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# Expansions of the exponential and the logarithm of power series and applications

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**Abstract** In the paper, the authors establish explicit formulas for asymptotic and power series expansions of the exponential and the logarithm of asymptotic and power series expansions. The explicit formulas for the power series expansions of the exponential and the logarithm of a power series expansion are applied to find explicit formulas for the Bell numbers and logarithmic polynomials in combinatorics and number theory.

**Mathematics Subject Classification** 34E05 · 11B73 · 11B75 · 11B83 · 30B10

## المخلص

في هذه الورقة يكرس المؤلفون صيغا صريحة لتمديدات مقاربية ومتسلسلات قوى لأس ولوغاريتم تمديدات مقاربية ومتسلسلات قوى. تم تطبيق الصيغ الصريحة لتمديدات متسلسلات القوى لأس ولوغاريتم تمديد متسلسلات قوى لإيجاد صيغ صريحة لأعداد بل وكثيرات الحدود اللوغاريتمية في نظرية التركيبات ونظرية الأعداد.

## 1 Introduction

Throughout this paper, we understand an empty sum to be 0 and regard an empty product as 1.

Let us recall definitions for an asymptotic expansion.

**Definition 1.1** [11, p. 31, Definition 2.1] Let  $F$  be a function of a real or complex variable  $z$ ; let  $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$  denote a convergent or divergent formal power series, of which the sum of the first  $n$  terms is denoted by  $S_n(z)$ ; and let

$$R_n(z) = F(z) - S_n(z).$$

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In other words,

$$F(z) = \sum_{k=0}^{n-1} \frac{a_k}{z^k} + R_n(z) = S_n(z) + R_n(z), \quad n \geq 0,$$

where we assume that when  $n = 0$  we have  $F(z) = R_0(z)$ . Next, assume that for each  $n \geq 0$  the relation

$$R_n(z) = O\left(\frac{1}{z^n}\right), \quad z \rightarrow \infty$$

holds in some unbounded domain  $\Delta$ . Then,  $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$  is called an asymptotic expansion of the function  $F(z)$  and we denote this by

$$F(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad z \rightarrow \infty, \quad z \in \Delta.$$

**Definition 1.2** [12, p. 151] A divergent series

$$\sum_{n=0}^{\infty} \frac{A_n}{z^n},$$

in which the sum of the first  $n + 1$  terms is  $S_n(z)$ , is said to be an asymptotic expansion of a function  $f(z)$  for a given range of values of  $\arg z$ , if the expression

$$R_n(z) = z^n[f(z) - S_n(z)]$$

satisfies the condition

$$\lim_{|z| \rightarrow \infty} R_n(z) = 0 \quad (n \text{ fixed}),$$

even though

$$\lim_{n \rightarrow \infty} |R_n(z)| = \infty \quad (z \text{ fixed}).$$

We denote the fact that the series is the asymptotic expansion of  $f(z)$  by writing

$$f(z) \sim \sum_{n=0}^{\infty} \frac{A_n}{z^n}.$$

On the asymptotic expansion of the composite of a function and an asymptotic expansion, we recite the following four results.

**Theorem 1.3** [6, p. 540, Theorem 2] *If  $g(z) = \sum_{n=0}^{\infty} \alpha_n z^n$  is a power series with positive radius  $r$ , if  $F(x)$  possesses the asymptotic representation*

$$F(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k},$$

and if  $|a_0| < r$ , then the function

$$\Phi(x) = g(F(x))$$

also possesses an asymptotic representation, and this is again calculated exactly as if  $\sum_{n=0}^{\infty} \frac{a_n}{x^n}$  were convergent, where, since  $F(x) \rightarrow a_0$  as  $x \rightarrow \infty$ , and since  $|a_0| < r$ , the function  $\Phi(x)$  is obviously defined for every sufficiently large  $x$ .



In [6, p. 541], it was stated that, when taking  $g(z) = e^z$  in Theorem 1.3, we obtain, without any restrictions,

$$e^{F(x)} \sim e^{a_0} \left( 1 + \frac{a_1}{x} + \frac{a_1^2/2 + a_2}{x^2} + \dots \right).$$

**Theorem 1.4** [9, Remark 3.2] *If*

$$f(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k}, \quad x \rightarrow \infty,$$

*then*

$$e^{f(x)} \sim \sum_{k=0}^{\infty} \frac{\alpha_k}{x^k}, \quad x \rightarrow \infty,$$

where  $\alpha_0 = e^{a_0}$  and

$$\alpha_k = e^{a_0} \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{\sum_{\ell=1}^j i_\ell = k, \\ 0 \leq i_\ell \leq k, 1 \leq \ell \leq j, \\ i_\ell \in \{0\} \cup \mathbb{N}}} \prod_{\ell=1}^j a_{i_\ell} \tag{1.1}$$

**Theorem 1.5** [1, Lemma 4] *Let*

$$A(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n} \tag{1.2}$$

*be a given asymptotic expansion. Then, the composition  $B(x) = e^{A(x)}$  has the asymptotic expansion*

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{x^n},$$

where  $b_0 = 1$  and

$$b_n = \frac{1}{n} \sum_{k=1}^n k a_k b_{n-k}, \quad n \in \mathbb{N}. \tag{1.3}$$

**Theorem 1.6** [1, Lemma 5] *Let  $c_0 \neq 0$  and*

$$C(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^n}$$

*be a given asymptotic expansion. Then, the composition  $A(x) = \ln C(x)$  has the expansion*

$$A(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n},$$

where

$$a_n = \frac{1}{c_0} \left( c_n - \frac{1}{n} \sum_{k=1}^{n-1} k a_k c_{n-k} \right), \quad n \in \mathbb{N}. \tag{1.4}$$

In [3], many useful conclusions on asymptotic expansions were obtained.

We note that no accurate and explicit references were given in [9] to cite the formula (1.1) in Theorem 1.4.

We observe that the constant term did not appear in (1.2) in Theorem 1.5 and that the formulas (1.3) and (1.4) are recurrences.

In Sect. 2 of this paper, we will add a constant term into (1.2) and acquire a modified version of Theorem 1.5. In Sect. 3, we will derive from the recurrences (1.3) and (1.4) explicit formulas for the above  $a_n$  and  $b_n$ . In Sect. 4, we will simply confirm explicit formulas for power series expansions of the exponential and the logarithm of a power series expansion. In Sect. 5, we will apply the explicit formulas for the power series expansions of the exponential and the logarithm of a power series expansion to find explicit formulas for the Bell numbers and logarithmic polynomials extensively studied in combinatorics and number theory.

## 2 A slightly modified version of Theorem 1.5

Now we are in a position to give a slightly modified version of Theorem 1.5.

**Theorem 2.1** *Let*

$$D(x) = \sum_{k=0}^{\infty} \frac{d_k}{x^k}$$

*be an asymptotic expansion. Then, the function  $E(x) = e^{D(x)}$  has the asymptotic expansion*

$$E(x) = \sum_{k=0}^{\infty} \frac{e_k}{x^k},$$

where

$$e_0 = e^{d_0} \tag{2.1}$$

and

$$e_k = \frac{1}{k} \sum_{\ell=1}^k \ell d_{\ell} e_{k-\ell} = \frac{1}{k} \sum_{\ell=0}^{k-1} (k - \ell) d_{k-\ell} e_{\ell}, \quad k \in \mathbb{N}. \tag{2.2}$$

*First proof* Differentiation yields

$$E'(x) = e^{D(x)} D'(x) = E(x) D'(x)$$

which can be written as:

$$\begin{aligned} \sum_{k=1}^{\infty} (-k) \frac{e_k}{x^{k+1}} &= \left( \sum_{k=0}^{\infty} \frac{e_k}{x^k} \right) \left( \sum_{k=1}^{\infty} (-k) \frac{d_k}{x^{k+1}} \right) \\ &= \frac{1}{x^2} \left( \sum_{k=0}^{\infty} \frac{e_k}{x^k} \right) \left( \sum_{k=0}^{\infty} (-k-1) \frac{d_{k+1}}{x^k} \right) = \frac{1}{x^2} \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^k (-\ell-1) d_{\ell+1} e_{k-\ell} \right) \frac{1}{x^k}, \end{aligned}$$

that is,

$$\sum_{k=1}^{\infty} k \frac{e_k}{x^{k+1}} = \sum_{k=1}^{\infty} \left( \sum_{\ell=0}^{k-1} (\ell+1) d_{\ell+1} e_{k-\ell-1} \right) \frac{1}{x^{k+1}} = \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^k \ell d_{\ell} e_{k-\ell} \right) \frac{1}{x^{k+1}}.$$

Equating coefficients of  $\frac{1}{x^{k+1}}$  results in (2.2).

Taking the limit  $x \rightarrow \infty$  on both sides of  $E(x) = e^{D(x)}$ , we arrive at (2.1). The proof of Theorem 2.1 is complete. □



*Second proof* Set  $D(x) = d_0 + D_1(x)$ , where  $D_1(x) = \sum_{k=1}^{\infty} \frac{d_k}{x^k}$ . Then, by virtue of Theorem 1.5, we have

$$E(x) = e^{D(x)} = e^{d_0} e^{D_1(x)} = e^{d_0} \sum_{k=0}^{\infty} \frac{\beta_k}{x^k},$$

where  $\beta_0 = 1$  and

$$\beta_k = \frac{1}{k} \sum_{\ell=1}^k \ell d_{\ell} \beta_{k-\ell}, \quad k \in \mathbb{N}.$$

This means that the Eq. (2.1) is valid and that  $e_k = e^{d_0} \beta_k$  for  $k \in \mathbb{N}$ . Hence, the sequence  $e_k$  for  $k \in \mathbb{N}$  satisfies

$$\frac{e_k}{e^{d_0}} = \frac{1}{k} \sum_{\ell=1}^k \ell d_{\ell} \frac{e_{k-\ell}}{e^{d_0}}, \quad k \in \mathbb{N}.$$

Consequently, the recurrence relation (2.2) follows. The proof of Theorem 2.1 is complete. □

### 3 Explicit formulas for $a_n$ and $e_n$

In this section, we derive explicit formulas for  $a_n$  and  $e_n$  from recurrence relations (1.4) and (2.2).

**Theorem 3.1** *Under the conditions of Theorem 1.6, we have*

$$a_n = \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-1} (-1)^j \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} m_j \prod_{i=0}^j \frac{c_{m_i}}{c_0}, \quad n \in \mathbb{N}. \tag{3.1}$$

*Proof* From the recurrence relation (1.4) and by induction, it follows that

$$\begin{aligned} a_n &= \frac{c_n}{c_0} - \frac{1}{c_0} \frac{1}{n} \sum_{k=1}^{n-1} a_k k c_{n-k} \\ &= \frac{c_n}{c_0} - \frac{1}{c_0} \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{c_k}{c_0} - \frac{1}{c_0} \frac{1}{k} \sum_{p=1}^{k-1} a_p p c_{k-p} \right) k c_{n-k} \\ &= \frac{c_n}{c_0} - \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} c_k k c_{n-k} + \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} a_p p c_{k-p} c_{n-k} \\ &= \frac{c_n}{c_0} - \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} c_k k c_{n-k} + \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} \left( \frac{c_p}{c_0} - \frac{1}{c_0} \frac{1}{p} \sum_{q=1}^{p-1} a_q q c_{p-q} \right) p c_{k-p} c_{n-k} \\ &= \frac{c_n}{c_0} - \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} c_k k c_{n-k} + \frac{1}{c_0^3} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} c_p p c_{k-p} c_{n-k} \\ &\quad - \frac{1}{c_0^3} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} \sum_{q=1}^{p-1} a_q q c_{p-q} c_{k-p} c_{n-k} \\ &= \frac{c_n}{c_0} - \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} c_k k c_{n-k} + \frac{1}{c_0^3} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} c_p p c_{k-p} c_{n-k} \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{c_0^3} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} \sum_{q=1}^{p-1} \left( \frac{c_q}{c_0} - \frac{1}{c_0} \frac{1}{q} \sum_{t=1}^{q-1} a_t t c_{q-t} \right) q c_{p-q} c_{k-p} c_{n-k} \\
 = & \frac{c_n}{c_0} - \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} c_k k c_{n-k} + \frac{1}{c_0^3} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} c_p p c_{k-p} c_{n-k} \\
 & - \frac{1}{c_0^4} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} \sum_{q=1}^{p-1} c_q q c_{p-q} c_{k-p} c_{n-k} + \frac{1}{c_0^4} \frac{1}{n} \sum_{k=1}^{n-1} \sum_{p=1}^{k-1} \sum_{q=1}^{p-1} \sum_{t=1}^{q-1} a_t t c_{q-t} c_{p-q} c_{k-p} c_{n-k} \\
 = & \frac{c_n}{c_0} - \frac{1}{c_0^2} \frac{1}{n} \sum_{k=1}^{n-1} c_k k c_{n-k} + \frac{1}{c_0^3} \frac{1}{n} \sum_{k=2}^{n-1} \sum_{p=1}^{k-1} c_p p c_{k-p} c_{n-k} \\
 & - \frac{1}{c_0^4} \frac{1}{n} \sum_{k=3}^{n-1} \sum_{p=2}^{k-1} \sum_{q=1}^{p-1} c_q q c_{p-q} c_{k-p} c_{n-k} + \frac{1}{c_0^4} \frac{1}{n} \sum_{k=4}^{n-1} \sum_{p=3}^{k-1} \sum_{q=2}^{p-1} \sum_{t=1}^{q-1} a_t t c_{q-t} c_{p-q} c_{k-p} c_{n-k} \\
 = & \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-2} \frac{(-1)^j}{c_0^{j+1}} \sum_{\ell_1=j}^{n-1} \sum_{\ell_2=j-1}^{\ell_1-1} \cdots \sum_{\ell_j=1}^{\ell_{j-1}-1} c_{\ell_j} \ell_j c_{n-\ell_1} \prod_{i=1}^{j-1} c_{\ell_i - \ell_{i+1}} \\
 & + \frac{1}{n} \frac{(-1)^{n-1}}{c_0^{n-1}} \sum_{\ell_1=n-1}^{n-1} \sum_{\ell_2=n-2}^{\ell_1-1} \cdots \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} a_{\ell_{n-1}} \ell_{n-1} c_{n-\ell_1} \prod_{i=1}^{n-2} c_{\ell_i - \ell_{i+1}} \\
 = & \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-2} \frac{(-1)^j}{c_0^{j+1}} \sum_{\ell_1=j}^{n-1} \sum_{\ell_2=j-1}^{\ell_1-1} \cdots \sum_{\ell_j=1}^{\ell_{j-1}-1} c_{\ell_j} \ell_j c_{n-\ell_1} \prod_{i=1}^{j-1} c_{\ell_i - \ell_{i+1}} \\
 & + \frac{(-1)^{n-1}}{c_0^{n-1}} \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \sum_{\ell_2=n-2}^{\ell_1-1} \cdots \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} a_1 \ell_{n-1} c_{n-\ell_1} \prod_{i=1}^{n-2} c_{\ell_i - \ell_{i+1}} \\
 = & \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-2} \frac{(-1)^j}{c_0^{j+1}} \sum_{\ell_1=j}^{n-1} \sum_{\ell_2=j-1}^{\ell_1-1} \cdots \sum_{\ell_j=1}^{\ell_{j-1}-1} c_{\ell_j} \ell_j c_{n-\ell_1} \prod_{i=1}^{j-1} c_{\ell_i - \ell_{i+1}} \\
 & + \frac{(-1)^{n-1}}{c_0^n} \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \sum_{\ell_2=n-2}^{\ell_1-1} \cdots \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} c_1 \ell_{n-1} c_{n-\ell_1} \prod_{i=1}^{n-2} c_{\ell_i - \ell_{i+1}} \\
 = & \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-2} \frac{(-1)^j}{c_0^{j+1}} \sum_{\ell_1=j}^{n-1} \sum_{\ell_2=j-1}^{\ell_1-1} \cdots \sum_{\ell_j=1}^{\ell_{j-1}-1} c_{\ell_j} \ell_j c_{n-\ell_1} \prod_{i=1}^{j-1} c_{\ell_i - \ell_{i+1}} \\
 & + \frac{(-1)^{n-1}}{c_0^n} \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \sum_{\ell_2=n-2}^{\ell_1-1} \cdots \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} c_{\ell_{n-1}} \ell_{n-1} c_{n-\ell_1} \prod_{i=1}^{n-2} c_{\ell_i - \ell_{i+1}} \\
 = & \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{(-1)^j}{c_0^{j+1}} \sum_{\ell_1=j}^{n-1} \sum_{\ell_2=j-1}^{\ell_1-1} \cdots \sum_{\ell_j=1}^{\ell_{j-1}-1} c_{\ell_j} \ell_j c_{n-\ell_1} \prod_{i=1}^{j-1} c_{\ell_i - \ell_{i+1}} \\
 & + \frac{(-1)^{n-1}}{c_0^n} \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \sum_{\ell_2=n-2}^{\ell_1-1} \cdots \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} c_{\ell_{n-1}} \ell_{n-1} c_{n-\ell_1} \prod_{i=1}^{n-2} c_{\ell_i - \ell_{i+1}} \\
 = & \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{(-1)^j}{c_0^{j+1}} \sum_{\substack{j \leq \ell_1 \leq n-1, 2 \leq k \leq j, \\ j-k+1 \leq \ell_k \leq \ell_{k-1}-1}} \ell_j c_{\ell_j} c_{n-\ell_1} \prod_{i=1}^{j-1} c_{\ell_i - \ell_{i+1}}.
 \end{aligned}$$

Let  $m_0 = n - \ell_1, m_j = \ell_j$ , and  $m_i = \ell_i - \ell_{i+1}$  for  $1 \leq i \leq j - 1$ . Then

$$\sum_{i=0}^j m_i = n, \quad \ell_k = n - \sum_{i=0}^{k-1} m_i, \quad 1 \leq k \leq j,$$

and

$$\begin{aligned} a_n &= \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{(-1)^j}{c_0^{j+1}} \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} m_j \prod_{i=0}^j c_{m_i} \\ &= \frac{c_n}{c_0} + \frac{1}{n} \sum_{j=1}^{n-1} (-1)^j \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} m_j \prod_{i=0}^j \frac{c_{m_i}}{c_0}. \end{aligned}$$

The proof of Theorem 3.1 is complete. □

**Theorem 3.2** Under the conditions of Theorem 2.1, we have

$$e_n = e^{d_0} \left( d_n + \sum_{j=1}^{n-1} \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} \prod_{i=0}^j \frac{m_i d_{m_i}}{n - \sum_{q=0}^{i-1} m_q} \right), \quad n \in \mathbb{N}. \tag{3.2}$$

*Proof* From the recurrence relation (2.2) and by induction, it follows that

$$\begin{aligned} e_n - e_0 d_n &= \frac{1}{n} \sum_{k=1}^{n-1} e_k (n - k) d_{n-k} \\ &= \frac{1}{n} \sum_{k=1}^{n-1} \left[ e_0 d_k + \frac{1}{k} \sum_{p=1}^{k-1} e_p (k - p) d_{k-p} \right] (n - k) d_{n-k} \\ &= \frac{e_0}{n} \sum_{k=1}^{n-1} d_k (n - k) d_{n-k} + \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} e_p (k - p) d_{k-p} (n - k) d_{n-k} \\ &= \frac{e_0}{n} \sum_{k=1}^{n-1} d_k (n - k) d_{n-k} + \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} \left[ e_0 d_p + \frac{1}{p} \sum_{q=1}^{p-1} e_q (p - q) d_{p-q} \right] (k - p) d_{k-p} (n - k) d_{n-k} \\ &= \frac{e_0}{n} \sum_{k=1}^{n-1} d_k (n - k) d_{n-k} + \frac{e_0}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} d_p (k - p) d_{k-p} (n - k) d_{n-k} \\ &\quad + \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} \frac{1}{p} \sum_{q=1}^{p-1} e_q (p - q) d_{p-q} (k - p) d_{k-p} (n - k) d_{n-k} \\ &= \frac{e_0}{n} \sum_{k=1}^{n-1} d_k (n - k) d_{n-k} + \frac{e_0}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} d_p (k - p) d_{k-p} (n - k) d_{n-k} \\ &\quad + \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} \frac{1}{p} \sum_{q=1}^{p-1} \left[ e_0 d_q + \frac{1}{q} \sum_{t=1}^{q-1} e_t (q - t) d_{q-t} \right] \\ &\quad \times (p - q) d_{p-q} (k - p) d_{k-p} (n - k) d_{n-k} \end{aligned}$$

$$\begin{aligned}
 &= \frac{e_0}{n} \sum_{k=1}^{n-1} d_k(n-k)d_{n-k} + \frac{e_0}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} d_p(k-p)d_{k-p}(n-k)d_{n-k} \\
 &\quad + \frac{e_0}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} \frac{1}{p} \sum_{q=1}^{p-1} d_q(p-q)d_{p-q}(k-p)d_{k-p}(n-k)d_{n-k} \\
 &\quad + \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} \frac{1}{p} \sum_{q=1}^{p-1} \frac{1}{q} \sum_{t=1}^{q-1} e_t(q-t)d_{q-t}(p-q)d_{p-q}(k-p)d_{k-p}(n-k)d_{n-k} \\
 &= \frac{e_0}{n} \sum_{k=1}^{n-1} d_k(n-k)d_{n-k} + \frac{e_0}{n} \sum_{k=2}^{n-1} \frac{1}{k} \sum_{p=1}^{k-1} d_p(k-p)d_{k-p}(n-k)d_{n-k} \\
 &\quad + \frac{e_0}{n} \sum_{k=3}^{n-1} \frac{1}{k} \sum_{p=2}^{k-1} \frac{1}{p} \sum_{q=1}^{p-1} d_q(p-q)d_{p-q}(k-p)d_{k-p}(n-k)d_{n-k} \\
 &\quad + \frac{1}{n} \sum_{k=4}^{n-1} \frac{1}{k} \sum_{p=3}^{k-1} \frac{1}{p} \sum_{q=2}^{p-1} \frac{1}{q} \sum_{t=1}^{q-1} e_t(q-t)d_{q-t}(p-q)d_{p-q}(k-p)d_{k-p}(n-k)d_{n-k} \\
 &= \sum_{j=1}^{n-2} \frac{e_0}{n} \sum_{\ell_1=j}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=j-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{j-1}=2}^{\ell_{j-2}-1} \frac{1}{\ell_{j-1}} \sum_{\ell_j=1}^{\ell_{j-1}-1} d_{\ell_j}(n-\ell_1)d_{n-\ell_1} \\
 &\quad \times \prod_{i=1}^{j-1} (\ell_i - \ell_{i+1})d_{\ell_i-\ell_{i+1}} + \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=n-2}^{\ell_1-1} \frac{1}{\ell_2} \\
 &\quad \cdots \sum_{\ell_{n-2}=2}^{\ell_{n-3}-1} \frac{1}{\ell_{n-2}} \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} e_{\ell_{n-1}}(n-\ell_1)d_{n-\ell_1} \prod_{i=1}^{n-2} (\ell_i - \ell_{i+1})d_{\ell_i-\ell_{i+1}} \\
 &= \sum_{j=1}^{n-2} \frac{e_0}{n} \sum_{\ell_1=j}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=j-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{j-1}=2}^{\ell_{j-2}-1} \frac{1}{\ell_{j-1}} \sum_{\ell_j=1}^{\ell_{j-1}-1} d_{\ell_j}(n-\ell_1)d_{n-\ell_1} \\
 &\quad \times \prod_{i=1}^{j-1} (\ell_i - \ell_{i+1})d_{\ell_i-\ell_{i+1}} + \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=n-2}^{\ell_1-1} \frac{1}{\ell_2} \\
 &\quad \cdots \sum_{\ell_{n-2}=2}^{\ell_{n-3}-1} \frac{1}{\ell_{n-2}} \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} e_1(n-\ell_1)d_{n-\ell_1} \prod_{i=1}^{n-2} (\ell_i - \ell_{i+1})d_{\ell_i-\ell_{i+1}} \\
 &= \sum_{j=1}^{n-2} \frac{e_0}{n} \sum_{\ell_1=j}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=j-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{j-1}=2}^{\ell_{j-2}-1} \frac{1}{\ell_{j-1}} \sum_{\ell_j=1}^{\ell_{j-1}-1} d_{\ell_j}(n-\ell_1)d_{n-\ell_1} \\
 &\quad \times \prod_{i=1}^{j-1} (\ell_i - \ell_{i+1})d_{\ell_i-\ell_{i+1}} + \frac{1}{n} \sum_{\ell_1=n-1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=n-2}^{\ell_1-1} \frac{1}{\ell_2} \\
 &\quad \cdots \sum_{\ell_{n-2}=2}^{\ell_{n-3}-1} \frac{1}{\ell_{n-2}} \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} e_0d_1(n-\ell_1)d_{n-\ell_1} \prod_{i=1}^{n-2} (\ell_i - \ell_{i+1})d_{\ell_i-\ell_{i+1}} \\
 &= \sum_{j=1}^{n-2} \frac{e_0}{n} \sum_{\ell_1=j}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=j-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{j-1}=2}^{\ell_{j-2}-1} \frac{1}{\ell_{j-1}} \sum_{\ell_j=1}^{\ell_{j-1}-1} d_{\ell_j}(n-\ell_1)d_{n-\ell_1}
 \end{aligned}$$



$$\begin{aligned}
 & \times \prod_{i=1}^{j-1} (\ell_i - \ell_{i+1}) d_{\ell_i - \ell_{i+1}} + \frac{e_0}{n} \sum_{\ell_1=n-1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=n-2}^{\ell_1-1} \frac{1}{\ell_2} \\
 & \cdots \sum_{\ell_{n-2}=2}^{\ell_{n-3}-1} \frac{1}{\ell_{n-2}} \sum_{\ell_{n-1}=1}^{\ell_{n-2}-1} d_{\ell_{n-1}} (n - \ell_1) d_{n-\ell_1} \prod_{i=1}^{n-2} (\ell_i - \ell_{i+1}) d_{\ell_i - \ell_{i+1}} \\
 & = \sum_{j=1}^{n-1} \frac{e_0}{n} \sum_{\ell_1=j}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=j-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{j-1}=2}^{\ell_{j-2}-1} \frac{1}{\ell_{j-1}} \sum_{\ell_j=1}^{\ell_{j-1}-1} d_{\ell_j} (n - \ell_1) d_{n-\ell_1} \prod_{i=1}^{j-1} (\ell_i - \ell_{i+1}) d_{\ell_i - \ell_{i+1}} \\
 & = \frac{e^{d_0}}{n} \sum_{j=1}^{n-1} \sum_{\substack{j \leq \ell_1 \leq n-1, 2 \leq k \leq j, \\ j-k+1 \leq \ell_k \leq \ell_{k-1}-1}} \left( \prod_{i=1}^j \frac{1}{\ell_i} \right) \ell_j d_{\ell_j} (n - \ell_1) d_{n-\ell_1} \prod_{i=1}^{j-1} (\ell_i - \ell_{i+1}) d_{\ell_i - \ell_{i+1}}.
 \end{aligned}$$

Let  $m_0 = n - \ell_1$ ,  $m_j = \ell_j$ , and  $m_i = \ell_i - \ell_{i+1}$  for  $1 \leq i \leq j - 1$ . Then

$$\sum_{i=0}^j m_i = n, \quad \ell_k = n - \sum_{i=0}^{k-1} m_i, \quad 1 \leq k \leq j,$$

and

$$\begin{aligned}
 e_n &= e^{d_0} d_n + \frac{e^{d_0}}{n} \sum_{j=1}^{n-1} \sum_{\substack{n-j \geq m_0 \geq 1, 2 \leq k \leq j, \\ n-j+k-1 - \sum_{i=0}^{k-2} m_i \geq m_{k-1} \geq 1, \\ \sum_{i=0}^j m_i = n}} \prod_{i=1}^j \frac{1}{n - \sum_{q=0}^{i-1} m_q} \prod_{i=0}^j m_i d_{m_i} \\
 &= e^{d_0} d_n + e^{d_0} \sum_{j=1}^{n-1} \sum_{\substack{1 \leq m_0 \leq n-j, 2 \leq k \leq j, \\ 1 \leq m_{k-1} \leq n-j+k-1 - \sum_{i=0}^{k-2} m_i, \\ \sum_{i=0}^j m_i = n}} \prod_{i=0}^j \frac{m_i d_{m_i}}{n - \sum_{q=0}^{i-1} m_q} \\
 &= e^{d_0} \left( d_n + \sum_{j=1}^{n-1} \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} \prod_{i=0}^j \frac{m_i d_{m_i}}{n - \sum_{q=0}^{i-1} m_q} \right).
 \end{aligned}$$

The proof of Theorem 3.2 is complete. □

#### 4 Expansions of the exponential and logarithm of power series

By similar arguments as in the proofs of Theorems 3.1 and 3.2, we can obtain the following power series expansions of the exponential and the logarithm of a power series expansion. For simplicity, we do not write down their proofs in details.

**Theorem 4.1** *Let*

$$D(x) = \sum_{k=0}^{\infty} d_k x^k$$

*be a power series expansion. Then, the function  $E(x) = e^{D(x)}$  has the power series expansion*

$$E(x) = \sum_{k=0}^{\infty} e_k x^k,$$



where the coefficients  $e_k$  for  $k \in \{0\} \cup \mathbb{N}$  satisfy the formulas (2.1), (2.2), (3.2), and

$$e_k = e^{d_0} \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{\sum_{\ell=1}^j i_\ell = k, \\ i_\ell \geq 1, 1 \leq \ell \leq j}} \prod_{\ell=1}^j d_{i_\ell}, \quad k \in \mathbb{N}. \tag{4.1}$$

**Theorem 4.2** Let  $c_0 \neq 0$  and

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

be a given power series expansion. Then, the composition  $A(x) = \ln C(x)$  has the expansion

$$A(x) = \sum_{n=1}^{\infty} a_n x^n,$$

where the coefficients  $a_n$  for  $n \in \mathbb{N}$  satisfy (1.4) and (3.1).

### 5 Applications of Theorems 4.1 and 4.2

In this section, we will apply Theorems 4.1 and 4.2, respectively, to the Bell numbers and logarithmic polynomials which are extensively studied in combinatorics and number theory.

#### 5.1 An application of Theorems 4.1 to the Bell numbers

In combinatorics, Bell numbers, usually denoted by  $B_n$  for  $n \in \{0\} \cup \mathbb{N}$ , count the number of ways a set with  $n$  elements can be partitioned into disjoint and nonempty subsets. Every Bell number  $B_n$  can be generated by

$$e^{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

The first few Bell numbers  $B_n$  are

$$\begin{aligned} B_0 = 1, \quad B_1 = 1, \quad B_2 = 2, \quad B_3 = 5, \quad B_4 = 15, \\ B_5 = 52, \quad B_6 = 203, \quad B_7 = 877, \quad B_8 = 4140, \quad B_9 = 21147. \end{aligned} \tag{5.1}$$

For more detailed information, refer to [2, pp. 210–212, Section 5.4].

**Theorem 5.1** The Bell numbers  $B_k$  for  $k \in \mathbb{N}$  can be computed by

$$B_k = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} B_\ell, \tag{5.2}$$

$$B_k = 1 + k! \sum_{j=1}^{k-1} \sum_{\substack{\sum_{i=0}^j m_i = k, \\ m_i \geq 1, 0 \leq i \leq j}} \frac{1}{\prod_{i=0}^j [(m_i - 1)! (k - \sum_{q=0}^{i-1} m_q)]}, \tag{5.3}$$

and

$$B_k = k! \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{\sum_{\ell=1}^j i_\ell = k, \\ i_\ell \geq 1, 1 \leq \ell \leq j}} \frac{1}{\prod_{\ell=1}^j i_\ell!}. \tag{5.4}$$

*Proof* Applying Theorem 4.1 to  $D(x) = e^x - 1 = \sum_{k=1}^{\infty} \frac{x^k}{k!}$  yields

$$d_0 = 0, \quad d_k = \frac{1}{k!}, \quad \text{and} \quad e_k = \frac{B_k}{k!}$$

for  $k \in \mathbb{N}$ . From (2.2) in Theorem 4.1, it follows that

$$\frac{B_k}{k!} = \frac{1}{k} \sum_{\ell=1}^k \frac{1}{(\ell-1)!} \frac{B_{k-\ell}}{(k-\ell)!} = \frac{1}{k} \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell-1)!} \frac{B_{\ell}}{\ell!}, \quad k \in \mathbb{N}.$$

This is equivalent to

$$\begin{aligned} B_k &= (k-1)! \sum_{\ell=1}^k \frac{1}{(\ell-1)!} \frac{B_{k-\ell}}{(k-\ell)!} = (k-1)! \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell-1)!} \frac{B_{\ell}}{\ell!} \\ &= \sum_{\ell=1}^k \binom{k-1}{\ell-1} B_{k-\ell} = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} B_{\ell} \end{aligned}$$

for  $k \in \mathbb{N}$ .

From (2.1) and (3.2) in Theorem 4.1, it follows that  $B_0 = 0!e_0 = 0!e^{d_0} = 1$  and

$$\begin{aligned} B_k &= k!e_k \\ &= k! \left\{ \frac{1}{k!} + \sum_{j=1}^{k-1} \sum_{\substack{\sum_{i=0}^j m_i = k, \\ m_i \geq 1, 0 \leq i \leq j}} \frac{1}{\prod_{i=0}^j [(m_i - 1)! (k - \sum_{q=0}^{i-1} m_q)]} \right\} \\ &= 1 + k! \sum_{j=1}^{k-1} \sum_{\substack{\sum_{i=0}^j m_i = k, \\ m_i \geq 1, 0 \leq i \leq j}} \frac{1}{\prod_{i=0}^j [(m_i - 1)! (k - \sum_{q=0}^{i-1} m_q)]} \end{aligned}$$

for  $k \in \mathbb{N}$ . The formula (5.3) follows.

From (4.1) in Theorem 4.1, it follows that

$$\frac{B_k}{k!} = e^0 \sum_{j=1}^k \frac{1}{j!} \sum_{\sum_{\ell=1}^j i_{\ell} = k} \prod_{\ell=1}^j \frac{1}{i_{\ell}!} = \sum_{j=1}^k \frac{1}{j!} \sum_{\sum_{\ell=1}^j i_{\ell} = k} \frac{1}{\prod_{\ell=1}^j i_{\ell}!}, \quad k \in \mathbb{N}.$$

This can be easily rewritten as (5.4). The required proof is complete. □

### 5.2 An application of Theorem 4.2 to logarithmic polynomials

According to the monograph [2, pp. 140–141], the logarithmic polynomials  $L_n$  can be defined by

$$\ln \left( \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} \right) = \sum_{n=1}^{\infty} L_n \frac{t^n}{n!}, \tag{5.5}$$

where  $g_0 = G(a) = 1$ ,  $g_n = G^{(n)}(a)$  for  $n \in \mathbb{N}$ , and  $G(x)$  is an infinitely differentiable function at  $x = a$ , and they are expressions for the  $n$ th derivative of  $\ln G(x)$  at the point  $x = a$ .

**Theorem 5.2** For  $n \in \mathbb{N}$ , the logarithmic polynomials  $L_n$  can be computed by

$$L_n = g_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} g_{n-k} L_k \quad (5.6)$$

and

$$L_n = g_n + (n-1)! \sum_{j=1}^{n-1} (-1)^j \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} m_j \prod_{i=0}^j \frac{g_{m_i}}{m_i!}. \quad (5.7)$$

*Proof* Applying Theorem 4.2 to the Eq. (5.5) gives

$$c_0 = g_0 = 1, \quad c_n = \frac{g_n}{n!}, \quad a_n = \frac{L_n}{n!}, \quad n \in \mathbb{N}.$$

Substituting these quantities into (1.4) and (3.1) produces

$$\frac{L_n}{n!} = \frac{g_n}{n!} - \frac{1}{n} \sum_{k=1}^{n-1} k \frac{L_k}{k!} \frac{g_{n-k}}{(n-k)!} = \frac{1}{n!} \left[ g_n - \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k)!} g_{n-k} L_k \right]$$

and

$$\frac{L_n}{n!} = \frac{g_n}{n!} + \frac{1}{n} \sum_{j=1}^{n-1} (-1)^j \sum_{\substack{\sum_{i=0}^j m_i = n, \\ m_i \geq 1, 0 \leq i \leq j}} m_j \prod_{i=0}^j \frac{g_{m_i}}{m_i!}.$$

Hence, the formulas (5.6) and (5.7) follow immediately. The proof of Theorem 5.2 is complete.  $\square$

## 6 Remarks

*Remark 6.1* When  $d_0 = 0$ , Theorem 2.1 becomes [1, Lemma 4]. Therefore, our Theorem 2.1 is a slight generalization of [1, Lemma 4]. The constant term  $d_0$  does not appear in the recursion formula (2.2), but it has a relation with the constant term  $e_0$ . On the other hand, the second proof of Theorem 2.1 reveals that Theorems 1.5 and 2.1 are equivalent to each other. However, it is clear that Theorem 2.1 can be used more conveniently.

*Remark 6.2* The formulas (3.2) and (4.1) are not that different. The difference is that in (4.1) we have the products  $d_{i_1} d_{i_2} \cdots d_{i_\ell}$  whereas in (3.2) we have these products with weight factors. But if we group terms which are obtained by permutation of the indices, then these weights add up to the expected value. For example, if we want to find the coefficient of  $z^6$  in

$$\left( \sum_{k=1}^{\infty} d_k z^k \right)^3,$$

then  $d_1 d_2 d_3$  appears six times which makes the total contribution  $6d_1 d_2 d_3$ . In the formula (3.2), these six terms are written as:

$$6 \sum \prod_{i=0}^2 \frac{m_i d_{m_i}}{\sum_{q=i}^2 m_q},$$

where the sum is over the six permutations of (1, 2, 3) for  $(m_0, m_1, m_2)$ . The total contribution of  $d_1 d_2 d_3$  is the same as before. Therefore, one could say that the formula (3.2) is not that different from (4.1).

Since the proofs of the formulas (5.3) and (5.4) based on the formulas (3.2) and (4.1), the formula (5.3) is not that different from (5.4) yet.



*Remark 6.3* Letting  $k = 3, 4$  in (5.4), respectively, gives

$$\begin{aligned} B_3 &= 3! \sum_{j=1}^3 \frac{1}{j!} \sum_{\sum_{\ell=1}^j i_\ell=3} \frac{1}{\prod_{\ell=1}^j i_\ell!} \\ &= 3! \left( \frac{1}{1!} \sum_{\sum_{\ell=1}^1 i_\ell=3} \frac{1}{\prod_{\ell=1}^1 i_\ell!} + \frac{1}{2!} \sum_{\sum_{\ell=1}^2 i_\ell=3} \frac{1}{\prod_{\ell=1}^2 i_\ell!} + \frac{1}{3!} \sum_{\sum_{\ell=1}^3 i_\ell=3} \frac{1}{\prod_{\ell=1}^3 i_\ell!} \right) \\ &= 3! \left[ \frac{1}{3!} + \frac{1}{2!} \left( \frac{1}{1!2!} + \frac{1}{2!1!} \right) + \frac{1}{3!} \frac{1}{1!1!1!} \right] \\ &= 5 \end{aligned}$$

and

$$\begin{aligned} B_4 &= 4! \sum_{j=1}^4 \frac{1}{j!} \sum_{\sum_{\ell=1}^j i_\ell=4} \frac{1}{\prod_{\ell=1}^j i_\ell!} \\ &= 4! \left( \frac{1}{1!} \sum_{\sum_{\ell=1}^1 i_\ell=4} \frac{1}{\prod_{\ell=1}^1 i_\ell!} + \frac{1}{2!} \sum_{\sum_{\ell=1}^2 i_\ell=4} \frac{1}{\prod_{\ell=1}^2 i_\ell!} \right. \\ &\quad \left. + \frac{1}{3!} \sum_{\sum_{\ell=1}^3 i_\ell=4} \frac{1}{\prod_{\ell=1}^3 i_\ell!} + \frac{1}{4!} \sum_{\sum_{\ell=1}^4 i_\ell=4} \frac{1}{\prod_{\ell=1}^4 i_\ell!} \right) \\ &= 4! \left[ \frac{1}{4!} + \frac{1}{2!} \left( \frac{1}{1!3!} + \frac{1}{2!2!} + \frac{1}{3!1!} \right) + \frac{1}{3!} \left( \frac{1}{1!1!2!} + \frac{1}{1!2!1!} + \frac{1}{2!1!1!} \right) + \frac{1}{4!} \frac{1}{1!1!1!1!} \right] \\ &= 15. \end{aligned}$$

These two concrete results are the same as corresponding ones listed in (5.1).

*Remark 6.4* Finding various expressions for a mathematical quantity is meaningful and significant in mathematics. For example, mathematicians find several infinite product expressions for  $\pi$ . From this point of view, the formulas (5.3) and (5.4) are all useful somewhat.

*Remark 6.5* In [2, pp. 140–141, Theorem A], it was given that the logarithmic polynomials  $L_n$  equal

$$L_n = L_n(g_1, g_2, \dots, g_n) = \sum_{k=1}^n (-1)^{k-1} (k-1)! B_{n,k}(g_1, g_2, \dots, g_{n-k+1}),$$

where  $B_{n,k}$  denotes the Bell polynomials of the second kind, which are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N}, \\ \sum_{i=1}^n i \ell_i = n, \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i}$$

for  $n \geq k \geq 0$ . This is different from the formulas (5.6) and (5.7).

*Remark 6.6* The recurrence (5.2) is a recovery of a known result listed in [4, p. 373, Exercise 15]. However, the explicit formula (5.3) is new.

*Remark 6.7* The Bell numbers  $B_k$  are connected with the Lah numbers  $L(n, k)$  and the Stirling numbers of the second kind  $S(n, k)$ . See, for example, the papers [5, 7, 8] and closely related references therein. Importantly, some inequalities for determinants and products of the Bell numbers  $B_k$  were established in [8].

*Remark 6.8* This paper is a revised version of the preprint [10].

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