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On non-expansivity of topical functions by a new pseudo-metric

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Abstract In this paper, we first define a new pseudo-metric d on a normed linear space X. We do this by introducing two different classes of elementary topical functions. Next, we use this pseudo-metric d to investigate the non-expansivity and some properties of topical functions. Finally, the characterizations of fixed points of topical functions are given, and a relation between the pseudo-metric d and the original norm of the normed linear space X is presented.

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الملخص

في هذا البحث، نعرّف أولا شبه مقاس جديد d على فضاء خطي معيّر X. نقوم بهذا بتقديم صنفين مختلفين من دوال طوبولوجية ابتدائية. بعد ذلك، وباستعمال هذا شبه المقاس d، نتحقق من عدم تمددية الدوال الطوبولوجية وبعض خواصها. وأخيرا، نعطي تمييزا للنقاط الثابتة للدوال الطوبولوجية ونقدّم علاقة بين شبه المقاس d والمعيار الأساسي للفضاء الخطي المعيّر X.

1 Introduction

The Cayley–Hilbert's metric (Hilbert's metric) is especially beneficial in proving the existence of a unique fixed point for a positive homogeneous operator defined on a Banach space X [7]. This metric has been defined on a closed convex, pointed cone and solid subset K of a Banach space X. In fact, in [7] Bushell restricted the domain of a particular type of a positive nonlinear operator and the existence of its unique fixed point was proved using the Banach contraction-mapping theorem. Moreover, in [4] Birkhoff obtained the other usefulness

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of this metric in algebra and analysis. For example, the Perron–Frobenius theorem for non-negative matrices has been proved by an application of the Banach contraction-mapping theorem in suitable metric spaces.

Birkhoff's version of Hilbert's metric is a distance between pairs of rays in a closed cone, and is closely related to Hilbert's classical cross-ratio metric [6]. A version of this metric is discussed by Bushell [5] which could be traced back to the work of Birkhoff [4] and Samelson [16]. It has found numerous applications in mathematical analysis, especially in the analysis of linear, and nonlinear, mappings on cones. Birkhoff's version of Hilbert's metric provides a different perspective on Hilbert geometries and naturally leads to infinite-dimensional generalizations. Recently, it has been given a strong convergence theorem for an iterative algorithm that approximates fixed points of those self-mappings of the Hilbert ball which are non-expansive with respect to the hyperbolic metric [10]. Moreover, in [9] an extension of the Banach Contraction Principle for best proximity points of a non-self mapping on the open unit Hilbert ball has been obtained, and this result also was established for best proximity points of non-expansive mappings and firmly non-expansive mappings.

This is a motivation for us to define a metric (or, a pseudo-metric) and investigate the non-expansivity, characterizations of fixed points and other properties of topical functions (i.e., plus-homogeneous and increasing). Indeed, we do this, by introducing two different classes of elementary topical functions, in particular, these two classes are of min-type and max-type functions (see, Example 3.1, below).

The structure of the paper is as follows. In Sect. 2, we provide definitions, notations and preliminary results related to topical functions. In Sect. 3, we first introduce two different classes of elementary topical functions, and define a new pseudo-metric d using these two classes. Next, we investigate the properties of the pseudo-metric d. The results on non-expansivity of topical functions and characterizations of fixed points of this class of functions are given in Sect. 4. Furthermore, we present a relation between the pseudo-metric d and the original norm of the normed linear space X.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real normed linear space. We assume that X is equipped with a closed convex pointed cone S. The cone S is called pointed if $S\cap(-S) = \{0\}$. We also assume that $int S \neq \emptyset$, where $int A$ denotes interior of a subset A of X. For each $x, y \in X$, we say that $x \leq y$ or $y \geq x$ if and only if $y - x \in S$. Also, we say that $x < y$ or $y > x$ if and only if $y - x \in S\backslash\{0\}$. It is easy to see that " $\leq$ " is a partial order on X, and so, $(X, \leq)$ is an ordered normed linear space.

Moreover, we assume that S is a normal cone. Recall [14] that the cone S is called *normal* if there exists a constant $m > 0$ such that $\|x\| \leq m\|y\|$, whenever $0 \leq x \leq y$ with $x, y \in X$. Let $\mathbf{1} \in$ int S (see Remark 2.5, below) and let

$$B := \{x \in X : -\mathbf{1} \leq x \leq \mathbf{1}\}. \tag{1}$$

It is well known and easy to check that B can be considered as the unit ball of a certain norm $\|\cdot\|_1$ on X, which is equivalent to the initial norm $\|\cdot\|$. Assume without loss of generality that $\|\cdot\| = \|\cdot\|_1$.

In the sequel, we will consider the ordered normed linear space $(X, \leq, \|\cdot\|)$ and the unit ball B as described above unless stated otherwise.

We recall from [11] the following definitions.

A function $p : X \longrightarrow \bar{\mathbb{R}} := [-\infty, +\infty]$ is called plus-homogeneous if $p(x + \lambda\mathbf{1}) = p(x) + \lambda$ for all $x \in X$ and all $\lambda \in \mathbb{R}$, where $\mathbf{1} \in int S$.

A function $p : X \longrightarrow \bar{\mathbb{R}}$ is called increasing if for each $x, y \in X$ with $x \leq y$, then $p(x) \leq p(y)$.

Definition 2.1 [15] A function $p : X \longrightarrow \bar{\mathbb{R}}$ is called topical if f is increasing and plus-homogeneous.

For more details and properties of topical functions, see, for example; [3,13,15].

Definition 2.2 [2,8] A function $T : X \longrightarrow X$ is called non-expansive with respect to a metric d (or, a pseudo-metric d) if

$$d(T(x), T(y)) \leq d(x, y), \ \forall\, x, y \in X.$$

Definition 2.3 [2,8] A function $T : X \longrightarrow X$ is called contractive with respect to a metric d (or, a pseudo-metric d) if

$$d(T(x), T(y)) < d(x, y), \ \forall\, x, y \in X, \text{ with } x \neq y.$$

Definition 2.4 [2,8] A function $T : X \longrightarrow X$ is called contraction with respect to a metric d (or, a pseudo-metric d) if there exists $\alpha \in (0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y),\ \forall\, x, y \in X.$$

Remark 2.5 Since S is a cone, it follows that $int S$ is also a cone. Therefore, there exists $0 \neq u \in int S$ because $int S \neq \emptyset$, and hence, $\frac{u}{\|u\|} \in int S$. Let $\mathbf{1} := \frac{u}{\|u\|}$. So, $\mathbf{1} \in int S$ and $\|\mathbf{1}\| = 1$.

Throughout the paper, let $\mathbf{1} \in int S$ be as in Remark 2.5.

3 A new pseudo-metric and its properties

In this section, using the elementary topical functions defined in [11,12], we first introduce a new pseudo-metric d, and then we investigate its properties. We start with the definition of the elementary topical functions.

In [11,12], the elementary topical function $\varphi : X \times X \longrightarrow \mathbb{R}$ is defined by

$$\varphi(x, y) := \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} + y \leq x\},\ \forall\, x, y \in X.$$

It should be noted that, in view of (1), the set $\{\lambda \in \mathbb{R} : \lambda\mathbf{1} + y \leq x\}$ is non-empty and bounded from above (by $\|x - y\|$). Clearly, this set is closed. So, in the definition of φ, we can use maximum instead of supremum. It follows from the definition of φ that:

$$-\|x - y\| \leq \varphi(x, y) \leq \|x - y\|,\ \forall\, x, y \in X. \tag{2}$$

$$\varphi(x, y)\mathbf{1} + y \leq x,\ \forall\, x, y \in X. \tag{3}$$

We enlist some properties of the function φ, which have been obtained in [11,12] (therefore, we state them without proof).

Proposition 3.1

$$(x_1, x_2 \in X \text{ with } x_1 \leq x_2) \Longrightarrow \varphi(x_1, y) \leq \varphi(x_2, y),\ \forall\, y \in X, \tag{4}$$
$$(y_1, y_2 \in X \text{ with } y_1 \leq y_2) \Longrightarrow \varphi(x, y_2) \leq \varphi(x, y_1),\ \forall\, x \in X, \tag{5}$$
$$\varphi(x, x) = 0,\ \forall\, x \in X, \tag{6}$$
$$\varphi(x + \lambda\mathbf{1}, y) = \varphi(x, y) + \lambda,\ \forall\, x, y \in X,\ \forall\, \lambda \in \mathbb{R}, \tag{7}$$
$$\varphi(x, y + \lambda\mathbf{1}) = \varphi(x, y) - \lambda,\ \forall\, x, y \in X,\ \forall\, \lambda \in \mathbb{R}, \tag{8}$$
$$\varphi(x, y) = \varphi(-y, -x),\ \forall\, x, y \in X, \tag{9}$$
$$\varphi(x, x + \lambda\mathbf{1}) = -\lambda,\ \forall\, x \in X,\ \forall\, \lambda \in \mathbb{R}. \tag{10}$$

Now, for each $y \in X$, we define the function $\varphi_y : X \longrightarrow \mathbb{R}$ by $\varphi_y(x) := \varphi(x, y)$ for all $x \in X$. Therefore, it is clear that, for each $y \in X$, the function φ_y satisfies the relations (2)–(10).

In the sequel, we also consider the elementary topical function (cf. [11,12]) $\psi : X \times X \rightarrow \mathbb{R}$ defined by

$$\psi(x, y) := \inf\{\lambda \in \mathbb{R} : x \leq \lambda\mathbf{1} + y\},\ \forall\, x, y \in X.$$

It is worth noting that, in view of (1), the set $\{\lambda \in \mathbb{R} : x \leq \lambda\mathbf{1} + y\}$ is non-empty and bounded from below (by $-\|x - y\|$). Clearly, this set is closed. So, in the definition of ψ, we can use minimum instead of infimum. It follows from the definition of ψ that:

$$-\|x - y\| \leq \psi(x, y) \leq \|x - y\|,\ \forall\, x, y \in X. \tag{11}$$
$$x \leq \psi(x, y)\mathbf{1} + y,\ \forall\, x, y \in X. \tag{12}$$

We present some properties (without proof) of the function ψ obtained in [11,12].

Proposition 3.2

$$(x_1, x_2 \in X \text{ with } x_1 \leq x_2) \Longrightarrow \psi(x_1, y) \leq \psi(x_2, y), \ \forall\, y \in X, \tag{13}$$
$$(y_1, y_2 \in X \text{ with } y_1 \leq y_2) \Longrightarrow \psi(x, y_2) \leq \psi(x, y_1), \ \forall\, x \in X, \tag{14}$$
$$\psi(x, x) = 0, \ \forall\, x \in X, \tag{15}$$
$$\psi(x + \lambda\mathbf{1}, y) = \psi(x, y) + \lambda, \ \forall\, x, y \in X, \ \forall\, \lambda \in \mathbb{R}, \tag{16}$$
$$\psi(x, y + \lambda\mathbf{1}) = \psi(x, y) - \lambda, \ \forall\, x, y \in X, \ \forall\, \lambda \in \mathbb{R}, \tag{17}$$
$$\psi(x, y) = \psi(-y, -x), \ \forall\, x, y \in X, \tag{18}$$
$$\psi(x, x + \lambda\mathbf{1}) = -\lambda, \ \forall\, x \in X, \ \forall\, \lambda \in \mathbb{R}. \tag{19}$$

For each $y \in X$, we define the function $\psi_y : X \longrightarrow \mathbb{R}$ by $\psi_y(x) := \psi(x, y)$ for all $x \in X$. Therefore, it is clear that, for each $y \in X$, the function ψ_y satisfies the relations (11)–(19).

We now give some crucial properties of φ_y and ψ_y $(y \in X)$.

Proposition 3.3 *For each $y \in X$, the functions φ_y and ψ_y are topical.*

Proof This is an immediate consequence of the relations (4), (7), (13) and (16). □

Lemma 3.4 *For each $x, y \in X$,*

$$\varphi_y(x) = -\psi_x(y). \tag{20}$$

Proof The result follows from the definitions of φ_y and ψ_y. □

Proposition 3.5

$$\varphi_y(x)\mathbf{1} + y \leq x \leq \psi_y(x)\mathbf{1} + y, \ \forall\, x, y \in X. \tag{21}$$

Proof This is an immediate consequence of the relations (3) and (12). □

Corollary 3.6

$$\varphi_y(x) \leq \psi_y(x), \ \forall\, x, y \in X. \tag{22}$$

Proof Assume if possible that there exist $x, y \in X$ such that $\varphi_y(x) > \psi_y(x)$. This together with $0 \neq \mathbf{1} \in S$ and the fact that S is a cone implies that

$$[\varphi_y(x) - \psi_y(x)]\mathbf{1} \in S\backslash\{0\},$$

and hence, $\psi_y(x)\mathbf{1} < \varphi_y(x)\mathbf{1}$. This contradicts Proposition 3.5 (see page 3, for the definition of the strict order " $<$ "). □

Now, using the elementary topical functions φ_y and ψ_y $(y \in X)$, we introduce a new pseudo-metric on X.

Definition 3.7 We define the function $d : X \times X \longrightarrow [0, +\infty)$ by

$$d(x, y) := \psi_y(x) - \varphi_y(x), \ \forall\, x, y \in X. \tag{23}$$

Theorem 3.8 *(X, d) is a pseudo-metric space.*

Proof First, it should be noted that, in view of (6), (15) and (23), $d(x, x) = 0$ for all $x \in X$. Also, it follows from Corollary 3.6 that $d(x, y) \geq 0$ for all $x, y \in X$. Now, we show that

$$d(x, y) = 0 \Longleftrightarrow x = y + \lambda\mathbf{1}, \quad \text{for some } \lambda \in \mathbb{R}. \tag{24}$$

Assume that $x = y + \lambda\mathbf{1}$ for some $\lambda \in \mathbb{R}$. So, by (6), (7), (15), (16), (23) and the fact that $\varphi_y(y) = 0 = \psi_y(y)$, we have

$$\begin{aligned} d(x, y) &= d(y + \lambda\mathbf{1}, y) \\ &= \psi_y(y + \lambda\mathbf{1}) - \varphi_y(y + \lambda\mathbf{1}) \\ &= \psi_y(y) + \lambda - \varphi_y(y) - \lambda \\ &= \psi_y(y) - \varphi_y(y) \end{aligned}$$

$$= 0.$$

Conversely, suppose that, for $x, y \in X$, we have $d(x, y) = 0$. So, it follows from (23) that $\varphi_y(x) = \psi_y(x)$. Therefore, by

$$\varphi_y(x) = \max\{\lambda \in \mathbb{R} : \lambda\mathbf{1} + y \leq x\},$$

and

$$\psi_y(x) = \min\{\lambda \in \mathbb{R} : x \leq \lambda\mathbf{1} + y\},$$

we conclude that there exists $\lambda \in \mathbb{R}$ such that $x = y + \lambda\mathbf{1}$.

For triangle inequality, by (3) and (12), for each $x, y, z \in X$, one has

$$x \leq \psi_z(x)\mathbf{1} + z,$$

and

$$x \geq \varphi_z(x)\mathbf{1} + z.$$

Therefore, it follows from (4), (7), (13) and (16) that

$$\psi_y(x) \leq \psi_z(x) + \psi_y(z),$$

and

$$\varphi_y(x) \geq \varphi_z(x) + \varphi_y(z).$$

Hence,

$$\begin{aligned} d(x, y) = \psi_y(x) - \varphi_y(x) &\leq \psi_z(x) + \psi_y(z) - \varphi_z(x) - \varphi_y(z) \\ &= [\psi_z(x) - \varphi_z(x)] + [\psi_y(z) - \varphi_y(z)] \\ &= d(x, z) + d(z, y). \end{aligned}$$

Finally, we show that $d(x, y) = d(y, x)$ for all $x, y \in X$. By Lemma 3.4,

$$\begin{aligned} d(x, y) &= \psi_y(x) - \varphi_y(x) = -\varphi_x(y) - (-\psi_x(y)) \\ &= \psi_x(y) - \varphi_x(y) = d(y, x), \ \forall\, x, y \in X. \end{aligned}$$

Therefore, in view of (24) (also, see Example 3.10, below), we conclude that d is a pseudo-metric on X, and so, (X, d) is a pseudo-metric space. □

From now on, we consider the pseudo-metric space (X, d) given by Theorem 3.8.

Lemma 3.9

$$\begin{aligned} &(1) d(x + \alpha\mathbf{1}, y + \beta\mathbf{1}) = d(x, y), \ \forall\, x, y \in X, \ \forall\, \alpha, \beta \in \mathbb{R}. \\ &(2) d(\alpha x, \alpha y) = \alpha d(x, y), \ \forall\, x, y \in X, \ \forall\, \alpha \geq 0. \end{aligned}$$

Proof (1). By (7), (8), (16), (17) and (23), we obtain

$$\begin{aligned} d(x + \alpha\mathbf{1}, y + \beta 1) &= \psi_{y+\beta\mathbf{1}}(x + \alpha\mathbf{1}) - \varphi_{y+\beta\mathbf{1}}(x + \alpha\mathbf{1}) \\ &= \psi_{y+\beta\mathbf{1}}(x) + \alpha - \varphi_{y+\beta\mathbf{1}}(x) - \alpha \\ &= \psi_{y+\beta\mathbf{1}}(x) - \varphi_{y+\beta\mathbf{1}}(x) \\ &= \psi_y(x) - \beta - \varphi_y(x) + \beta \\ &= \psi_y(x) - \varphi_y(x) \\ &= d(x, y). \end{aligned}$$

In view of (23) and using the definitions of φ_y and ψ_y, the relation (2) follows. □

Example 3.10 Let $X := \mathbb{R}^n$ with the norm $\|x\| := \max_{1\le i\le n} |x_i|$ for each $x := (x_1, \dots, x_n) \in \mathbb{R}^n$. Let $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^n$, $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, and $y := (y_1, \dots, y_n) \in \mathbb{R}^n$. Therefore,

$$\varphi_y(x) = \min_{1\le i\le n} \{x_i - y_i\},$$
$$\psi_y(x) = \max_{1\le i\le n} \{x_i - y_i\}.$$

So, in view of (23), one has

$$d(x, y) = \max_{1\le i\le n} \{x_i - y_i\} + \max_{1\le i\le n} \{y_i - x_i\}. \tag{25}$$

Now, it is easy to see that if $x = y + \lambda\mathbf{1}$ $(0 \ne \lambda \in \mathbb{R})$, then $d(x, y) = 0$, but $x \ne y$. In particular, let $x, y \in \mathbb{R}^2$ be such that $x := (x_1, 0)$ with $0 \ne x_1 \in \mathbb{R}$ and $y := (0, -x_1)$. Let $\mathbf{1} := (1, 1) \in \mathbb{R}^2$ and $\lambda := x_1$. Then, $x = y + \lambda\mathbf{1}$, and hence, $d(x, y) = 0$, but $x \ne y$. Consequently, d is a pseudo-metric.

Definition 3.11 A subset $D \subseteq X$ is called null set if the diameter of D is equal to 0, i.e.,

$$\sup\{d(x, y) : x, y \in D\} = 0,$$

where we define the diameter of D (diam(D)) by

$$\mathrm{diam}(D) := \sup\{d(x, y) : x, y \in D\},$$

with the convention $\sup \emptyset := 0$.

Example 3.12 Let X be as in Example 3.10, and let $E := \{(x, \dots, x) \in X : x \in \mathbb{R}\}$. Then, E is a null set, because in view of (25), one has $d(u, v) = 0$ for all $u, v \in E$.

The following example shows that the pseudo-metric d may be a metric on some subspace W of X, and moreover, (W, d) is a complete metric space.

Example 3.13 Let $X := \mathbb{R}^2$ and $\mathbf{1} := (1, 1) \in \mathbb{R}^2$. Let d be as in Definition 3.7. Assume that $m \in \mathbb{N}$ with $m \ge 2$ is fixed. Let $E_m := \{(x, mx) : x \in \mathbb{R}\} \subset X$. Then, (E_m, d) is a complete metric subspace of X.

We only show that if $u, v \in E_m$ are such that $d(u, v) = 0$, then $u = v$. This together with Theorem 3.8 implies that d is a metric on E_m, and hence, (E_m, d) is a metric space. To this end, assume that $u, v \in E_m$ are such that $d(u, v) = 0$. Therefore, $u = (x, mx)$ and $v = (y, my)$ for some $x, y \in \mathbb{R}$. Since $d(u, v) = 0$, it follows from (24) that there exists $\lambda \in \mathbb{R}$ such that $u = v + \lambda\mathbf{1}$. This implies that

$$x - y = \lambda, \text{ and } m(x - y) = \lambda. \tag{26}$$

We claim that $\lambda = 0$. Otherwise, in view of (26), we obtain $m = 1$, which is a contradiction because $m \ge 2$. So, it follows from (26) that $x = y$, and so, $u = v$. Thus, by Theorem 3.8, d is a metric on E_m.

Now, let $\{u_k\}_{k\ge1}$ be a Cauchy sequence in E_m. Then there exists a sequence $\{x_k\}_{k\ge1}$ in $\mathbb{R}$ such that $u_k = (x_k, mx_k)$ $(k = 1, 2, \dots)$. So, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(u_k, u_n) < \varepsilon$ for all $k, n \ge N$. In view of (25), we have

$$d(u_k, u_n) = \max\{x_k - x_n, mx_k - mx_n\} + \max\{x_n - x_k, mx_n - mx_k\}. \tag{27}$$

This implies that

$$\begin{aligned} |x_k - x_n| &= \frac{1}{m-1}\{\max\{x_k - x_n, mx_k - mx_n\} + \max\{x_n - x_k, mx_n - mx_k\}\} \\ &= \frac{1}{m-1} d(u_k, u_n) < \varepsilon, \ \forall\, k, n \ge N \text{ (note that } m \ge 2). \end{aligned}$$

Therefore, the sequence $\{x_k\}_{k\ge1}$ is Cauchy in $\mathbb{R}$, and so, there exists $x \in \mathbb{R}$ such that $x_k \longrightarrow x$ with respect to the Euclidean metric. Let $u := (x, mx) \in E_m$. But,

$$d(u_k, u) = \max\{x_k - x, mx_k - mx\} + \max\{x - x_k, mx - mx_k\}.$$

This together with the fact that $x_k \longrightarrow x$ with respect to the Euclidean metric implies that $d(u_k, u) \longrightarrow 0$ as $k \longrightarrow +\infty$, i.e., $u_k \longrightarrow u$ with respect to the metric d. Then, (E_m, d) is a complete metric space.

4 Non-expansivity and characterizations of fixed points of topical functions

In this section, using the pseudo-metric d, we obtain some results on non-expansivity of topical functions. In fact, we show that each topical function is non-expansive with respect to the pseudo-metric d. Also, we present the characterizations of fixed points of topical functions. We denote by $FixT$ the set of all fixed points of a function $T : X \longrightarrow X$, and is defined by

$$FixT := \{x \in X : T(x) = x\},$$

(see, [1,2]). It should be noted that a topical function $T : X \longrightarrow X$ has not necessarily a fixed point. In the following, we give an example. We first give the following definition of a topical function.

Definition 4.1 A function $T : X \longrightarrow X$ is called topical if T is increasing ($x, y \in X$ and $x \leq y \Longrightarrow T(x) \leq T(y)$) and plus-homogeneous ($T(x + \lambda \mathbf{1}) = T(x) + \lambda \mathbf{1}$ for all $x \in X$ and all $\lambda \in \mathbb{R}$).

Example 4.2 Let $0 \neq a \in \mathbb{R}$ be fixed, and let $\mathbf{1} := 1 \in \mathbb{R}$. Define $T_a : \mathbb{R} \longrightarrow \mathbb{R}$ by $T_a(x) := x + a$ for all $x \in \mathbb{R}$. It is easy to show that T_a is a topical function in the sense of Definition 4.1. Also, by Lemma 3.9 (1), we have

$$d(T_a(x), T_a(y)) = d(x + a, y + a) = d(x, y), \ \forall\, x, y \in X.$$

So, T_a is also non-expansive with respect to the pseudo-metric d. Clearly, T_a has no fixed point, i.e., $FixT_a = \emptyset$.

Definition 4.3 A subset G of X is called a generator for X, if for each $x \in X$, there exist $y \in G$ and $\lambda \in \mathbb{R}$ such that $x = y + \lambda \mathbf{1}$, and we write $X = \langle G \rangle$.

Example 4.4 Assume that $X := \mathbb{R}^2$ and $\mathbf{1} := (1, 1) \in \mathbb{R}^2$. Let $G := \{(x, -x) \in X : x \in \mathbb{R}\}$. Then, $X = \langle G \rangle$, i.e., G is a generator for X. Indeed, for each $u := (x, y) \in X$, put

$$v := \left(\frac{x - y}{2}, \frac{y - x}{2}\right), \text{ and } \lambda := \frac{x + y}{2}.$$

Then, $v \in G$, $\lambda \in \mathbb{R}$ and $u = v + \lambda \mathbf{1}$.

Theorem 4.5 *Let $T : X \longrightarrow X$ be an arbitrary function, and let $G \subseteq X$ be a generator for X. Suppose that T is plus-homogeneous on G (see Definition 4.1). If $T : G \longrightarrow G$ is non-expansive, then $T : X \longrightarrow X$ is also non-expansive.*

Proof Let $x_1, x_2 \in X$ be arbitrary. Since G is a generator for X, in view of Definition 4.3, there exist $y_1, y_2 \in G$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $x_1 = y_1 + \lambda_1 \mathbf{1}$ and $x_2 = y_2 + \lambda_2 \mathbf{1}$. This together with Lemma 3.9 and the fact that T is non-expansive and plus-homogeneous on G implies that

$$\begin{aligned} d(T(x_1), T(x_2)) &= d(T(y_1 + \lambda_1 \mathbf{1}), T(y_2 + \lambda_2 \mathbf{1})) \\ &= d(T(y_1) + \lambda_1 \mathbf{1}, T(y_2) + \lambda_2 \mathbf{1}) \\ &= d(T(y_1), T(y_2)) \\ &\leq d(y_1, y_2) \\ &= d(y_1 + \lambda_1 \mathbf{1}, y_2 + \lambda_2 \mathbf{1}) \\ &= d(x_1, x_2). \end{aligned}$$

This completes the proof. □

We now show that if a topical function $T : X \longrightarrow X$ has at least one fixed point, then T has infinitely many fixed points. We first give the following definition.

Definition 4.6 Let $y_0 \in X$ be fixed. A subset C of X is called pseudo-plus cone in the direction y_0, if $x \in C$, then $x + \lambda y_0 \in C$ for all $\lambda \in \mathbb{R}_+$.

Proposition 4.7 *Let $T : X \longrightarrow X$ be a topical function. Then, $FixT$ is a pseudo-plus cone in the direction* $\mathbf{1}$.

Proof If $FixT = \emptyset$, then it is clear that $FixT$ is a pseudo-plus cone in the direction $\mathbf{1}$. Assume that $FixT \neq \emptyset$. Let $x_0 \in FixT$ and $\lambda \in \mathbb{R}_+$ be arbitrary. Since $T(x_0) = x_0$ and T is topical, it follows that

$$T(x_0 + \lambda\mathbf{1}) = T(x_0) + \lambda\mathbf{1} = x_0 + \lambda\mathbf{1}.$$

This implies that $x_0 + \lambda\mathbf{1} \in FixT$ for all $\lambda \in \mathbb{R}_+$, and hence, $FixT$ is a pseudo-plus cone in the direction $\mathbf{1}$. Consequently, if $FixT \neq \emptyset$, then $FixT$ has infinitely many points. □

The following result exerts that there does not exist any topical function which is contractive with respect to the pseudo-metric d.

Theorem 4.8 *There does not exist any topical function $T : X \longrightarrow X$ that is contractive, i.e., there does not exist a topical function $T : X \longrightarrow X$, which satisfies the following strict inequality:*

$$d(Tx, Ty) < d(x, y), \ \forall\, x, y \in X \text{ with } x \neq y. \tag{28}$$

Proof Assume if possible that there exists a topical function $T : X \longrightarrow X$, which satisfies the strict inequality (28). Now, let $x_0 \in X$ be arbitrary, and let $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ be such that $\alpha \neq \beta$. Put $x := x_0+\alpha\mathbf{1}$ and $y := x_0+\beta\mathbf{1}$ in (28). Then,

$$d(T(x_0 + \alpha\mathbf{1}), T(x_0 + \beta\mathbf{1})) < d(x_0 + \alpha\mathbf{1}, x_0 + \beta\mathbf{1}).$$

This together with the fact that T is topical implies that

$$d(T(x_0) + \alpha\mathbf{1}, T(x_0) + \beta\mathbf{1}) < d(x_0 + \alpha\mathbf{1}, x_0 + \beta\mathbf{1}).$$

Therefore, by Lemma 3.9, we obtain

$$d(T(x_0), T(x_0)) < d(x_0, x_0),$$

which is a contradiction because $d(z, z) = 0$ for all $z \in X$. This completes the proof. □

Definition 4.9 Let $p \in \mathbb{R}$ be fixed and arbitrary. A function $T : X \longrightarrow X$ is called p-topical if T is increasing in the sense of Definition 4.1 and $T(x + \lambda\mathbf{1}) = T(x) + p\lambda\mathbf{1}$ for all $x \in X$ and all $\lambda \in \mathbb{R}$.

Note that if $p = 1$ in Definition 4.9, then T is topical.

Theorem 4.10 *Let the function $T : X \longrightarrow X$ be p-topical $(p > 0)$. Then,*

$$d(T(x), T(y)) \leq pd(x, y), \ \forall\, x, y \in X. \tag{29}$$

Proof Let $x, y \in X$ be arbitrary. By (21), we have

$$\varphi_y(x)\mathbf{1} + y \leq x \leq \psi_y(x)\mathbf{1} + y.$$

This together with the fact that T is p-topical implies that

$$p\varphi_y(x)\mathbf{1} + T(y) \leq T(x) \leq p\psi_y(x)\mathbf{1} + T(y). \tag{30}$$

Now, in view of the definitions of φ_y and ψ_y and (30), we get

$$p\varphi_y(x) \leq \varphi_{T(y)}(T(x)), \text{ and } \psi_{T(y)}(T(x)) \leq p\psi_y(x). \tag{31}$$

Therefore, it follows from (23) and (31) that

$$\begin{aligned} d(T(x), T(y)) &= \psi_{T(y)}(T(x)) - \varphi_{T(y)}(T(x)) \\ &\leq p\psi_y(x) - p\varphi_y(x) \\ &= pd(x, y). \end{aligned}$$

□

Remark 4.11 In view of Theorem 4.10, every p-topical function $(p > 0)$ is continuous with respect to the pseudo-metric d.

Remark 4.12 In Theorem 4.10, if $p = 1$, then T is topical, and so, by (29), we conclude that T is non-expansive. Consequently, every topical function is non-expansive with respect to the pseudo-metric d.

Remark 4.13 In Theorem 4.10, if $0 < p < 1$, then it follows from (29) that T is a contraction with respect to the pseudo-metric d.

Example 4.14 Let $X := \mathbb{R}^2$ and $\mathbf{1} := (1, 1) \in \mathbb{R}^2$. Define the function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(x_1, x_2) := (\frac{x_1}{2}, \frac{x_2}{2})$ for all $(x_1, x_2) \in \mathbb{R}^2$. It is easy to check that the function T is $\frac{1}{2}$-topical, and so, in view of Remark 4.13, T is a contraction with the unique fixed point zero.

Now, we give the following result on p-topical functions.

Theorem 4.15 *Assume that* (X, d) *is complete. Let* $T : X \longrightarrow X$ *be a* p*-topical function* $(0 < p < 1)$, *and let* $x_0 \in X$ *be fixed. Define* $x_{n+1} := T(x_n)$, $n = 0, 1, 2, \ldots$. *Then the sequence* $\{x_n\}_{n\geq 0}$ *converges to some point* $x \in X$ (*it should be noted that* x *is not necessarily unique, and also,* x *is not necessarily a fixed point of* T, *see Example 4.16, below*). *Furthermore,*

$$d(T(x), x) = 0, \text{ and hence, } d(T^n(x), T^m(x)) = 0, \ \forall\, n, m \in \mathbb{N}.$$

Proof We first show that the sequence $\{x_n\}_{n\geq 0}$ converges to some point $x \in X$. To this end, by the hypothesis and Theorem 4.10, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq p d(x_n, x_{n-1}) \\ &\ \vdots \\ &\leq p^n d(x_1, x_0). \end{aligned}$$

Thus, for each $m, n \in \mathbb{N}$ with $m > n$, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (p^n + p^{n+1} + p^{n+2} + \cdots + p^{m-1}) d(x_1, x_0) \\ &\leq (p^n + p^{n+1} + p^{n+2} + \cdots) d(x_1, x_0) \\ &= \frac{p^n}{1 - p} d(x_1, x_0). \end{aligned}$$

So, the sequence $\{x_n\}_{n\geq 0}$ is Cauchy in X. Since X is a complete pseudo-metric space, then there exists $x \in X$ such that $x_n \longrightarrow x$ with respect to the pseudo-metric d. Furthermore, by Theorem 4.10 and the fact that $x_{n+1} = T(x_n)$, $n = 0, 1, 2, \ldots$, it follows that $x_n \longrightarrow T(x)$ with respect to the pseudo-metric d. Therefore, for every $\varepsilon > 0$, we conclude that

$$d(x, T(x)) \leq d(x, x_n) + d(x_n, T(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all sufficiently large n. This implies that $d(T(x), x) = 0$ (note that in this case, in view of (24), there exists $\lambda \in \mathbb{R}$ such that $T(x) = x + \lambda\mathbf{1}$), and hence, by Theorem 4.10, we deduce that $d(T^n(x), T^m(x)) = 0$ for all $m, n \in \mathbb{N}$. □

Example 4.16 Let $X := \mathbb{R}^k$ with the norm $\|x\| := \max_{1\leq i\leq k} |x_i|$ for each $x := (x_1, \ldots, x_k) \in \mathbb{R}^k$. Let $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^k$, and let d be the pseudo-metric defined in Definition 3.7. Define the function $T : X \longrightarrow X$ by

$$T(x_1, \ldots, x_k) := \frac{1}{2}(x_1, \ldots, x_k), \ \forall\, (x_1, \ldots, x_k) \in \mathbb{R}^k.$$

Then,

(1) (X, d) is a complete pseudo-metric space.

(2) There exists a sequence $\{z_n\}_{n\geq 0} \subset X$ with $z_{n+1} = T(z_n)$ $(n = 0, 1, 2, \ldots)$ such that $z_n \longrightarrow \alpha\mathbf{1}$ with respect to the pseudo-metric d for each $\alpha \in \mathbb{R}$. Moreover,

$$FixT = \{(0, \ldots, 0)\}.$$

We now prove the assertions (1) and (2). To this end, first note that in view of Definition 4.9, it is easy to see that T is a $\frac{1}{2}$-topical function. Also, by (25),

$$d(x, y) = \max_{1\leq i\leq k}\{x_i - y_i\} + \max_{1\leq i\leq k}\{y_i - x_i\}, \tag{32}$$

for all $x = (x_1, \ldots, x_k),\ y = (y_1, \ldots, y_k) \in \mathbb{R}^k$. Now, let

$$x_{n,i} := \frac{1}{2^n},\ \text{for each } i = 1, 2, \ldots, k-1,\ x_{n,k} := -\frac{1}{2^n}, \text{ and}$$
$$z_n := (x_{n,1}, \ldots, x_{n,k-1}, x_{n,k}) \in \mathbb{R}^k,\ n = 1, 2, \ldots.$$

Let $z_0 := (x_{0,1}, \ldots, x_{0,k-1}, x_{0,k}) := (1, \ldots, 1, -1) \in \mathbb{R}^k$.

Solution (1). Assume that $\{(t_n^1, \ldots, t_n^k)\}_{n\geq 0}$ is a Cauchy sequence in X with respect to the pseudo-metric d. So, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d((t_n^1, \ldots, t_n^k), (t_m^1, \ldots, t_m^k)) < \varepsilon,\ \forall\, m, n \geq N.$$

Thus, by (32), one has

$$\max_{1\leq i\leq k}\{t_n^i - t_m^i\} + \max_{1\leq i\leq k}\{t_m^i - t_n^i\} < \varepsilon,\ \forall\, m, n \geq N.$$

This implies (not difficult to check) that

$$|t_n^i - t_m^i| < \varepsilon,\ \forall\, m, n \geq N,\ i = 1, \ldots, k.$$

Hence, $\{t_n^i\}_{n\geq 0}$ is a Cauchy sequence in $\mathbb{R}$ with respect to the Euclidean metric for each $i = 1, \ldots, k$. Then, there exists $t^i \in \mathbb{R}$ such that $t_n^i \longrightarrow t^i$ $(i = 1, \ldots, k)$. Now, we show that $\{(t_n^1, \ldots, t_n^k)\}_{n\geq 0}$ converges to $(t^1, \ldots, t^k)$ with respect to the pseudo-metric d. In view of (32), one has

$$\begin{aligned} d((t_n^1, \ldots, t_n^k), (t^1, \ldots, t^k)) &= \max_{1\leq i\leq k}\{t_n^i - t^i\} + \max_{1\leq i\leq k}\{t^i - t_n^i\} \\ &\leq \max_{1\leq i\leq k}\{|t_n^i - t^i|\} + \max_{1\leq i\leq k}\{|t_n^i - t^i|\} \\ &= 2\max_{1\leq i\leq k}\{|t_n^i - t^i|\} \longrightarrow 0,\ \text{as } n \longrightarrow +\infty, \end{aligned}$$

which completes the solution of (1).

Solution (2). In view of the definition of the sequence $\{z_n\}_{n\geq 0}$, one has

$$\begin{aligned} z_{n+1} &= (x_{n+1,1}, \ldots, x_{n+1,k-1}, x_{n+1,k}) \\ &= \left(\frac{1}{2^{n+1}}, \ldots, \frac{1}{2^{n+1}}, -\frac{1}{2^{n+1}}\right) \\ &= T(z_n),\ n = 0, 1, 2, \ldots. \end{aligned}$$

Moreover, by (32), we get

$$\begin{aligned} d(z_n, \alpha\mathbf{1}) &= d\left(\left(\frac{1}{2^n}, \ldots, \frac{1}{2^n}, -\frac{1}{2^n}\right), (\alpha, \ldots, \alpha)\right) \\ &= \max\left\{\frac{1}{2^n} - \alpha, \ldots, \frac{1}{2^n} - \alpha, -\frac{1}{2^n} - \alpha\right\} + \max\left\{\alpha - \frac{1}{2^n}, \ldots, \alpha - \frac{1}{2^n}, \alpha + \frac{1}{2^n}\right\} \\ &= \frac{1}{2^n} - \alpha + \alpha + \frac{1}{2^n} = \frac{1}{2^{n-1}} \longrightarrow 0,\ \text{as } n \longrightarrow +\infty. \end{aligned}$$

So, $z_n \longrightarrow \alpha\mathbf{1}$ with respect to the pseudo-metric d for each $\alpha \in \mathbb{R}$. It is easy to see that $(0, \ldots, 0)$ is the only fixed point of T, while $\alpha\mathbf{1}$ $(0 \neq \alpha \in \mathbb{R})$ is not a fixed point of T. This completes the solution of (2).

Definition 4.17 Let $T : X \longrightarrow X$ be a function. Suppose that $p \in \mathbb{R}$ and $n \in \mathbb{N}$. We say that T is a p^n-topical function if T^n is increasing and

$$T^n(x + \alpha\mathbf{1}) = T^n(x) + p^n\alpha\mathbf{1},\ \forall\, x \in X,\ \forall\, \alpha \in \mathbb{R}.$$

Proposition 4.18 *Let $T : X \longrightarrow X$ be a p-topical function $(p \in \mathbb{R})$, and let $n \in \mathbb{N}$. Then, T is a p^n-topical function.*

Proof By induction on n. For $n = 1$, by the assumption T is p-topical, so the result follows. Assume that for $n = k$, the function T is p^k-topical. We show that the result holds for $n = k + 1$. To this end, since T is p-topical, we have

$$\begin{aligned} T^{k+1}(x + \alpha\mathbf{1}) &= T(T^k(x + \alpha\mathbf{1})) \\ &= T(T^k(x) + p^k\alpha\mathbf{1}) \\ &= T^{k+1}(x) + p^{k+1}\alpha\mathbf{1},\ \forall\, x \in X,\ \forall\, \alpha \in \mathbb{R}. \end{aligned}$$

Also, for each $x, y \in X$ with $x \leq y$, by the hypothesis of induction, one has $T^k(x) \leq T^k(y)$. As T is increasing, so we conclude that

$$T^{k+1}(x) = T(T^k(x)) \leq T(T^k(y)) = T^{k+1}(y),$$

and hence, T is a p^n-topical function. □

Theorem 4.19 *Let the function $T : X \longrightarrow X$ be p^n-topical $(p > 0,\ n \in \mathbb{N})$. Then,*

$$d(T^n(x), T^n(y)) \leq p^n d(x, y),\ \forall\, x, y \in X.$$

Proof This follows by an argument similar to the proof of Theorem 4.3. □

By Theorem 4.19, the proof of the following theorem is similar to that of Theorem 4.4, and therefore, we omit its proof.

Theorem 4.20 *Assume that (X, d) is complete. Let the function $T : X \longrightarrow X$ be p-topical (and hence, a p^n-topical function) $(0 < p < 1,\ n \in \mathbb{N})$, and let $x_0 \in X$ be fixed. Define $x_{n+1} := T^n(x_0)$, $n = 0, 1, 2, \ldots$. Then the sequence $\{x_n\}_{n\geq 0}$ converges to some point $x \in X$ (it should be noted that x is not necessarily unique, and also, x is not necessarily a fixed point of T).*

Lemma 4.21 *Let $T : X \longrightarrow X$ be a p^n-topical function $(0 < p < 1,\ n \in \mathbb{N})$. Then the set of fixed points of T $(FixT)$ is a null set in the sense of Definition 3.2.*

Proof Let $x, y \in FixT$ be arbitrary. Then, $T(x) = x$ and $T(y) = y$, and so, for each $n \in \mathbb{N}$, one has $T^n(x) = x,\ T^n(y) = y$. This together with Theorem 4.19 implies that

$$d(x, y) \leq p^n d(x, y),\ (n \in \mathbb{N}).$$

Since $0 < p < 1$, we conclude that $d(x, y) = 0$. So, $diam(FixT) = 0$, and hence, $FixT$ is a null set. □

The definition of a firmly non-expansive operator on a Hilbert space has been given in [2, Chapter 4]. We adopt it in the pseudo-metric space (X, d) as follows.

Definition 4.22 Let $T : X \longrightarrow X$ be a function. We say that T is firmly non-expansive if

$$d^2(T(x), T(y)) + d^2((Id - T)(x), (Id - T)(y)) \leq d^2(x, y),\ \forall\, x, y \in X.$$

Remark 4.23 It should be noted that in view of Definition 4.22, T is firmly non-expansive if and only if $Id - T$ is firmly non-expansive.

Theorem 4.24 *Let $T : X \longrightarrow X$ be an arbitrary function, and let $G \subseteq X$ be a generator for X. Suppose that T is plus-homogeneous on G (see Definition 4.1). If $T : G \longrightarrow G$ is firmly non-expansive, then $T : X \longrightarrow X$ is also firmly non-expansive.*

Proof Let $x_1, x_2 \in X$ be arbitrary. Since G is a generator for X, in view of Definition 4.3, there exist $y_1, y_2 \in G$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $x_1 = y_1 + \alpha_1 \mathbf{1}$ and $x_2 = y_2 + \alpha_2 \mathbf{1}$. This together with Lemma 3.9 and the fact that T is plus-homogeneous on G implies that

$$\begin{aligned} d(T(x_1), T(x_2)) &= d(T(y_1 + \alpha_1 \mathbf{1}), T(y_2 + \alpha_2 \mathbf{1})) \\ &= d(T(y_1) + \alpha_1 \mathbf{1}, T(y_2) + \alpha_2 \mathbf{1}) \\ &= d(T(y_1), T(y_2)), \end{aligned} \tag{33}$$

and also,

$$\begin{aligned} &d(x_1 - T(x_1), x_2 - T(x_2)) \\ &= d(y_1 + \alpha_1 \mathbf{1} - T(y_1 + \alpha_1 \mathbf{1}), y_2 + \alpha_2 \mathbf{1} - T(y_2 + \alpha_2 \mathbf{1})) \\ &= d(y_1 + \alpha_1 \mathbf{1} - T(y_1) - \alpha_1 \mathbf{1}, y_2 + \alpha_2 \mathbf{1} - T(y_2) - \alpha_2 \mathbf{1}) \\ &= d(y_1 - T(y_1), y_2 - T(y_2)). \end{aligned} \tag{34}$$

Moreover, by Lemma 3.9, we have

$$d(x_1, x_2) = d(y_1 + \alpha_1 \mathbf{1}, y_2 + \alpha_2 \mathbf{1}) = d(y_1, y_2). \tag{35}$$

Hence, (33), (34) and (35) together with the fact that T is firmly non-expansive on G imply that

$$\begin{aligned} &d^2(T(x_1), T(x_2)) + d^2((Id - T)(x_1), (Id - T)(x_2)) \\ &= d^2(T(y_1), T(y_2)) + d^2((Id - T)(y_1), (Id - T)(y_2)) \\ &\le d^2(y_1, y_2) \\ &= d^2(x_1, x_2), \ \forall\, x_1, x_2 \in X. \end{aligned}$$

So, T is firmly non-expansive on X, and hence, the proof is complete. □

In the following, we give the relation between the norm of X and the pseudo-metric d.

Theorem 4.25 *The following inequality holds.*

$$d(x, y) \le 2\|x - y\|, \ \forall\, x, y \in X. \tag{36}$$

Proof As B is the unit ball of X, so it follows from (1) that

$$-\|x\|\mathbf{1} \le x \le \|x\|\mathbf{1}, \ \forall\, x \in X. \tag{37}$$

Now, let $y \in X$ be arbitrary. Since the function $\varphi_y : X \longrightarrow \mathbb{R}$ is increasing, we conclude from (37) that

$$\varphi_y(-\|x\|\mathbf{1}) \le \varphi_y(x) \le \varphi_y(\|x\|\mathbf{1}), \ \forall\, x \in X. \tag{38}$$

On the other hand, the function $\varphi_y : X \longrightarrow \mathbb{R}$ is plus-homogeneous, so it follows from (38) that

$$\varphi_y(0) - \|x\| \le \varphi_y(x) \le \varphi_y(0) + \|x\|, \ \forall\, x \in X. \tag{39}$$

By an argument similar to the above and the fact that the function $\psi_y : X \longrightarrow \mathbb{R}$ is topical (increasing and plus-homogeneous), we obtain

$$\psi_y(0) - \|x\| \le \psi_y(x) \le \psi_y(0) + \|x\|, \ \forall\, x \in X. \tag{40}$$

By adding the inequalities (39) and (40), we have

$$[\psi_y(0) - \varphi_y(0)] - 2\|x\| \le \psi_y(x) - \varphi_y(x) \le [\psi_y(0) - \varphi_y(0)] + 2\|x\|, \ \forall\, x, y \in X. \tag{41}$$

In view of the definition of the pseudo-metric d, it follows from (41) that

$$|d(x, y) - d(0, y)| \le 2\|x\|, \ \forall\, x, y \in X. \tag{42}$$

Replacing x and y by $x-y$ in (42), we get

$$d(0, x-y) = |d(0, x-y)| \leq 2\|x-y\|, \ \forall\, x, y \in X. \tag{43}$$

By the definition of the pseudo-metric d, and the definitions of φ_y and ψ_y, it is easy to check that

$$d(0, x-y) = \psi_x(y) - \varphi_x(y) = d(y, x) = d(x, y), \ \forall\, x, y \in X.$$

This together with (43) implies that

$$d(x, y) \leq 2\|x-y\|, \ \forall\, x, y \in X.$$

□

Theorem 4.26 *Suppose that $T : X \longrightarrow X$ is a topical function. Then, $T : (X, \|\cdot\|) \longrightarrow (X, d)$ is Lipschitz continuous with the Lipschitz constant 2.*

Proof As B is the unit ball of X, so it follows from (1) that

$$-\|x-y\|\mathbf{1} \leq x-y \leq \|x-y\|\mathbf{1}, \ \forall\, x, y \in X.$$

This implies that

$$y - \|x-y\|\mathbf{1} \leq x \leq y + \|x-y\|\mathbf{1}, \ \forall\, x, y \in X. \tag{44}$$

Since T is a topical function, it follows from (44) that

$$T(y) - \|x-y\|\mathbf{1} \leq T(x) \leq T(y) + \|x-y\|\mathbf{1}, \ \forall\, x, y \in X.$$

So,

$$-\|x-y\|\mathbf{1} \leq T(x) - T(y) \leq \|x-y\|\mathbf{1}, \ \forall\, x, y \in X.$$

This together with (1) implies that

$$\|T(x) - T(y)\| \leq \|x-y\|, \ \forall\, x, y \in X. \tag{45}$$

Now, by Theorem 4.25 and (45), we conclude that

$$d(T(x), T(y)) \leq 2\|x-y\|, \ \forall\, x, y \in X.$$

□

Corollary 4.27 *Let $T : X \longrightarrow X$ be a topical function. Then, for each $n \in \mathbb{N}$, the function $T^n : (X, \|\cdot\|) \longrightarrow (X, d)$ is Lipschitz continuous with the Lipschitz constant 2.*

Proof In view of Remark 4.2 and Theorem 4.3, we have

$$d(T(x), T(y)) \leq d(x, y), \ \forall\, x, y \in X.$$

This together with Theorem 4.26 implies that

$$d(T^n(x), T^n(y)) \leq 2\|x-y\|, \ \forall\, x, y \in X, \ \forall\, n \in \mathbb{N},$$

i.e., for each $n \in \mathbb{N}$, $T^n : (X, \|\cdot\|) \longrightarrow (X, d)$ is a Lipschitz continuous function with the Lipschitz constant 2. □

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