an open subset of $(0,T)\times G$ such that*

$$\left|\nabla u(t,x)\right| \leq 1-\nu, \qquad \textit{for almost all } (t,x) \in \mathcal{O}$$

for some $0<\nu<1$. Then, $\nabla\cdot\xi = 0$ in $\mathcal{O}$ in the sense of distributions.

Proof Let $\mathfrak{V}$ be a nonempty open set such that

$$\overline{\mathfrak{V}} \subset \mathcal{O} \cap \{(0,s)\times G\}, \qquad \text{for some } 0<s<T$$

Choose any $\phi \in C_c^\infty(\mathfrak{V})$. Then, there is $\lambda \neq 0$ such that

$$\left|\lambda\nabla\phi(t,x)\right| < \nu, \qquad \forall (t,x)$$

Let us define

$$v = u - \lambda\phi$$

Then, $v(t) \in \mathfrak{K}$, for almost all $t \in (0,T)$, and we can apply Lemma 3.5 so that

$$\underline{\lim}_{\epsilon\to 0} < -\nabla\xi * \rho_\epsilon, u * \rho_\epsilon - v * \rho_\epsilon >_{L^2(0,s;H^{-1}(G)),L^2(0,s;H_0^1(G))} \geq 0$$

and thus,

$$\underline{\lim}_{\epsilon\to 0} < -\nabla\xi * \rho_\epsilon, \lambda\phi * \rho_\epsilon >_{L^2(0,s;H^{-1}(G)),L^2(0,s;H_0^1(G))} \geq 0$$

But

$$\begin{aligned} 0 &\leq \underline{\lim}_{\epsilon\to 0} < -\nabla\cdot\xi * \rho_\epsilon, \lambda\phi * \rho_\epsilon >_{L^2(0,s;H^{-1}(G)),L^2(0,s;H_0^1(G))} \\ &= \underline{\lim}_{\epsilon\to 0} < -\nabla\cdot\xi * \rho_\epsilon, \lambda\phi * \rho_\epsilon >_{C_s^*,C_s} \\ &= < -\nabla\cdot\xi, \lambda\phi >_{C_s^*,C_s}, \qquad \text{by Lemma 2.3} \end{aligned}$$

By considering $\pm\lambda$, we conclude that

$$< \nabla \cdot \xi, \phi >_{C_s^*, C_s} \quad = 0$$

This implies that $\nabla \cdot \xi = 0$ in $\mathfrak{V}$, and hence, in $\mathcal{O}$ in the sense of distributions. $\square$

Remark 3.8 For a solution u of (3.3) and (3.4), the corresponding measure ξ is not determined uniquely, because we can add to ξ any divergence-free smooth vector field with compact support in $(0, T) \times G$. We also note that Theorem 3.7 is related to the general fact that $\partial I_{\mathfrak{K}}(u) = 0$ for $u \in$ interior of $\mathfrak{K}$ if it exists.

4 Stochastic problem

Throughout this section, $\big(\Omega, \mathcal{F}, P\big)$ is a given complete probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration on $\big(\Omega, \mathcal{F}\big)$ such that $\mathcal{F}_t$ is right-continuous for all t, and $\mathcal{F}_0$ contains all P-negligible sets in $\mathcal{F}$. For general information on stochastic calculus, see [6,10,13].

We set

$$M(t) = \sum_{j=1}^{\infty} \int_0^t g_j dB_j \tag{4.1}$$

where $\{B_j\}_{j=1}^{\infty}$ is a sequence of mutually independent standard Brownian motions on $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\big)$, and each $g_j = g_j(\omega, t, x)$ is $H_0^1(G) \cap H^k(G)$-valued progressively measurable with $k > \frac{d}{2} + 1$ such that

$$\sum_{j=1}^{\infty} E\left(\int_0^T \|g_j\|^2_{H_0^1(G)\cap H^k(G)} dt\right) < \infty. \tag{4.2}$$

Under these assumptions, $M(t) \in C([0, T]; C_0^1(\overline{G}))$, for P-almost all $\omega \in \Omega$.

We first address the issue of existence of a solution, and then we will discuss the Markov property of the solution and prove the existence of an invariant measure.

4.1 Existence

Definition 4.1 A stochastic process u is a solution of (1.1) and (1.3) if it is adapted to $\{\mathcal{F}_t\}$ and for P-almost all $\omega \in \Omega$, $u(\omega)$ is a solution of (3.3) and (3.4) according to Definition 3.1.

Theorem 4.2 *Let $T > 0$ be given. Suppose that u_0 is $\mathcal{F}_0$-measurable, and $u_0(\omega) \in \mathfrak{K}$, P-almost all $\omega \in \Omega$. Under the conditions (4.1) and* (4.2), *there is a pathwise unique solution to (1.1)* and *(1.3).*

Our strategy of the proof is as follows.

By means of the penalty method discussed in [11], we consider the following initial boundary value problem for each $\epsilon > 0$.

$$\frac{\partial u}{\partial t} - \Delta u - \frac{1}{\epsilon}\nabla\cdot\left(\left(|\nabla u|^2 - 1\right)^+ \nabla u\right) = \frac{\partial M}{\partial t}, \qquad (t,x) \in (0,T)\times G \tag{4.3}$$

$$u(t,x) = 0, \qquad (t,x) \in [0,T]\times \partial G \tag{4.4}$$

$$u(0,x) = u_0(x), \qquad x \in G \tag{4.5}$$

In conjunction with the existence of a unique solution, we will obtain basic stochastic estimates of a solution independent of $\epsilon > 0$. By means of these estimates, we can construct a pathwise solution for P-almost all $\omega \in \Omega$, and use Theorem 3.6 to show that this is the desired stochastic process.

We now present the technical details.

The above problem (4.3) and (4.5) can be resolved by direct application of Theorems 4.2.4 and 4.2.5 of [13]. For this, we consider the following operator for $\epsilon > 0$.

$$A_\epsilon(w) = \Delta w + \frac{1}{\epsilon}\nabla\cdot\left(\left(|\nabla w|^2 - 1\right)^+ \nabla w\right)$$

Our Gelfand triple is

$$V = W_0^{1,4}(G) \subset L^2(G) \subset W^{-1,\frac{4}{3}}(G) = V^*$$

It is easy to see the following properties of the operator $A_\epsilon = A_\epsilon(w)$.

[I] For all $w_1, w_2, w_3 \in V$, the map

$$\lambda \mapsto < A_\epsilon(w_1 + \lambda w_2), w_3 >_{V^*,\,V}$$

is continuous $\mathbb{R} \to \mathbb{R}$.

[II] For all $w_1, w_2 \in V$,

$$< A_\epsilon(w_1) - A_\epsilon(w_2), w_1 - w_2 >_{V^*,V} \ \leq\ 0$$

[III] For some constant $C_\epsilon > 0$,

$$< A_\epsilon(w), w >_{V^*,V} \leq -\frac{1}{2\epsilon}\|w\|_V^4 + C_\epsilon$$

for all $w \in V$.

[IV] For all $w \in V$,

$$\|A_\epsilon(w)\|_{V^*} \leq C_\epsilon \|w\|_V^3 + C$$

Here C and C_ϵ are some positive constants.

For the property [II], we consider a convex functional $J_\epsilon(\cdot)$ on $W_0^{1,4}(G)$ defined by

$$J_\epsilon(w) = \frac{1}{2}\int_G |\nabla w|^2\, dx + \frac{1}{4\epsilon}\int_G \left(\left(|\nabla w|^2 - 1\right)^+\right)^2 dx$$

Then, $-A_\epsilon(w)$ is the Gâteaux differential at w of J_ϵ.

For [III] and [IV], we note that

$$\begin{aligned}\frac{1}{\epsilon}\int_G \left(|\nabla w|^2 - 1\right)^+ |\nabla w|^2 dx &\geq \frac{1}{\epsilon}\int_G |\nabla w|^4 dx - \frac{1}{\epsilon}\int_G |\nabla w|^2 dx\\ &\geq \frac{1}{2\epsilon}\int_G |\nabla w|^4 dx - \frac{1}{2\epsilon}\int_G dx\end{aligned}$$

$$\begin{aligned}\left|\int_G \nabla w\cdot\nabla v dx\right| \leq \|\nabla w\|_{L^{\frac{4}{3}}(G)}\|\nabla v\|_{L^4(G)} &\leq C\|\nabla w\|_{L^4(G)}\|\nabla v\|_{L^4(G)}\\ &\leq \left(C\|\nabla w\|^3_{L^4(G)} + C\right)\|\nabla v\|_{L^4(G)}\end{aligned}$$

and

$$\frac{1}{\epsilon}\int_G \left(|\nabla w|^2 - 1\right)^+ \nabla w\cdot\nabla v\, dx \leq \frac{1}{\epsilon}\left(\int_G |\nabla w|^4 dx\right)^{\frac{3}{4}}\left(\int_G |\nabla v|^4 dx\right)^{\frac{1}{4}}$$

for all $v, w \in W_0^{1,4}(G)$.

For later use, we also point out the following fact which can be easily proved.

Lemma 4.3 *Let $f \in L^2(0, T; H^{-1}(G))$, $h \in L^2(0, T; V)$, and $\{g_k\}$ be a sequence in $L^2(0, T; V)$ such that $g_k \to g$, weakly in $L^2(0, T; H_0^1(G))$. Then,*

$$\begin{aligned}&\lim_{k\to\infty}\int_0^T < f, g_k >_{V^*,V} dt = \int_0^T < f, g >_{H^{-1}(G),H_0^1(G)} dt\\ &\underline{\lim}_{k\to\infty}\int_0^T < -\Delta g_k, g_k >_{V^*,V} dt \geq \int_0^T \|g\|^2_{H_0^1(G)} dt\\ &\lim_{k\to\infty}\int_0^T < -\Delta g_k, h >_{V^*,V} dt = \int_0^T < -\Delta g, h >_{H^{-1}(G),H_0^1(G)} dt\end{aligned}$$

According to Theorems 4.2.4 and 4.2.5 of [13] with help of [I]–[IV], there is a pathwise unique solution u of (4.3) and (4.5) which satisfies the following properties.

$$u(\omega) \in C([0,T]; L^2(G)), \quad \text{for P-almost all } \omega \in \Omega, \text{ and is progressively measurable} \tag{4.6}$$

$$u \in L^4\big([0,T] \times \Omega, dt \otimes dP; W_0^{1,4}(G)\big) \tag{4.7}$$

and it holds for P-almost all $\omega \in \Omega$ that

$$\begin{aligned} &\|u(t)\|^2_{L^2(G)} + 2\int_0^t \|u(s)\|^2_{H_0^1(G)} ds + \frac{2}{\epsilon}\int_0^t \int_G \big(|\nabla u|^2 - 1\big)^+ |\nabla u(s)|^2 dx ds \\ &= \|u_0\|^2_{L^2(G)} + 2\sum_{j=1}^{\infty} \int_0^t < u(s), g_j(s) >_{L^2(G)} dB_j \\ &\quad + \sum_{j=1}^{\infty} \int_0^t \|g_j(s)\|^2_{L^2(G)} ds, \qquad \forall t \in [0,T]. \end{aligned} \tag{4.8}$$

By the Burkholder-Davis-Gundy inequality, we can derive from (4.8) that

$$E\Big(\sup_{0 \le t \le T} \|u(t)\|^2_{L^2(G)}\Big) \le C, \tag{4.9}$$

$$E\Big(\int_0^T \|u(t)\|^2_{H_0^1(G)} dt\Big) \le C, \tag{4.10}$$

$$\begin{aligned} &E\Big(\frac{1}{\epsilon}\int_0^T \int_G \big(|\nabla u|^2 - 1\big)^+ |\nabla u(s)| ds\, dx\Big) \\ &\le E\Big(\frac{1}{\epsilon}\int_0^T \int_G \big(|\nabla u|^2 - 1\big)^+ |\nabla u(s)|^2 ds\, dx\Big) \le C \end{aligned} \tag{4.11}$$

for some positive constants C independent of ϵ. Let us denote by u_k the solution when $\epsilon = \frac{1}{k}, \quad k = 1, 2, \ldots$, and define

$$\xi_k = k\big(|\nabla u_k|^2 - 1\big)^+ \nabla u_k$$

Since we have

$$E\Big(\sup_{0 \le t \le T} \|u_k(t)\|^2_{L^2(G)} + \|u_k\|^2_{L^2(0,T;H_0^1(G))} + \big\||\xi_k|\big\|_{L^1([0,T]\times\overline{G})}\Big) \le C$$

for all $k \geq 1$, it follows that

$$P\Bigg(\bigcup_{L=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\Bigg\{\omega \in \Omega \Bigg| \sup_{0\leq t\leq T}\|u_k(t)\|^2_{L^2(G)} + \|u_k\|^2_{L^2(0,T;H^1_0(G))} + \||\xi_k|\|_{L^1([0,T]\times\overline{G})} \leq L\Bigg\}\Bigg) = 1$$

Hence, there is $\hat{\Omega} \subset \Omega$ such that $P(\Omega \setminus \hat{\Omega}) = 0$, and for each $\omega \in \hat{\Omega}$, there is a subsequence $\{k_m\}$ depending on ω and a positive constant $L(\omega)$ such that

$$\sup_{0\leq t\leq T}\|u_{k_m}(t)\|^2_{L^2(G)} + \|u_{k_m}\|^2_{L^2(0,T;H^1_0(G))} + \||\xi_{k_m}|\|_{L^1([0,T]\times\overline{G})} \leq L(\omega) < \infty \tag{4.12}$$

for all k_m. Hence, there is a subsequence still denoted by $\{(u_{k_m}, \xi_{k_m})\}$ such that

$$u_{k_m} \to u \qquad \text{weak star in } L^\infty(0,T;L^2(G)) \tag{4.13}$$

$$u_{k_m} \to u \qquad \text{weakly in } L^2(0,T;H^1_0(G)) \tag{4.14}$$

$$\xi_{k_m} \to \xi \qquad \text{weak star in } \mathcal{M}\big([0,T]\times\overline{G}\big)^d \tag{4.15}$$

For each $k \geq 1$, it holds for P-almost all $\omega \in \Omega$ that

$$\frac{\partial}{\partial t}\big(u_k - M\big) - \Delta u_k - \nabla\cdot\xi_k = 0 \tag{4.16}$$

in the sense of distributions over $(0,T)\times G$, and

$$\int_0^s < \frac{\partial v}{\partial t} - \frac{\partial(u_k - M)}{\partial t}, v - u_k + M >_{V^*,V} dt = \frac{1}{2}\|v(s) - u_k(s) + M(s)\|^2_{L^2(G)} - \frac{1}{2}\|v(0) - u_0\|^2_{L^2(G)} \tag{4.17}$$

$$\int_0^s < \frac{\partial(u_k - M)}{\partial t}, v - u_k + M >_{V^*,V} dt = \int_0^s < \Delta u_k, v - u_k + M >_{V^*,V} dt + \int_0^s < \nabla\cdot\xi_k, v - u_k >_{V^*,V} dt - \int_0^s\int_G \nabla M\cdot\xi_k\, dx\, dt \tag{4.18}$$

for all $s \in [0, T]$, and all $v \in L^4(0, T; V)$ such that $\frac{\partial v}{\partial t} \in L^2(0, T; H^{-1}(G))$. Here, we note that

$$u_k \in L^4(0, T; V), \qquad \nabla \cdot \xi_k \in L^{\frac{4}{3}}(0, T; V^*),$$

and

$$\frac{\partial}{\partial t}\big(u_k - M\big) \in L^{\frac{4}{3}}(0, T; V^*),$$

for P-almost all $\omega \in \Omega$.

By adding (4.17) and (4.18), we have

$$\begin{aligned}
&\int_0^s < \frac{\partial v}{\partial t}, v - u_k + M >_{V^*,V} dt - \int_0^s < \Delta u_k, v - u_k + M >_{V^*,V} dt \\
&- \int_0^s < \nabla \cdot \xi_k, v - u_k >_{V^*,V} dt + \int_0^s \int_G \nabla M \cdot \xi_k \, dx \, dt \\
&= \frac{1}{2}\big\| v(s) - u_k(s) + M(s) \big\|^2_{L^2(G)} - \frac{1}{2}\big\| v(0) - u_0 \big\|^2_{L^2(G)}
\end{aligned} \tag{4.19}$$

But

$$\Big\langle -\nabla \cdot \Big(\big(\big|\nabla v\big|^2 - 1\big)^+ \nabla v \Big) + \nabla \cdot \Big(\big(\big|\nabla u_k\big|^2 - 1\big)^+ \nabla u_k \Big), \quad v - u_k \Big\rangle_{V^*,V} \geq 0$$

for almost all $t \in [0, T]$, and

$$\big(\big|\nabla v(t)\big|^2 - 1\big)^+ = 0 \qquad \text{if } v(t) \in \mathfrak{K}$$

Thus, if $v(t) \in \mathfrak{K}$, for almost all $t \in [0, T]$,

$$- < \nabla \cdot \xi_k, \quad v - u_k >_{V^*,V} \leq 0, \qquad \text{for almost all } t \in [0, T]$$

and hence, (4.19) yields

$$\begin{aligned}
&\int_0^s < \frac{\partial v}{\partial t}, v - u_k + M >_{V^*,V} dt - \int_0^s < \Delta u_k, v - u_k + M >_{V^*,V} dt \\
&+ \int_0^s \int_G \nabla M \cdot \xi_k dx \, dt \geq \frac{1}{2}\big\| v(s) - u_k(s) + M(s) \big\|^2_{L^2(G)} - \frac{1}{2}\big\| v(0) - u_0 \big\|^2_{L^2(G)}
\end{aligned} \tag{4.20}$$

for all $s \in [0, T]$.

It follows that

$$\int_0^T \psi(s)\Big(\int_0^s < \frac{\partial v}{\partial t}, v - u_k + M >_{V^*,V} dt\Big) ds$$
$$- \int_0^T \psi(s)\Big(\int_0^s < \Delta u_k, v - u_k + M >_{V^*,V} dt\Big) ds$$
$$+ \int_0^T \psi(s)\Big(\int_0^s \int_G \nabla M \cdot \xi_k \, dx \, dt\Big) ds$$
$$\geq \int_0^T \psi(s)\Big(\frac{1}{2}\big\|v(s) - u_k(s) + M(s)\big\|^2_{L^2(G)} - \frac{1}{2}\big\|v(0) - u_0\big\|^2_{L^2(G)}\Big) ds \tag{4.21}$$

for all $\psi \in C([0, T])$ such that $\psi(t) \geq 0, \forall t$.

There is $\Omega^\dagger \subset \hat{\Omega}$ such that $P\big(\Omega \setminus \Omega^\dagger\big) = 0$ and for each $\omega \in \Omega^\dagger$, (4.16) and (4.21) hold for all k, all $\psi \in C([0, T])$ such that $\psi(t) \geq 0, \forall t$, and all v such that $\dfrac{\partial v}{\partial t} \in L^2(0, T; H^{-1}(G))$ and $v(t) \in \mathfrak{K}$, for almost all $t \in [0, T]$. For $\omega \in \Omega^\dagger$, let $\big(u, \xi\big)$ be determined by (4.13)–(4.15). Then, by Lemma 4.3, (4.21) implies that

$$\int_0^T \psi(s)\Big(\int_0^s < \frac{\partial v}{\partial t}, v - u + M >_{H^{-1}(G), H_0^1(G)} dt\Big) ds$$
$$- \int_0^T \psi(s)\Big(\int_0^s < \Delta u, v - u + M >_{H^{-1}(G), H_0^1(G)} dt\Big) ds$$
$$+ \int_0^T \psi(s)\Big(\int_{[0,s]\times\overline{G}} \nabla M \cdot d\xi\Big) ds$$
$$\geq \int_0^T \psi(s)\Big(\frac{1}{2}\big\|v(s) - u(s) + M(s)\big\|^2_{L^2(G)} - \frac{1}{2}\big\|v(0) - u_0\big\|^2_{L^2(G)}\Big) ds \tag{4.22}$$

Hence, it holds that

$$\int_0^s < \frac{\partial v}{\partial t}, v - u + M >_{H^{-1}(G), H_0^1(G)} dt - \int_0^s < \Delta u, v - u + M >_{H^{-1}(G), H_0^1(G)} dt$$
$$+ \int_{[0,s]\times\overline{G}} \nabla M \cdot d\xi \geq \frac{1}{2}\left\| v(s) - u(s) + M(s) \right\|_{L^2(G)}^2 - \frac{1}{2}\left\| v(0) - u_0 \right\|_{L^2(G)}^2 \tag{4.23}$$

for almost all $s \in [0, T]$, for each v such that $\frac{\partial v}{\partial t} \in L^2(0, T; H^{-1}(G))$ with $v(t) \in \mathfrak{K}$, for almost all $t \in [0, T]$. It also follows from (4.16) that

$$\frac{\partial}{\partial t}\big(u - M\big) - \Delta u - \nabla \cdot \xi = 0 \tag{4.24}$$

in the sense of distributions over $(0, T) \times G$. We now modify the measure ξ.

$$\hat{\xi}\big(F\big) \stackrel{\text{def}}{=} \xi\big(F \cap \{(0, T] \times \overline{G}\}\big), \qquad \forall F \in \mathcal{B}\big([0, T] \times \overline{G}\big)$$

Then,

$$\left\|\hat{\xi}\right\|\big(\{0\} \times \overline{G}\big) = 0 \tag{4.25}$$

and, (4.23) and (4.24) are still valid with ξ replaced by $\hat{\xi}$, because $M(0, x) = 0$, for all $x \in \overline{G}$. Since the functional

$$w \mapsto \int_0^T \int_G \big(\left|\nabla w\right|^2 - 1\big)^+ dx\, dt$$

is convex on $L^2(0, T; H_0^1(G))$, it follows from (4.12) and (4.14) that

$$\begin{aligned}
\int_0^T \int_G \big(\left|\nabla u\right|^2 - 1\big)^+ dx\, dt &\leq \underline{\lim}_{m\to\infty} \int_0^T \int_G \big(\left|\nabla u_{k_m}\right|^2 - 1\big)^+ dx\, dt \\
&\leq \underline{\lim}_{m\to\infty} \int_0^T \int_G \big(\left|\nabla u_{k_m}\right|^2 - 1\big)^+ \left|\nabla u_{k_m}\right| dx\, dt \\
&\leq \underline{\lim}_{m\to\infty} \frac{L(\omega)}{k_m} = 0
\end{aligned}$$

Therefore,

$$u(t) \in \mathfrak{K}, \qquad \text{for almost all } t \in [0, T] \tag{4.26}$$

and u and $\hat{\xi}$ satisfy all the conditions of Definition 3.1. Also, by Lemma 3.2, we see that

$$u \in C_r([0,T); W^{s,p}(G)), \qquad \text{for all } 0 \le s < 1, 1 \le p < \infty$$

and, by Lemma 2.4,

$$u(t) \in \mathfrak{K}, \qquad \text{for all } t \in [0,T)$$

For each $\omega \in \Omega^\dagger$, a solution $u(\omega)$ of (3.3) and (3.4) has been obtained as the limit of a convergent subsequence satisfying (4.13)–(4.15). By Theorem 3.6, the limit is independent of the choice of such a subsequence. Based on this, we will establish measurability of u.

Fix any $t_* \in (0,T)$. As above, let u_k be the solution of (4.3)–(4.5) for $\epsilon = \frac{1}{k}$, $k \ge 1$, with corresponding ξ_k. We set

$$\Pi(u_k) = \sup_{0 \le t \le t_*} \left\| u_k(t) \right\|^2_{L^2(G)} + \left\| u_k \right\|^2_{L^2(0,t_*;H^1_0(G))} + \left\| |\xi_k| \right\|_{L^1([0,t_*]\times\overline{G})}$$

and choose any $\psi \in L^2(0,t_*;L^2(G))$ and $\lambda \in \mathbb{R}$. It holds that

$$\begin{aligned}&\left\{ \omega \in \Omega^\dagger \middle| < u, \psi >_{L^2(0,t_*;L^2(G))} \le \lambda \right\} \\ &= \bigcap_{\nu=1}^{\infty} \bigcup_{L=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \left\{ \omega \in \Omega^\dagger \middle| < u_k, \psi >_{L^2(0,t_*;L^2(G))} \le \lambda + \frac{1}{\nu}, \qquad \Pi(u_k) \le L \right\}\end{aligned}$$

Since the set on the right-hand side belongs to $\mathcal{F}_{t_*}$ and $L^2(0,t_*;L^2(G))$ is separable, u is $L^2(0,t_*;L^2(G))$-valued $\mathcal{F}_{t_*}$-measurable, which is valid for each $t_* \in (0,T)$. Let ρ_ϵ be a mollifier satisfying (2.6). Then, $\big(u * \rho_\epsilon\big)(t_*)$ is $L^2(G)$-valued $\mathcal{F}_{t_*+\epsilon}$-measurable. Since $u \in C_r([0,T); L^2(G))$, for each $\omega \in \Omega^\dagger$,

$$\big(u * \rho_\epsilon\big)(t_*) \to u(t_*) \qquad \text{strongly in } L^2(G)$$

as $\epsilon \to 0$. Since $\mathcal{F}_{t_*+} = \mathcal{F}_{t_*}$, $u(t_*)$ is $L^2(G)$-valued $\mathcal{F}_{t_*}$-measurable. Also, $u(t_*)$ is $H^1_0(G)$-valued $\mathcal{F}_{t_*}$-measurable, because $\mathcal{B}\big(H^1_0(G)\big) \subset \mathcal{B}\big(L^2(G)\big)$. Hence, u is adapted to $\{\mathcal{F}_t\}$. Pathwise uniqueness is a direct consequence of Theorem 3.6. Now the proof of Theorem 4.2 is complete.

4.2 Markov property and invariant measure

We further assume that $g_j = g_j(x)$, $\forall j$, in (4.1), and (4.2) holds for each $0 < T < \infty$.

Let $X(t;s,x)$, $0 \le s \le t < \infty$, be the solution of (1.1) with the initial condition (1.3) replaced by

$$X(s;s,x) = x \in \mathfrak{K} \tag{4.27}$$

Then, for each $x \in \mathfrak{K}$, we have $X(t, s; x) \in \mathfrak{K}$, for all $t \in [s, \infty)$, for P-almost all $\omega \in \Omega$.

For $0 \leq s < t < \infty$, we construct a σ-algebra $\mathcal{G}_{t,s}$ as follows:
$\mathcal{H}_{t,s}$ is the σ-algebra generated by $\{B_j(z) - B_j(s)\}$, $s \leq z \leq t$, $j = 1, 2, \ldots$, and P-negligible sets
and

$$\mathcal{G}_{t,s} = \bigcap_{\epsilon > 0} \mathcal{H}_{t+\epsilon,s}$$

Then, $\mathcal{G}_{t,s} \subset \mathcal{F}_t$, and $\mathcal{G}_{t,s}$ is independent of $\mathcal{F}_s$. Let $X_k(t, s; x)$ be the solution on the interval $[s, \infty)$ of (4.3), (4.4) and

$$X_k(s, s; x) = x \tag{4.28}$$

Then, $X_k(t, s; x)$ is adapted to $\{\mathcal{G}_{t,s}\}_{t \geq s}$. By the same proof of the fact that $u(t_*)$ is $\mathcal{F}_{t_*}$-measurable, $X(t, s; x)$ is $\mathcal{G}_{t,s}$-measurable, and hence, independent of $\mathcal{F}_s$.

Lemma 4.4 *For each $0 \leq s \leq z \leq t < \infty$, and $x \in \mathfrak{K}$, it holds that*

$$X(t; s, x) = X\big(t; z, X(z; s, x)\big), \qquad \text{for } P\text{-almost all } \omega \in \Omega$$

Proof This follows from pathwise uniqueness of a solution. □

Lemma 4.5 *Let f be $H_0^1(G)$-valued $\mathcal{F}_s$-measurable such that $f(\omega) \in \mathfrak{K}$, for P-almost all $\omega \in \Omega$. Then, for each $\epsilon > 0$, there is a function f_ϵ such that*

$$f_\epsilon(\omega) \in \mathfrak{K}, \quad \forall \omega, \tag{4.29}$$

$$f_\epsilon(\omega) = \sum_{k=1}^{N(\epsilon)} a_k \chi_{F_k}(\omega) \tag{4.30}$$

where $a_k \in \mathfrak{K}, \forall k$, and F_k's are disjoint $\mathcal{F}_s$-measurable subsets such that $\Omega = \bigcup_{k=1}^{N(\epsilon)} F_k$, and

$$\big\| f(\omega) - f_\epsilon(\omega) \big\|_{L^2(G)} < \epsilon, \qquad \text{for } P\text{-almost all } \omega \in \Omega. \tag{4.31}$$

Here, $\chi_{F_k}(\cdot)$ denotes the characteristic function of the set F_k.

Proof Since $\mathfrak{K}$ is a compact metric space with the metric of $L^2(G)$, it is easy to see (4.30) and (4.31). Since $\Omega = \bigcup_{k=1}^{N(\epsilon)} F_k$, (4.30) implies (4.29). □

Lemma 4.6 *Let $0 \leq s \leq t < \infty$, For $i = 1, 2$, let h_i be $L^2(G)$-valued $\mathcal{F}_s$-measurable and $h_i(\omega) \in \mathfrak{K}$, for P-almost all $\omega \in \Omega$. Then, it holds that*

$$\big\| X(t, s; h_1) - X(t, s; h_2) \big\|^2_{L^2(G)} \leq \big\| h_1 - h_2 \big\|^2_{L^2(G)}, \qquad \text{for } P\text{-almost all } \omega \in \Omega$$

Proof This follows by the same argument as for (3.16) □

By means of the above Lemmas 4.4–4.6, we can repeat the standard argument to show the following Markov property of the process $X = X(t; s, x)$. For details, see [6,13].

Theorem 4.7 *Let F be a bounded continuous function on $L^2(G)$. For any $0 \leq s \leq z \leq t < \infty$, and $x \in \mathfrak{K}$, it holds that*

$$E\Big(F\big(X(t; s, x)\big)\Big|\mathcal{F}_z\Big) = E\Big(F\big(X(t; z, y)\big)\Big)\Big|_{y=X(z;s,x)}, \qquad \text{for } P\text{-almost all } \omega \in \Omega.$$

Theorem 4.8 *There is an invariant measure μ on $\mathfrak{K}$ such that*

$$\int\limits_{\mathfrak{K}} E\Big(F\big(X(t; 0, x)\big)\Big)\, d\mu = \int\limits_{\mathfrak{K}} F(x)\, d\mu$$

for all $t \geq 0$, and all bounded continuous function F on $L^2(G)$.

Proof By virtue of Lemma 2.5 and Theorem 4.7, we can use the method of Krylov and Bogoliubov to prove the existence of an invariant measure. We omit the details, which are well-known. □

References

1. Barbu, V., Da Prato, G., Tubaro, L.: Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space. Ann. Probab. **37**, 1427–1458 (2009)
2. Barbu, V., Da Prato, G., Tubaro, L.: The stochastic reflection problem in Hilbert spaces. Commun. Partial Differ. Equ. **37**, 352–367 (2012)
3. Benssousan, A., Rascanu, A.: Stochastic variational inequalities in infinite dimensional spaces. Numer. Funct. Anal. Optim. **18**, 19–54 (1997)
4. Brezis, H.: Problèmes unilatéraux. J. Math. Pures Appl. **51**, 1–164 (1972)
5. Cépa, E.: Problème de Skorohod multivoque. Ann. Probab. **26**, 500–532 (1998)
6. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992)
7. Donati-Martin, C., Pardoux, É.: White noise driven SPDEs with reflection Probab. Theory Related Fields **95**, 1–24 (1993)
8. Haussmann, U.G.: Stochastic PDEs with unilateral constraints in higher dimensions in "Stochastic partial differential equations and applications" (Trento, 1990). Pitman Res. Notes Math. Ser. **268**, 204–215 (1992)
9. Haussmann, U.G., Pardoux, E.: Stochastic variational inequalities of parabolic type. Appl. Math. Optim. **20**, 163–192 (1989)
10. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, Berlin (1997)
11. Lions, J.L.: Quelques méthodes de résolution des problémes aux limites non linéaires. Dunod, Paris (1969)
12. Nualart, D., Pardoux, E.: White noise driven quasilinear SPDEs with reflection. Probab. Theory Relat. Fields **93**, 77–89 (1992)
13. Prévôt, C., Röckner, M.: A Concise Course on Stochastic Partial Differential Equations, Lecture Notes in Mathematics, vol. 1905. Springer, Berlin (2007)
14. Röckner, M., Zhu, R., Zhu, X.: The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple. Ann. Probab. **40**, 1759–1794 (2012)

15. Skorohod, A.V.: Stochastic equations for diffusion processes in a bounded region. Theory Probab. Appl. **6**, 264–274 (1961)
16. Tanaka, H.: Stochastic differential equations with reflecting boundary condition in convex regions Hiroshima. Math. J. **9**, 163–177 (1979)
17. Zhang, X.: Skorohod problem and multivalued stochastic evolution equations in Banach spaces. Bull. Sci. Math. **131**, 175–217 (2007)