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On a class of stretch metrics in Finsler Geometry

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Abstract The class of stretch metrics contains the class of Landsberg metrics and the class of R-quadratic metrics. In this paper, we show that a regular non-Randers type (α, β)-metric with vanishing S-curvature is stretchian if and only if it is Berwaldian. Let F be an almost regular non-Randers type (α, β)-metric. Suppose that F is not a Berwald metric. Then, we find a family of stretch (α, β)-metrics which is not Landsbergian. By presenting an example, we show that the mentioned facts do not hold for the Randers-type metrics. It follows that every regular (α, β)-metric with isotropic S-curvature is R-quadratic if and only if it is a Berwald metric.

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الملخص

تحتوي فئة قياسات التمدد على فئة قياسات لاندسبيرج وفئة القياسات R-تربيعية. في هذه المقالة نبيّن أن (α,β)-قياس منتظم من نوع غير-راندورس ذات S-انحناء منعدم تكون تمددية إذا وفقط إذا كانت بروالدية. لتكن F (α,β)-قياسا منتظما غير-راندورس تقريبا. لنفرض أن F ليس قياساً بروالدياً. إذن سنجد عائلة من (α,β)-قياسات تمدد والتي ليست لاندسبيرجية. وبتقديم مثال، نبيّن أن الحقائق المذكورة لا تتوفر في القياسات من نوع راندورس، ومنه يتبع أن أي (α,β)-قياس منتظم ذا S-انحناء متماثل يكون R-تربيعيا إذا وفقط إذا كان قياساً بروالدياً.

1 Introduction

It is a long-existing open problem in Finsler geometry to finding *unicorns*, i.e., Landsberg metrics which are not Berwaldian [2,21]. In [1], Asanov found a special family of unicorns in the class of non-regular (α, β)-metrics. In [13], Shen proved that unicorn does not exist in the class of regular (α, β)-metrics. He found a more complicated family of unicorns in the class of non-regular (α, β)-metrics which contains the Asanov's metrics. Let us explain some details about the obtained unicorns. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an almost

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regular (α, β)-metric on a manifold M defined:

$$\phi(s) = \exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}}\, dt\right], \tag{1}$$

where $q > 0$ and k are real constants. Suppose that β satisfies

$$r_{ij} = c(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0, \tag{2}$$

where $c = c(x)$ is a scalar function on M. If $c \neq 0$, then F is a Landsberg metric which is not a Berwald metric. In this case, F is a unicorn [16]. If $c = 0$, then F reduces to a Berwald metric. If $k = 0$ and $c \neq 0$, then we get the family of unicorns obtained by Asanov in [1].

The class of Finsler metrics (1) appeared in other studies of almost regular (α, β)-metrics which are not related to unicorns. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an almost regular non-Berwaldian (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is not a Finsler metric of Randers-type. In [20], it is proved that F is a generalized Douglas-Weyl metric with vanishing S-curvature if and only if ϕ is given by (1).

Let (M, F) be a Finsler manifold. The third-order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear form $\mathbf{C}_y$ on T_xM which is called Cartan torsion. The rate of change of Cartan torsion $\mathbf{C}$ along geodesics is called the Landsberg curvature $\mathbf{L}$. A Finsler metric satisfies $\mathbf{L} = 0$ is called a Landsberg metric. As a generalization of Landsberg curvature, Berwald introduced a non-Riemannian curvature so-called stretch curvature and denoted it by $\mathbf{\Sigma}_y$ [3]. F is said to be stretch metric if $\mathbf{\Sigma} = 0$. From the geometric point of view, it is proved that a stretch curvature vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram [9]. This curvature has been investigated by Matsumoto and Shibata in [7,8,15].

In order to find explicit examples of stretch metrics, we consider the class of (α, β)-metrics. An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. For example, $\phi = c_1\sqrt{1 + c_2 s^2} + c_3 s$ is called a Randers type metric, where $c_1 > 0$, c_2, and c_3 are real constants [10,19]. In [13], Shen proved that every regular Landsberg (α, β)-metric is a Berwald metric. Every Landsberg metric is a stretch metric. Then, it is natural to study the class of stretch (α, β)-metrics. In this paper, we characterize the stretch (α, β)-metrics with vanishing S-curvature.

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β)-metric with vanishing S-curvature on a manifold M of dimension $n \geq 3$. Suppose that F is a stretch metric. Then one of the following holds:*

(i) *If F is a regular metric, then it reduces to a Berwald metric;*
(ii) *If F is an almost regular metric which is not Berwaldian, then ϕ is given by*

$$\phi = c\exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} dt\right], \tag{3}$$

where $c > 0$, $q > 0$ and k are real constants. In this case, F is not a Landsberg metric.

The condition of vanishing of S-curvature in Theorem 1.1 can not be dropped—See the following:

Example 1.2 Let us consider the following Finsler metric on the unit ball $\mathbb{B}^n$

$$F := \frac{\left(\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\right)^2}{(1 - |x|^2)^2\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}},$$

where $|.|$ and $<, >$ denote the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively. F is a stretch metric that satisfies $\mathbf{S} \neq 0$ which is not Berwaldian.

In [22], it is proved that every Douglas-Randers metric (equivalently, Randers metric with closed 1-form) with vanishing stretch curvature is a Berwald metric. But Theorem 1.1 does not hold for Finsler metrics of Randers-type, generally as shown in the following example:

Example 1.3 Let us consider the well-known Shen's fish tank metric as follows. Let $X = (x, y, z) \in \mathbb{B}^3(1) \subset \mathbb{R}^3$ and $Y = (u, v, w) \in T_x\mathbb{B}^3(1)$. Put

$$F = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2} + \frac{xv - yu}{1 - x^2 - y^2}.$$

The Shen's fish tank metric F is a stretch metric with vanishing S-curvature while it is not a Berwald metric [14].

For a Finsler manifold (M, F), the Riemann curvature is a family of linear transformations $\mathbf{R}_y : T_xM \to T_xM$, where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y$, $\forall\lambda > 0$. F is said to be R-quadratic if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$ [11].

Corollary 1.4 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular non-Randers type (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is of isotropic S-curvature. Then F is an R-quadratic metric if and only if it is a Berwald metric.*

In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2 Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \otimes T_xM \to \mathbb{R}$ is called a fundamental tensor:

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y + ru + sv + tw)\Big]_{r=s=t=0},$$

where $u, v, w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion [18]. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

For $y \in T_xM_0$, define $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(q, u, v, w) := \Sigma_{ijkl}(y)q^iu^jv^kw^l$, where

$$\Sigma_{ijkl} := L_{ijk|l} - L_{ijl|k}. \tag{4}$$

The family $\mathbf{\Sigma} := \{\mathbf{\Sigma}_y\}_{y\in TM_0}$ is called the stretch curvature. F is called a stretch metric if $\mathbf{\Sigma} = 0$ [3]. By definition, every Landsberg metric is a stretch metric.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2G^i(x, y)$, $\lambda > 0$, and given by

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic of F. F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$. Equivalently, for a non-zero vector $y \in T_xM_0$, let us define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

B is called the Berwald curvature. Then F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$ (see [6]).

For a Finsler metric F on an n-dimensional manifold M, the Busemann–Hausdorff volume form $\mathrm{d}V_F = \sigma_F(x)\mathrm{d}x^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \ \Big|\ F\big(y^i \frac{\partial}{\partial x^i}|_x\big) < 1\Big\}},$$

where $\mathbb{B}^n(1)$ denotes the unit ball in $\mathbb{R}^n$. Let $G^i(x, y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. F is called of isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M. If F is a Berwald metric then $\mathbf{S} = 0$ (see [17]).

Given a Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$, a 1-form $\beta = b_i y^i$ on a manifold M, and a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o := \sup_{x\in M} \|\beta\|_x$, one can define a function on TM by $F := \alpha\phi(s)$, $s = \beta/\alpha$. If ϕ and b_o satisfy (5) and (6) below, then F is a Finsler metric on M. Finsler metrics in this form are called (α, β)-metrics. Let $b_o > 0$. Then, $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric on a manifold M for any pair $\{\alpha, \beta\}$ with $\sup_{x\in M} \|\beta\|_x \le b_o$ if and only if $\phi = \phi(s)$ satisfies the following conditions:

$$\phi(s) > 0, \qquad (|s| \le b_o) \tag{5}$$
$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b \le b_o). \tag{6}$$

Let $\phi = \phi(s)$ satisfy (5) and (6), where $|s| \le b \le b_o$. A function $F = \alpha\phi(s)$ is called an almost regular (α, β)-metric if β satisfies $||\beta_x||_\alpha \le b_0$, $\forall x \in M$ [13]. An almost regular (α, β)-metric $F = \alpha\phi(s)$ might be singular (even not defined) in the two extremal directions $y \in T_xM$ with $\beta(x, y) = \pm b_0\alpha(x, y)$.

For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, let us represent

$$r_{ij} := \tfrac{1}{2}\big(b_{i;j} + b_{j;i}\big), \ \ s_{ij} := \tfrac{1}{2}\big(b_{i;j} - b_{j;i}\big),$$
$$r_j := b^i r_{ij}, \quad r := b^i b^j r_{ij}, \quad s_j := b^i s_{ij}, \quad r_0 := r_j y^j, \ \ s_0 := s_j y^j,$$
$$r_{i0} := r_{ij}y^j, \ r_{00} := r_{ij}y^i y^j, \ s_{i0} := s_{ij}y^j, \ s^i{}_j := a^{im}s_{mj}, \ r^i{}_j := a^{im}r_{mj}, \ s^i{}_0 := s^i{}_j y^j.$$

Let $G^i = G^i(x, y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x, y)$ denote the coefficients of F and α, respectively, in the same coordinate system. Then, we have

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + (-2Q\alpha s_0 + r_{00})\left(\Theta\frac{y^i}{\alpha} + \Psi b^i\right), \tag{7}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \ \ \Delta := 1 + sQ + (b^2 - s^2)Q', \ \ \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}.$$

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''.$$

In [5], Cheng–Shen characterize (α, β)-metrics with isotropic S-curvature as follows:

Lemma 2.1 ([5]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α, β)-metric on a manifold of dimension $n \geq 3$ and $b := \|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S-curvature, $\mathbf{S} = (n + 1)cF$, if and only if β satisfies one of the following:*

(a)

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{8}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n + 1)k \frac{\phi\Delta^2}{b^2 - s^2}, \tag{9}$$

where k is a constant. In this case, $\mathbf{S} = (n + 1)cF$ with $c = k\varepsilon$.

(b)

$$r_{ij} = 0, \qquad s_j = 0. \tag{10}$$

In this case, $\mathbf{S} = 0$, regardless of choices of a particular ϕ.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For this aim, we need the following:

Lemma 3.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular non-Randers type (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has vanishing S-curvature. Then the following holds:*

$$y_i s^i{}_0 = 0, \quad y_i s^i{}_{0|0} = 0, \quad y_i b^j s^i{}_{j|0} = \phi(\phi - s\phi')s^j{}_0 s_{j0}, \tag{11}$$

where $y_i := g_{ij} y^j$.

Proof For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the following holds:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j, \tag{12}$$

where $\alpha_i := \alpha^{-1} a_{ij} y^j$ and

$$\rho := \phi(\phi - s\phi'), \quad \rho_0 := \phi\phi'' + \phi'\phi'$$
$$\rho_1 := -\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big], \quad \rho_2 := s\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big].$$

Then, we get

$$y_i := \rho\bar{y}_i + \rho_0 b_i\beta + \rho_1(b_i\alpha + \beta\alpha_i) + \rho_2\alpha\alpha_i, \tag{13}$$

where $\bar{y}_i := a_{ij} y^j$. By (10), we have $b_i s^i{}_0 = 0$. Since $\bar{y}_i s^i{}_0 = 0$, from (13) it follows that

$$y_i s^i{}_0 = 0. \tag{14}$$

By considering $y_{i|0} = 0$, from (14) we get

$$y_i s^i{}_{0|0} = 0.$$

Taking a horizontal derivation of $s_j = b^j s^i{}_j = 0$ implies that

$$0 = (b^j s^i{}_j)_{|0} = b^j{}_{|0} s^i{}_j + b^j s^i{}_{j|0} = (r^j{}_0 + s^j{}_0)s^i{}_j + b^j s^i{}_{j|0}. \tag{15}$$

Since $r^j{}_0 = 0$, (15) reduces to the following:

$$b^j s^i{}_{j|0} = -s^j{}_0 s^i{}_j. \tag{16}$$

By (13) and (16), we get

$$y_i b^j s^i{}_{j|0} = -(\rho + \rho_1 s + \rho_2) s^j{}_0 s^0{}_j = (\rho + \rho_1 s + \rho_2) s^j{}_0 s_{j0}. \tag{17}$$

Since $\rho_1 s + \rho_2 = 0$, we have

$$y_i b^j s^i{}_{j|0} = \rho s^j{}_0 s_{j0} = \phi(\phi - s\phi') s^j{}_0 s_{j0}. \tag{18}$$

This completes the proof. □

Proof of Theorem 1.1 Since F has vanishing S-curvature, by (10) we have $r_{ij} = s_j = 0$. Plugging these relations in (7) yields

$$G^i = G^i_\alpha + \alpha Q s^i{}_0. \tag{19}$$

Taking third-order vertical derivations of (19) with respect to y^j, y^k, and y^l implies that

$$\begin{aligned} B^i{}_{jkl} &= s^i{}_l\Big[Q\alpha_{jk} + Q_k\alpha_j + Q_j\alpha_k + \alpha Q_{jk}\Big] + s^i{}_j\Big[Q\alpha_{lk} + Q_k\alpha_l + Q_l\alpha_k + \alpha Q_{lk}\Big] \\ &\quad + s^i{}_k\Big[Q\alpha_{jl} + Q_j\alpha_l + Q_l\alpha_j + \alpha Q_{jl}\Big] + s^i{}_0\Big[\alpha_{jkl}Q + \alpha_{jk}Q_l + \alpha_{lk}Q_j \\ &\quad + \alpha_{lj}Q_k + \alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} + \alpha_k Q_{jl}\Big]. \end{aligned} \tag{20}$$

By assumption, F is stretch metric. Then contracting (4) with y^l yields

$$L_{jkl|s}y^s = -\frac{1}{2} y_i B^i{}_{jkl|s} y^s = 0.$$

By (20) we get

$$\begin{aligned} L_{jkl|s}y^s &= -\frac{1}{2} y_i s^i{}_{0|0}\Big[\alpha_{jkl}Q + \alpha_{jk}Q_l + \alpha_{lk}Q_j + \alpha_{lj}Q_k + \alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} \\ &\quad + \alpha_k Q_{jl}\Big] - \frac{1}{2} y_i s^i{}_0\Big[\alpha_{jkl}Q_{|0} + \alpha_{jk}Q_{l|0} + \alpha_{lk}Q_{j|0} + \alpha_{lj}Q_{k|0} + \alpha Q_{jkl|0} \\ &\quad + \alpha_l Q_{jk|0} + \alpha_j Q_{lk|0} + \alpha_k Q_{jl|0}\Big] - \frac{1}{2} y_i s^i{}_l\Big[Q_{|0}\alpha_{jk} + Q_{k|0}\alpha_j + Q_{j|0}\alpha_k \\ &\quad + \alpha Q_{jk|0}\Big] - \frac{1}{2} y_i s^i{}_j\Big[Q_{|0}\alpha_{lk} + Q_{k|0}\alpha_l + Q_{l|0}\alpha_k + \alpha Q_{lk|0}\Big] \\ &\quad - \frac{1}{2} y_i s^i{}_k\Big[Q_{|0}\alpha_{jl} + Q_{j|0}\alpha_l + Q_{l|0}\alpha_j + \alpha Q_{jl|0}\Big] \\ &\quad - \frac{1}{2} y_i s^i{}_{l|0}\Big[Q\alpha_{jk} + Q_k\alpha_j + Q_j\alpha_k + \alpha Q_{jk}\Big] \\ &\quad - \frac{1}{2} y_i s^i{}_{j|0}\Big[Q\alpha_{lk} + Q_k\alpha_l + Q_l\alpha_k + \alpha Q_{lk}\Big] \\ &\quad - \frac{1}{2} y_i s^i{}_{k|0}\Big[Q\alpha_{jl} + Q_j\alpha_l + Q_l\alpha_j + \alpha Q_{jl}\Big]. \end{aligned} \tag{21}$$

Contracting (21) with $b^j b^k b^l$ and using (11) implies that

$$\phi(\phi - s\phi')(Q\alpha_2 + 2\alpha_1 Q_1 + \alpha Q_2) s^m{}_0 s_{m0} = 0, \tag{22}$$

where

$$\alpha_1 := b^i \alpha_{y^i}, \quad \alpha_2 := b^i b^j \alpha_{y^i y^j}, \quad Q_1 := b^i Q_{y^i}, \quad Q_2 := b^i b^j Q_{y^i y^j}.$$

Since $\phi(\phi - s\phi') \neq 0$, (22) reduces to the following:

$$(Q\alpha_2 + 2\alpha_1 Q_1 + \alpha Q_2) s^m{}_0 s_{m0} = 0. \tag{23}$$

By (23), we have two main cases as follows:

Case (i): Let $s^m_{\ 0} s_{m0} = 0$. Since α is a positive-definite metric, it implies that $s^i_{\ j} = 0$. Therefore, β is a closed 1-form. Since F is a regular (α, β)-metric, (20) implies that F is a Berwald metric.

Case (ii): Let the following hold

$$2\alpha_1 Q_1 + \alpha Q_2 + Q\alpha_2 = 0. \tag{24}$$

We have

$$\alpha_{y^i} = \alpha^{-1} y_i, \quad \alpha_{y^j y^k} = \alpha^{-3} A_{jk},$$

where

$$A_{jk} := \alpha^2 a_{jk} - y_j y_k.$$

So we get

$$\alpha_1 = s, \tag{25}$$
$$\alpha_2 = (b^2 - s^2)\alpha^{-1}, \tag{26}$$
$$Q_1 = Q'(b^2 - s^2)\alpha^{-1}, \tag{27}$$
$$Q_2 = (b^2 - s^2)\big[(b^2 - s^2)Q'' - 3sQ'\big]\alpha^{-2}. \tag{28}$$

Plugging (25), (26), (27), and (28) into (24) implies that

$$(b^2 - s^2)Q'' - sQ' + Q = 0. \tag{29}$$

By solving (29), we get

$$Q = ks + q\sqrt{b^2 - s^2},$$

where k and q are real constants. It results the following:

$$\phi = c \exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} dt\right], \tag{30}$$

where $c > 0$, $q > 0$, and k are real constants. (30) is an almost regular (α, β)-metric (for more details, see [13]). Since $s_{ij} \neq 0$, by Theorem 1.2 in [13], it follows that F is not a Landsberg metric. □

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a non-zero vector $y \in T_x M_0$, the Riemann curvature $\mathbf{R}_y : T_x M \to T_x M$ is defined by $\mathbf{R}_y(u) := R^i_{\ k}(y) u^k \frac{\partial}{\partial x^i}$, where

$$R^i_{\ k}(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature.

In [4], Cheng considers regular (α, β)-metrics with isotropic S-curvature and proves the following:

Theorem 3.2 ([4]) *A regular (α, β)-metric $F := \alpha\phi(\beta/\alpha)$, of non-Randers type on an n-dimensional manifold M is of isotropic S-curvature, $\mathbf{S} = (n+1)\sigma F$, if and only if β satisfies $r_{ij} = 0$ and $s_j = 0$. In this case, $\mathbf{S} = 0$, regardless of the choice of a particular $\phi = \phi(s)$.*

Proof of Corollary 1.4 The following Bianchi identity holds:

$$B^i_{\ jkl|m} - B^i_{\ jmk|l} = R^i_{\ jml,k}, \tag{31}$$

where

$$R^i_{\ jkl} = \frac{1}{3}\left\{\frac{\partial^2 R^i_{\ k}}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_{\ l}}{\partial y^j \partial y^k}\right\}.$$

For more details, see page 136 in [12]. Multiplying (31) with y_i implies that

$$\begin{aligned} y_i R^i{}_{jkl,m} &= y_i B^i{}_{jml|k} - y_i B^i{}_{jkm|l} \\ &= -2L_{jml|k} + 2L_{jkm|l} \\ &= 2\Sigma_{jkml}. \end{aligned} \tag{32}$$

By (32), it follows that every R-quadratic metric is a stretch metric. Then by Theorems 1.1 and 3.2, the proof follows. □

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