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Maja Fošner · Nadeem ur Rehman · Tarannum Bano

A note on generalized derivations on prime rings

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Abstract Let R be a prime ring with the extended centroid C and symmetric Martindale quotient ring $Q_s(R)$. In this paper we prove the following result. Let $F : R \rightarrow R$ be a generalized derivation associated with a non-zero derivation d on R and let h be an additive map of R such that $F(x)x = xh(x)$ for all $x \in R$. Then either R is commutative or $F(x) = xp$ and $h(x) = px$ where $p \in Q_s(R)$.

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الملخص

لتكن R حلقة أولية بمركز متوسط ممدّد C وحلقة حاصل قسمة مارتيندال متناظرة Q_s(R). في هذا البحث، نثبت النتيجة التالية: ليكن $F : R \rightarrow R$ اشتقاقا معمّما ومرتبطا باشتقاق غيرصفري (غير معدوم) d على R وليكن h تطبيقا جمعيا من R بحيث أن $F(x)x = xh(x)$ لكل $x \in R$. إذن إمّا R تبديلية أو $F(x) = x$ و $h(x) = px$ حيث $p \in Q_s$(R).

1 Introduction

Throughout the paper, R will be an associative ring with center Z. Recall that R is prime if for any $a, b \in R$, $aRb = 0$ implies that $a = 0$ or $b = 0$. By $Q_l(R)$ and $Q_r(R)$ we denote the left Martindale ring of quotients of R and the right Martindale ring of quotients of R, respectively. Further, we denote by $Q_s(R)$ the symmetric Martindale quotients ring of ring R. The center C of $Q_s(R)$ is a field and it is the center of both $Q_l(R)$ and $Q_r(R)$. It is called the extended centroid of R. Also it is easily seen that C is the centralizer of R in both $Q_r(R)$ and $Q_l(R)$. In particular, $Z \subseteq C$. The subring of $Q_r(R)$ (or $Q_l(R)$) generated by R and C is called the central closure of R and will be denoted by R_C. Another subring of $Q_r(R)$ is $Q_s(R) = \{q \in Q_r | Iq \subseteq R$

M. Fošner (✉)
Faculty of Logistics, University of Maribor, Mariborska cesta 7, 3000 Celje, Slovenia
E-mail: maja.fosner@um.si

N. ur Rehman · T. Bano
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail: rehman100@gmail.com

T. Bano
E-mail: tarannumdlw@gmail.com

for some nonzero ideal I of $R\}$. It is called the symmetric Martindale ring of quotients. We point out that $R \subseteq R_C \subseteq Q_s(R) \subseteq Q_r(R)$. Note that $q_1 R q_2 = 0$, where $q_1, q_2 \in Q_l(R)$ or $q_1, q_2 \in Q_r(R)$ implies that $q_1 = 0$ or $q_2 = 0$. In particular, this shows that all R_C, $Q_s(R)$, $Q_l(R)$, and $Q_r(R)$ are prime rings, so that one can construct (left, right, symmetric) Martindale ring of quotients and the central closure of each of these rings.

Let $R_C *_C C\{X\}$ be the free product over C of R_C and the free algebra over C on an infinite set X of indeterminates. A typical element in $R_C *_C C\{X\}$ is a sum of monomials of the form $\lambda a_{i_0} x_{j_1} a_{i_1} x_{j_2} \dots x_{j_n} a_{i_n}$ where $\lambda \in C$, $a_{i_k} \in R_C$ and $x_{j_k} \in X$. R satisfy a generalized polynomial identity over C (simply R is a GPI ring) if there exists a nonzero polynomial $p(x_1, x_2, \dots, x_n) \in R_C *_C C\{X\}$ such that $p(r_1, r_2, \dots, r_n) = 0$ for all $r_1, r_2, \dots, r_n \in R$. We refer the reader to [2,3] for more details.

An additive map $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, for a fixed $a \in R$, the map $I_a : R \to R$ given by $I_a(x) = [x, a]$ is a derivation called an inner derivation. Let S be a non-empty subset of R. A map $f : R \to R$ is said to be centralizing on S if $[f(x), x] \in Z$ for every $x \in S$. In special case where $[f(x), x] = 0$ for every $x \in S$, the map f is called commuting on S. The study of centralizing maps was initiated by a well-known theorem of Posner [15] which states that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. A number of authors have extended Posner's theorem in several ways. They have showed that nonzero derivations cannot be centralizing on various subsets of noncommutative prime rings (see [12] for probably the most general results of the kind), and similar conclusion hold for some other maps. In [5] Brešar studied maps that are centralizing and additive, and no further assumption was required. The main result of [5] characterizes commuting additive maps on prime rings R: every such map is of the form $x \mapsto \lambda x + h(x)$ where $\lambda \in C$, and h is an additive map of R into C. Later, Lanski [13] dealt with the situation where a nonzero derivation d of a prime ring R satisfies $c_1 x d(y) + c_2 d(x) y + c_3 y d(x) + c_4 d(y) x \in C$ for some $c_i \in C$ and all $x, y \in S$, where S is a subset of R. Neglecting rings of characteristic 2, the conclusion was: either all $c_i = 0$ or R satisfies S_4, the standard identity of degree 4 (however, the exact statements are much more precise). The condition considered by Lanski clearly covers the case of centralizing derivations, namely a linearization of $[d(x), x] \in Z$ gives $xd(y) - d(x)y + yd(x) - d(y)x \in Z$. The same is true for the case skew-centralizing on S if $f(x)x + xf(x) \in Z$ for all $x \in S$. In the special case where $f(x)x + xf(x) = 0$ for all $x \in S$, the map f is called skew-commuting on S. In [6] Brešar proved that there is no nonzero additive maps that are skew-commuting on ideals of prime rings of characteristic not 2.

An additive map $F : R \to R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a map F, it is easy to see that $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$ for all $x, y \in R$. This observation leads to the following definition, given in [4] and [9]; an additive map $F : R \to R$ is called a generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Familiar examples of generalized derivations are derivations and generalized inner derivations and the later includes left multiplier, i.e., an additive map $F : R \longrightarrow R$ satisfying $F(xy) = F(x)y$ for all $x, y \in R$. In [11], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R, U is the Utumi quotient ring (i.e., the maximal right quotient ring) of R and d is a derivation from I to U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and thus all generalized derivations of R will implicitly assumed to be defined on the whole of U. Lee obtained the following: every generalized derivation F on a dense left ideal of R can be extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.

Motivated by the work of Brešar [5] and Lanski [13], in this paper we consider F as a generalized derivation and h as an additive map of R such that $F(x)x = xh(x)$ for all $x \in R$. In fact, our result extends Posner's Theorem [15], Brešar's Theorem [5] and Ashraf et al. [1].

2 Main result

In this section we shall prove our main theorem. Before that we need some known results:

Lemma 2.1 [14, Theorem 3] *A prime ring R satisfies a GPI if and only if R_C is a primitive ring with nonzero socle and eR_Ce is a finite-dimensional division algebra over C for each primitive idempotent e in R_C.*

Lemma 2.2 [7, Main Theorem] *Let R be prime ring and let n, m, k, l be positive integers. Suppose that*

$$\Sigma_{i=1}^{n} F_i(y) x a_i + \Sigma_{i=1}^{m} G_i(x) y b_i + \Sigma_{i=1}^{k} c_i y H_i(x) + \Sigma_{i=1}^{l} d_i x K_i(y) = 0$$

for all $x, y \in R$*, where* $F_i, G_i, H_i, K_i : R \to R_C$ *are additive maps and* $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\}, \{c_1, \ldots, c_k\}, \{d_1, \ldots, d_l\}$ *are* C*-independent subsets of* R*. Then one of the two possibilities holds:*

(i) R_C *is a primitive ring with nonzero socle and* eR_Ce *is a finite-dimensional division algebra over* C *for each primitive idempotent* e *in* R_C*(that is,* R *is a GPI ring),*
(ii) *There exists elements* $q_{ij} \in Q_s(R_C), i = 1, \ldots, l, j = 1, \ldots, m,\ p_{ij} \in Q_s(R_C), i = 1, \ldots, k, j = 1, \ldots, n$ *and additive maps* $\lambda_{ij} : R \to C, i = 1, \ldots, l, j = 1, \ldots, n, \mu_{ij} : R \to C, i = 1, \ldots, m, j = 1, \ldots, k$*, such that*

$$\begin{aligned} F_i(y) &= \Sigma_{j=1}^{k} c_j y p_{ji} + \Sigma_{j=1}^{l} \lambda_{ji}(y) d_j, \ \textit{for all } y \in R, i = 1, \ldots, n, \\ G_i(x) &= \Sigma_{j=1}^{l} d_j x q_{ji} - \Sigma_{j=1}^{k} \mu_{ij}(x) c_j, \ \textit{for all } x \in R, i = 1, \ldots, m, \\ H_i(x) &= -\Sigma_{j=1}^{n} p_{ij} x a_j + \Sigma_{j=1}^{m} \mu_{ji}(x) b_j, \ \textit{for all } x \in R, i = 1, \ldots, k, \\ K_i(y) &= -\Sigma_{j=1}^{m} q_{ij} y b_j - \Sigma_{j=1}^{l} \lambda_{ji}(y) a_j, \ \textit{for all } y \in R, i = 1, \ldots, l. \end{aligned}$$

Lemma 2.3 [10, Theorem 2] *Let* R *be a prime ring,* U *its maximal right quotient ring and* I_R *a dense* R*-submodule of* U_R*. Then* I *and* U *satisfy the same differential identities*

Now we are well equipped to prove our theorem:

Theorem 2.4 *Let* R *be a prime ring and* $F : R \to R$ *be a generalized derivation associated with a non-zero derivation* d*. Further let* h *be an additive map of* R *such that* $F(x)x = xh(x)$ *for all* $x \in R$*. Then either* R *is commutative or* $F(x) = xp$ *and* $h(x) = px$ *where* $p \in U$*.*

Proof We have $F(x)x = xh(x)$ for all $x \in R$. Linearizing this relation we have

$$F(y)x + F(x)y - yh(x) - xh(y) = 0 \tag{1}$$

for all $x, y \in R$. We solve this functional identity in two different cases.

Case I: R **is not a GPI ring.** Using Lemma 2.2, we get from (1)

$$F(y) = yp + \lambda(y), \tag{2}$$
$$F(x) = xq + \mu(x), \tag{3}$$
$$h(x) = px + \mu(x), \tag{4}$$
$$h(y) = qy + \lambda(y), \tag{5}$$

where $\lambda, \mu : R \to C$ additive maps. From (2), we have $F(y) - yp = \lambda(y) \in C$. Let G be the additive map defined as $G(y) = F(y) - yp$ for any element $y \in R$. Since F is a generalized derivation with associated derivation d, first we prove that G is a generalized derivation of R.

$$\begin{aligned} G(xy) &= F(xy) - xyp = F(x)y + xd(y) - xyp \\ &= F(x)y + xd(y) - xyp + (xpy - xpy) \\ &= (F(x) - xp)y + x(d(y) + [p, y]) \\ &= G(x)y + xg(y), \end{aligned}$$

where $g(x) = d(x) + [p, x]$ is the associated derivation of G. Hence G is a generalized derivation. Thus, by (2) $G(y)$ is central in R, for any element $y \in R$. Hence by Hvala [9, Lemma 3] either R is commutative or $G = 0$, which imply $F(y) - yp = 0$ and hence $F(y) = yp$ for any $y \in R$. Similarly from (3), we find that either R is commutative or $F(x) = xq$. These two relations imply that $p = q$ and $\lambda = \mu = 0$ and hence $h(x) = px$ where $p \in U$.

Case II: R **is a GPI ring.** If there exists a nonzero idempotent e in R_C. If there exists $e^2 = e \neq \{0, 1\}$ in $Q_s(R)$. Therefore, we can find a nonzero ideal I of R satisfying $eI + Ie \subseteq R$. Then from (1), we get

$$F(ey)ex + F(ex)ey = exh(ey) + eyh(ex), \text{ for all } x, y \in I.$$

Thus

$$F(ey)ex + F(ex)ey = e\{F(ey)ex + F(ex)ey\}, \text{ for all } x, y \in I.$$

This implies that

$$(1-e)F(ey)ex + (1-e)F(ex)ey = 0, \text{ for all } x, y \in I.$$

By Lemma 2.3, $Q_s(R)$ and I satisfy the same differential identity. Thus we have

$$(1-e)F(ey)ex + (1-e)F(ex)ey = 0, \text{ for all } x, y \in Q_s(R).$$

This can be written as

$$H(x)y + H(y)x = 0, \tag{6}$$

where $H(x) = (1-e)F(ex)e$. Replacing x by xz in (6), we get

$$H(xz)y + H(y)xz = 0. \tag{7}$$

Multiplying (6) by z from the right, we find that

$$H(x)yz + H(y)xz = 0. \tag{8}$$

Comparing (7) and (8), we find that

$$H(xz)y = H(x)yz. \tag{9}$$

Replacing y by yu in (9), we get

$$H(xz)yu = H(x)yuz. \tag{10}$$

Right multiplication of (9) by u, we get

$$H(xz)yu = H(x)yzu. \tag{11}$$

Comparing (10) and (11), we get $H(x)y[z, u] = 0$ for all $x, y, z, u \in Q_s(R)$. Since R is prime and $Q_s(R)$ is also prime, we get from the last relation either R is commutative or $H(x) = 0$ for all $x \in Q_s(R)$. If $H(x) = 0$, we get $(1-e)F(ex)e = 0$ and considering $F(x) = ax + d(x)$ for $a \in U$, we find that

$$(1-e)[aex + d(ex)]e = 0.$$

This implies

$$(1-e)aexe + (1-e)d(e)xe = 0.$$

Since R is prime and e is non-trivial, we get $(1-e)ae + (1-e)d(e) = 0$ for all non-trivial idempotents $e \in Q_s(R)$. Replacing e by $1-e$ we get $ea(1-e) + ed(-e) = 0$. Combining these two relations we get $d(e)+[a, e] = 0$. Let E be an additive subgroup generated by idempotents in $Q_s(R)$. Therefore, $d(u)+[a, u] = 0$ for all $u \in E$. Now, for all $u, v \in E$, we get $d(uv) + [a, uv] = 0 = (d(u) + [a, u])v + u(d(v) + [a, v])$. That is, $d(u) + [a, u] = 0$ for all $u \in \overline{E} = [E, E]$. Now $[E, E] \neq 0$, since $[e, e + ex(1-e)] \neq 0$ for some $x \in Q_s(R)$. By Herstein's arguments [8, page 4] $0 \neq Q_s[E, E]Q_s \subseteq \overline{E}$, $W = Q_s[E, E]Q_s$ is a nonzero ideal of Q_s. Therefore, $d(u) + [a, u] = 0$ for all $u \in W$ and hence $d(x) + [a, x] = 0$ for all $x \in Q_s(R)$ by Lemma 2.3. Thus $F(x) = ax + d(x) = ax - [a, x] = xa$.

By Martindale's theorem [14, Theorem 3] we know that R_C is a primitive ring and $H = \text{soc}(R_C) \neq 0$ and eR_Ce is a finite dimensional for any minimal idempotent e. If H contains no non-trivial idempotent, then H is a finite-dimensional division algebra over C. If soc (R_C) contains no nontrivial idempotent, then soc (R_C) must be a finite-dimensional division algebra over C, by [14, Theorem 3]. Since soc (R_C) is a nonzero ideal of R_C, it follows R_C = soc (R_C) is a division algebra. Then for $x \neq 0 \in R$ we have from the given condition $h(x) = x^{-1}F(x)x$ in R_C. For any $x, y \neq 0 \in R$ we get from (1)

$$F(x)y + F(y)x = xy^{-1}F(y)y + yx^{-1}F(x)x. \tag{12}$$

Let $xy^{-1} = u$ in (12), we get

$$F(uy)y + F(y)uy = uF(y)y + u^{-1}F(uy)uy \text{ for all } u, y(\neq 0) \in R,$$

and this implies that

$$uF(uy)y + uF(y)uy = u^2F(y)y + F(uy)uy \text{ for all } 0 \neq u, y \in R.$$

This can be written as

$$uF(uy) + uF(y)u = u^2F(y) + F(uy)u \text{ for all } u, y \in R.$$

Since $F(x) = ax + d(x)$ for $a \in Q_s(R)$, we get from the last relation

$$uauy + ud(uy) + uayu + ud(y)u = u^2ay + u^2d(y) + auyu + d(uy)u.$$

The above relation for $y = 1$ gives us $uau + ud(u) + uau = u^2a + au^2 + d(u)u$. This implies that $[u, d(u)] = [u, [u, a]]$, hence $d(u) = [u, a]$; now we get $F(x) = ax + xa - ax = xa$ and by our assumption that $F(x)x = xh(x)$, we get $h(x) = ax$, which completes the proof. □

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