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Evolution of Yamabe constant along the Ricci–Bourguignon flow

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Abstract In this article, with an essential assumption, we provide an evolution formula for the Yamabe constant along of the Ricci–Bourguignon flow of an n -dimensional closed Riemannian manifold for $n \geq 3$. In particular, we show that Yamabe constant is increasing on $[0, \delta]$ for some $\delta > 0$.

Mathematics Subject Classification 58C40 · 53E20 · 53C21

1 Introduction

Recently, the geometric flows such as the Ricci flow [15], Ricci–Bourguignon flow [6, 7], Cotton flow [18] and Yamabe flow [3, 11, 16] have been widely studied owing to their numerous applications in both mathematics and physics. In this paper, we consider the Ricci–Bourguignon flow introduced by Bourguignon in [6]. Let M be an n -dimensional closed Riemannian manifold with metric g . The family $g(t)$ of Riemannian metrics on manifold M is called the Ricci–Bourguignon flow whenever it satisfies the equation

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), \quad (1.1)$$

with the initial condition

$$g(0) = g_0.$$

Here and in the sequel, Ric is the Ricci tensor of $g(t)$, R is the scalar curvature and ρ is a real constant. As a way of quick remark, we mention that the Ricci–Bourguignon flow contains quite a number of other geometric flows. This could be seen for special values of the constant ρ appearing in (1.1). Specifically, when $\rho = 0$ the tensor $Ric - \rho Rg$ corresponds to the Ricci tensor and the Ricci–Bourguignon flow (1.1) then becomes Ricci flow. Other situations include when $\rho = \frac{1}{n}$, the case where the tensor $Ric - \rho Rg$ corresponds to the traceless Ricci tensor and the flow (1.1) becomes the normalized Ricci flow, when $\rho = 1/2(n - 1)$ and (1.1) reduces to Schouten tensor since $Ric - \rho Rg$, in this case, is a multiple of Schouten tensor. Ricci–Bourguignon flow also interpolates between the Ricci flow and the Yamabe flow when ρ is nonpositive after appropriate time rescaling. The short time existence and uniqueness for solution to the Ricci–Bourguignon flow 1.1 as a system

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of partial differential equations on $[0, T)$ have been established by Catino et al. ([7]) for the case $\rho < \frac{1}{2(n-1)}$. Other recent studies around this flow include [1, 9, 21].

Yamabe problem which is more or less the generalization of uniformization of compact surfaces to higher dimensional smooth manifolds is well known in the literature since the 60s and remains an active research area to date. An important geometric quantity associated with this problem is an invariant quantity known as Yamabe constant or Yamabe invariant. Detail descriptions of Yamabe constant and Yamabe invariant are given in Sect. 2. Yamabe invariant of smooth manifolds carries several important geometric and topological consequences or information. For instance, Yamabe invariant is positive if and only if the underlying manifold admits a metric of positive scalar curvature. Due to Perelman's resolution of Poincaré conjecture, it follows that a simply connected manifold can have negative Yamabe invariant only if it is of 4-dimensional. Our interest at this time is to study the behaviors of Yamabe invariant of manifolds evolving by certain geometric flow, which will enable us to reach some useful conclusions.

The aim of this paper is to provide an evolution formula for the Yamabe constant, which is defined as (2.2), under the Ricci–Bourguignon flow of an n -dimensional closed Riemannian manifold for $n \geq 3$. The evolution of subcritical Yamabe constant was studied by Chang and Lu [10] under the Ricci flow and they established a differential inequality of constant under some technical assumption. Later Chang–Lu's results were extended to the relative subcritical Yamabe constant under the Ricci flow with boundary under the condition that the mean curvature of the boundary vanishes by Botvinnik and Lu [5]. Danesvar Pip and Razavi [13] extended the same results to the case of Bernhard List's flow. See also [8] and [22] for similar results under Cotton flow and conformal Ricci flow, respectively. Also, in [23, 24] have been investigated the evolution of some geometric constants along the geometric flows.

Motivated by the above works we are concerned with the evolution of the Yamabe constant under Ricci–Bourguignon flow, as an application, we show that under some conditions, the initial metric is an Einstein metric if and only if the Yamabe flow constant is nondecreasing along the Ricci–Bourguignon flow. The rest of this paper is, therefore, planned as follows: Sect. 2 gives some basics and preliminary results on the Yamabe constant vis-a-vis Yamabe problem. Section 3 is devoted to the main results and their applications. In Sect. 4, we give three examples of the evolution of Yamabe constant on Einstein metrics, Ricci–Bourguignon soliton, and 3-dimensional Heisenberg Lie group.

2 Preliminaries

Given a smooth manifold M of dimension $n \geq 3$, we consider \mathcal{M} to be the set of Riemannian metrics on M . Recall that the normalized Einstein–Hilbert functional $\mathcal{E} : \mathcal{M} \rightarrow \mathbb{R}$ is given by

$$\mathcal{E}(g) = \frac{\int_M R_g d\mu}{\text{Vol}(M, g)^{\frac{n-2}{n}}}, \quad (2.1)$$

where R_g and $d\mu$ are the scalar curvature and the volume element of metric g , respectively. It is well known that every compact surface has a conformal metric of constant Gaussian curvature. A generalization of this is Yamabe problem, which asks if any Riemannian metric g on a compact smooth manifold M^n of dimension $n \geq 3$ is conformal to a metric with constant scalar curvature. In 1960, Yamabe [27] attempted to solve this problem, but his proof contained some error, discovered in 1968 by Trudinger [26]. Trudinger [26], Aubin [2] and Schoen [25] solved the Yamabe problem with a rather restrictive assumption on the manifold M . They proved that a minimum value of $\mathcal{E}(g)$ is attained in each conformal class of metrics and that this minimum is achieved by a metric of constant scalar curvature. Note that any metric conformal to g can be written as $\tilde{g} = e^{2f}g$, where f is smooth real-valued function on M .

Now recall that the Yamabe constant of a smooth metric g on a closed manifold M is given by

$$Y(g) = \inf_{u \in C^\infty(M), u > 0} \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R_g u^2 \right) d\mu}{\left(\int_M u^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}}}, \quad (2.2)$$

where ∇ is the Riemannian connection on M . A function u for which $Y(g)$ get its infimum is called the Yamabe minimizer (see [11, 20, 22, 27]).



In the next, we denote $\square := -a\Delta + R$, where $a = 4\frac{n-1}{n-2}$, Δ is Laplace–Beltrami operator and R is the scalar curvature of M . Yamabe problem is reduced to saying that $\tilde{g} = u\frac{4}{n-2}g$ has constant scalar curvature \mathcal{Y} if and only if u satisfies the Yamabe equation

$$\square u = \mathcal{Y}u^{p-1}, \quad \int_M u^p d\mu = 1, \quad \text{with } p = \frac{2n}{n-2}. \tag{2.3}$$

The metric $u\frac{4}{n-2}g$ is called the Yamabe metric and has constant scalar curvature. It happens that the exponent $q = p - 1 = (n + 2)/(n - 2)$ in (2.3) is precisely the critical value, below (subcritical) which the equation is easy to solve and above which may be delicate. The existence of solution to (2.3) follows from direct method in the calculus of variation (cf. [20]). It is also observed that equation (2.3) is the Euler–Lagrange equation for the functional $\mathcal{E}(\tilde{g})$. Thus, for a positive smooth function u satisfying $\tilde{g} = u\frac{4}{n-2}g$, we have infimums in (2.1) and (2.2) being equal, that is,

$$Y(g) = \inf_{u>0} \mathcal{E}(g).$$

Set

$$Y(M) = Y(g) = \inf\{\mathcal{E}(\tilde{g}) : \tilde{g} \text{ conformal to } g\}.$$

This constant $Y(M)$ is an invariant of the conformal classes of (M, g) which is usually called the Yamabe (or Yamabe invariant) constant. Aubin [2] showed that the Yamabe problem can be solved on any compact manifold M with $Y(M) \leq Y(S^n)$, where S^n is the sphere endowed with its standard metric. The sigma invariant of M is defined by

$$\sigma(M) := \sup_g Y(g), \tag{2.4}$$

where sup is taken over all smooth metrics on M . Therefore, the Yamabe minimizer is u (cf. [20]). In the critical and supercritical, the existence of solutions becomes a delicate issue. A useful way to handle supercritical problems consists in reducing the problem to a more general elliptic critical or subcritical problem, either by considering rotational symmetries or by means of maps preserving Laplace operator, or a combination of both (cf. [12] for instance).

Chang and Lu [10] assumed that Yamabe minimizer is C^1 -differentiable with respect to variable t and then they investigated the evolution of the subcritical Yamabe constant under the Ricci flow. They also showed that, if $g(0)$ is a Yamabe metric at time $t = 0$ and $\frac{R_g}{n-1}$ is not a positive eigenvalue of the Laplacian Δ_{g_α} for any Yamabe metric g_α in the conformal class $[g_0]$, then $\frac{d}{dt}|_{t=0} Y(g(t)) \geq 0$. Recently, Daneshvar Pip and Razavi in [13] studied the evolution of the Yamabe constant under Bernhard List’s flow. The results in this paper generalize and extend the aforementioned results [10, 13] as highlighted in the introduction.

3 Variation of Yamabe constant

In this section, we will find evolution formulae for $Y(t)$ along the flow (1.1). First, we recall some evolution formulae for geometric structure along the Ricci–Bourguignon flow. Next, we will present a useful proposition about the variation of Yamabe constant under the flow (1.1). From [7], we have the following lemma.

Lemma 3.1 *Under the Ricci–Bourguignon flow equation (1.1), we get*

- (1) $\frac{\partial}{\partial t} g^{ij} = 2(R^{ij} - \rho Rg^{ij}),$
- (2) $\frac{\partial}{\partial t} (d\mu) = (n\rho - 1)Rd\mu,$
- (3) $\frac{\partial}{\partial t} (\Gamma_{ij}^k) = -\nabla_j R_i^k - \nabla_i R_j^k + \nabla^k R_{ij} + \rho(\nabla_j R\delta_i^k + \nabla_i R\delta_j^k - \nabla^k Rg_{ij}),$
- (4) $\frac{\partial}{\partial t} R = [1 - 2(n - 1)\rho]\Delta R + 2|Ric|^2 - 2\rho R^2,$

where R is scalar curvature.

As a consequence of Lemma 3.1 we obtain the following result.

Lemma 3.2 *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the flow (1.1) on a closed oriented Riemannian manifold for $\rho < \frac{1}{2(n-1)}$. Let $u \in C^\infty(M)$ be a smooth function on $(M, g(t))$. Then we have the following evolutions:*

$$\frac{\partial}{\partial t} |\nabla u|^2 = 2R^{ij} \nabla_i u \nabla_j u - 2\rho R |\nabla u|^2 + 2g^{ij} \nabla_i u \nabla_j u_t, \tag{3.1}$$

$$\frac{\partial}{\partial t} (\Delta u) = 2R^{ij} \nabla_i \nabla_j u + \Delta u_t - 2\rho R \Delta u - (2 - n)\rho \nabla^k R \nabla_k u, \tag{3.2}$$

where $u_t = \frac{\partial u}{\partial t}$.

Proof By direct computation in local coordinates we have

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla u|^2 &= \frac{\partial}{\partial t} (g^{ij} \nabla_i u \nabla_j u) \\ &= \frac{\partial g^{ij}}{\partial t} \nabla_i u \nabla_j u + 2g^{ij} \nabla_i u \nabla_j u_t \\ &= 2R^{ij} \nabla_i u \nabla_j u - 2\rho R |\nabla u|^2 + 2g^{ij} \nabla_i u \nabla_j u_t, \end{aligned}$$

where we used Lemma 3.1 (1) and gives exactly (3.1). Next, by using again Lemma 3.1 and the twice-contracted second Bianchi identity $2\nabla^i R_{ij} = \nabla_j R$ we infer

$$\begin{aligned} \frac{\partial}{\partial t} (\Delta u) &= \frac{\partial}{\partial t} [g^{ij} (\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k})] \\ &= \frac{\partial g^{ij}}{\partial t} (\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}) + g^{ij} (\frac{\partial^2 u_t}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u_t}{\partial x^k}) - g^{ij} \frac{\partial}{\partial t} (\Gamma_{ij}^k) \frac{\partial u}{\partial x^k} \\ &= 2R^{ij} \nabla_i \nabla_j u + \Delta u_t - 2\rho R \Delta u - (2 - n)\rho \nabla^k R \nabla_k u. \end{aligned}$$

So, the proof of is complete. □

In the sequel, we shall state our main result.

Proposition 3.3 *Let $(M, g(t))$, $t \in [0, T)$ be a solution of the flow (1.1) on the smooth closed oriented Riemannian manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$. For any $p \in (2, \frac{2n}{n-2}]$, if there is a C^1 -family of smooth functions $u(t) > 0$, $t \in [0, T)$, which satisfies*

$$\square u(t) = \tilde{Y}(t)u(t)^{p-1} \tag{3.3}$$

with the condition

$$\int_M u^p d\mu = 1, \tag{3.4}$$

where $\tilde{Y}(t)$ is a function of t only. Then

$$\begin{aligned} \frac{d}{dt} \tilde{Y}(t) &= \int_M 2aR^{ij} \nabla_i u \nabla_j u d\mu + 2 \int_M |\text{Ric}|^2 u^2 d\mu \\ &\quad + \left(1 - 2(n-1)\rho + \frac{a(n\rho - 1)}{p}\right) \int_M u^2 \Delta R d\mu \\ &\quad - \left[\left(1 - \frac{2}{p}\right)(1 - n\rho) + 2\rho\right] \int_M (a|\nabla u|^2 + Ru^2) R d\mu. \end{aligned} \tag{3.5}$$

Proof Equation (3.3) results in

$$-a\Delta u + Ru = \tilde{Y}(t)u^{p-1}. \tag{3.6}$$

Now, multiplying (3.6) by u and upon integrating we use (3.4) to obtain

$$\tilde{Y}(t) = \int_M (a|\nabla u|^2 + Ru^2) d\mu \tag{3.7}$$



since $\int_M u^p d\mu = 1$. Taking time derivative of (3.7) and using Lemmas 3.1 and 3.2 yields

$$\begin{aligned} \frac{d}{dt} \tilde{Y}(t) &= a \int_M (2R^{ij} \nabla_i u \nabla_j u - 2\rho R |\nabla u|^2 + 2g^{ij} \nabla_i u \nabla_j u_t) d\mu \\ &+ \int_M ((1 - 2(n - 1)\rho) \Delta Ru^2 + 2|\text{Ric}|^2 u^2 - 2\rho R^2 u^2 + 2Ruu_t) d\mu \\ &+ \int_M (a|\nabla u|^2 + Ru^2)(n\rho - 1) R d\mu. \end{aligned} \tag{3.8}$$

Integrating by parts we obtain

$$\begin{aligned} \int_M (2ag^{ij} \nabla_i u \nabla_j u_t + 2Ruu_t) d\mu &= \int_M (-2au_t \Delta u + 2Ruu_t) d\mu \\ &= 2\tilde{Y}(t) \int_M u_t u^{p-1} d\mu. \end{aligned} \tag{3.9}$$

On the other hand, taking time derivative of the condition (3.4) gives

$$\int_M pu_t u^{p-1} d\mu + \int_M u^p (n\rho - 1) R d\mu = 0$$

which implies

$$\int_M u_t u^{p-1} d\mu = -\frac{(n\rho - 1)}{p} \int_M u^p R d\mu. \tag{3.10}$$

Also, multiplying (3.6) by Ru and integrating by parts we have

$$\begin{aligned} \tilde{Y}(t) \int_M Ru^p d\mu &= \int_M (-aRu \Delta u + R^2 u^2) d\mu \\ &= -\frac{a}{2} \int_M u^2 \Delta R d\mu + \int_M (a|\nabla u|^2 + Ru^2) R d\mu. \end{aligned} \tag{3.11}$$

Hence, using (3.9), (3.10) and (3.11) we have

$$\begin{aligned} \int_M (2ag^{ij} \nabla_i u \nabla_j u_t + 2Ruu_t) d\mu &= \frac{a(n\rho - 1)}{p} \int_M u^2 \Delta R d\mu \\ &- \frac{2(n\rho - 1)}{p} \int_M (a|\nabla u|^2 + Ru^2) R d\mu. \end{aligned} \tag{3.12}$$

Combining (3.8) and (3.12) yields the expected evolution formula. □

Remark 3.4 The traceless Ricci tensor of Riemannian manifold (M^n, g) is defined by $S_{ij} = R_{ij} - \frac{R}{n} g_{ij}$. So, we can write $R_{ij} = S_{ij} + \frac{R}{n} g_{ij}$ and $|R_{ij}|^2 = |S_{ij}|^2 + \frac{R^2}{n}$. Substituting these into the formula (3.5) with assumptions of Proposition 3.3 we can rewrite the evolution of $\tilde{Y}(t)$ along the Ricci–Bourguignon flow as follows:

$$\begin{aligned} \frac{d}{dt} \tilde{Y}(t) &= 2a \int_M S^{ij} \nabla_i u \nabla_j u d\mu + 2 \int_M |S_{ij}|^2 u^2 d\mu \\ &+ \left(1 - 2(n - 1)\rho + \frac{a(n\rho - 1)}{p} \right) \int_M u^2 \Delta R d\mu \\ &- \left[\left(1 - \frac{2}{p} \right) (1 - n\rho) + 2\rho - \frac{2}{n} \right] \int_M (a|\nabla u|^2 + Ru^2) R d\mu. \end{aligned} \tag{3.13}$$

Taking $p = \frac{2n}{n-2}$ in (3.13), then we obtain

$$\frac{d}{dt} \tilde{Y}(t) = 2a \int_M S^{ij} \nabla_i u \nabla_j u d\mu + 2 \int_M |S_{ij}|^2 u^2 d\mu - \frac{2}{p} \int_M u^2 \Delta R d\mu, \tag{3.14}$$

which contains Chang–Lu’s results [10], Proposition 1 under the Ricci flow (i.e., when $\rho = 0$).

We also observe that the evolution of Yamabe constant remains the same both under the Ricci flow (at point $\frac{n+2}{n-2}$) and Ricci–Bourguignon flow (at point $\frac{2n}{n-2}$). This can be seen from (3.14) which is equivalent to Chang–Lu’s results [10], Proposition 1 under the Ricci flow without necessarily setting $\rho = 0$.

Corollary 3.5 *Let $g(t)$ be the solution of the Ricci–Bourguignon flow on closed n -dimensional Riemannian manifold M with $g(0) = g_0$, where g_0 is a metric of constant scalar curvature. Assume that $\frac{R_{g_0}}{n-1}$ is not a positive eigenvalue of the Laplacian Δ_{g_0} . Then $\frac{d}{dt}|_{t=0}\tilde{Y}(t) \geq 0$ and the equality holds if and only if g_0 is an Einstein metric.*

Proof By Koiso’s decomposition theorem (Corollary 2.9 in [19] or Theorem 4.44 in [4]), there exists a C^1 -family of smooth positive functions $u(t)$ on $[0, \epsilon)$ for some $\epsilon > 0$ with constant $u(0)$, which satisfies the assumption of Proposition 3.3 for $p = \frac{2n}{n-2}$. Obviously, $\tilde{Y}(t) = \tilde{Y}_p(t)$ is the scalar curvature of $u(t)\frac{4}{n-2}g(t)$. Since $u(0)$ and R_{g_0} are constant, $\nabla u(0) = 0$ and $\nabla R_{g_0} = 0$. Hence, we have

$$\frac{d}{dt}\tilde{Y}(t)|_{t=0} = 2(u(0))^2 \int_M |S_{ij}(g_0)|^2 d\mu \geq 0. \tag{3.15}$$

Since the right-hand side of (3.15) is nonnegative, if $\frac{d}{dt}\tilde{Y}(t)|_{t=0} = 0$ then the trace Ricci tensor $S_{ij}(g_0)$ vanishes identically. Consequently, g_0 is an Einstein metric. \square

Notice that, $\tilde{Y}(t)$ in Corollary 3.5 cannot be equal to the Yamabe constant $Y(g(t))$ even if g_0 satisfies $\tilde{Y}(0) = Y(g(0))$. If we suppose that $u(t)\frac{4}{n-2}g(t)$ has unit volume and constant scalar curvature $Y(g(t))$, then we can conclude as follows, which says that infinitesimally the Ricci–Bourguignon flow will try to increase the Yamabe constant.

Corollary 3.6 *Let $g(t)$ be the solution of the Ricci–Bourguignon flow on closed n -dimensional Riemannian manifold M with $g(0) = g_0$, where g_0 is a metric of constant scalar curvature. Assume that there is a C^1 -family of smooth positive functions $u(t)$ on $[0, \epsilon)$ for some $\epsilon > 0$ with constant $u(0)$, such that $u(t)\frac{4}{n-2}g(t)$ has unit volume and constant scalar curvature $Y(g(t))$. Then*

- i $\frac{d}{dt}|_{t=0}\tilde{Y}(t) \geq 0$ and the equality holds if and only if g_0 is an Einstein metric.
- ii If g_0 further satisfies $Y(g_0) = \sigma(M)$, then g_0 is an Einstein metric,

where $\sigma(M)$ is defined in (2.4).

In what follows, we consider n -dimensional Riemannian manifold M whose sigma invariant is realized by some metric, the assumption is a little different from that of Corollary 3.6.

Corollary 3.7 *Let $g(t)$ be the solution of the Ricci–Bourguignon flow on closed n -dimensional Riemannian manifold M with $g(0) = g_0$. Suppose that sigma invariant of M is realized by g_0 , $Y(g_0) = \sigma(M)$. Let $\{\tilde{g}_\alpha\}$ be set the metrics in the conformal class $[g_0]$ with $Y(\tilde{g}_\alpha) = \sigma(M)$. Assume that for each α , $\frac{R_{\tilde{g}_\alpha}}{n-1}$ is not a positive eigenvalue of the Laplacian $\Delta_{\tilde{g}_\alpha}$, then g_0 is an Einstein metric and $\tilde{g}_\alpha = g_0$ for all α .*

4 Examples

In this section, we give some examples about the evolution of Yamabe constant along the Ricci–Bourguignon flow. First, we consider the initial Riemannian manifold (M^n, g_0) is Einstein manifold and then we find the evolving Yamabe constant along the Ricci–Bourguignon flow.

Example 4.1 Let (M^n, g_0) be an Einstein manifold, i.e. there exists a constant b such that $Ric(g_0) = bg_0$. Assume that a solution to the Ricci–Bourguignon flow is of the form

$$g(t) = c(t)g_0, \quad c(0) = 1,$$

where $c(t)$ is a positive function. By a straightforward computation, we have

$$\frac{\partial g}{\partial t} = c'(t)g_0, \quad Ric(g(t)) = Ric(g_0) = bg_0 = \frac{b}{c(t)}g(t), \quad R_{g(t)} = \frac{bn}{c(t)},$$

for this to be a solution of the Ricci–Bourguignon flow, we require

$$c'(t)g_0 = -2Ric(g(t)) + 2\rho R_{g(t)}g(t) = (-2b + 2\rho bn)g_0$$

this shows that

$$c(t) = (-2b + 2\rho bn)t + 1,$$

so $g(t)$ is an Einstein metric. Using formula (3.5) for evolution of Yamabe constant along the Ricci–Bourguignon flow, we obtain

$$\frac{d}{dt}\tilde{Y}(t) = -\left[\left(1 - \frac{2}{p}\right)(1 - n\rho) + 2\rho - \frac{2}{n}\right]\frac{bn}{c(t)}\tilde{Y}(t).$$

Integrating of the last inequality with respect to t on $[0, t]$, we get

$$\tilde{Y}(t) = \tilde{Y}(0)((-2b + 2\rho bn)t + 1)^{-\left[\left(1 - \frac{2}{p}\right)(1 - n\rho) + 2\rho - \frac{2}{n}\right]\frac{bn}{-2b + 2\rho bn}}.$$

In the following example, we determine the behaviour of the evolving Yamabe constant on self similar solutions to the Ricci–Bourguignon flow, which are called Ricci–Bourguignon soliton (see [14]).

Example 4.2 Let $(M^n, g(t))$ be a solution to the Ricci–Bourguignon flow with initial condition $g(0) = g_0$. The solution $g(t)$ is called self similar solution if there is a smooth function $c(t)$ and a 1-parameter family of diffeomorphism $\phi_t : M \rightarrow M$ such that $g(t) = c(t)\phi_t^*(g_0)$ with $c(0) = 1$ and $\phi_0 = id_M$. Since $\Delta_{\phi_t^*g_0} \circ \phi_t^* = \phi_t^* \circ \Delta_{g_0}$ and $Ric(\phi_t^*g_0) = \phi_t^* Ric(g_0)$ we have $R_{\phi_t^*g_0} = R_{g_0}$ and

$$\square_{\phi_t^*g_0} \circ \phi_t^* = (-a\Delta_{\phi_t^*g_0} + R_{\phi_t^*g_0}) \circ \phi_t^* = \phi_t^* \circ (-a\Delta_{g_0} + R) = \phi_t^* \circ \square_{g_0}.$$

Therefore, the operators $\square_{\phi_t^*g_0}$ and \square_{g_0} have the same eigenvalues with eigenfunctions u and ϕ_t^*u , respectively. Hence, if $g(t)$ is a self similar solution to the Ricci–Bourguignon flow on (M^n, g_0) then $\tilde{Y}(t) = \frac{1}{c(t)}\tilde{Y}(0)$.

In the next example, we consider 3-dimensional Heisenberg group.

Example 4.3 The 3-dimensional Heisenberg group is isomorphic to the set of upper-triangle 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

endowed with usual the matrix multiplication. For any given metric g_0 on Heisenberg group we choose a Milnor frame $\{X_1, X_2, X_3\}$ in which

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$

the metric g_0 is diagonal and we denote by

$$g_0 = A_0(\theta^1)^2 + B_0(\theta^2)^2 + C_0(\theta^3)^2,$$

where $\{\theta^1, \theta^2, \theta^3\}$ is the dual coframe to the Milnor frame $\{X_1, X_2, X_3\}$. We assume that

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2$$

be a solution of Ricci–Bourguignon flow. According to [17] under the normalization $A_0B_0C_0 = 1$, we get

$$R_{11} = \frac{1}{2}A^3, \quad R_{22} = -\frac{1}{2}A^2B, \quad R_{33} = -\frac{1}{2}A^2C, \quad R = -\frac{1}{2}A^2, \quad |Ric|^2 = \frac{3}{4}A^4.$$

The Ricci–Bourguignon flow equations are then

$$\begin{cases} \frac{d}{dt}A = -(1 + \rho)A^3, \\ \frac{d}{dt}B = (1 - \rho)A^2B, \\ \frac{d}{dt}C = (1 - \rho)A^2C. \end{cases}$$

Starting with the equation for A , these can be integrated directly to have

$$\begin{cases} A = A_0 (1 + 2(1 + \rho)A_0^2 t)^{-\frac{1}{2}}, \\ B = B_0 (1 + 2(1 + \rho)A_0^2 t)^{\frac{1-\rho}{2(1+\rho)}}, \\ C = C_0 (1 + 2(1 + \rho)A_0^2 t)^{\frac{1-\rho}{2(1+\rho)}}. \end{cases}$$

Therefore, for $\rho \neq -1$ Eq. (3.5) yields

$$\begin{aligned} \frac{d}{dt} \tilde{Y}(t) &= 2a \int_M \left[-(1 + \rho)A^3 \nabla_1 u \nabla_1 u + (1 - \rho)A^2 B \nabla_2 u \nabla_2 u + (1 - \rho)A^2 C \nabla_3 u \nabla_3 u \right] d\mu \\ &\quad + \frac{3}{2}A^4 + \frac{1}{2} \left[\left(1 - \frac{2}{\rho}\right) (1 - 3\rho) + 2\rho \right] A^2 \tilde{Y}(t) \\ &\leq 2(1 - \rho)A^2 \tilde{Y}(t) + (1 - \rho)A^4 + \frac{3}{2}A^4 + \frac{1}{2} \left[\left(1 - \frac{2}{\rho}\right) (1 - 3\rho) + 2\rho \right] A^2 \tilde{Y}(t) \\ &= \left[\frac{5}{2}(1 - \rho) - \frac{1}{\rho}(1 - 3\rho) \right] A^2 \tilde{Y}(t) + \left(\frac{5}{2} - \rho \right) A^4. \end{aligned}$$

Let $a = \frac{5}{2}(1 - \rho) - \frac{1}{\rho}(1 - 3\rho)$ and $b = \left(\frac{5}{2} - \rho\right)$. If $a \neq -2(1 + \rho)$ then we have

$$\frac{d}{dt} \left((1 + 2(1 + \rho)A_0^2 t)^{-\frac{a}{2(1+\rho)}} \tilde{Y}(t) + \frac{bA_0^4}{1 + \frac{a}{2(1+\rho)}} (1 + 2(1 + \rho)A_0^2 t)^{-(1 + \frac{a}{2(1+\rho)})} \right) \leq 0.$$

Thus, the quantity

$$\left(\frac{A}{A_0} \right)^{\frac{a}{1+\rho}} \left(\tilde{Y}(t) + \frac{bA_0^2}{1 + \frac{a}{2(1+\rho)}} A^2 \right)$$

is nonincreasing along the Ricci–Bourguignon flow. If $a = -2(1 + \rho)$ then we get

$$\frac{d}{dt} \left((1 + 2(1 + \rho)A_0^2 t)^{-\frac{a}{2(1+\rho)}} \tilde{Y}(t) - \frac{bA_0^2}{2(1 + \rho)} \ln(1 + 2(1 + \rho)A_0^2 t) \right) \leq 0.$$

This shows that the quantity

$$(1 + 2(1 + \rho)A_0^2 t)^{-\frac{a}{2(1+\rho)}} \tilde{Y}(t) - \frac{bA_0^2}{2(1 + \rho)} \ln(1 + 2(1 + \rho)A_0^2 t)$$

is nonincreasing under the Ricci–Bourguignon flow.

5 Conclusion

We have obtained the evolution formula for Yamabe constant $\tilde{Y}(t)$ under the Ricci–Bourguignon flow as (3.5) and (3.13) on closed n -dimensional Riemannian manifolds with initial metric g_0 . We assume that g_0 is a metric of constant scalar curvature and we conclude that $\frac{d}{dt}|_{t=0} \tilde{Y}(t) \geq 0$ and the equality holds if and only if g_0 is an Einstein metric, whenever one of the following condition holds:

- (1) $\frac{R_{g_0}}{n-1}$ is not a positive eigenvalue of the Laplacian Δ_{g_0} ,
- (2) there is a C^1 -family of smooth positive functions $u(t)$ on $[0, \epsilon)$ for some $\epsilon > 0$ with constant $u(0)$, such that $u(t)^{\frac{4}{n-2}} g(t)$ has unit volume and constant scalar curvature $Y(g(t))$.



Also, we give examples on Einstein manifold, Ricci–Bourguignon soliton, and Heisenberg group in support of our results.

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Declarations

Conflict of interest We declare that there is no conflict of interest between the authors.

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