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# Zero-dimensional complete intersections and their linear span in the Veronese embeddings of projective spaces

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**Abstract** Let $\nu_{d,n} : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n}$, be the order $d$ Veronese embedding. For any $d_n \ge \cdots \ge d_1 > 0$ let $\check{\eta}(n, d; d_1, \dots, d_n) \subseteq \mathbb{P}^r$ be the union of all linear spans of $\nu_{d,n}(S)$ where $S \subset \mathbb{P}^n$ is a finite set which is the complete intersection of hypersurfaces of degree $d_1, \dots, d_n$. For any $q \in \check{\eta}(n, d; d_1, \dots, d_n)$, we prove the uniqueness of the set $\nu_{d,n}(S)$ if $d \ge d_1 + \cdots + d_{n-1} + 2d_n - n$ and $q$ is not spanned by a proper subset of $\nu_{d,n}(S)$. We compute $\dim \check{\eta}(2, d; d_1, d_1)$ when $d \ge 2d_1$.

**Mathematics Subject Classification** 14N05 · 15A69

## الملخص

لتكن $\nu_{d,n}:\mathbb{P}^r \to \mathbb{P}^r$، $r = \binom{n+d}{n}$. ترتيبا $d$ لاحتواء فيرونيز. لتكن $\check{\eta}(n;\ d;\ d_1,\dots,d_n) \subseteq \mathbb{P}^r$ اتحادا لكل مولّدات $\nu_{d,n}(S)$ و $d_n \ge \cdots \ge d_1$، حيث أن $S \subset \mathbb{P}^r$ مجموعة منتهية تكمّل تقاطع فوق السطوح بدرجات $d_1$،...، $d_n$. لكل $q \ni \check{\eta}(n;\ d;\ d_1,\dots,d_n)$، نثبت وحدانية المجموعة $\nu_{d,n}(S)$ إذا كان $d \ge d_1 + \cdots + d_{n-1} + 2d_n - n$ و $q$ غير مولّد بمجموعة جزئية فعلية من $\nu_{d,n}(S)$. نقوم بحساب بعد $\check{\eta}(2;\ d; d_1,\dots,d_n)$ عندما يكون $d \ge 2d_1$.

Let $\mathbb{K}$ be an algebraically closed field. The vector space $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ parameterizes the degree $d$ homogeneous polynomials in $n+1$ variables. Let $\nu_{d,n} : \mathbb{P}^n \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{P}^r$, $r := \binom{n+d}{n} - 1$, denote the Veronese embedding of $\mathbb{P}^n$. For any scheme, $A \subset \nu_{n,d}(\mathbb{P}^n)$ let $\langle A\rangle$ denote the linear span of $A$ in $\mathbb{P}^r$. For any finite set $S \subset \mathbb{P}^n$, we have $q \in \langle \nu_{d,n}(S)\rangle$ if and only if the homogeneous polynomial associated to $q$ is a linear combination of the $d$-powers of $|S|$ linear forms $\ell_p$, $p \in S$ ([13]). Sometimes it is cheaper to describe the set $S$ than to describe each of the point of $S$ and then add $|S|$ such descriptions. This comes handy if we only need to describe the linear space $\langle \nu_{d,n}(S)\rangle$, not a set of generators for it. We do the description taking as $S$ only the complete intersection finite sets (or the complete intersection zero-dimensional schemes).

Fix positive integers $d_1 \le \cdots \le d_n$. Let $W(n; d_1, \dots, d_n)$ (resp. $M(n; d_1, \dots, d_n)$) denote the set of all finite sets (resp. zero-dimensional schemes) of $\mathbb{P}^n$ which are the complete intersection of $n$ hypersurfaces of degree $d_1, \dots, d_n$. The set $M(n; d_1, \dots, d_n)$ is an irreducible quasi-projective variety and

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$W(n; d_1, \dots, d_n)$ is a non-empty Zariski open subset of it. The dimension $\alpha(n; d_1, \dots, d_n)$ of $M(n; d_1, \dots, d_n)$ and $W(n; d_1, \dots, d_n)$ depends only on the integers $n, d_1, \dots, d_n$ and it can easily be computed. In the particular case $d_i = d_1$ for all $i$ it is the dimension of the Grassmannian of all $n$-dimensional linear subspaces of the $\binom{n+d_1}{n}$-dimensional vector space $H^0(\mathcal{O}_{\mathbb{P}^n}(d_1))$ and hence $\alpha(n; d_1, \dots, d_n) = n(\binom{n+d_1}{n} - n)$. A general formula for complete intersections of dimension at least 2 is in [14, §2] and this case in $\mathbb{P}^{n+2}$ helps to get $\alpha(n; d_1, \dots, d_n)$. Fix any $Z \in M(n; d_1, \dots, d_n)$ and set $\beta(n, d; d_1, \dots, d_n) := \dim\langle \nu_{d,n}(Z)\rangle$. We have $\beta(n, d; d_1, \dots, d_n) = \binom{n+d}{n} - h^0(\mathcal{I}_Z(d))$ and hence the integer $\beta(n, d; d_1, \dots, d_n)$ may be computed using the Koszul complex of forms $f_1, \dots, f_n$ with $Z$ as their scheme-theoretic zero locus and it does not depend from the choice of $Z$. Set $\mathbb{I}(n, d; d_1, \dots, d_n)$ denote the subset of $W(n; d_1, \dots, d_n) \times \mathbb{P}^r$ formed by all pairs $(S, q)$ with $S \in W(n; d_1, \dots, d_n)$ and $q \in \langle \nu_{d,n}(S)\rangle$. $\mathbb{I}(n, d; d_1, \dots, d_n)$ is an irreducible quasi-projective variety of dimension $\alpha(n; d_1, \dots, d_n) + \beta(n, d; d_1, \dots, d_n)$. Let $\check{\eta}(n, d; d_1, \dots, d_n)$ denote the image of $\mathbb{I}(n, d; d_1, \dots, d_n)$ by the projection $W(d_1, \dots, d_n) \times \mathbb{P}^r \to \mathbb{P}^r$. Call $\eta(n, d; d_1, \dots, d_n)$ the closure of $\check{\eta}(n, d; d_1, \dots, d_n)$ in $\mathbb{P}^r$. By a theorem of Chevalley ([11, Exercises II.3.18, II.3.19]), $\check{\eta}(n, d; d_1, \dots, d_n)$ is constructible. Since $\mathbb{I}(n, d; d_1, \dots, d_n)$ is irreducible, $\check{\eta}(n, d; d_1, \dots, d_n)$ and $\eta(n, d; d_1, \dots, d_n)$ are irreducible. They obviously have at most dimension $\alpha(n; d_1, \dots, d_n) + \beta(n, d; d_1, \dots, d_n)$. We call the integer

$$\min\{r, \alpha(n; d_1, \dots, d_n) + \beta(n, d; d_1, \dots, d_n)\}$$

the *expected dimension* of $\eta(n, d; d_1, \dots, d_n)$.

A Koszul complex shows that $\beta(n, d; d_1, \dots, d_n) = d_1 \cdots d_n - 1$ (i.e., $h^1(\mathcal{I}_Z(d)) = 0$ for any $Z \in M(n; d_1, \dots, d_n)$) if and only if $d \ge d_1 + \cdots + d_n - n$.

**Question 0.1** *Assume $\alpha(n; d_1, \dots, d_n) + \beta(n, d; d_1, \dots, d_n) < r$. Find conditions assuring that for a general $q \in \check{\eta}(n, d; d_1, \dots, d_n)$ there is a unique $S \in W(n; d_1, \dots, d_n)$ such that $q \in \langle \nu_{d,n}(S)\rangle$?*

Obviously, we need $d \ge d_n$, because $\eta(n, d; d_1, \dots, d_n) = \emptyset$ if $d < d_n$.

Under the following strong assumption on $d$ we prove the following uniqueness theorem.

**Theorem 0.2** *Assume $d \ge d_1 + \cdots + d_{n-1} + 2d_n - n$. Take $q \in \check{\eta}(n, d; d_1, \dots, d_n)$ and assume the existence of $A \in W(n; d_1, \dots, d_n)$, $B \in M(n; d_1, \dots, d_n)$ such that $q \in \langle \nu_{d,n}(A)\rangle \cap \langle \nu_{d,n}(B)\rangle$. Then, there exists $E \subseteq A \cap B$ such that $q \in \langle \nu_{d,n}(E)\rangle$.*

In the set-up of Theorem 0.2, we have $|S| = \prod_{i=1}^n d_i$, which is often much higher than $d/2$. Thus, Theorem 0.2 is not a by-product of other uniqueness theorems for secant varieties of Veronese embedding ([7, Theorem 1.18]). Example 1.1 shows that, in general, the assumption $d \ge d_1 + \cdots + d_{n-1} + 2d_n - n$ in Theorem 0.2 cannot be improved, but this is a very specific example with $n = d_1 = 2$ and we do not know if (under certain assumptions on $n, d_1, \dots, d_n$) we may take a lower value of $d$.

*Remark 0.3* Take $Z \in W(n; d_1, \dots, d_n)$ and a general $q \in \langle \nu_{d,n}(Z)\rangle$. If $d \ge d_1 + \cdots + d_n - n$, then $\nu_{d,n}(Z)$ is linearly independent, i.e., $\dim\langle \nu_{d,n}(Z)\rangle = \deg(Z) - 1$. Since $q$ is general in $\langle \nu_{d,n}(Z)\rangle$, we have $q \notin \langle \nu_{d,n}(Z')\rangle$ for any $Z' \subsetneq Z$. Thus, when $d \ge d_1 + \cdots + d_{n-1} + 2d_n - n$ Theorem 0.2 implies that $Z$ is the only $A \in M(n; d_1, \dots, d_n)$ such that $q \in \langle \nu_{d,n}(Z)\rangle$.

*Remark 0.4* Take $n = 1$. Fix positive integers $d$ and $d_1$. We have $r = d$ and $\eta(1, d; d_1)$ is just the classical secant variety $\sigma_{d_1}(\nu_{d,1}(\mathbb{P}^1))$. Thus, $\dim \eta(1, d; d_1) = \min\{d, 2d_1 - 1\}$. Sylvester's theorem shows that both Question 0.1 and the statement of Theorem 0.2 are true for $(d_1, d)$ if and only if $d \ge 2d_1 - 1$ ([12, Theorem 1.40]).

We prove the following result concerning $\dim \eta(n, d; d_1, \dots, d_n)$.

**Theorem 0.5** *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *Fix integers $d \ge 2b \ge 4$. Then, $\eta(2, d; b, b)$ has the expected dimension $b^2 + \binom{b+2}{2} - 3$.*

Suppose you may write the given homogenous degree $d$ polynomial $f$ as a sum

$$f = g_1 + \cdots + g_k \tag{1}$$

with $k$ very low, and the homogeneous polynomials $g_1, \dots, g_k$ " simple ", but not $d$-powers of linear forms, or at least not all $d$-powers of linear forms. Our idea is that perhaps it helps even if we only find very different addenda $g_1, \dots, g_k$, in the sense that each $g_i$ is simple for a very different reason and some of them may be given by a complete intersection, even with different multidegrees.

Concerning an additive decomposition like (1), we stress again that the addenda $g_i$ may be simple for very different reasons. In [4,5], all addenda except one are $d$-powers of a linear forms, while the other one is of the form $L^{d-1}M$ with $L$ and $M$ non-proportional linear forms. The polynomial $L^{d-1}M$ is in the linear span of $\nu_{d,n}(Z)$, where $Z$ is a connected complete intersection of multidegree $(a_1,\dots,a_n)$ with $a_1=\cdots=a_{n-1}=1$, $a_n=2$, but $Z$ is assumed to be connected. We have $L^{d-1}M\in\eta(n,d;1,\dots,1,2)$. E. Carlini fixed a positive integer $s\le n$ and considered the case in which each $g_i$ only depends on $s$ homogeneous coordinates (each $g_i$ with respect to a different set of $s$ linearly independent linear forms). Starting with R. Fröberg, G. Ottaviani and B. Shapiro ([10]) there is a lot of work in the case in which (for a fixed proper divisor $k$ of $d$) each $g_i$ is a $k$-power of a homogeneous form of degree $d/k$ ([3,6,8,9,15,17]).

Now assume $g_1\in\langle\nu_{d,n}(S)\rangle$ with $S$ a complete intersection of multidegree $(d_1,\dots,d_n)$, say $S=\{f_1=\cdots=f_n=0\}$ with $\deg(f_i)=d_i$. The set $S$ depends with continuity on the coefficients of $f_1,\dots,f_n$ and so if we only know approximatively $g_1$ (but we are assured that $g_1\in\check{\eta}(n,d;d_1,\dots,d_n)$) there is hope to recover a good approximation of $f_1,\dots,f_n$ and of $S$. For different $g_i$ in (1) we may use different multidegrees.

## 1 Proof of Theorem 0.2

*Proof of Theorem 0.2* Since $A$ is a finite set, the scheme $A\cap B$ is a finite set contained in $A$. Since $\deg(A)=\deg(B)$, either $A=B$ or $A\cap B\subsetneq A$. Assume $q\notin\langle\nu_{n,d}(A\cap B)\rangle$. Since $q\notin\langle\nu_{n,d}(A\cap B)\rangle$, the existence of $q$ implies $h^1(\mathcal{I}_{A\cup B}(d))>0$. Since $d_1\le\cdots\le d_n$ and $B$ is a complete intersection of hypersurfaces of degree $d_1,\dots,d_n$, $\mathcal{I}_B(d_n)$ is globally generated. Since $A\ne B$ and $A$ is a finite set, there is $Y\in|\mathcal{I}_B(d_n)|$ such that $Y\cap A=A\cap B$. Consider the residual exact sequence

$$0\to\mathcal{I}_{A\setminus A\cap B}(d-d_n)\to\mathcal{I}_{A\cup B}(d)\to\mathcal{I}_{B,Y}(d)\to 0 \tag{2}$$

Since $d\ge d_1+\cdots+d_n-n$, we have $h^1(\mathcal{I}_B(d))=0$. Hence $h^1(Y,\mathcal{I}_{B,Y}(d))=0$. Since $d-d_n\ge d_1+\cdots+d_n-n$, we have $h^1(\mathcal{I}_A(d-d_n))=0$. Hence, $h^1(\mathcal{I}_{A\setminus A\cap Y}(d-d_n))=0$. The exact sequence (2) gives $h^1(\mathcal{I}_{A\cup B}(d))=0$, a contradiction. □

*Example 1.1* Assume $n\ge 2$ and fix integers $2\le d_1\le\cdots\le d_n$ and an integer $d$ such that $d_1+\cdots+d_n-n\le d\le d_1+\cdots+d_{n-1}+2d_n-n-1$. Take an integral $D\in|\mathcal{O}_{\mathbb{P}^n}(d_n)|$ and call $A$, $B$ the complete intersection of $D$ with general hypersurfaces of degree $d_1,\dots,d_{n-1}$. Since these hypersurfaces are general, we have $A,B\in W(n;d_1,\dots,d_n)$ and $A\cap B=\emptyset$. Since $d\ge d_1+\cdots+d_n-n$, we have $\dim\langle\nu_{d,n}(B)\rangle=\dim\langle\nu_{d,n}(A)\rangle=\deg(A)-1$, i.e. $h^1(\mathcal{I}_A(d))=h^1(\mathcal{I}_B(d))=0$. To prove that Theorem 0.2 cannot be extended to the data $d,d_1,\dots,d_n$ it is sufficient to find $A$, $B$ such that $\langle\nu_{d,n}(A)\rangle\cap\langle\nu_{d,n}(B)\rangle\ne\emptyset$, i.e., (since $A\cap B=\emptyset$ and $h^1(\mathcal{I}_A(d))=h^1(\mathcal{I}_B(d))=0$) it is sufficient to find $A$, $B$ such that $h^1(\mathcal{I}_{A\cup B}(d))\ne 0$. Since $A\cup B\subset D$, we have the residual exact sequence of $D$ in $\mathbb{P}^n$:

$$0\to\mathcal{O}_{\mathbb{P}^n}(d-d_n)\to\mathcal{I}_{A\cup B}(d)\to\mathcal{I}_{A\cup B,D}(d)\to 0 \tag{3}$$

Since $d-d_n\ge 0$, we have $h^1(\mathcal{O}_{\mathcal{O}_{\mathbb{P}^n}}(d-d_n))=h^2(\mathcal{O}_{\mathcal{O}_{\mathbb{P}^n}}(d-d_n))=0$. Thus by (3) it is sufficient to find $A$, $B$ such that $h^1(D,\mathcal{I}_{A\cup B,D}(d))\ne 0$. Take $n=2$, $d_1=2$, $D$ smooth and $d=d_1+2d_2-3=2d_2-1$. We have $D\cong\mathbb{P}^1$ and $\deg(\mathcal{O}_D(d))=4d_2-2$. Thus, $h^0(\mathcal{O}_D(d))=4d_2-1$. Since $\deg(A\cup B)=4d_2$, we have $h^1(D,\mathcal{I}_{A\cup B,D}(d))\ne 0$.

## 2 Proof of Theorem 0.5

We are only able to do the case $d_n=d_1$. We set $b:=d_1$. Thus $b$ is a positive integer and (taking a minimal $n$) we may assume $b\ge 2$. We also assume $n\ge 2$, because Remark 0.4 covers the case $n=1$.

For any positive integer $k$ let $\sigma_k(\nu_{d,n}(\mathbb{P}^n))\subseteq\mathbb{P}^r$ denote the $k$-secant variety of the Veronese variety $\nu_{d,n}(\mathbb{P}^n)$, i.e., the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle\nu_{d,n}(S)\rangle$ with $S$ a finite subset of $\mathbb{P}^n$ with cardinality $k$. All integers $\dim\sigma_k(\nu_{d,n}(\mathbb{P}^n))$ are known by the Alexander–Hirschowitz theorem ([2,6]).

*Remark 2.1* For any $Z\in W(n;b,\dots,b)$, we have $\deg(Z)=b^n$ and $h^0(\mathcal{I}_Z(b))=n$, i.e., $h^1(\mathcal{I}_Z(b))=b^n+n-\binom{n+b}{n}$. Since $Z$ is a finite set, there is $Z'\subset Z$ such that $|Z'|=\binom{n+b}{n}-n$, $h^0(\mathcal{I}_{Z'}(b))=n$ and $h^1(\mathcal{I}_{Z'}(b))=0$. Let $S\subset\mathbb{P}^n$ be a general set with $|S|=\binom{n+b}{n}-n$. Since $S$ is general, we have $h^0(\mathcal{I}_S(b))=n$, i.e. $h^1(\mathcal{I}_S(b))=0$. Let $\mathcal{E}$ be the scheme-theoretic base locus of $|\mathcal{I}_S(b)|$. The case of $Z$ just discussed shows that $\mathcal{E}$ is a finite set with cardinality $b^n$.

For any $q \in \mathbb{P}^n$ let $2q$ denote the first infinitesimal neighborhood of $q$ in $\mathbb{P}^n$, i.e., the closed subscheme of $\mathbb{P}^n$ with $(\mathcal{I}_q)^2$ as its ideal sheaf. The scheme $2q$ is a zero-dimensional scheme with $\deg(2q) = n+1$ and $(2q)_{\mathrm{red}} = \{q\}$.

**Proposition 2.2** *Take $d \ge b \ge 2$ and $n \ge 2$. Set $a := \binom{n+b}{n} - n$. Take a general $S \subset \mathbb{P}^n$ such that $|S| = a$. Let $S \cup A$ with $A \cap S = \emptyset$ and $|A| = b^n + n - \binom{n+b}{n}$ be the scheme-theoretic base locus of $|\mathcal{I}_S(b)|$ (Remark 2.1). Set $E := \cup_{q\in S} 2q$ and $F := A \cup E$. Then, $\dim \eta(n, d; b, \dots, b) \ge \dim\langle \nu_{d,n}(F)\rangle$.*

*Proof* Set $a := \binom{n+b}{n} - n$. Fix a general $q \in \check{\eta}(n, d; b, \dots, b)$ and take $Z \in W(n; b, \dots, b)$ such that $q \in \langle \nu_{d,n}(Z)\rangle$. By the generality of $q$, we may assume that $Z$ is a general element of $W(n; b, \dots, b)$ and that $q$ is a general element of $\langle \nu_{d,n}(Z)\rangle$. Take $S \subset Z$ such that $|S| = a$ and $h^1(\mathcal{I}_S(b)) = 0$ (Remark 2.1). Set $A := Z \setminus S$. Take a maximal $A' \subseteq A$ such that $\nu_{d,n}(S \cup A')$ is linearly independent, i.e., a minimal $A' \subseteq A$ such that $\langle \nu_{d,n}(S \cup A')\rangle = \langle \nu_{d,n}(Z)\rangle$. Take an ordering of the points of $S$ and then an ordering of the points of $A$ with the points of $A'$ coming first. Call $q_1, \dots, q_{|Z|}$ the points of $Z$ in this order. For $i \in \{1, \dots, |Z|\}$ take regular systems of parameters $z_{ij}$, $1 \le i \le |Z|$, $1 \le j \le n$, of the local ring $\mathcal{O}_{\mathbb{P}^n, q_i}$. Set $m := |S \cup A'|$. Instead of $\mathbb{I}(n, d; d_1, \dots, d_n) \subseteq W(n; d_1, \dots, d_n) \times \mathbb{P}^r$, we consider the map $u : (\mathbb{P}^n)^a \times G(m, r+1) \to \eta(n, d; b, \dots, b)$ defined in a neighborhood of $(q_1, \dots, q_a, \langle \nu_{d,n}(Z)\rangle)$ identifying the points in this neighborhood with the points $(q_1(\lambda), \dots, q_a(\lambda), \langle \nu_{d,n}(\cup_{i=1}^{a+|A'|} q_i(\lambda))\rangle)$ with $q_i(\lambda)$ varying near $q_i$ (essentially, we use the ordering of the points of $Z$ and use the product $(\mathbb{P}^n)^a$ instead of the symmetric product of $\mathbb{P}^n$). It is sufficient to prove that the Jacobian matrix $M$ of $u$ at $(q_1, \dots, q_a, \langle \nu_{d,n}(Z)\rangle)$ has rank at least $\dim\langle \nu_{d,n}(F)\rangle$.

Since $q$ is general in $\langle \nu_{d,n}(S \cup A')\rangle$ and $\nu_{d,n}(S \cup A')$ is a linearly independent set, there is a unique $o \in \langle \nu_{d,n}(S)\rangle$ such that $q \in \langle \{o\} \cup \langle \nu_{n,d}(A')\rangle$. By Terracini's lemma ([1, Corollary 1.10]), the top $na \times na$ principal minor of the Jacobian matrix $M$ of $u$ has rank $na - h^1(\mathcal{I}_E(d))$. The restriction of $u$ to $\langle \nu_{d,n}(Z)\rangle$ is essentially the identity matrix (or use that the Zariski tangent space of $\check{\eta}(n, d; b, \dots, b)$ at $q$ contains every linear subspace contained in $\eta(n, d; b, \dots, b)$ and containing $q$ and in particular it contains $\langle \nu_{d,n}(A')\rangle$). Thus, the first $na + |A'|$ columns of $M$ have rank $\dim\langle \nu_{n,d}(E \cup A')\rangle$. Since, $\langle \nu_{d,n}(S \cup A')\rangle = \langle \nu_{d,n}(S \cup A)\rangle$ and $E \supset S$, we have $\langle \nu_{d,n}(E \cup A')\rangle = \langle \nu_{d,n}(E \cup A)\rangle$. □

**Lemma 2.3** *Assume* $\mathrm{char}(\mathbb{K}) \ne 2$. *Fix a general $Z \in W(n, d; b, \dots, b)$. Then, $h^1(\mathcal{I}_A(b)) = 0$ for all $A \subset Z$ such that $|A| = \binom{n+d}{n} - n$.*

*Proof* Since $Z$ is general, the complete intersection of $n-1$ different elements of $|\mathcal{I}_A(b)|$ is an integral curve, $C$. It is sufficient to prove the lemma for a general effective divisor $Z'$ of $C$, which is the complete intersection of $C$ and a degree $b$ hypersurface. Call $V \subseteq H^0(\mathcal{O}_C(b))$ the image of the restriction map. Since $\mathcal{O}_{\mathbb{P}^n}(b)$ is very ample, in characteristic 0 it is easy to see that the monodromy group of the embedding $j$ of $C$ induced by $V$ is the full symmetric group. In characteristic $\ne 2$ we use [16, Corollary 2.2] and that $b \ge 2$ to see the reflexivity of the curve $j(C)$. □

*Proof of Theorem 0.5* Fix a general $Z \in W(2; b, b)$, take $S \subset Z$ with $|S| = \binom{b+2}{2} - 2$ and with $h^1(\mathcal{I}_S(b)) = 0$ and set $A := Z \setminus S$, $E := \cup_{q\in S} 2q$ and $F := A \cup E$. By proposition 2.2, it is sufficient to prove that $h^1(\mathcal{I}_F(d)) = 0$. Take a general $C \in |\mathcal{I}_Z(b)|$. Since $Z$ is general, $C$ is irreducible (take the complete intersection of two general members of $|\mathcal{O}_{\mathbb{P}^2}(b)|$). We have $\mathrm{Res}_C(F) = S$. Since $Z$ is general, we may assume that $S$ is a general subset of $C$ with cardinality $\binom{b+2}{2} - 2$. Since $d \ge 2b$, we have $h^0(\mathcal{O}_C(d-b)) \ge h^0(\mathcal{O}_C(b)) = \binom{b+2}{2} > |S|$ and $S$ is general in $C$, we have $h^1(C, \mathcal{I}_{S,C}(d-b)) = 0$. Since the restriction map $H^0(\mathcal{O}_{\mathbb{P}^2}(d-b)) \to H^0(\mathcal{O}_C(d-b))$ is surjective, we get $h^1(\mathcal{I}_S(d-b)) = 0$. By the residual exact sequence

$$0 \to \mathcal{I}_S(d-b) \to \mathcal{I}_F \to \mathcal{I}_{F\cap C,C}(d) \to 0$$

it is sufficient to prove that $h^1(C, \mathcal{I}_{F\cap C,C}(d)) = 0$. Since $d \ge 2b$, it is sufficient to prove that $h^1(C, \mathcal{I}_{F\cap C,C}(2b)) = 0$. Let $G$ be the degree $2b^2$ divisors $2Z$ of $C$ (each point of $Z$ counts with multiplicity 2). Since $Z \in |\mathcal{O}_C(b)|$, we have $G \in |\mathcal{O}_C(2b)|$. Thus, $h^0(C, \mathcal{I}_{G,C}(2b)) = 1$. Recall that $h^0(\mathcal{O}_C(2b)) = \binom{2b+2}{2} - \binom{b+2}{2} = (3b^2 + 3b)/2$. Since $Z \in |\mathcal{O}_C(b)|$, we have $\mathcal{I}_{Z,C}(2b) \cong \mathcal{O}_C(b)$. Since $\omega_C \cong \mathcal{O}_C(b-3)$, we have $h^1(C, \mathcal{I}_{Z,C}(2b)) = 0$, i.e. $h^0(C, \mathcal{I}_{Z,C}(2b)) = (b^2 + 3b)/2$. Since $h^0(C, \mathcal{I}_{G,C}(2b)) = 1$, and $\binom{b+2}{2} - 2 = (b^2 + 3b - 2)/2$, there is a divisor $G'$ with $Z \subset G' \subset F$ with $\deg(G') = \binom{b+2}{2} - 2 + \deg(Z)$ (i.e., in $G'$ exactly $\binom{b+2}{2} - 2$ of the points of $Z$ appear with multiplicity 2) and $h^1(C, \mathcal{I}_{G',C}(2b)) = 0$. Lemma 2.3 says that we may take $G'$ as $F$. □

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