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Hyponormality of Toeplitz operators with polynomial symbols on the weighted Bergman space

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Abstract This paper gives the complete proof of the Conjecture given by Hazarika and this author jointly which deals with a necessary and sufficient condition for the hyponormality of Toeplitz operator, T_φ on the weighted Bergman space with certain polynomial symbols under some assumptions about the Fourier coefficients of the symbol φ .

الملخص.

تقدم هذه الورقة برهاناً كاملاً على التخمين الذي قدمه م. هازاريكا مع هذا المؤلف والذي يتعامل مع الشرط اللازم والكافي لمفهوم فوق المعيارية (فوق الناظمية) لمؤثرات توبليتز، T_φ على فضاءات بيرغمان الموزونة والمعرفة برموز كثيرة حدود معينة في ظل بعض الافتراضات حول معاملات فورييه للرمز φ .

Mathematics Subject Classification 47B35 · 47B20

1 Introduction

Let \mathbb{D} be the open unit disc in the complex plane \mathbb{C} . For $-1 < \alpha < \infty$, let $L^2(\mathbb{D}, dA_\alpha)$ denote the Hilbert space consisting of functions on \mathbb{D} which are square integrable with respect to the measure $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, where dA denotes the normalized Lebesgue area measure on \mathbb{D} . The inner product on $L^2(\mathbb{D}, dA_\alpha)$ is defined as $\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z)g(z)dA_\alpha(z)$, $f, g \in L^2(\mathbb{D}, dA_\alpha)$. The weighted Bergman space $A_\alpha^2(\mathbb{D})$ is then defined as the closed subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of analytic functions on \mathbb{D} . For any nonnegative integer n , if $e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} z^n$, ($z \in \mathbb{D}$) with the usual Gamma function $\Gamma(s)$, then the set $\{e_n\}$ forms an orthonormal basis for $A_\alpha^2(\mathbb{D})$ [12, 13]. The reproducing kernel in $A_\alpha^2(\mathbb{D})$ is defined as $K_z^{(\alpha)}(w) = \frac{1}{(1-\bar{z}w)^{2+\alpha}}$, for $z, w \in \mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D}, dA_\alpha)$, the multiplication operator M_φ on $A_\alpha^2(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi \cdot f$. The orthogonal projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto $A_\alpha^2(\mathbb{D})$ is given by $(P_\alpha f)(z) = \langle f, K_z^{(\alpha)} \rangle_\alpha = \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^{2+\alpha}} dA_\alpha(w)$, for $f \in L^2(\mathbb{D}, dA_\alpha)$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ with symbol φ is defined on $A_\alpha^2(\mathbb{D})$ by $T_\varphi f = P_\alpha(\varphi \cdot f)$. We, thus, have $T_\varphi f(z) = \int_{\mathbb{D}} \frac{\varphi(w)f(w)}{(1-z\bar{w})^{2+\alpha}} dA_\alpha(w)$, $f \in A_\alpha^2(\mathbb{D})$ and $w \in \mathbb{D}$. The Hankel operator on $A_\alpha^2(\mathbb{D})$ is defined by $H_\varphi f = (I - P_\alpha)(\varphi \cdot f)$.

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A bounded linear operator T on a Hilbert space is said to be hyponormal if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive semi-definite. The hyponormality of Toeplitz operators T_φ on the Hardy space $H^2(\mathbb{T})$ where $\mathbb{T} = \partial\mathbb{D}$ was first characterised by Cowen [2] which was simplified by Nakazi and Takahashi [9]. For $\varphi \in L^\infty(\mathbb{T})$, we write

$$\mathcal{E}(\varphi) = \{k \in H^\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty. The solution was given on the basis of a dilation theorem of Sarason [11]. As for the Bergman space, no such dilation theorem exists [3]; so, the characterization for the hyponormality of Toeplitz operators on the Bergman space is still remained an open question. For $\varphi = f + \bar{g}$ where f, g are bounded analytic functions, Sadraoui [10] gave a necessary and sufficient condition for the hyponormality of Toeplitz operators on the Bergman space which is equivalently true on the weighted Bergman space too. His theorem is stated below:

Theorem 1.1 [10] *Let f, g be bounded and analytic in $L^2(\mathbb{D}, dA)$. The following statements are equivalent:*

- (i) $T_{f+\bar{g}}$ is hyponormal;
- (ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}$;
- (iii) $H_{\bar{g}} = CH_{\bar{f}}$, where C is of norm less than or equal to one.

Recently, Hwang and Lee [5], Hwang, Lee and Park [6], Lu and Liu [7], Lu and Shi [8] and Hazarika along with this author in [4] gave some necessary and sufficient conditions for the hyponormality of Toeplitz operators on weighted Bergman space with the class of functions $\varphi = f + \bar{g}$ where $f, g \in L^2(\mathbb{D}, dA_\alpha)$. In [4], the authors gave a Conjecture in which it was made an attempt to give a complete criteria for the hyponormality of T_φ for this class of functions in the weighted Bergman space. The Conjecture is:

Conjecture 1.2 *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^{-m} + a_{-N} z^{-N}$ ($1 \leq m < N$). If $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$ and α is sufficiently large, then T_φ on $A_\alpha^2(\mathbb{D})$ is hyponormal*

$$\iff \begin{cases} \prod_{j=m}^{N-1} (\alpha + 2 + j)(|a_{-m}|^2 - |a_m|^2) \leq \prod_{j=0}^{N-(m+1)} (N - j)(|a_N|^2 - |a_{-N}|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2 (|a_{-N}|^2 - |a_N|^2) \leq m^2 (|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}| \end{cases}$$

In this paper, we give the proof of this Conjecture for all $\alpha > -1$. Since, the hyponormality of operators is translation invariant, we may assume that $f(0) = g(0) = 0$. We have the following properties of Toeplitz operators: If $f, g \in L^\infty(\mathbb{D}, dA_\alpha)$, then

- (i) $T_{f+g} = T_f + T_g$
- (ii) $T_f^* = T_{\bar{f}}$
- (iii) $T_{\bar{f}} T_g = T_{\bar{f}g}$ if f or g is analytic.

Also, for the projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto $A_\alpha^2(\mathbb{D})$, we have

$$P_\alpha(\bar{z}^t z^s) = \begin{cases} \frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t \end{cases}$$

where s and t are nonnegative integers. Again, for $\gamma_k = \|z^k\|_\alpha$, we have

$$\begin{aligned} \gamma_k^2 &= \|z^k\|_\alpha^2 = \langle z^k, z^k \rangle_\alpha = \int_{\mathbb{D}} z^k \bar{z}^k dA_\alpha(z) = (\alpha + 1) \int_{\mathbb{D}} |z|^{2k} (1 - |z|^2)^\alpha dA(z) \\ &= (\alpha + 1) \int_0^1 t^k (1 - t)^\alpha dt = (\alpha + 1) B(k + 1, \alpha + 1) = \frac{\Gamma(k + 1)\Gamma(\alpha + 2)}{\Gamma(\alpha + k + 2)} \end{aligned}$$

2 The proof of the Conjecture

To prove the Conjecture we need some specific lemmas.

Lemma 2.1 [1, 8] *Fix $m \geq 1$. Then for $\alpha \geq -1$,*

- (i) $H_{\bar{z}^m}(z^k)(\xi) = \begin{cases} \bar{\xi}^m \xi^k & \text{if } 0 \leq k < m \\ \bar{\xi}^m \xi^k - \frac{\gamma_k^2}{\gamma_{k-m}^2} \xi^{k-m} & \text{if } m \leq k; \end{cases}$
(ii) the functions $\{H_{\bar{z}^m}(z^k)\}_{k=0}^{\infty}$ are orthogonal in $L^2(\mathbb{D}, dA_\alpha)$;
(iii) $H_{\bar{z}^m}^* H_{\bar{z}^m}(z^k)(\xi) = \omega_{mk}^2 \xi^k$, $k = 0, 1, 2, \dots$, where

$$\omega_{mk}^2 = \begin{cases} \frac{\gamma_{k+m}^2}{\gamma_k^2} & \text{if } 0 \leq k < m \\ \frac{\gamma_{k+m}^2}{\gamma_k^2} - \frac{\gamma_k^2}{\gamma_{k-m}^2} & \text{if } m \leq k; \end{cases}$$

- (iv) $\|H_{\bar{z}^m}(z^k)\|_\alpha = \omega_{mk} \gamma_k$.

Lemma 2.2 [5] For $0 \leq m \leq N$ and let $K_i = \{k_i \in A_\alpha^2(\mathbb{D}) : k_i(z) = \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i}\}$ where, $i = 0, 1, 2, \dots, N-1$, we have

$$\|P_\alpha(\bar{z}^m)k_i(z)\|_\alpha^2 = \begin{cases} \sum_{k=0}^{\infty} \frac{(Nk+i)^2 \Gamma(Nk+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 & \text{if } m \leq i \\ \sum_{k=1}^{\infty} \frac{(Nk+i)^2 \Gamma(Nk+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 & \text{if } m > i \end{cases}$$

Lemma 2.3 Let $k \geq 1$ and $1 \leq m < N$. Let

$$\begin{aligned} \xi(k) &= \frac{(k+m)!}{\Gamma(k+m+\alpha+2)} - \frac{k!^2 \Gamma(k-m+\alpha+2)}{\Gamma(k+\alpha+2)^2 \Gamma(k-m+1)} \\ \zeta(k) &= \frac{(k+N)!}{\Gamma(k+N+\alpha+2)} - \frac{k!^2 \Gamma(k-N+\alpha+2)}{\Gamma(k+\alpha+2)^2 \Gamma(k-N+1)} \end{aligned}$$

Then, $\lim_{k \rightarrow \infty} \frac{\xi(k)}{\zeta(k)} = \frac{m^2}{N^2}$

Proof By simplifying,

$$\begin{aligned} \frac{\xi(k)}{\zeta(k)} &= \frac{\frac{\prod_{j=1}^m (k+j)}{\prod_{j=0}^{m-1} (k+\alpha+2+j)} - \frac{\prod_{j=0}^{m-1} (k-j)}{\prod_{j=1}^m (k+\alpha+2-j)}}{\frac{\prod_{j=1}^N (k+j)}{\prod_{j=0}^{N-1} (k+\alpha+2+j)} - \frac{\prod_{j=0}^{N-1} (k-j)}{\prod_{j=1}^N (k+\alpha+2-j)}} \\ &= \frac{\frac{\prod_{j=1}^m (k+j) \prod_{j=1}^m (k+\alpha+2-j) - \prod_{j=0}^{m-1} (k-j) \prod_{j=0}^{m-1} (k+\alpha+2+j)}{\prod_{j=-m}^{m-1} (k+\alpha+2+j)}}{\frac{\prod_{j=1}^N (k+j) \prod_{j=1}^N (k+\alpha+2-j) - \prod_{j=0}^{N-1} (k-j) \prod_{j=0}^{N-1} (k+\alpha+2+j)}{\prod_{j=-N}^{N-1} (k+\alpha+2+j)}} \end{aligned}$$

If $Q_n(k)$ is a polynomial of degree n in k , then $\prod_{j=1}^m (k+j) \prod_{j=1}^m (k+\alpha+2-j) - \prod_{j=0}^{m-1} (k-j) \prod_{j=0}^{m-1} (k+\alpha+2+j)$ is a polynomial $Q_{2m-2}(k)$. Now, $\prod_{j=1}^m (k+j) = k^m + \sum_{j=1}^m j k^{m-1} + L k^{m-2} + \dots + \prod_{j=1}^m j$ where, $L = 1.2 + (1+2) \cdot 3 + (1+2+3) \cdot 4 + \dots + (1+2+3+\dots+m-1) \cdot m$; and

$$\begin{aligned} &\prod_{j=1}^m (k+\alpha+2-j) \\ &= (k+\alpha+2)^m - \sum_{j=1}^m j (k+\alpha+2)^{m-1} + L (k+\alpha+2)^{m-2} - \dots \\ &= \left(k^m + m(\alpha+2) k^{m-1} + \sum_{j=1}^{m-1} j (\alpha+2)^2 k^{m-2} + \dots \right) \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^m j \left(k^{m-1} + (m-1)(\alpha+2)k^{m-2} + \dots \right) + Lk^{m-2} + \dots \\
& = k^m + \left(m(\alpha+2) - \sum_{j=1}^m j \right) k^{m-1} + \left((\alpha+2)^2 \sum_{j=1}^{m-1} j - (m-1)(\alpha+2) \sum_{j=1}^m j + L \right) k^{m-2} + \dots
\end{aligned}$$

Similarly,

$$\prod_{j=0}^m (k-j) = k^m - \sum_{j=1}^{m-1} j k^{m-1} + L_1 k^{m-2} + \dots + \prod_{j=1}^{m-1} j$$

where, $L_1 = 1.2 + (1+2).3 + (1+2+3).4 + \dots + (1+2+3+\dots+\overline{m-2}).(m-1)$
And,

$$\begin{aligned}
\prod_{j=0}^{m-1} (k+\alpha+2+j) & = k^m + \left(m(\alpha+2) + \sum_{j=1}^{m-1} j \right) k^{m-1} \\
& + \left(\sum_{j=1}^{m-1} j(\alpha+2)^2 + (m-1)(\alpha+2) \sum_{j=1}^{m-1} j + L_1 \right) k^{m-2} + \dots
\end{aligned}$$

If we denote the coefficients of $k^{(n)}$'s of the polynomial $Q_n(k)$ by C_n 's, then we have $C_{2m} = 0$, $C_{2m-1} = 0$.

$$\begin{aligned}
C_{2m-2} & = (\alpha+2)^2 \sum_{j=1}^{m-1} j - (m-1)(\alpha+2) \sum_{j=1}^m j + 2L + \sum_{j=1}^m j \left(m(\alpha+2) - \sum_{j=1}^m j \right) \\
& - (\alpha+2)^2 \sum_{j=1}^{m-1} j - (m-1)(\alpha+2) \sum_{j=1}^{m-1} j - 2L_1 + \sum_{j=1}^{m-1} j \left(m(\alpha+2) + \sum_{j=1}^{m-1} j \right) \\
& = (\alpha+2) \left(m + 2 \sum_{j=1}^{m-1} j \right) + 2m \sum_{j=1}^{m-1} j - \left(\sum_{j=1}^m j \right)^2 + \left(\sum_{j=1}^{m-1} j \right)^2 \\
& = m^2(\alpha+1)
\end{aligned}$$

Thus there exists a polynomial $Q_{2m-3}(k)$ such that $\xi(k) = m^2(\alpha+1)k^{2m-2} + Q_{2m-3}(k)$. Similarly, $\zeta(k) = N^2(\alpha+1)k^{2N-2} + Q_{2N-3}(k)$.

Therefore,

$$\lim_{k \rightarrow \infty} \frac{\xi(k)}{\zeta(k)} = \lim_{k \rightarrow \infty} \frac{m^2(\alpha+1)k^{2m-2} + Q_{2m-3}(k)}{N^2(\alpha+1)k^{2N-2} + Q_{2N-3}(k)} \cdot \frac{\prod_{j=-N}^{N-1} (k+\alpha+2+j)}{\prod_{j=-m}^{m-1} (k+\alpha+2+j)} = \frac{m^2}{N^2}.$$

□

Lemma 2.4 Let $f(z) = a_m z^m + a_N z^N$ and $g(z) = a_{-m} z^m + a_{-N} z^N$, with $1 \leq m < N$. Let $\alpha > -1$ and $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$. Then for $i \neq j$, we have $\langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z) \rangle_\alpha = \langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z) \rangle_\alpha$.

Proof

$$\begin{aligned}
\langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z) \rangle_\alpha & = \langle \bar{a}_m H_{\bar{z}^m} k_i(z) + \bar{a}_N H_{\bar{z}^N} k_i(z), \bar{a}_m H_{\bar{z}^m} k_j(z) + \bar{a}_N H_{\bar{z}^N} k_j(z) \rangle_\alpha \\
& = |a_m|^2 \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z) \rangle_\alpha + |a_N|^2 \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^N} k_j(z) \rangle_\alpha \\
& \quad + a_m \bar{a}_N \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z) \rangle_\alpha + \bar{a}_m a_N \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z) \rangle_\alpha \\
& = a_m \bar{a}_N \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z) \rangle_\alpha + \bar{a}_m a_N \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z) \rangle_\alpha.
\end{aligned}$$



Since,

$$\begin{aligned} \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z) \rangle_\alpha &= \left\langle H_{\bar{z}^m} \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i}, H_{\bar{z}^m} \sum_{k=0}^{\infty} c_{Nk+j} z^{Nk+j} \right\rangle_\alpha \\ &= \sum_{k=0}^{\infty} c_{Nk+i} \bar{c}_{Nk+j} \langle H_{\bar{z}^m} z^{Nk+i}, H_{\bar{z}^m} z^{Nk+j} \rangle_\alpha \\ &= 0 \text{ by Lemma 2.1.} \end{aligned}$$

And, $\langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^N} k_j(z) \rangle_\alpha = 0$.

Similarly, $\langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z) \rangle_\alpha = a_{-m} \bar{a}_{-N} \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z) \rangle_\alpha + \bar{a}_{-m} a_{-N} \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z) \rangle_\alpha$. Since, $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$, the proof is complete. \square

Now we are ready to prove Conjecture 1.2.

Proof Let $K_i = \{k_i \in A_\alpha^2(\mathbb{D}) : k_i(z) = \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i}\}$, where $i = 0, 1, 2, \dots, N-1$. By Theorem 1.1, T_φ is hyponormal if and only if

$$\left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \right\rangle_\alpha \geq 0. \quad (1)$$

That is, if and only if

$$\left\langle H_{\bar{f}} \sum_{i=0}^{N-1} k_i(z), H_{\bar{f}} \sum_{i=0}^{N-1} k_i(z) \right\rangle_\alpha - \left\langle H_{\bar{g}} \sum_{i=0}^{N-1} k_i(z), H_{\bar{g}} \sum_{i=0}^{N-1} k_i(z) \right\rangle_\alpha \geq 0.$$

That is, if and only if

$$\begin{aligned} &\sum_{i=0}^{N-1} \langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_i(z) \rangle_\alpha + \sum_{i \neq j; i, j \geq 0}^{N-1} \langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_i(z) \rangle_\alpha \\ &- \sum_{i \neq j; i, j \geq 0}^{N-1} \langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z) \rangle_\alpha \geq 0. \end{aligned} \quad (2)$$

Thus, using Lemma 2.4 in (2), we have that T_φ is hyponormal if and only if

$$\sum_{i=0}^{N-1} \langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_i(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_i(z) \rangle_\alpha \geq 0. \quad (3)$$

We have, $\sum_{i=0}^{N-1} \langle M_{\bar{f}} k_i(z), M_{\bar{f}} k_i(z) \rangle_\alpha = \sum_{i=0}^{N-1} \langle (\bar{a}_m \bar{z}^m + \bar{a}_N \bar{z}^N) k_i(z), (\bar{a}_m \bar{z}^m + \bar{a}_N \bar{z}^N) k_i(z) \rangle_\alpha = |a_m|^2 \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^m k_i(z) \rangle_\alpha + |a_N|^2 \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + \bar{a}_m a_N \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_m \bar{a}_N \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^m k_i(z) \rangle_\alpha$.

Similarly,

$$\begin{aligned} &\sum_{i=0}^{N-1} \langle M_{\bar{g}} k_i(z), M_{\bar{g}} k_i(z) \rangle_\alpha \\ &= |a_{-m}|^2 \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^m k_i(z) \rangle_\alpha + |a_{-N}|^2 \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^N k_i(z) \rangle_\alpha \\ &+ \bar{a}_{-m} a_{-N} \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_{-m} \bar{a}_{-N} \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^m k_i(z) \rangle_\alpha. \end{aligned}$$

Therefore, using the assumption $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$, we have

$$\begin{aligned}
& \sum_{i=0}^{N-1} \langle M_{\bar{f}} k_i(z), M_{\bar{f}} k_i(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle M_{\bar{g}} k_i(z), M_{\bar{g}} k_i(z) \rangle_\alpha \\
&= (|a_m|^2 - |a_{-m}|^2) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} |c_{Nk+i}|^2 \langle \bar{z}^m z^{Nk+i}, \bar{z}^m z^{Nk+i} \rangle_\alpha \\
&\quad + (|a_N|^2 - |a_{-N}|^2) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} |c_{Nk+i}|^2 \langle \bar{z}^N z^{Nk+i}, \bar{z}^N z^{Nk+i} \rangle_\alpha \\
&= (|a_m|^2 - |a_{-m}|^2) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} \frac{(Nk+i+m)! \Gamma(\alpha+2)}{\Gamma(Nk+i+m+\alpha+2)} |c_{Nk+i}|^2 \\
&\quad + (|a_N|^2 - |a_{-N}|^2) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} \frac{(N(k+1)+i)! \Gamma(\alpha+2)}{\Gamma(N(k+1)+i+\alpha+2)} |c_{Nk+i}|^2. \tag{4}
\end{aligned}$$

Also, we have

$$\begin{aligned}
\langle T_{\bar{g}} k_i(z), T_{\bar{g}} k_i(z) \rangle_\alpha &= \langle \bar{a}_{-m} T_{\bar{z}^m} k_i(z) + \bar{a}_{-N} T_{\bar{z}^N} k_i(z), \bar{a}_{-m} T_{\bar{z}^m} k_i(z) \rangle_\alpha \\
&\quad + \bar{a}_{-N} T_{\bar{z}^N} k_i(z) \rangle_\alpha = |a_{-m}|^2 \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha \\
&\quad + |a_{-N}|^2 \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha + \bar{a}_{-m} a_{-N} \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha \\
&\quad + a_{-m} \bar{a}_{-N} \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha.
\end{aligned}$$

And,

$$\begin{aligned}
\langle T_{\bar{f}} k_i(z), T_{\bar{f}} k_i(z) \rangle_\alpha &= |a_m|^2 \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha \\
&\quad + |a_N|^2 \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha + \bar{a}_m a_N \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha \\
&\quad + a_m \bar{a}_N \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha.
\end{aligned}$$

Therefore, using the assumption $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$ and Lemma 2.2, we have

$$\begin{aligned}
& \sum_{i=0}^{N-1} \langle T_{\bar{g}} k_i(z), T_{\bar{g}} k_i(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle T_{\bar{f}} k_i(z), T_{\bar{f}} k_i(z) \rangle_\alpha \\
&= (|a_{-m}|^2 - |a_m|^2) \sum_{i=0}^{N-1} \|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2 + (|a_{-N}|^2 - |a_N|^2) \sum_{i=0}^{N-1} \|P_\alpha(\bar{z}^N k_i(z))\|_\alpha^2 \\
&= (|a_{-m}|^2 - |a_m|^2) \left(\sum_{i=0}^{m-1} \|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2 + \sum_{i=m}^{N-1} \|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2 \right) \\
&\quad + (|a_{-N}|^2 - |a_N|^2) \sum_{i=0}^{N-1} \|P_\alpha(\bar{z}^N k_i(z))\|_\alpha^2 \\
&= (|a_{-m}|^2 - |a_m|^2) \sum_{i=0}^{m-1} \sum_{k=1}^{\infty} \frac{(Nk+i)!^2 \Gamma(Nk+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 \\
&\quad + (|a_{-m}|^2 - |a_m|^2) \sum_{i=m}^{N-1} \sum_{k=0}^{\infty} \frac{(Nk+i)!^2 \Gamma(Nk+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 \\
&\quad + (|a_{-N}|^2 - |a_N|^2) \sum_{i=0}^{N-1} \sum_{k=1}^{\infty} \frac{(Nk+i)!^2 \Gamma(Nk+i-N+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-N+1)} |c_{Nk+i}|^2. \tag{5}
\end{aligned}$$



Thus by applying (4), (5) in (3) and simplifying, we have that T_φ is hyponormal if and only if

$$\begin{aligned} &(|a_m|^2 - |a_{-m}|^2) \left(\sum_{i=0}^{m-1} \left(\frac{(i+m)!\Gamma(\alpha+2)}{\Gamma(i+m+\alpha+2)} |c_i|^2 + \sum_{k=1}^{\infty} \left(\frac{(Nk+i+m)!\Gamma(\alpha+2)}{\Gamma(Nk+i+m+\alpha+2)} \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{(Nk+i)!\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)} \right) |c_{Nk+i}|^2 + \sum_{i=m}^{N-1} \sum_{k=0}^{\infty} \left(\frac{(Nk+i+m)!\Gamma(\alpha+2)}{\Gamma(Nk+i+m+\alpha+2)} \right. \right. \\ &\quad \left. \left. - \frac{(Nk+i)!\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)} \right) |c_{Nk+i}|^2 \right) \right) + (|a_N|^2 - |a_{-N}|^2) \sum_{i=0}^{N-1} \left(\frac{(N+i)!\Gamma(\alpha+2)}{\Gamma(N+i+\alpha+2)} |c_i|^2 \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \left(\frac{(N(k+1)+i)!\Gamma(\alpha+2)}{\Gamma(N(k+1)+i+\alpha+2)} - \frac{(Nk+i)!\Gamma(N(k-1)+i+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(N(k-1)+i+1)} \right) |c_{Nk+i}|^2 \right) \geq 0 \end{aligned}$$

or equivalently,

$$\begin{aligned} &(|a_m|^2 - |a_{-m}|^2) \left(\sum_{k=0}^{m-1} \frac{(k+m)!\Gamma(\alpha+2)}{\Gamma(k+m+\alpha+2)} |c_k|^2 + \sum_{k=m}^{\infty} \left(\frac{(k+m)!\Gamma(\alpha+2)}{\Gamma(k+m+\alpha+2)} \right. \right. \\ &\quad \left. \left. - \frac{k!^2\Gamma(k-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(k+\alpha+2)^2\Gamma(k-m+1)} \right) |c_k|^2 \right) + (|a_N|^2 - |a_{-N}|^2) \left(\sum_{k=0}^{N-1} \frac{(k+N)!\Gamma(\alpha+2)}{\Gamma(k+N+\alpha+2)} |c_k|^2 \right. \\ &\quad \left. + \sum_{k=N}^{\infty} \left(\frac{(k+N)!\Gamma(\alpha+2)}{\Gamma(k+N+\alpha+2)} - \frac{k!^2\Gamma(k-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(k+\alpha+2)^2\Gamma(k-N+1)} \right) |c_k|^2 \right) \geq 0. \end{aligned} \quad (6)$$

Let for all $k \geq 1$

$$\begin{aligned} \xi(k) &= \frac{\frac{(k+m)!}{\Gamma(k+m+\alpha+2)} - \frac{k!^2\Gamma(k-m+\alpha+2)}{\Gamma(k+\alpha+2)^2\Gamma(k-m+1)}}{\frac{(k+N)!}{\Gamma(k+N+\alpha+2)} - \frac{k!^2\Gamma(k-N+\alpha+2)}{\Gamma(k+\alpha+2)^2\Gamma(k-N+1)}} \\ &= \frac{\frac{\prod_{j=1}^m (k+j)}{\prod_{j=0}^{m-1} (k+\alpha+2+j)} - \frac{\prod_{j=0}^{m-1} (k-j)}{\prod_{j=1}^m (k+\alpha+2-j)}}{\frac{\prod_{j=1}^N (k+j)}{\prod_{j=0}^{N-1} (k+\alpha+2+j)} - \frac{\prod_{j=0}^{N-1} (k-j)}{\prod_{j=1}^N (k+\alpha+2-j)}}. \end{aligned} \quad (7)$$

Now, if $|a_N| \geq |a_{-N}|$ then,

$$\frac{|a_N|^2 - |a_{-N}|^2}{|a_{-m}|^2 - |a_m|^2} \geq \xi(k), \quad \text{for all } k \geq 1. \quad (8)$$

For all $k = 1, 2, \dots, N-1$, we have

$$\frac{\frac{m!}{\Gamma(m+\alpha+2)}}{\frac{N!}{\Gamma(N+\alpha+2)}} \geq \frac{\frac{(k+m)!}{\Gamma(k+m+\alpha+2)}}{\frac{(k+N)!}{\Gamma(k+N+\alpha+2)}} \geq \xi(N). \quad (9)$$

Hence from (8) and (9), T_φ is hyponormal if and only if

$$\frac{|a_N|^2 - |a_{-N}|^2}{|a_{-m}|^2 - |a_m|^2} \geq \frac{\prod_{j=m}^{N-1} (\alpha+2+j)}{\prod_{j=0}^{N-(m+1)} (N-j)}. \quad (10)$$

By Lemma 2.3, we get

$$\lim_{k \rightarrow \infty} \xi(k) = \frac{m^2}{N^2}. \quad (11)$$

If $|a_{-N}| \geq |a_N|$, then

$$\frac{|a_{-N}|^2 - |a_N|^2}{|a_m|^2 - |a_{-m}|^2} \leq \xi(k) \quad \text{for all } k \geq 1. \quad (12)$$

Hence in this case T_φ is hyponormal if and only if

$$\frac{|a_{-N}|^2 - |a_N|^2}{|a_m|^2 - |a_{-m}|^2} \leq \frac{m^2}{N^2}. \quad (13)$$

Thus the results follow from (10) and (13). \square

3 Conclusion

This theorem may give a clue to the readers to think about the generalised form of the hyponormality of Toeplitz operators with a class of polynomial symbols relaxing some of the restrictions to the Fourier coefficients.

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