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Congruences modulo 8 for $(2,\ k)$-regular overpartitions for odd $k > 1$

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Abstract In this paper, we study various arithmetic properties of the function $\overline{p}_{2,\ k}(n)$, which denotes the number of $(2,\ k)$-regular overpartitions of n with odd $k > 1$. We prove several infinite families of congruences modulo 8 for $\overline{p}_{2,\ k}(n)$. For example, we find that for all non-negative integers β, n and $k \equiv 1 \pmod 8$, $\overline{p}_{2,\ k}(2^{1+\beta}(16n+14)) \equiv 0 \pmod 8$.

الملخص

في هذه المقالة ندرس عدة خصائص حسابية للدالة $\bar{p}_{2,k}(n)$ التي تشير للعدد $(2,k)$- منتظم على تجزيئات n بفردي $k > 1$. نثبت عدة أسر لا نهائية من التطابقات قياس 8 للقيمة $\bar{p}_{2,k}(n)$. فمثلا، نجد أن لكل الأعداد الصحيحة غير السالبة β, n و $k \equiv 1 \pmod 8$ ، $\bar{p}_{2,k}(2^{1+\beta}(16n+14)) \equiv 0 \pmod 8$

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1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A partition of an integer $n \in \mathbb{N}$ is a non-increasing sequence of positive integers that sum to n. For a positive integer $\ell > 1$, a partition is called ℓ-regular if none of the parts is divisible by ℓ. For example, the 3-regular partitions of 5 are

$$5, 4+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

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An overpartition of $n \in \mathbb{N}_0$ is a partition of n in which the first occurrence of a number may be overlined. For example, the overpartitions of 3 are

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

The generating function for $\overline{p}(n)$, the number of overpartitions of n with $\overline{p}(0)=1$ is given by [5]

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + \cdots. \tag{1.1}$$

An extensive study of overpartitions can be found in the work of Corteel and Lovejoy [5].

Let $\overline{po}(n)$ denote the number of overpartitions of n into odd parts. The generating function for $\overline{po}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{po}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^{2n+1})}{(1-q^{2n-1})} = 1 + 2q + 2q^2 + 4q^3 + \cdots. \tag{1.2}$$

Many mathematicians have extensively studied the arithmetic properties of $\overline{po}(n)$ and they have also established several Ramanujan-type congruences satisfied by $\overline{po}(n)$ (for example, one can see [4,7]).

Let $\overline{A}_\ell(n)$ denote the number of ℓ-regular overpartitions of n. The generating function for $\overline{A}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{A}_\ell(n)q^n = \frac{f_2}{f_1^2}\frac{f_\ell^2}{f_{2\ell}}, \quad \ell > 1, \tag{1.3}$$

where

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots, \quad |q| < 1$$

and for $m \in \mathbb{N}$

$$f_m := (q^m; q^m)_\infty.$$

This function was introduced and investigated by Lovejoy [9]. Later, Shen [13] discovered several Ramanujan-like congruences for $\overline{A}_3(n)$ and $\overline{A}_4(n)$. Since then, a number of congruence properties for various ℓ-regular overpartition functions have been proved. (For example, one can see [2,3,10,12].) Very recently, the arithmetic properties of ℓ-regular overpartition pairs have been studied in [11].

Definition 1.1 A partition of n is said to be a $(2,k)$-regular overpartition of n, if it is both 2 and k-regular overpartition of n.

Motivated by the above works, in this paper we prove infinite families of congruences modulo 8 for $\overline{p}_{2,\,k}(n)$ which enumerate the number of $(2,\ k)$-regular overpartitions of n, for infinitely many values of odd $k > 1$. For example, we prove the following theorems:

Theorem 1.2 *If $n, \alpha \in \mathbb{N}_0$ and $k \equiv 1 \pmod 4$, then*

$$\overline{p}_{2,\,k}(2^{2+\alpha}n) \equiv \overline{p}_{2,\,k}(2^2 n) \pmod 8,$$
$$\overline{p}_{2,\,k}(2^{1+\alpha}(4n+2)) \equiv 7\overline{p}_{2,\,k}(4n+2) \pmod 8.$$

Theorem 1.3 *Let $p \equiv 5, 7 \pmod 8$ and $k \equiv 5 \pmod 8$ such that $\left(\frac{k}{p}\right) = 1$. Then for all $n, \beta \in \mathbb{N}_0$, we have*

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 6p^{2\beta+2}\right) \equiv 0 \pmod 8.$$

It is easy to see that the generating function for $\overline{p}_{2,\,k}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(n)q^n = \frac{(-q;q^2)_\infty (q^k;q^{2k})_\infty}{(q;q^2)_\infty(-q^k;q^{2k})_\infty} = \frac{f_2^3 f_k^2 f_{4k}}{f_1^2 f_4 f_{2k}^3}. \tag{1.4}$$

For example, setting $k = 3$ in (1.4), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,3}(n)q^n = \frac{(-q;q^2)_\infty (q^3;q^6)_\infty}{(q;q^2)_\infty (-q^3;q^6)_\infty} = 1 + 2q + 2q^2 + 2q^3 + 2q^4 + 4q^5 + 6q^6 + \cdots .$$

The $(2,\ 3)$-regular overpartitions of the integer 6 are

$$5+1,\overline{5}+1,5+\overline{1},\overline{5}+\overline{1},1+1+1+1+1+1,\overline{1}+1+1+1+1+1.$$

We note that work of Lin [8] on overpartition pairs into odd parts gives numerous congruences for $\overline{p}_{2,3}(n)$ modulo 3.

2 Set of preliminary results

In order to prove the main congruences of this paper, we collect some dissection formulas in this section.

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined by [1]

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

The following lemma is a consequence of Entry 25 of (v) and (vi) in [1, pp. 35–36].

Lemma 2.1 *The following* 2*-dissection formulas are true:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8} \tag{2.1}$$

and

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.2}$$

Ranganatha [12] established the p-dissection formula for $\frac{f_1^5}{f_2^2}$ which can be stated as follows:

Lemma 2.2 [12, Theorem 3.2] *If $p \geq 5$ is a prime and*

$$\frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6, \end{cases}$$

then

$$\frac{f_1^5}{f_2^2} = \sum_{\substack{i=-\frac{p-1}{2} \\ i \not\equiv \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{3i^2+i}{2}} \sum_{n=-\infty}^{\infty} (6pn + 6i + 1) q^{\frac{pn(3pn+6i+1)}{2}} \pm pq^{\frac{p^2-1}{24}} \frac{f_{p^2}^5}{f_{2p^2}^2}.$$

Furthermore, if $-\frac{p-1}{2} \leq i \leq \frac{p-1}{2}$, $i \neq \frac{\pm p-1}{6}$, we have $\frac{3i^2+i}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

From [6], we recall the following p-dissection formula for $\frac{f_2^2}{f_1}$:

Lemma 2.3 [6, Theorem 2.1] *For any odd prime p, we have*

$$\frac{f_2^2}{f_1} = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \frac{f_{2p^2}^2}{f_{p^2}}. \tag{2.3}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ for $0 \leq m \leq \frac{p-3}{2}$.

3 Congruences modulo 8 for $\overline{p}_{2,\,k}(n)$

For notational convenience, in this section, we assume that all congruences are modulo 8, $k \in \mathbb{N}$ is odd and p is a prime, unless stated otherwise. We first establish the following generating function for $\overline{p}_{2,\,k}(2n)$ modulo 8.

Lemma 3.1 *For any odd $k > 1 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(2n)q^n \equiv \frac{f_4 f_{4k} f_8^3 f_{8k}^3 f_2^2 f_{2k}^2}{f_{16}^2 f_{16k}^2} + 2q^k \frac{f_{4k} f_8 f_{8k}^3}{f_4 f_{2k}^2 f_2^2} + 2q \frac{f_4 f_{8k} f_8^3}{f_{4k} f_2^2 f_{2k}^2} + 4q^{k+1} f_8^3 f_{8k}^3 + 4q^{\frac{k+1}{2}} f_4^3 f_{4k}^3 \pmod 8. \tag{3.1}$$

Proof Substituting (2.1) and (2.2) in (1.4), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(n)q^n &= \frac{f_2^3 f_{4k}}{f_4 f_{2k}^3}\left(\frac{f_{2k} f_{8k}^5}{f_{4k}^2 f_{16k}^2} - 2q^k \frac{f_{2k} f_{16k}^2}{f_{8k}}\right)\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)\\ &= \frac{f_8^5 f_{8k}^5}{f_2^2 f_{4k} f_4 f_{2k}^2 f_{16}^2 f_{16k}^2} - 2q^k \frac{f_{4k} f_8^5 f_{16k}^2}{f_2^2 f_4 f_{2k}^2 f_{16}^2 f_{8k}} + 2q \frac{f_4 f_{16}^2 f_{8k}^5}{f_2^2 f_{4k} f_{2k}^2 f_8 f_{16k}^2}\\ &\quad - 4q^{k+1} \frac{f_{4k} f_4 f_{16}^2 f_{16k}^2}{f_2^2 f_{2k}^2 f_8 f_{8k}}.\end{aligned} \tag{3.2}$$

Extracting the terms involving even powers of q in the above identity where $k > 1 \in \mathbb{N}$ is odd, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(2n)q^n = \frac{f_4^5 f_{4k}^5}{f_{2k} f_2 f_8^2 f_{8k}^2}\frac{1}{f_1^2 f_k^2} - 4q^{(k+1)/2}\frac{f_{2k} f_2 f_8^2 f_{8k}^2}{f_4 f_{4k}}\frac{1}{f_1^2 f_k^2}. \tag{3.3}$$

In view of (2.2), we have for any $s \in \mathbb{N}$

$$\frac{1}{f_1^2 f_s^2} = \frac{f_8^5 f_{8s}^5}{f_2^5 f_{16}^2 f_{2s}^5 f_{16s}^2} + 2q^s \frac{f_8^5 f_{4s}^2 f_{16s}^2}{f_2^5 f_{16}^2 f_{2s}^5 f_{8s}} + 2q \frac{f_4^2 f_{16}^2 f_{8s}^5}{f_2^5 f_8 f_{2s}^5 f_{16s}^2} + 4q^{s+1} \frac{f_4^2 f_{16}^2 f_{4s}^2 f_{16s}^2}{f_2^5 f_8 f_{2s}^5 f_{8s}}. \tag{3.4}$$

Using (3.4), we can rewrite (3.3) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(2n)q^n &\equiv \frac{f_4^5 f_{4k}^5 f_8^3 f_{8k}^3}{f_{2k}^6 f_2^6 f_{16}^2 f_{16k}^2} + 2q^k \frac{f_4^5 f_{4k}^7 f_8^3 f_{16k}^2}{f_{2k}^6 f_2^6 f_{8k}^3 f_{16}^2} + 2q \frac{f_4^7 f_{4k}^5 f_{8k}^3 f_{16}^2}{f_{2k}^6 f_2^6 f_8^3 f_{16k}^2}\\ &\quad + 4q^{k+1} \frac{f_4^7 f_{4k}^7 f_{16}^2 f_{16k}^2}{f_{2k}^6 f_2^6 f_8^3 f_{8k}^3} + 4q^{\frac{k+1}{2}} \frac{f_8^7 f_{8k}^7}{f_{2k}^4 f_2^4 f_4 f_{4k} f_{16}^2 f_{16k}^2}.\end{aligned} \tag{3.5}$$

It is easy to check that, for any $m, \ell \in \mathbb{N}$, we have

$$f_\ell^{2m} \equiv f_{2\ell}^m \pmod 2, \tag{3.6}$$

$$f_\ell^{4m} \equiv f_{2\ell}^{2m} \pmod 4, \tag{3.7}$$

$$f_\ell^{8m} \equiv f_{2\ell}^{4m} \pmod 8. \tag{3.8}$$

Employing (3.6)–(3.8) into (3.5), we obtain (3.1). □

Lemma 3.2 *For any odd $k > 1 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(2n+1)q^n \equiv 2\frac{f_8 f_4 f_{8k}}{f_{4k} f_{2k}^2} + 4q^k f_4^3 f_{8k}^3 + 4q f_4^9 + 6q^{\frac{k-1}{2}} \frac{f_{8k} f_{4k} f_8}{f_4 f_2^2} + 4q^{\frac{3k-1}{2}} f_{4k}^9 + 4q^{\frac{k+1}{2}} f_8^3 f_{4k}^3 \pmod 8. \tag{3.9}$$

Proof From (3.2) with odd $k > 1$, we have

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(2n+1)q^n = 2\frac{f_2 f_8^2 f_{4k}^5}{f_1^2 f_{2k} f_k^2 f_4 f_{8k}^2} - 2q^{\frac{k-1}{2}} \frac{f_{2k} f_4^5 f_{8k}^2}{f_1^2 f_2 f_k^2 f_8^2 f_{4k}}. \tag{3.10}$$

By (3.4), we find that

$$\begin{aligned}\left(\frac{f_2 f_8^2 f_{4k}^5}{f_{2k} f_4 f_{8k}^2} - q^{\frac{k-1}{2}} \frac{f_{2k} f_4^5 f_{8k}^2}{f_2 f_8^2 f_{4k}}\right)\frac{1}{f_1^2 f_k^2} &\equiv \frac{f_8^7 f_{4k}^5 f_{8k}^3}{f_2^4 f_{2k}^6 f_4 f_{16}^2 f_{16k}^2} + 2q^k \frac{f_8^7 f_{4k}^7 f_{16k}^2}{f_2^4 f_{2k}^6 f_4 f_{8k}^3 f_{16}^2} + 2q \frac{f_8 f_{4k}^5 f_4 f_{8k}^3 f_{16}^2}{f_2^4 f_{2k}^6 f_{16k}^2}\\ &\quad + 3q^{\frac{k-1}{2}} \frac{f_4^5 f_{8k}^7 f_8^3}{f_{2k}^4 f_2^6 f_{4k} f_{16}^2 f_{16k}^2} + 2q^{\frac{3k-1}{2}} \frac{f_4^5 f_{8k} f_8^3 f_{4k} f_{16k}^2}{f_{2k}^4 f_2^6 f_{16}^2}\\ &\quad + 2q^{\frac{k+1}{2}} \frac{f_4^7 f_{8k}^7 f_{16}^2}{f_{2k}^4 f_2^6 f_8^3 f_{4k} f_{16k}^2} \pmod 4.\end{aligned} \tag{3.11}$$

In view of (3.6), (3.7), (3.10) and (3.11), we obtain the required congruence. □

Theorem 3.3 *If $n, \alpha \in \mathbb{N}_0$ and $k \equiv 1 \pmod 4$, then*

$$\overline{p}_{2,\,k}(2^{2+\alpha}n) \equiv \overline{p}_{2,\,k}(2^2 n) \pmod 8, \tag{3.12}$$

$$\overline{p}_{2,\,k}(2^{1+\alpha}(4n+2)) \equiv 7\overline{p}_{2,\,k}(4n+2) \pmod 8. \tag{3.13}$$

Proof Extracting the odd and even powers of q in (3.1), where $k \equiv 1 \pmod 4$, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n)q^n \equiv \frac{f_2 f_{2k} f_4^3 f_{4k}^3}{f_8^2 f_{8k}^2} f_1^2 f_k^2 + 4q^{\frac{k+1}{2}} f_4^3 f_{4k}^3, \tag{3.14}$$

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+2)q^n \equiv 2q^{\frac{k-1}{2}} \frac{f_{2k} f_4 f_{4k}^3}{f_2 f_k^2 f_1^2} + 2\frac{f_2 f_{4k} f_4^3}{f_{2k} f_1^2 f_k^2} + 4q^{\frac{k-1}{4}} f_2^3 f_{2k}^3. \tag{3.15}$$

Applying (2.1), we deduce that

$$f_1^2 f_s^2 = \frac{f_{2s} f_{8s}^5 f_2 f_8^5}{f_{4s}^2 f_{16s}^2 f_4^2 f_{16}^2} - 2q \frac{f_{2s} f_{8s}^5 f_2 f_{16}^2}{f_{4s}^2 f_{16s}^2 f_8} - 2q^s \frac{f_{2s} f_{16s}^2 f_2 f_8^5}{f_{8s} f_4^2 f_{16}^2} + 4q^{s+1} \frac{f_{2s} f_{16s}^2 f_2 f_{16}^2}{f_{8s} f_8}. \tag{3.16}$$

Substituting (3.16) into (3.14) and then using (3.6)–(3.8) in the resulting congruence, we can rewrite (3.14) as

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n)q^n \equiv \frac{f_4 f_{4k} f_8^3 f_{8k}^3}{f_{16}^2 f_{16k}^2} f_2^2 f_{2k}^2 + 6q \frac{f_4 f_{8k} f_8^3}{f_{4k} f_2^2 f_{2k}^2} + 6q^k \frac{f_{4k} f_8 f_{8k}^3}{f_4 f_{2k}^2 f_2^2} + 4q^{k+1} f_8^3 f_{8k}^3 + 4q^{\frac{k+1}{2}} f_4^3 f_{4k}^3,$$

which yields

$$\overline{p}_{2,\,k}(8n) \equiv \overline{p}_{2,\,k}(4n),$$

$$\overline{p}_{2,\,k}(8n+4) \equiv 7\overline{p}_{2,\,k}(4n+2).$$

Congruences (3.12) and (3.13) follow from the above two congruences and by induction on $\alpha \geq 0$. □

Lemma 3.4 *For any $k \equiv 1 \pmod 8$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+6)q^n \equiv 4q^{\frac{3k-3}{4}} \frac{f_{16k}^5}{f_{32k}^2} \frac{f_{2k}^5}{f_{4k}^2} + 4q^{\frac{k-1}{4}} f_4^3 f_{2k}^3 + 4q^{\frac{k-1}{2}} f_4^3 f_{2k}^3 + 4\frac{f_{16}^5}{f_{32}^2} \frac{f_2^5}{f_4^2} \pmod 8. \tag{3.17}$$

Proof Substituting (3.4) into (3.15), we see that

$$\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(4n+2)q^n \equiv 2q^{\frac{k-1}{2}}\frac{f_4 f_{4k}^3 f_8^5 f_{8k}^5}{f_{2k}^4 f_2^6 f_{16}^2 f_{16k}^2} + 4q^{\frac{3k-1}{2}}\frac{f_4 f_{4k}^5 f_8^5 f_{16k}^2}{f_{2k}^4 f_2^6 f_{16}^2 f_{8k}} + 4q^{\frac{k+1}{2}}\frac{f_4^3 f_{4k}^3 f_{16}^2 f_{8k}^5}{f_{2k}^4 f_2^6 f_8 f_{16k}^2}$$
$$+ 4q^{\frac{k-1}{4}} f_2^3 f_{2k}^3 + 2\frac{f_{4k} f_4^3 f_8^5 f_{8k}^5}{f_2^4 f_{2k}^6 f_{16}^2 f_{16k}^2} + 4q^k \frac{f_{4k}^3 f_4^3 f_8^5 f_{16k}^2}{f_2^4 f_{2k}^6 f_{16}^2 f_{8k}}$$
$$+ 4q\frac{f_{4k} f_4^5 f_{16}^2 f_{8k}^5}{f_2^4 f_{2k}^6 f_8 f_{16k}^2}. \tag{3.18}$$

Extracting the odd powers of q in (3.18) for $k \equiv 1 \pmod 8$, we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,\,k}(8n+6)q^n \equiv 4q^{\frac{3k-3}{4}}\frac{f_2 f_{2k}^5 f_4^5 f_{8k}^2}{f_k^4 f_1^6 f_8^2 f_{4k}} + 4q^{\frac{k-1}{4}}\frac{f_2^3 f_{2k}^3 f_8^2 f_{4k}^5}{f_k^4 f_1^6 f_4 f_{8k}^2} + 4q^{\frac{k-1}{2}}\frac{f_{2k}^3 f_2^3 f_4^5 f_{8k}^2}{f_1^4 f_k^6 f_8^2 f_{4k}}$$
$$+ 4\frac{f_{2k} f_2^5 f_8^2 f_{4k}^5}{f_1^4 f_k^6 f_4 f_{8k}^2}. \tag{3.19}$$

Employing (3.6), modulo 2, we have

$$\frac{f_2 f_{2k}^5 f_4^5 f_{8k}^2}{f_k^4 f_1^6 f_8^2 f_{4k}} \equiv \frac{f_{16k}^5}{f_{32k}^2}\frac{f_{2k}^5}{f_{4k}^2}, \quad \frac{f_2^3 f_{2k}^3 f_8^2 f_{4k}^5}{f_k^4 f_1^6 f_4 f_{8k}^2} \equiv f_4^3 f_{2k}^3, \quad \frac{f_{2k}^3 f_2^3 f_4^5 f_{8k}^2}{f_1^4 f_k^6 f_8^2 f_{4k}} \equiv f_2^3 f_{4k}^3 \tag{3.20}$$

and

$$\frac{f_{2k} f_2^5 f_8^2 f_{4k}^5}{f_1^4 f_k^6 f_4 f_{8k}^2} \equiv \frac{f_{16}^5}{f_{32}^2}\frac{f_2^5}{f_4^2}. \tag{3.21}$$

Invoking (3.20) and (3.21) into (3.19), we obtain the required congruence. □

In order to state the congruence properties for $\overline{p}_{2,\,k}(n)$, we need the following definition: Let $p \geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo p and } p \nmid a, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo p and } p \nmid a, \\ 0 & \text{if } p \mid a. \end{cases}$$

Theorem 3.5 *Let $p \equiv 5, 7 \pmod 8$ and $k \equiv 1 \pmod 8$ such that $\left(\frac{k}{p}\right) = 1$. Then for all n and $\beta \in \mathbb{N}_0$, we have*

$$\overline{p}_{2,\,k}\Big(2^{1+\beta}(16n+14)\Big) \equiv 0 \pmod 8 \tag{3.22}$$

and for $1 \leq j \leq p-1$,

$$\overline{p}_{2,\,k}\Big(16p^{2\beta+2}n + 16p^{2\beta+1}j + 6p^{2\beta+2}\Big) \equiv 0 \pmod 8. \tag{3.23}$$

Proof If $k \equiv 1$, then terms appearing on the right side of (3.17) are powers of q^2 and thus equating the odd powers of q, we obtain

$$\overline{p}_{2,k}(16n+14) \equiv 0.$$

Congruence (3.22) follows from the above congruence and (3.13).

Next, we turn to prove (3.23). From (3.17), we have

$$\overline{p}_{2,k}(16n+6) \equiv 4a(n), \tag{3.24}$$

where $k \equiv 1$ and

$$\sum_{n=0}^{\infty} a(n)q^n = q^{\frac{3k-3}{8}} \frac{f_{8k}^5}{f_{16k}^2} \frac{f_k^5}{f_{2k}^2} + q^{\frac{k-1}{8}} f_2^3 f_k^3 + q^{\frac{k-1}{4}} f_1^3 f_{2k}^3 + \frac{f_8^5}{f_{16}^2} \frac{f_1^5}{f_2^2}. \tag{3.25}$$

Now, consider the following congruence relations:

$$4 \cdot (3m_1^2 + m_1) + \frac{3r_1^2 + r_1}{2} \equiv 3 \cdot \frac{p^2-1}{8} \pmod{p},$$
$$m_2^2 + m_2 + k \cdot \frac{r_2^2 + r_2}{2} \equiv (k+2) \cdot \frac{p^2-1}{8} \pmod{p},$$

where $-\frac{p-1}{2} \le m_1, r_1 \le \frac{p-1}{2}$, $0 \le m_2, r_2 \le \frac{p-3}{2}$. Because $\left(\frac{-2}{p}\right) = -1$, the first congruence holds if and only if $m_1 = r_1 = \frac{\pm p - 1}{6}$ and the second congruence holds if and only if $m_2 = r_2 = \frac{p-1}{2}$, since $\left(\frac{-2k}{p}\right) = -1$. Substituting Lemmas 2.2 and 2.3 into (3.25), extracting the terms involving $q^{pn+3\frac{p^2-1}{8}}$ in the resulting identity, canceling $q^{3\frac{p^2-1}{8}}$ on both sides and then replacing q^p by q, we deduce

$$\sum_{n=0}^{\infty} a\Big(pn + 3 \cdot \frac{p^2-1}{8}\Big)q^n = p^2 q^{p\frac{3k-3}{8}} \frac{f_{8kp}^5}{f_{16kp}^2} \frac{f_{kp}^5}{f_{2kp}^2} + q^{p\frac{k-1}{8}} f_{2p}^3 f_{kp}^3 + q^{p\frac{k-1}{4}} f_p^3 f_{2kp}^3 + p^2 \frac{f_{8p}^5}{f_{16p}^2} \frac{f_p^5}{f_{2p}^2}.$$

The terms appearing on the right side are powers of q^p and thus for $1 \le j \le p-1$, we have

$$a\Big(p^2 n + 3 \cdot \frac{p^2-1}{8}\Big) \equiv a(n) \pmod{2} \tag{3.26}$$

and

$$a\Big(p^2 n + pj + 3 \cdot \frac{p^2-1}{8}\Big) = 0. \tag{3.27}$$

From (3.26) and by induction on $\beta \in \mathbb{N}_0$, we find that for all $n \in \mathbb{N}_0$

$$a\Big(p^{2\beta} n + 3 \cdot \frac{p^{2\beta}-1}{8}\Big) \equiv a(n) \pmod{2}. \tag{3.28}$$

Replacing n by $p^2 n + pj + 3 \cdot \frac{p^2-1}{8}$ in (3.28) and then using (3.27), we obtain

$$a\Big(p^{2\beta+2} n + p^{2\beta+1} j + 3 \cdot \frac{p^{2\beta+2}-1}{8}\Big) \equiv 0 \pmod{2}. \tag{3.29}$$

Finally, replacing n by $p^{2\beta+2} n + p^{2\beta+1} j + 3 \cdot \frac{p^{2\beta+2}-1}{8}$ in (3.24) and then using (3.29), we obtain (3.23). This completes the proof. □

Theorem 3.6 *Let* $p \equiv 5, 7 \pmod{8}$ *and* $k \equiv 5 \pmod{8}$ *such that* $\left(\frac{k}{p}\right) = 1$. *Then for all* $n, \beta \in \mathbb{N}_0$, *we have*

$$\overline{p}_{2,\,k}\Big(8p^{2\beta+2} n + 8p^{2\beta+1} j + 6p^{2\beta+2}\Big) \equiv 0 \pmod{8}.$$

Proof Extracting the terms involving odd powers of q in (3.18) where $k \equiv 5$, we see that

$$\overline{p}_{2,\,k}(8n+6)q^n \equiv 4b(n), \tag{3.30}$$

where $b(n)$ is defined by

$$\sum_{n=0}^{\infty} b(n)q^n = q^{\frac{3k-3}{8}} \frac{f_{16k}^5}{f_{32k}^2} \frac{f_{2k}^5}{f_{4k}^2} + q^{\frac{k-1}{4}} f_4^3 f_{2k}^3 + q^{\frac{k-5}{8}} f_1^3 f_k^3 + q^{\frac{k-1}{2}} f_2^3 f_{4k}^3 + \frac{f_{16}^5}{f_{32}^2} \frac{f_2^5}{f_4^2}. \tag{3.31}$$

Now, consider the following congruence relations:

$$8\cdot(3m_1^2+m_1)+(3r_1^2+r_1)\equiv 3\cdot\frac{p^2-1}{4}\pmod p,$$
$$2\cdot(m_2^2+m_2)+k\cdot(r_2^2+r_2)\equiv (k+2)\cdot\frac{p^2-1}{4}\pmod p,$$
$$(m_2^2+m_2)+k\cdot(r_2^2+r_2)\equiv (k+1)\cdot\frac{p^2-1}{4}\pmod p,$$

where $-\frac{p-1}{2}\le m_1,r_1\le\frac{p-1}{2}$, $0\le m_2,r_2\le\frac{p-3}{2}$. Since $\left(\frac{-2}{p}\right)=-1$, the first congruence holds only if $m_1=r_1=\frac{\pm p-1}{6}$. The second and third congruences hold if and only if $m_2=r_2=\frac{p-1}{2}$, because $\left(\frac{-2k}{p}\right)=-1$. Substituting Lemmas 2.2 and 2.3 into (3.31), extracting the terms involving $q^{pn+3\cdot\frac{p^2-1}{4}}$, we obtain

$$\sum_{n=0}^{\infty}b\Big(pn+3\cdot\frac{p^2-1}{4}\Big)q^n=p^2q^{p\frac{3k-3}{8}}\frac{f_{16kp}^5}{f_{32kp}^2}\frac{f_{2kp}^5}{f_{4kp}^2}+q^{p\frac{k-1}{4}}f_{4p}^3f_{2kp}^3+q^{p\frac{k-5}{8}}f_p^3f_{kp}^3$$
$$+q^{p\frac{k-1}{2}}f_{2p}^3f_{4kp}^3+p^2\frac{f_{16p}^5}{f_{32p}^2}\frac{f_{2p}^5}{f_{4p}^2},\tag{3.32}$$

which implies that

$$b\Big(p^2n+3\cdot\frac{p^2-1}{4}\Big)\equiv b(n)\pmod 2\tag{3.33}$$

and

$$b\Big(p^2n+pj+3\cdot\frac{p^2-1}{4}\Big)=0,\tag{3.34}$$

where $1\le j\le p-1$. From (3.33) and by induction on β, we find that for $n\in\mathbb{N}_0$,

$$b\Big(p^{2\beta}n+3\cdot\frac{p^{2\beta}-1}{4}\Big)\equiv b(n)\pmod 2.\tag{3.35}$$

By (3.35) and (3.34), we find that

$$b\Big(p^{2\beta+2}n+p^{2\beta+1}j+3\cdot\frac{p^{2\beta+2}-1}{4}\Big)\equiv 0\pmod 2.\tag{3.36}$$

Finally, the congruence follows from (3.30) and (3.36). □

Theorem 3.7 *Let $p\ge 3$ and $k\equiv 3\pmod 8$ such that $\left(\frac{-k}{p}\right)=-1$. Then for $\alpha,\beta,n\in\mathbb{N}_0$ and $1\le j\le p-1$, we have*

$$\overline{p}_{2,\,k}(2^{2+\alpha}n)\equiv\overline{p}_{2,\,k}(2^2n)\pmod 8\tag{3.37}$$

and

$$\overline{p}_{2,\,k}\Big(2^{2+\alpha}\Big(2p^{2\beta+2}n+2p^{2\beta+1}j+p^{2\beta+2}\Big)\Big)\equiv 3\overline{p}_{2,k}\Big(4p^{2\beta+2}n+4p^{2\beta+1}j+2p^{2\beta+2}\Big)\pmod 8.\tag{3.38}$$

Proof Extracting the even and odd powers of q in (3.1), where $k \equiv 3 \pmod 4$, we find that

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n)q^n \equiv \frac{f_2 f_{2k} f_4^3 f_{4k}^3}{f_8^2 f_{8k}^2} f_1^2 f_k^2 + 4q^{\frac{k+1}{2}} f_4^3 f_{4k}^3 + 4q^{\frac{k+1}{4}} f_2^3 f_{2k}^3, \tag{3.39}$$

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+2)q^n \equiv 2q^{\frac{k-1}{2}} \frac{f_{2k} f_4 f_{4k}^3}{f_2 f_k^2 f_1^2} + 2\frac{f_2 f_{4k} f_4^3}{f_{2k} f_1^2 f_k^2}. \tag{3.40}$$

In view of (3.16), we can rewrite (3.39) as follows:

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n)q^n \equiv \frac{f_2^2 f_{2k}^2 f_4 f_{4k} f_{8k}^3 f_8^3}{f_{16k}^2 f_{16}^2} + 6q \frac{f_2^2 f_{2k}^2 f_4^3 f_{4k} f_{8k}^3 f_{16}^2}{f_8^3 f_{16k}^2} + 6q^k \frac{f_2^2 f_{2k}^2 f_4 f_{4k}^3 f_8^3 f_{16k}^2}{f_{8k}^3 f_{16}^2}$$
$$+ 4q^{k+1} \frac{f_2^2 f_{2k}^2 f_4^3 f_{4k}^3 f_{16k}^2 f_{16}^2}{f_{8k}^3 f_8^3} + 4q^{\frac{k+1}{2}} f_4^3 f_{4k}^3 + 4q^{\frac{k+1}{4}} f_2^3 f_{2k}^3. \tag{3.41}$$

Extracting the even and odd powers of q in (3.41) where $k \equiv 3$ and then using (3.39) and (3.40), we obtain

$$\overline{p}_{2,\,k}(8n) \equiv \overline{p}_{2,\,k}(4n), \tag{3.42}$$

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+4)q^n \equiv 3\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+2)q^n + 4q^{\frac{k-3}{8}} f_1^3 f_k^3. \tag{3.43}$$

Congruence (3.37) follows from (3.42) and by induction on $\alpha \in \mathbb{N}_0$.

Let $c(n)$ be defined by

$$\sum_{n=0}^{\infty} c(n)q^n = q^{\frac{k-3}{8}} f_1^3 f_k^3. \tag{3.44}$$

Now, consider the congruence relation

$$\frac{m^2+m}{2} + k \cdot \frac{r^2+r}{2} \equiv (k+1) \cdot \frac{p^2-1}{8} \pmod p,$$

where $0 \le m, r \le \frac{p-3}{2}$ and $p \ge 3$. The above congruence relation holds if and only if $m = r = \frac{p-1}{2}$, since $\left(\frac{-k}{p}\right) = -1$. Applying Lemma 2.3 into (3.44) and then extracting the terms of the form $q^{pn+\frac{p^2-1}{2}}$, we obtain

$$\sum_{n=0}^{\infty} c\Big(pn + \frac{p^2-1}{2}\Big)q^n = q^{p\frac{k-3}{8}} f_p^3 f_{kp}^3, \tag{3.45}$$

which implies that

$$c\Big(p^2 n + \frac{p^2-1}{2}\Big) = c(n),$$
$$c\Big(p^2 n + pj + \frac{p^2-1}{2}\Big) = 0,$$

where $1 \le j \le p-1$. From the above two identities and by induction, we find that

$$c\Big(p^{2\beta+2}n + p^{2\beta+1}j + \frac{p^{2\beta+2}-1}{2}\Big) = 0. \tag{3.46}$$

From (3.43), (3.44) and (3.46), we have

$$\overline{p}_{2,\,k}\Big(8p^{2\beta+2}n + 8p^{2\beta+1}j + 4p^{2\beta+2}\Big) \equiv 3\overline{p}_{2,\,k}\Big(4p^{2\beta+2}n + 4p^{2\beta+1}j + 2p^{2\beta+2}\Big).$$

Finally, replacing n by $2p^{2\beta+2}n + 2p^{2\beta+1}j + p^{2\beta+2}$ in (3.37) and then using the above congruence, we arrive at (3.38). □

Lemma 3.8 *If* $k \equiv 3 \pmod 8$*, then for all* $n \in \mathbb{N}_0$*, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(16n+10)q^n \equiv 4q^{\frac{k-3}{8}} f_2^3 f_k^3 + 4q^{\frac{k-1}{2}} f_1^3 f_{4k}^3 \pmod 8 \tag{3.47}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(16n+14)q^n \equiv 4q^{\frac{k-3}{4}} f_1^3 f_{2k}^3 + 4q^{\frac{k-3}{8}} f_k^3 f_4^3 \pmod 8. \tag{3.48}$$

Proof In view of (3.4), (3.6) and (3.7), we deduce that

$$\begin{aligned}
&\left(\frac{f_2 f_{4k} f_4^3}{f_{2k}} + q^{(k-1)/2}\frac{f_{2k} f_4 f_{4k}^3}{f_2}\right)\frac{1}{f_1^2 f_k^2}\\
&\equiv q^{\frac{k-1}{2}} \frac{f_4 f_{4k}^3 f_8^5 f_{8k}^5}{f_{2k}^4 f_2^6 f_{16}^2 f_{16k}^2} + 2q^{\frac{3k-1}{2}} \frac{f_4 f_{4k}^5 f_8^5 f_{16k}^2}{f_{2k}^4 f_2^6 f_{16}^2 f_{8k}} + 2q^{\frac{k+1}{2}} \frac{f_4^3 f_{4k}^3 f_{16}^2 f_{8k}^5}{f_{2k}^4 f_2^6 f_8 f_{16k}^2}\\
&\quad + \frac{f_{4k} f_4^3 f_8^5 f_{8k}^5}{f_2^4 f_{2k}^6 f_{16}^2 f_{16k}^2} + 2q^k \frac{f_{4k}^3 f_4^3 f_8^5 f_{16k}^2}{f_2^4 f_{2k}^6 f_{16}^2 f_{8k}} + 2q \frac{f_{4k} f_4^5 f_{16}^2 f_{8k}^5}{f_2^4 f_{2k}^6 f_8 f_{16k}^2} \pmod 4\\
&\equiv q^{\frac{k-1}{2}} \frac{f_{4k} f_{8k} f_8}{f_4 f_2^2} + 2q^{\frac{3k-1}{2}} f_{4k}^9 + 2q^{\frac{k+1}{2}} f_8^3 f_{4k}^3 + \frac{f_4 f_8 f_{8k}}{f_{4k} f_{2k}^2} + 2q^k f_4^3 f_{8k}^3 + 2q f_4^9 \pmod 4.
\end{aligned} \tag{3.49}$$

Combining (3.40) and (3.49) and then extracting the odd and even powers of q where $k \equiv 3 \pmod 4$, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+2)q^n \equiv 4q^{\frac{3k-1}{4}} f_{2k}^9 + 4q^{\frac{k+1}{4}} f_4^3 f_{2k}^3 + 2\frac{f_2 f_4 f_{4k}}{f_{2k} f_k^2}, \tag{3.50}$$

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+6)q^n \equiv 4f_2^9 + 4q^{\frac{k-1}{2}} f_2^3 f_{4k}^3 + 2q^{\frac{k-3}{4}} \frac{f_{2k} f_{4k} f_4}{f_2 f_1^2}. \tag{3.51}$$

In view of (2.1), (3.6) and (3.7), we have

$$\frac{f_2 f_4 f_{4k}}{f_{2k}}\frac{1}{f_k^2} = \frac{f_2 f_4 f_{4k} f_{8k}^5}{f_{2k}^6 f_{16k}^2} + 2q^k \frac{f_2 f_4 f_{4k}^3 f_{16k}^2}{f_{2k}^6 f_{8k}} \equiv \frac{f_2 f_4 f_{4k} f_{8k}}{f_{2k}^6} + 2q^k f_2^3 f_{8k}^3 \pmod 4, \tag{3.52}$$

$$\frac{f_{2k} f_{4k} f_4}{f_2}\frac{1}{f_1^2} = \frac{f_{2k} f_{4k} f_4 f_8^5}{f_2^6 f_{16}^2} + 2q \frac{f_{2k} f_{4k} f_4^3 f_{16}^2}{f_2^6 f_8} \equiv \frac{f_{2k} f_{4k} f_4 f_8}{f_2^6} + 2q f_{2k}^3 f_8^3 \pmod 4. \tag{3.53}$$

Substituting (3.52) in (3.50) and then extracting the terms involving odd powers of q where $k \equiv 3$, we obtain (3.47). In a similar way, (3.48) follows from (3.53) and (3.51). □

Lemma 3.9 *For all* $n \in \mathbb{N}_0$ *and* $k \equiv 7 \pmod 8$*, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(16n+6)q^n \equiv 4f_8 f_1 + 4q^{\frac{k+1}{8}} f_k^3 f_4^3 \pmod 8 \tag{3.54}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(16n+10)q^n \equiv 4q^{\frac{3k-5}{8}} f_{8k} f_k + 4q^{\frac{k-1}{2}} f_1^3 f_{4k}^3 \pmod 8. \tag{3.55}$$

Proof Substituting (3.53) in (3.51) and then extracting the terms involving the even powers of q where $k \equiv 7$, we obtain (3.54). Congruence (3.55) follows from (3.52) and (3.50). □

Theorem 3.10 *Let* $p \equiv 1, 7 \pmod 8$ *and* $k \equiv 3 \pmod 8$ *such that* $\left(\frac{-k}{p}\right) = -1$. *Then for all* $n, \beta \in \mathbb{N}_0$ *and* $1 \le j \le p-1$, *we have*

$$\overline{p}_{2,\,k}\Big(16p^{2\beta+2}n + 16p^{2\beta+1}j + 10p^{2\beta+2}\Big) \equiv 0 \pmod 8 \tag{3.56}$$

and

$$\overline{p}_{2,\,k}\Big(16p^{2\beta+2}n + 16p^{2\beta+1}j + 14p^{2\beta+2}\Big) \equiv 0 \pmod 8. \tag{3.57}$$

Proof From (3.47), we have

$$\overline{p}_{2,\,k}(16n+10) \equiv 4d(n), \tag{3.58}$$

where $k \equiv 3$ and

$$\sum_{n=0}^{\infty} d(n)q^n = q^{\frac{k-3}{8}} f_2^3 f_k^3 + q^{\frac{k-1}{2}} f_1^3 f_{4k}^3. \tag{3.59}$$

For $p \equiv 1, 7$ and $0 \le m, r \le \frac{p-3}{2}$, the following congruences

$$m^2 + m + k \cdot \frac{r^2+r}{2} \equiv (k+2) \cdot \frac{p^2-1}{8} \pmod p,$$
$$\frac{m^2+m}{2} + 2k \cdot (r^2+r) \equiv (4k+1) \cdot \frac{p^2-1}{8} \pmod p,$$

hold true if and only if $m = r = \frac{p-1}{2}$, since $\left(\frac{-2k}{p}\right) = -1$. Substituting Lemma 2.3 into (3.59) and then extracting the terms involving $q^{pn+5\cdot\frac{p^2-1}{8}}$, we obtain

$$\sum_{n=0}^{\infty} d\Big(pn + 5\cdot\frac{p^2-1}{8}\Big)q^n = q^{p\frac{k-3}{8}} f_{2p}^3 f_{kp}^3 + q^{p\frac{k-1}{2}} f_p^3 f_{4kp}^3,$$

and so we have

$$d\Big(p^2n + 5\cdot\frac{p^2-1}{8}\Big) = d(n), \tag{3.60}$$

$$d\Big(p^2n + pj + 5\cdot\frac{p^2-1}{8}\Big) = 0, \tag{3.61}$$

where $1 \le j \le p-1$. By (3.60) and (3.61), we have

$$d\Big(p^{2\beta+2}n + p^{2\beta+1}j + 5\cdot\frac{p^{2\beta+2}-1}{8}\Big) = 0. \tag{3.62}$$

The congruence (3.56) follows from (3.58) and (3.62). The proof of the congruence (3.57) is analogous to the proof of (3.56), except that in place of (3.47), (3.48) is used. □

Theorem 3.11 *Let* $p \equiv 5, 7 \pmod 8$ *and* $k \equiv 7 \pmod 8$ *such that* $\left(\frac{k}{p}\right) = 1$. *Then for all* $n, \beta \in \mathbb{N}_0$ *and* $1 \le j \le p-1$, *we have*

$$\overline{p}_{2,\,k}\Big(16p^{2\beta+2}n + 16p^{2\beta+1}j + 6p^{2\beta+2}\Big) \equiv 0 \pmod 8 \tag{3.63}$$

and

$$\overline{p}_{2,\,k}\Big(16p^{2\beta+2}n + 16p^{2\beta+1}j + 10p^{2\beta+2}\Big) \equiv 0 \pmod 8. \tag{3.64}$$

Proof Rewrite (3.54) in the form

$$\overline{p}_{2,\,k}(16n+6) \equiv 4f(n), \tag{3.65}$$

where $k \equiv 7$ and

$$\sum_{n=0}^{\infty} f(n)q^n = f_8 f_1 + q^{\frac{k+1}{8}} f_k^3 f_4^3. \tag{3.66}$$

Now, consider the following congruence relations:

$$4\cdot(3m_1^2+m_1) + \frac{3r_1^2+r_1}{2} \equiv 3\cdot\frac{p^2-1}{8} \pmod{p},$$
$$k\cdot\frac{m_2^2+m_2}{2} + 2\cdot(r_2^2+r_2) \equiv (k+4)\cdot\frac{p^2-1}{8} \pmod{p},$$

where $-\frac{p-1}{2} \le m_1, r_1 \le \frac{p-1}{2}$, $0 \le m_2, r_2 \le \frac{p-3}{2}$. The first congruence relation holds true if and only if $m_1 = r_1 = \frac{\pm p-1}{6}$, since $p \equiv 5, 7$. Since $\left(\frac{k}{p}\right) = 1$, the second congruence relation holds true if and only if $m_2 = r_2 = \frac{p-1}{2}$. Substituting Lemmas 2.2, 2.3 into (3.66) and then extracting the terms involving $q^{pn+3\cdot\frac{p^2-1}{8}}$, we obtain

$$\sum_{n=0}^{\infty} f\Big(pn + 3\cdot\frac{p^2-1}{8}\Big)q^n = p^2 f_{8p} f_p + q^{p\frac{k+1}{8}} f_{kp}^3 f_{4p}^3,$$

which implies that

$$f\Big(p^2 n + 3\cdot\frac{p^2-1}{8}\Big) \equiv f(n) \pmod{2},$$
$$f\Big(p^2 n + pj + 3\cdot\frac{p^2-1}{8}\Big) = 0,$$

where $1 \le j \le p-1$. From the above two identities, we deduce that

$$f\Big(p^{2\beta+2}n + p^{2\beta+1}j + 3\cdot\frac{p^{2\beta+2}-1}{8}\Big) \equiv 0 \pmod{2}.$$

Finally, replacing n by $p^{2\beta+2}n + p^{2\beta+1}j + 3\cdot\frac{p^{2\beta+2}-1}{8}$ in (3.65) and then using the above identity, we arrive at (3.63). The proof of the congruence (3.64) is similar to the proof of (3.63), except that in place of (3.54), (3.55) is used. □

Theorem 3.12 *Let $p \equiv 5, 7 \pmod{8}$ and $k \equiv 1 \pmod{8}$ such that $\left(\frac{k}{p}\right) = 1$. Then for all $n, \beta \in \mathbb{N}_0$, we have*

$$\overline{p}_{2,\,k}(8n+7) \equiv 0 \pmod{8} \tag{3.67}$$

and for $1 \le j \le p-1$,

$$\overline{p}_{2,\,k}\Big(8p^{2\beta+2}n + p^{2\beta+1} + 3p^{2\beta+2}\Big) \equiv 0 \pmod{8}. \tag{3.68}$$

Proof Extracting the terms involving the odd powers of q in (3.9) where $k \equiv 1 \pmod{4}$, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+3)q^n \equiv 4q^{\frac{k-1}{2}} f_2^3 f_{4k}^3 + 4f_2^9 + 4q^{\frac{3k-3}{4}} f_{2k}^9 + 4q^{\frac{k-1}{4}} f_4^3 f_{2k}^3. \tag{3.69}$$

For $k \equiv 1$, equating the coefficients of odd powers of q, we arrive at (3.67). Again, from (3.69), we have

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+3)q^n \equiv 4q^{\frac{k-1}{4}} f_1^3 f_{2k}^3 + 4f_1^9 + 4q^{\frac{3k-3}{8}} f_k^9 + 4q^{\frac{k-1}{8}} f_2^3 f_k^3. \tag{3.70}$$

Applying (3.6) repeatedly, we find that

$$\frac{f_{8k}^5}{f_{16k}^2}\frac{f_k^5}{f_{2k}^2} \equiv \frac{f_k^{40}}{f_k^{32}}\frac{f_k^5}{f_k^4} \equiv f_k^9 \pmod 2.$$

Employing above congruence in (3.70), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+3)q^n \equiv 4\Big(q^{\frac{k-1}{4}} f_1^3 f_{2k}^3 + \frac{f_8^5}{f_{16}^2}\frac{f_1^5}{f_2^2} + q^{\frac{3k-3}{8}} \frac{f_{8k}^5}{f_{16k}^2}\frac{f_k^5}{f_{2k}^2} + q^{\frac{k-1}{8}} f_2^3 f_k^3\Big). \tag{3.71}$$

From (3.25) and (3.71), we have

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+3)q^n \equiv 4\sum_{n=0}^{\infty} a(n)q^n,$$

which implies that

$$\overline{p}_{2,\,k}(8n+3) \equiv 4a(n).$$

Now, the congruence (3.68) follows from the above congruence and (3.29). □

If we extract the even and odd powers of q in (3.69) where $k \equiv 5$, we obtain the following lemma:

Lemma 3.13 *If* $k \equiv 5 \pmod 8$ *and* $n \in \mathbb{N}_0$, *then*

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+3)q^n \equiv 4q^{\frac{k-1}{4}} f_1^3 f_{2k}^3 + 4f_1^9 \pmod 8 \tag{3.72}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+7)q^n \equiv 4q^{\frac{3k-7}{8}} f_k^9 + 4q^{\frac{k-5}{8}} f_2^3 f_k^3 \pmod 8. \tag{3.73}$$

Lemma 3.14 *If* $k \equiv 1 \pmod 8$ *and for* $n \in \mathbb{N}_0$, *then*

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+5)q^n \equiv 4q^{\frac{k-1}{2}} f_1^3 f_{4k}^3 + 4q^{(k-1)/8} f_k^3 f_4^3 \pmod 8. \tag{3.74}$$

Proof Let $k \equiv 1 \pmod 4$. By (3.9), we see that

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+1)q^n \equiv 2\frac{f_4 f_2 f_{4k}}{f_{2k} f_k^2} + 6q^{\frac{k-1}{4}} \frac{f_{4k} f_{2k} f_4}{f_2 f_1^2}. \tag{3.75}$$

In view of (3.52), (3.53) and (3.75), we have

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+1)q^n \equiv 2\frac{f_2 f_4 f_{4k} f_{8k}}{f_{2k}^6} + 4q^k f_2^3 f_{8k}^3 + 6q^{\frac{k-1}{4}} \frac{f_{2k} f_{4k} f_4 f_8}{f_2^6} + 4q^{\frac{k+3}{4}} f_{2k}^3 f_8^3. \tag{3.76}$$

Extracting the terms involving q^{2n+1} in (3.76) where $k \equiv 1$, we obtain the required congruence. □

Theorem 3.15 *Let* $p \equiv 1, 7 \pmod 8$ *and* $k \equiv 3 \pmod 8$ *such that* $\left(\frac{-k}{p}\right) = -1$. *Then for* $1 \leq j \leq p-1$ *and* $n, \beta \in \mathbb{N}_0$, *we have*

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 5p^{2\beta+2}\right) \equiv 0 \pmod 8 \tag{3.77}$$

and

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 7p^{2\beta+2}\right) \equiv 0 \pmod 8. \tag{3.78}$$

Proof Let $k \equiv 3 \pmod 4$. In view of (3.9) and (3.50), we have

$$\overline{p}_{2,\,k}(8n+2) \equiv \overline{p}_{2,\,k}(4n+1). \tag{3.79}$$

From (3.56) and (3.79), we arrive at (3.77).

If we pick out those terms in which the power of q is congruent to 1 mod 2, we find that

$$\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+3)q^n \equiv 4q^{\frac{k-1}{2}} f_2^3 f_{4k}^3 + 4f_2^9 + 6q^{\frac{k-3}{4}} \frac{f_{4k} f_{2k} f_4}{f_2 f_1^2}. \tag{3.80}$$

By (3.53) and (3.80), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(4n+3)q^n &\equiv 4q^{\frac{k-1}{2}} f_2^3 f_{4k}^3 + 4f_2^9 + 6q^{\frac{k-3}{4}} \left(\frac{f_{2k} f_{4k} f_4 f_8}{f_2^6} + 2q f_{2k}^3 f_8^3\right) \\ &= 4q^{\frac{k-1}{2}} f_2^3 f_{4k}^3 + 4f_2^9 + 6q^{\frac{k-3}{4}} \frac{f_{2k} f_{4k} f_4 f_8}{f_2^6} + 4q^{\frac{k+1}{4}} f_{2k}^3 f_8^3.\end{aligned} \tag{3.81}$$

It follows from (3.81) with $k \equiv 3$ and (3.48),

$$\overline{p}_{2,\,k}(16n+14) \equiv \overline{p}_{2,\,k}(8n+7).$$

In view of (3.57) and the above congruence, we obtain (3.78). □

Theorem 3.16 *Let* $p \equiv 5, 7 \pmod 8$ *and* $k \equiv 7 \pmod 8$ *such that* $\left(\frac{k}{p}\right) = 1$. *Then for* $1 \leq j \leq p-1$ *and* $n, \beta \in \mathbb{N}_0$, *we have*

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 3p^{2\beta+2}\right) \equiv 0 \pmod 8 \tag{3.82}$$

and

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 5p^{2\beta+2}\right) \equiv 0 \pmod 8. \tag{3.83}$$

Proof Let $k \equiv 7$. From (3.6), (3.54) and (3.81), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_{2,\,k}(8n+3)q^n &\equiv 4f_1^9 + 4q^{\frac{k+1}{8}} f_k^3 f_4^3 \equiv 4f_8 f_1 + 4q^{\frac{k+1}{8}} f_k^3 f_4^3 \\ &\equiv \sum_{n=0}^{\infty} \overline{p}_{2,\,k}(16n+6)q^n,\end{aligned}$$

which yields

$$\overline{p}_{2,\,k}(8n+3) \equiv \overline{p}_{2,\,k}(16n+6). \tag{3.84}$$

Changing n to $2n+1$ in (3.79), we obtain

$$\overline{p}_{2,\,k}(16n+10) \equiv \overline{p}_{2,\,k}(8n+5). \tag{3.85}$$

Finally, from (3.63), (3.84) and (3.64), (3.85), we obtain (3.82) and (3.83), respectively. □

We can easily prove the following theorem with the help of Lemmas 3.13, 2.2 and 2.3:

Theorem 3.17 *Let* $p \equiv 5, 7 \pmod 8$ *and* $k \equiv 5 \pmod 8$ *such that* $\left(\frac{k}{p}\right) = 1$. *Then for* $n, \beta \in \mathbb{N}_0$ *and* $1 \le j \le p-1$, *we have*

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 3p^{2\beta+2}\right) \equiv 0 \pmod 8$$

and

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 7p^{2\beta+2}\right) \equiv 0 \pmod 8.$$

In view of Lemmas 2.3 and 3.14, we can prove the following congruence:

Theorem 3.18 *Let* $p \equiv 1, 7 \pmod 8$ *and* $k \equiv 1 \pmod 8$ *such that* $\left(\frac{2}{p}\right) = 1$. *Then for* $n, \beta \in \mathbb{N}_0$ *and* $1 \le j \le p-1$, *we have*

$$\overline{p}_{2,\,k}\left(8p^{2\beta+2}n + 8p^{2\beta+1}j + 5p^{2\beta+2}\right) \equiv 0 \pmod 8.$$

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