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## Inferences using type-II progressively censored data with binomial removals

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**Abstract** This paper considers the estimation problem for Burr type-X model, when the lifetimes are collected under type-II progressive censoring with random removals, where the number of units removed at each failure time follows a binomial distribution. The methods of maximum likelihood as well as the Bayes procedure to derive both point and interval estimates of the parameters are used. The expected test time to complete the censoring test is computed and analyzed for different censoring schemes. The effect of the binomial distribution parameter  $p$  on the expected test time under progressive censoring and the relative expected test time over the complete sample are investigated. Monte Carlo simulations are performed to compare and evaluate the performance of different methods. Furthermore, an example with a real data set is presented for illustrative purposes.

**Mathematics Subject Classification** 62N05 · 62F10

### الملخص

يستعرض هذا البحث مسألة تقدير معلمة نموذج بر العاشر، عندما تكون العينة المتاحة هي عينة ذات مراقبة تتابعية من النوع الثاني مع الحذف العشوائي، بحيث يكون عدد المجموعات المستبعدة من التجربة عند أول سقوط هو متغير عشوائي يتبع قانون ذات الحدين بالمعلمة  $p$ ، وباستخدام طريقة الإمكان الأكبر و طريقة بيزر تم إيجاد تقديرات النقطة والفترة لمعلمة توزيع بر العاشر ومعلمة توزيع ذات الحدين  $p$ . أيضا تم حساب وقت الاختبار المتوقع لإنهاء التجربة للعينات ذات المراقبة التتابعية من النوع الثاني مع الحذف العشوائي وتم مناقشة تأثير  $p$  على الوقت المتوقع للاختبار ومقارنة ذلك مع نتائج العينة الكاملة، كما تم انجاز دراسة المحاكاة للنتائج بهدف المقارنة بين المقدرات المختلفة، وعلاوة على ذلك، تم تقديم مثلا عدديا باستخدام مجموعة من البيانات الحقيقية بغرض التوضيح.

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## 1 Introduction

In many life test studies, it is common that the lifetimes of test units may not be able to record exactly. An experimenter may terminate the life test before all  $n$  products fail to save time or cost. Therefore, the test is considered to be censored in which data collected are the exact failure times on those failed units and the running times on those non-failed units. A generalization of type-II censoring is progressive type-II censoring which is useful when the loss of life test units at points other than the termination point is unavoidable. For the theory methods and applications of progressive censoring, one can refer to the monograph by Balakrishnan and Aggarwala [1] and the survey paper by Balakrishnan [2]. The progressively type-II censoring scheme is described as follows. First, the experimenter places  $n$  units on test. If the first failure is observed,  $r_1$  of surviving units are randomly removed. When the  $i$ th failure unit is observed,  $r_i$  of surviving units are randomly selected and removed,  $i = 2, \dots, m$ . This experiment terminates when the  $m$ th failure unit is observed and  $r_m = n - m - r_1 - \dots - r_{m-1}$  of surviving units are all removed. Note that, in this scheme,  $r_1, r_2, \dots, r_m$ , the number of units  $n$  and the number of observed failure times  $m$  are all prefixed. However, in some practical situations, these numbers may occur at random. For example, in some reliability experiments, an experimenter may decide that it is inappropriate to carry on the testing on some of the tested units even though these units have not failed. In such cases, the pattern of removal at each failure is random.

Inference, sampling design and generalization based on progressively censored samples were studied by many authors, see among others, Balasooriya et al. [3], Ng et al. [4], Balakrishnan et al. [5], Fernandez [6], Soliman [7], Asgharzadeh [8], Ku and Kaya [9], Wu et al. [10], Banerjee and Kundu [11] and Raqab et al. [12]. However, a little work is introduced in the Bayesian context. Amin [13] considered the Bayes estimation and Bayes prediction problems for Pareto distribution based on the progressive type-II censoring with random removals. Also, in Bayesian setting, Sarhan and Abuammoh [14] discussed some statistical inference for the exponential distribution using progressively censoring sample with random removals. Wu [15] has studied estimation for Pareto distribution under progressive censoring with uniform removals. A uniform removal pattern may not seem very realistic as it assumes that each removal event occurs with an equal probability regardless of the number of units removed. A more realistic alternative to describe the number of occurrences of an event out of  $n$  trials is the binomial distribution as suggested by Tse et al. [16]. Classical and Bayesian procedures are developed in this paper in the context of parameter estimation and estimated the expected test time for Burr-X model under progressive censoring with binomial removals.

The rest of this paper is organized as follows. The model formulation and the corresponding likelihood function under type-II progressive censoring with binomial removals are discussed in Sect. 2. In Sect. 3, the procedures of obtaining the maximum likelihood estimates of the parameters  $\theta$  and  $p$  are discussed. Both point and interval estimations of the parameters are derived. Point and interval estimations using Bayesian procedures are presented in Sect. 4. In Sect. 5, we discuss the expected test time under progressive type-II censoring with the effect of various  $p$ . Finally, we will give an example with a real data set to illustrate our proposed methods. Results from simulation studies assessing the performance of our proposed methods are included in Sect. 6.

## 2 The model

Let the lifetime of a particular unit have a Burr type-X distribution with probability density function (pdf)

$$f(x; \theta) = 2\theta x \exp(-x^2) (1 - \exp(-x^2))^{\theta-1}, \quad x > 0, \theta > 0. \quad (1)$$

The corresponding cumulative distribution function(cdf) is

$$F_X(x) = (1 - \exp(-x^2))^\theta, \quad x > 0, \theta > 0. \quad (2)$$

Let  $(X_1, R_1), (X_2, R_2), \dots, (X_m, R_m)$  denote a progressively type-II censored sample, where  $X_1 < X_2 < \dots < X_m$  with predetermined number of removals, say  $R_1 = r_1, R_2 = r_2, \dots, R_m = r_m$ . Then, the conditional likelihood function can be written as (see Cohen [17])

$$L(\theta; x | R = r) = c \prod_{i=1}^m f(x_i) [1 - F(x_i)]^{r_i}, \quad (3)$$



where  $c = n(n - r_1 - 1) \cdots (n - r_1 - r_2 - \cdots - r_{m-1} - m + 1)$ , and  $r_i$  can be any integer value between 0 and  $(n - m - r_1 - \cdots - r_{i-1})$  for  $i = 1, 2, 3, \dots, m - 1$ . Substituting (1) and (2), into (3) gives

$$L(\theta; x | R = r) = c T(x) \theta^m \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \exp\left(\theta \sum_{i=1}^m (j_i + 1) \ln U_i\right), \tag{4}$$

where

$$T(x) = \prod_{i=1}^m 2x_i \exp(-x_i^2) (U_i)^{-1}, U_i = (1 - \exp(-x_i^2)),$$

$$G = (-1)^{j_1 + \cdots + j_m} \binom{r_1}{j_1} \cdots \binom{r_m}{j_m}. \tag{5}$$

Suppose that an individual unit being removed from the test at the  $i$ th failure,  $i = 1, 2, \dots, m - 1$  is independent of the others, but with same probability  $p$ . Then, the number  $R_i$  of units removed at the  $i$ th failure,  $i = 1, 2, \dots, m - 1$ , follows a binomial distribution with parameters  $n - m - \sum_{l=1}^{i-1} r_l$  and  $p$ .  $\text{bin}\left(n - m - \sum_{l=1}^{i-1} r_l, p\right)$ . Therefore,

$$P(R_1 = r_1) = \binom{n - m}{r_1} p^{r_1} (1 - p)^{n - m - r_1}, \tag{6}$$

and for  $i = 1, 2, 3, \dots, m - 1$

$$P(R_i = r_i | R_{i-1} = r_{i-1}, \dots, R_1 = r_1) = \binom{n - m - \sum_{l=1}^{i-1} r_l}{r_i} p^{r_i} (1 - p)^{n - m - \sum_{l=1}^{i-1} r_l - r_i}, \tag{7}$$

where  $0 \leq r_1 \leq n - m$  and  $0 \leq r_i \leq n - m - \sum_{l=1}^{i-1} r_l$  for  $i = 1, 2, 3, \dots, m - 1$ . Furthermore, suppose that  $R_i$  is independent of  $x_i$  for all  $i$ . Then, the likelihood function takes the following form,

$$L(\theta, p; x, r) = L(\theta; x | R = r) P(R = r), \tag{8}$$

where

$$P(R = r) = P(R_1 = r_1) \times P(R_2 = r_2 | R_1 = r_1) \times P(R_3 = r_3 | R_2 = r_2, R_1 = r_1) \times \cdots \times P(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \dots, R_1 = r_1). \tag{9}$$

Then,

$$P(R = r) = \frac{(n - m)!}{\left(n - m - \sum_{l=1}^{m-1} r_l\right)! \prod_{i=1}^{m-1} r_i!} p^{\sum_{i=1}^{m-1} r_i} (1 - p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i)r_i}. \tag{10}$$

Using (4), (8) and (10), the likelihood function takes the following form

$$L(\theta, p; x, r) = c^* T(x) L_1(\theta) L_2(p), \tag{11}$$

where

$$L_1(\theta) = \theta^m \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \exp\left(\theta \sum_{i=1}^m (j_i + 1) \ln U_i\right), \tag{12}$$

and

$$L_2(p) = p^{\sum_{l=1}^{m-1} r_l} (1 - p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i)r_i}, \tag{13}$$

with  $c^* = \frac{c(n - m)!}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)! \prod_{i=1}^{m-1} r_i!}$  and  $T(x) = \prod_{i=1}^m 2x_i \exp(-x_i^2) (U_i)^{-1}$ . It should be noted that both  $c^*$  and  $T(x)$  do not depend on the parameters  $\theta$  and  $p$ .

### 3 Maximum likelihood estimation

This section discusses the procedures of obtaining the maximum likelihood estimates of the parameters  $\theta$  and  $p$  based on progressively type-II censoring data with binomial removals. Both point and interval estimations of the parameters are derived.

#### 3.1 Point estimation

It is obvious that  $L_1$  in (12) does not involve  $p$ . Thus, the maximum likelihood estimate (MLE) of  $\theta$  can be derived by maximizing (12) directly. On the other hand,  $L_2$  in (13) does not depend on the parameter  $\theta$ , then the MLE of  $p$  can be obtained directly by maximizing (13). In particular, after taking the logarithms of  $L_1(\theta)$  and  $L_2(p)$ , the MLEs of  $\theta$  and  $p$  can be found by solving the following equations

$$\frac{m}{\hat{\theta}} + \sum_{j_i=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \sum_{i=1}^m (j_i + 1) \ln U_i = 0, \quad (14)$$

$$\frac{\sum_{i=1}^{m-1} r_i}{\hat{p}} - \frac{\left( (m-1)(n-m) - \sum_{i=1}^{m-1} (m-i-1)r_i \right)}{1 - \hat{p}} = 0. \quad (15)$$

Making use of (14) and (15) yields

$$\hat{\theta}_{ML} = (-m) \left( \sum_{j_i=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \sum_{i=1}^m (j_i + 1) \ln U_i \right)^{-1}, \quad (16)$$

and

$$\hat{p}_{ML} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i-1)r_i}. \quad (17)$$

#### 3.2 Interval estimation

##### 3.2.1 Bootstrap confidence intervals

In this subsection, we use the parametric bootstrap percentile method suggested by Efron [18]. The algorithms for estimating confidence intervals of the parameters  $\theta$  and  $p$  are illustrated as follows

1. From the original data  $X = X_1, X_2, \dots, X_m$  with the corresponding values  $R = r_i, i = 1, 2, \dots, m$ , compute the ML estimates  $\hat{\theta}$  and  $\hat{p}$  of the parameters using (16) and (17).
2. Use  $\hat{p}$  to generate a bootstrap sample  $R^* = r_i^*, i = 1, 2, \dots, m$  using binomial distribution, where  $r_1^*$  follows the bin  $(n - m, \hat{p})$  distribution and the variables  $r_i^* | r_1^*, r_2^*, \dots, r_{i-1}^*$  follow the bin  $(n - m - \sum_{j=1}^{i-1} r_j, \hat{p})$  distribution for  $i = 2, 3, \dots, m - 1$ .
3. Use  $\hat{\theta}$  in step 1, with the binomial progressive censoring scheme obtained in step 2, generate a bootstrap sample  $X^* = X_1^*, X_2^*, \dots, X_m^*$  using algorithm presented in Balakrishnan and Sandhu [19].
4. As in step 1, based on  $X^*$ , compute the bootstrap sample estimates of  $\theta$  and  $p$ , say  $\hat{\theta}^*$  and  $\hat{p}^*$ .
5. Repeat steps 2–4  $N$  BOOT times.
6. Arrange all  $\hat{\theta}^*$ 's and  $\hat{p}^*$ 's, in an ascending order to obtain the bootstrap sample  $(\varphi_l^{[1]}, \varphi_l^{[2]}, \dots, \varphi_l^{[N]})$ ,  $l = 1, 2$  (where  $\varphi_1 \equiv \hat{\theta}^*, \varphi_2 \equiv \hat{p}^*$ ).

Let  $G(z) = P(\varphi_l \leq z)$  be the cumulative distribution function (cdf) of  $\varphi_l$ .

Define  $\varphi_{l\text{boot}} = G^{-1}(z)$  for given  $z$ . The  $100(1 - \gamma)\%$  approximate bootstrap confidence interval of  $\varphi_l$  is given by

$$\left[ \varphi_{l\text{boot}} \left( \frac{\gamma}{2} \right), \varphi_{l\text{boot}} \left( 1 - \frac{\gamma}{2} \right) \right]. \quad (18)$$



### 3.2.2 Approximate interval estimation

The approximate confidence intervals based on asymptotic distributions of the MLE of the parameters  $\theta$  and  $p$  are derived. The asymptotic variances and covariances of the MLE for parameters  $\theta$  and  $p$  are given by elements of the inverse of Fisher information matrix

$$\mathbf{I}_{ij} = E \left[ -\frac{\partial^2 L}{\partial \theta \partial p} \right]; \quad i, j = 1, 2. \tag{19}$$

Unfortunately, the exact mathematical expressions for the above expectations are difficult to obtain. Therefore, we give the approximate (observed) asymptotic variance–covariance matrix for the MLE, which is obtained by dropping the expectation operator  $E$  and it can be written as

$$\left[ \begin{array}{cc} -\frac{\partial^2 L}{\partial \theta^2} & -\frac{\partial^2 L}{\partial \theta \partial p} \\ -\frac{\partial^2 L}{\partial p \partial \theta} & -\frac{\partial^2 L}{\partial p^2} \end{array} \right]_{(\hat{\theta}, \hat{p})}^{-1} = \left[ \begin{array}{cc} \text{var}(\hat{\theta}) & \text{cov}(\hat{\theta}, \hat{p}) \\ \text{cov}(\hat{\theta}, \hat{p}) & \text{var}(\hat{p}) \end{array} \right], \tag{20}$$

with

$$\frac{\partial^2 L}{\partial \theta^2} = -\frac{m}{\theta^2}, \quad \frac{\partial^2 L}{\partial \theta \partial p} = \frac{\partial^2 L}{\partial p \partial \theta} = 0, \tag{21}$$

and

$$\frac{\partial^2 L}{\partial p^2} = \frac{\sum_{i=1}^{m-1} r_i}{p^2} + \frac{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i)r_i}{(1-p)^2}. \tag{22}$$

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for parameters  $\theta$  and  $p$ .

Therefore,  $(1 - \alpha)100\%$  approximate confidence intervals for parameters  $\theta$  and  $p$  become

$$\hat{\theta} \pm Z_{\alpha/2} \sqrt{\text{var}(\hat{\theta})} \quad \text{and} \quad \hat{p} \pm Z_{\alpha/2} \sqrt{\text{var}(\hat{p})}, \tag{23}$$

where  $Z_{\alpha/2}$  is the percentile of the standard normal distribution with right-tail probability  $\alpha/2$ .

## 4 Bayes estimation

It is well known that choice of loss function is an integral part of Bayesian estimation procedures. A wide variety of loss functions has been developed in the literature to describe various types of loss structures. The symmetric square error loss SEL is one of the most popular loss functions. It is widely employed in the inference, but its application is motivated by its good mathematical properties, not by its applicability for representing a true loss structure. A loss function should represent the consequences of different errors. There are situations where overestimation and underestimation can lead to different consequences. Such conditions are very common in engineering, medical and biomedical sciences. For example, when we estimate the average reliable working life of the components, overestimation is usually more serious than underestimation. In this case, an asymmetric loss function might be more appropriate. A number of asymmetric loss functions may be found in the literature, but among these asymmetric losses, LINEX loss function (LLF) is dominantly and widely used because it is a natural extension of SEL.

The mathematical form of LLF may simply be expressed as

$$L(\Delta) \propto e^{c\Delta} - c\Delta - 1; \quad c \neq 0, \tag{24}$$

where  $\Delta = (\tilde{u} - u)$ ,  $\tilde{u}$  is an estimate of  $u$ .

It is easy to verify that the value of  $\tilde{u}$  that minimizes  $E_u(L(\tilde{u} - u))$  in (24) is

$$\tilde{u}_{BL} = -\frac{1}{c} \log(E_u[\exp(-cu)]), \tag{25}$$

another useful asymmetric loss function is the General Entropy loss (GEL) function

$$L_2(\tilde{u}, u) \propto (\tilde{u}/u)^q - q \log(\tilde{u}/u) - 1, \quad (26)$$

whose minimum occurs at  $\tilde{u} = u$ . The Bayes predictive estimate  $\tilde{u}_{BG}$  of  $u$  under GEL (26) is

$$\tilde{u}_{BG} = (E_u[u^{-q}])^{-1/q}. \quad (27)$$

For more details, see Soliman [7].

#### 4.1 Point estimation

We assume that the parameters  $\theta$  and  $p$  behave as independent random variables. As prior distributions in derivation of Bayes estimators of the parameters, we use conjugate gamma prior distribution with known parameters  $\alpha, \beta$  for  $\theta$ . Namely, the prior pdf of  $\theta$  takes the following form

$$\pi_1(\theta | \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{(\alpha-1)} \exp(-\beta\theta), \quad \theta > 0, \alpha > 0, \beta > 0, \quad (28)$$

while  $p$  has conjugate Beta prior distribution with known parameters  $\gamma, \lambda$ . That is, the prior pdf of  $p$  is given by

$$\pi_2(p) = \frac{1}{B(\gamma, \lambda)} p^{(\gamma-1)} (1-p)^{(\lambda-1)}, \quad 0 < p < 1, \gamma, \lambda > 0. \quad (29)$$

The joint prior (pdf) of  $\theta$  and  $p$  is

$$\begin{aligned} \pi(\theta, p) &= \pi_1(\theta) \pi_2(p), \theta > 0, 0 < p < 1 \\ &= \frac{\beta^\alpha}{B(\gamma, \lambda) \Gamma(\alpha)} \theta^{(\alpha-1)} \exp\{-\beta\theta\} p^{(\gamma-1)} (1-p)^{(\lambda-1)}. \end{aligned} \quad (30)$$

Therefore, the joint posterior (pdf) of  $\theta$  and  $p$  is

$$\pi^*(\theta, p | x, r) = \frac{p^{(\gamma^*-1)} (1-p)^{(\lambda^*-1)} \sum_{j_1=0}^{r_1} \sum_{j_m=0}^{r_m} G \theta^{m+\alpha-1} \exp\{-q_j \theta\}}{B(\gamma^*, \lambda^*) \Gamma(m+\alpha) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-m-\alpha}}, \quad (31)$$

where

$$q_j = \beta - \sum_{i=1}^m (j_i + 1) \ln U_i, \quad \gamma^* = \gamma + \sum_{l=1}^{m-1} r_l \quad \text{and} \quad \lambda^* = \lambda + (m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i. \quad (32)$$

Therefore, the marginal posterior (pdf) of  $\theta$  and  $p$  is given by

$$\pi_1^*(\theta | x, r) = \frac{\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \theta^{(m+\alpha-1)} \exp(-\theta q_j)}{\Gamma(m+\alpha) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-m-\alpha}}, \quad (33)$$

and

$$\pi_2^*(p | x, r) = \frac{1}{B(\gamma^*, \lambda^*)} p^{(\gamma^*-1)} (1-p)^{(\lambda^*-1)}. \quad (34)$$

It is noted that the posterior distribution of  $\theta$  is Gamma with parameters  $(m+\alpha)$  and  $q_j$ , while the posterior distribution of  $p$  is Beta with parameters  $\gamma^*$  and  $\lambda^*$ .



4.1.1 Symmetric Bayes estimation

*SEL function* Under *SEL* function (symmetric), the estimator of a parameter is the posterior mean. Thus, Bayes estimators of the parameters are obtained using the posterior densities (33) and (34).

The Bayes estimators  $\tilde{\theta}_{BS}$  and  $\tilde{p}_{BS}$  of the parameters  $\theta$  and  $p$  are

$$\begin{aligned} \tilde{\theta}_{BS} &= \int_0^\infty \theta \pi_1^*(\theta | x, r) \, d\theta \\ &= \frac{(m + \alpha) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-(m+\alpha+1)}}{\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-m-\alpha}}, \end{aligned} \tag{35}$$

and

$$\tilde{p}_{BS} = \int_0^1 p \pi_2^*(p | x \cdot r) \, dp = \frac{\gamma^*}{\gamma^* + \lambda^*}. \tag{36}$$

4.1.2 Asymmetric Bayes estimation

- (i) *LL function* If we put  $u = \theta$  in (25), then the Bayes estimate  $\tilde{\theta}_{BL}$  of parameter  $\theta$  relative to the LLF using (33) is

$$\begin{aligned} \tilde{\theta}_{BL} &= -\frac{1}{a} \log \left[ \int_0^\infty \exp(-a\theta) \pi_1^*(\theta | \underline{x}) \, d\theta \right] \\ &= -\frac{1}{a} \log \left[ \frac{\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G (a + q_j)^{-(m+\alpha)}}{\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-(m+\alpha)}} \right]. \end{aligned} \tag{37}$$

Similarly, if in (25),  $u = p$ , then the Bayes estimate  $\tilde{p}_{BL}$  of the parameter  $p$  relative to the LLF using (34) is

$$\begin{aligned} \tilde{p}_{BL} &= -\frac{1}{a} \log \left[ \int_0^1 \exp(-ap) \pi_2^*(p | x, r) \, dp \right] \\ &= -\frac{1}{a} \log \left[ \frac{1}{B(\gamma^*, \lambda^*)} \int_0^1 \exp(-ap) p^{(\gamma^*-1)} (1-p)^{(\lambda^*-1)} \, dp \right]. \end{aligned} \tag{38}$$

One can use a numerical integration technique to get the above integration (38).

- (ii) *GEL function* Let  $u = \theta$  in (27), then the Bayes estimate  $\tilde{\theta}_{BG}$  of parameter  $\theta$  relative to the GEL function (27) is

$$\begin{aligned} \tilde{\theta}_{BG} &= [E_\theta(\theta^{-b} | \underline{x})]^{(-1/b)} \\ &= \left[ \int_0^\infty \theta^{-b} \pi_1^*(\theta | x, r) \, d\theta \right]^{(-1/b)}. \end{aligned} \tag{39}$$

From (33) resulting in

$$\tilde{\theta}_{BG} = \left[ \frac{\Gamma(m + \alpha - b) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-(m+\alpha-b)}}{\Gamma(m + \alpha) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G q_j^{-(m+\alpha)}} \right]^{(-1/b)}. \tag{40}$$

Let  $u = p$ , in (27), then the Bayes estimate  $\tilde{p}_{BG}$  of parameter  $p$  relative to the GEL function is

$$\begin{aligned} \tilde{p}_{BG} &= \left[ \int_0^1 p^{-b} \pi_2^*(p | \underline{x}) \, dp \right]^{(-1/b)} \\ &= \left[ \frac{\Gamma(\gamma^* + \lambda^*) \Gamma(\gamma^* - b)}{\Gamma(\gamma^*) \Gamma(\gamma^* + \lambda^* - b)} \right]^{(-1/b)}. \end{aligned} \tag{41}$$

## 4.2 Interval estimation

### 4.2.1 Highest posterior density interval (HPDI)

In general, the Bayesian method to interval estimation is much more direct than the maximum likelihood method. Now, having obtained the posterior distribution  $p(\omega \mid \text{Data})$ , we ask, “How likely is it that the parameter  $\omega$  lies within the specified interval  $[\omega_L, \omega_U]$ ?”. Bayesian call this interval based on the posterior distribution a ‘credible interval’. The interval  $[\omega_L, \omega_U]$  is said to be a  $(1 - \alpha)100\%$  credible interval for  $\theta$  if

$$\int_{\omega_L}^{\omega_U} p(\omega \mid \text{Data}) \, d\theta = 1 - \alpha. \quad (42)$$

For the shortest credible interval, we have to minimize the interval  $[\omega_L, \omega_U]$  subject to the condition (42) which requires

$$p(\omega_L \mid \text{Data}) = p(\omega_U \mid \text{Data}). \quad (43)$$

As interval  $[\omega_L, \omega_U]$  which simultaneously satisfies (42) and (43) is called the ‘shortest’  $(1 - \alpha)100\%$  credible interval. A highest posterior density interval (HPDI) is such that the posterior density for every point inside the interval is greater than that for every point outside of it. For a unimodal, but not necessarily symmetrical, posterior density and the shortest credible and the HPD intervals are identical.

We now proceed to obtain the  $(1 - \alpha)100\%$  HPD intervals for the parameters  $\theta$  and  $p$ . Consider the posterior distribution of  $\theta$  in (33). Due to the unimodality of (33), the  $(1 - \alpha)100\%$  HPDI  $[\theta_L, \theta_U]$  for the parameter  $\theta$  is given by the simultaneous solution of the equations

$$\int_{\theta_L}^{\theta_U} \pi_1^*(\theta \mid x, r) \, d\theta = (1 - \alpha) \quad \text{and} \quad \pi_1^*(\theta_L \mid x, r) = \pi_1^*(\theta_U \mid x, r). \quad (44)$$

Similarly, using the posterior pdf of  $p$  in (34), the  $(1 - \alpha)100\%$  HPDI  $[p_L, p_U]$  for the parameter  $p$  is given by the simultaneous solution of the equations

$$\int_{p_L}^{p_U} \pi_2^*(p \mid x, r) \, dp = (1 - \alpha) \quad \text{and} \quad \pi_2^*(p_L \mid x, r) = \pi_2^*(p_U \mid x, r). \quad (45)$$

Now, we have to use a mathematical package to get the numerical solution to solve these two systems.

## 5 Expected test time

In practical application, an experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an appropriate sampling plan because the time required to complete a test is directly related to the cost. Under a type-II censoring plan, the time required to complete a test is the time to observe the  $m$ th failure in a sample of  $n$  test units. The time is given by  $\mathbf{X}_m$ , which denotes the  $m$ th order statistics in a sample of size  $n$ . Similarly, under progressive type-II censoring sampling plan with random or binomial removals conditioning on  $R$ , the expected value of  $\mathbf{X}_m$  (see Balakrishnan and Aggarwala [1]) is given by

$$\begin{aligned} E[\mathbf{X}_m \mid R = r] &= C(r) \sum_{l_1=0}^{r_1} \cdots \sum_{l_m=0}^{r_m} (-1)^{al} \frac{\binom{r_1}{l_1} \cdots \binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h(l_i)} \int_0^\infty x f(x) F^{h(l_i)-1}(x) \, dx \\ &= 2\theta C(r) \sum_{l_1=0}^{r_1} \cdots \sum_{l_m=0}^{r_m} (-1)^{al} \frac{\binom{r_1}{l_1} \cdots \binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h(l_i)} \sum_{k=0}^{\theta h(l_m)-1} (-1)^k \binom{\theta h(l_m)-1}{k} \\ &\quad \times \left( \frac{\sqrt{\pi}}{4(k+1)^{3/2}} \right), \end{aligned} \quad (46)$$

where



$$al = \sum_{i=1}^m l_i, C(r) = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots \left[ n - \sum_{i=1}^{m-1} (r_i - 1) \right], \tag{47}$$

and  $h(l_i) = l_1 + l_2 + \cdots + l_i + i$ . Furthermore, the expected time of a type-II censoring test without removals can be found by setting  $r_i = 0$  for all  $i = 1, \dots, m - 1$  and  $r_m = n - m$  in (46). It is given by

$$E[X_m^*] = 2m\theta \binom{n}{m} \sum_{k=0}^{m\theta-1} (-1)^k \binom{m\theta - 1}{k} \left( \frac{\sqrt{\pi}}{4(k + 1)^{3/2}} \right). \tag{48}$$

Similarly, the expected time of a complete sampling case with  $n$  test units can also be obtained by setting  $m = n$  and  $r_i = 0$  for all  $i = 1, \dots, m$  in (46). It is given by

$$E[X_n^{**}] = 2n\theta \sum_{k=0}^{n\theta-1} (-1)^k \binom{n\theta - 1}{k} \left( \frac{\sqrt{\pi}}{4(k + 1)^{3/2}} \right). \tag{49}$$

The expected termination point for progressively type-II censoring with binomial removals is evaluated by taking expectation on both sides (46) with respect to the  $R$ . That is,

$$E[X_m] = E_R[E[\mathbf{X}_m | R = r]] = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_1)-r_2} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} P(R, p) E[\mathbf{X}_m | R = r], \tag{50}$$

where  $g(r_1) = n - m$ ,  $g(r_i) = n - m - r_1 - \cdots - r_{i-1}$ ,  $i = 2, \dots, m - 1$ , and  $P(R, p)$  is given in (10).

The ratio of the expected time under different schemes to the expected time under complete sampling namely, ratio of expected experiment times (REET) is

$$REET = \frac{\text{Expected experiment time under different schemes}}{\text{Expected experiment time under complete sample}} \tag{51}$$

Suppose that an experimenter wants to observe the failure of at least  $m$  complete failures when the test is anticipated to be conducted under different schemes. Then, the REET provides important information in determining whether the experiment time can be shortened significantly if a much larger sample of  $n$  test units is used and the test is stopped once  $m$  failures are observed.

To compare (49) and (50), we use:  $REET = \frac{E[X_m]}{E[X_n^{**}]}$  which define the ratio of the expected termination point under type-II progressive censoring with binomial removals and the expected termination point for complete sample. It is clear that when the value of REET closer to 1, the termination point is closer to the complete sample.

### 6 Illustrative example

*Example 6.1 (Real life data)* To illustrate the inferential procedures developed in the preceding sections, we choose the real data set which was also used in Lawless ([20], pp. 185). These data are from Nelson [21] concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34kv (minutes). The 19 times to breakdown are

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

Consider a life test where 19 units of lifetimes are put in test, simultaneously. The test is terminated at the time of the thirteenth failure. The number of surviving items removed from the experiment at the failure of each units denoted by  $r_i$  is generated from the binomial distribution as follows:  $r_1$  from bin  $(6, p = 0.4)$  distribution and the variables  $r_i | r_1, r_2, \dots, r_{i-1}$  from bin  $(6 - \sum_{j=1}^{i-1} r_j, 0.4)$  distributions for  $i = 2, 3, \dots, 12$ . We set  $r_m$  according to the following relation:  $r_m = n - m - \sum_{l=1}^{i-1} r_l$  if  $n - m - \sum_{l=1}^{i-1} r_l > 0$  and  $r_m = 0$ , otherwise. The observed failure times of the first 13 units measured in an informative experiment with the corresponding values of  $r_i$  are:  $(x_i, r_i) = (0.19, 2), (0.78, 1), (0.96, 1), (1.31, 1), (2.78, 1), (3.16, 0), (4.67, 0), (4.85, 0), (6.5, 0), (7.35, 0), (8.01, 0), (8.27, 0), (12.06, 0)$ . These data generated with  $n = 19$  and  $m = 13$  using algorithm presented

**Table 1** Different point estimates for  $\theta$  and  $p$

Parameters	( $\cdot$ ) <sub>MI</sub>	( $\cdot$ ) <sub>Boot</sub>	( $\cdot$ ) <sub>BS</sub>	( $\cdot$ ) <sub>BL</sub>			( $\cdot$ ) <sub>BG</sub>		
				a			b		
				-1	0.0001	1	-1	0.0001	1
$\theta$	3.1761	3.1960	3.1822	3.5397	3.1822	2.9094	3.1822	3.086	2.9890
$p$	0.3750	0.3822	0.3750	0.3820	0.3750	0.3682	0.3750	0.3553	0.3333

**Table 2** 95% confidence intervals for  $\theta$  and  $p$

Parameters	ACI	Length	BCI	Length	HPDI	Length
$\theta$	[1.6386,4.7136]	3.075	[1.8347,5.0574]	3.2227	[1.8391,4.9024]	3.0634
$p$	[0.1378,0.6122]	0.4744	[0.1475,0.6615]	0.5140	[0.1634,0.6162]	0.4528

**Table 3** The MSE of the estimate of  $\theta$  for  $(\theta, p) = (0.321, 0.4)$ . with CVP of 95% ACI and 95% HPDI

$n$	$m$	ML	SE	LINEX		GE		CVP	CVP
				a		b			
				-1	1	-1	1		
30	27	0.0645	0.0644	0.0656	0.0633	0.0644	0.0609	0.950	0.971
	24	0.0685	0.0683	0.0696	0.0671	0.0683	0.0646	0.944	0.970
	18	0.0727	0.0730	0.0747	0.0714	0.0730	0.0679	0.957	0.972
40	12	0.0785	0.0791	0.0811	0.0773	0.0791	0.0734	0.939	0.966
	36	0.0573	0.0575	0.0585	0.0566	0.0575	0.0542	0.956	0.965
	32	0.0578	0.0580	0.0589	0.0571	0.0580	0.0550	0.958	0.976
60	24	0.0617	0.0620	0.0631	0.0610	0.0620	0.0589	0.941	0.971
	16	0.0657	0.0663	0.0676	0.0651	0.0663	0.0626	0.945	0.966
	54	0.0455	0.0455	0.0459	0.0451	0.0455	0.0442	0.948	0.972
80	48	0.0479	0.0480	0.0485	0.0475	0.0480	0.0464	0.937	0.967
	36	0.0506	0.0509	0.0516	0.0503	0.0509	0.0489	0.944	0.974
	24	0.0528	0.0534	0.0541	0.0526	0.0534	0.0511	0.957	0.968
100	72	0.0400	0.0401	0.0404	0.0398	0.0401	0.0391	0.940	0.981
	64	0.0411	0.0413	0.0416	0.0409	0.0413	0.0400	0.947	0.978
	48	0.0431	0.0434	0.0438	0.0430	0.0434	0.0420	0.956	0.975
100	32	0.0479	0.0485	0.0490	0.0479	0.0485	0.0466	0.951	0.971
	90	0.0338	0.0338	0.0340	0.0337	0.0338	0.0332	0.951	0.977
	80	0.0362	0.0363	0.0365	0.0361	0.0363	0.0356	0.941	0.972
100	60	0.0363	0.0365	0.0367	0.0362	0.0365	0.0357	0.961	0.986
	40	0.0399	0.0403	0.0407	0.0400	0.0403	0.0391	0.952	0.985

in Balakrishnan and Sandhu [19]. We use our results of the previous sections and the above data to derive different estimates of the parameters  $\theta$  and  $P$ . The Bayes estimates are derived under the non-informative prior ( $\alpha = \beta = \gamma = \lambda = 0$ ) provides prior distributions which are not proper. The Bayes point estimates relative to squared error, LINEX and general entropy loss functions are denoted, respectively, by: ( $\cdot$ )<sub>BS</sub>, ( $\cdot$ )<sub>BL</sub>, ( $\cdot$ )<sub>BG</sub>. The results of different point estimates are shown in Table 1. Also, we compute 95% approximate confidence interval (ACI), 95% Bootstrap confidence interval (BCI) and 95% highest posterior density interval (HPDI) for the parameters  $\theta$  and  $P$ . The results are given in Table 2.

### 7 A Simulation study

*Example 7.1* In this example, we consider tests with  $n = 30, 40, 60$  and  $100$  units generated from Burr-X distribution with parameter  $\theta$ . The test is terminated when the number of failed subjects achieves or exceeds a certain value  $m$ , where  $m/n = 40, 60, 80$  and  $90\%$ . Data for the dropouts  $r_i$  were generated from the binomial distribution as described in example 1. 1000 samples are generated for the parameter value  $(\theta, p) = (0.321, 0.4)$ . The samples were generated using the algorithm described in Balakrishnan and Sandhu [19]. We used different sample sizes ( $n$ ), different sampling schemes (i.e., different  $R_i$  values) and informative priors  $(\alpha, \beta) = (\gamma, \lambda) = (1, 2)$  with the values of previous parameters  $\theta$  and  $p$ , respectively. For each simulated data set, the MLE, symmetric Bayes and asymmetric Bayes estimates of  $\theta$  and  $p$  and their mean square error

**Table 4** The MSE of the estimator of  $p$  for  $(\theta, p) = (0.321, 0.4)$ . with CVP of 95 % ACI and 95 % HPDI

$n$	$m$	ML	SE	LINEX		GE		CVP ACI	CVP HPDI
				a		b			
				-1	1	-1	1		
30	27	0.1713	0.1030	0.1070	0.1001	0.1030	0.1140	0.973	0.982
	24	0.1329	0.0995	0.1019	0.0974	0.0995	0.1006	0.936	0.978
	18	0.1000	0.0864	0.0877	0.0853	0.0864	0.0857	0.928	0.966
40	12	0.0766	0.0695	0.0703	0.0683	0.0695	0.0685	0.953	0.982
	36	0.1571	0.1026	0.1062	0.0997	0.1026	0.1026	0.922	0.982
	32	0.1161	0.0937	0.0955	0.0920	0.0937	0.0937	0.958	0.979
60	24	0.0858	0.0764	0.0774	0.0753	0.0764	0.0747	0.942	0.972
	16	0.0653	0.0611	0.0616	0.0607	0.0611	0.0607	0.952	0.984
	54	0.1368	0.1018	0.1045	0.0994	0.1018	0.1012	0.937	0.976
80	48	0.0991	0.0854	0.0868	0.0842	0.0854	0.0842	0.930	0.963
	36	0.0649	0.0604	0.0610	0.0598	0.0604	0.0596	0.956	0.981
	24	0.0527	0.0505	0.0507	0.0502	0.0505	0.0504	0.950	0.982
100	72	0.1232	0.0984	0.1005	0.0964	0.0984	0.0966	0.945	0.983
	64	0.0818	0.0737	0.0745	0.0730	0.0737	0.0734	0.943	0.977
	48	0.0571	0.0543	0.0546	0.0540	0.0543	0.0539	0.952	0.975
100	32	0.0455	0.0437	0.0439	0.0435	0.0437	0.0434	0.952	0.978
	90	0.1102	0.0919	0.0935	0.0904	0.0919	0.0907	0.928	0.955
	80	0.0740	0.0682	0.0690	0.0675	0.0682	0.0673	0.940	0.980
	60	0.0505	0.0484	0.0487	0.0482	0.0484	0.0481	0.948	0.976
	40	0.0423	0.0412	0.0413	0.0410	0.0412	0.0410	0.941	0.965

**Table 5** Expected test time  $E[X_m]$  for type-II progressive censoring with binomial removals: for  $\theta = 1$

$n$	$m$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
6	6	1.5203	1.5203	1.5203	1.5203	1.5203	1.5203	1.5203	1.5203	1.5203
	5	1.2537	1.3162	1.3618	1.3941	1.4160	1.4304	1.4394	1.4449	1.4482
	4	1.0043	1.0635	1.1235	1.1825	1.2363	1.2810	1.3144	1.3366	1.3498
	3	0.7925	0.8365	0.8844	0.9367	0.9931	1.0514	1.1080	1.1580	1.1970
10	10	1.6757	1.6757	1.6757	1.6757	1.6757	1.6757	1.6757	1.6757	1.6757
	9	1.5057	1.5771	1.6114	1.6271	1.6341	1.6373	1.6390	1.6399	1.6406
	8	1.3358	1.4549	1.5256	1.5631	1.5817	1.5908	1.5955	1.5981	1.5999
	7	1.1586	1.2927	1.4013	1.4719	1.5114	1.5317	1.5421	1.5479	1.5516
	6	1.0060	1.1082	1.1981	1.2714	1.3305	1.3809	1.4259	1.4637	1.4884
15	5	0.8667	0.9684	1.0799	1.1854	1.2719	1.3346	1.3757	1.4009	1.4161
	15	1.7914	1.7914	1.7914	1.7914	1.7914	1.7914	1.7914	1.7914	1.7914
	14	1.4617	1.6105	1.7098	1.7521	1.7657	1.7692	1.7700	1.7704	1.7706
	13	1.2286	1.4340	1.6046	1.6928	1.7288	1.7416	1.7458	1.7473	1.7479
	12	1.1335	1.3766	1.5755	1.6725	1.7072	1.7174	1.7204	1.7219	1.7228
	11	1.0939	1.3261	1.5268	1.6313	1.6724	1.6862	1.6910	1.6934	1.6950
	10	0.9943	1.0772	1.2417	1.3968	1.5140	1.5909	1.6344	1.6552	1.6631
	9	0.9039	0.9929	1.1738	1.3372	1.4513	1.5268	1.5781	1.6109	1.6264
8	0.6164	0.7063	0.9518	1.2154	1.4048	1.5085	1.5565	1.5772	1.5859	
	7	0.5360	0.5979	0.8036	1.0596	1.2787	1.4213	1.4943	1.5245	1.5362

(MSE) are calculated. Using results in Sects. 3.2 and 4.2, the approximate confidence interval (ACI) based on asymptotic distributions of the MLEs and HPDI of the parameters is calculated and compared based on their coverage probabilities (CVP). All of the computations were performed by (mathematica 7.0) using a Pentium 4 processor, so we cannot compute the BCI and CVP to the bootstrap method. Tables 3 and 4 present the simulation results for  $\theta$  and  $p$ , respectively.

The following results emerge from the simulation results

1. The ML method provides better estimates of  $\theta$  in the sense of having smaller MSE, while the Bayes estimates of  $p$  have smaller MSE.
2. The MSE associated with both MLE and Bayes estimates of the parameters decrease with increasing the sample size  $n$ . Also, it decreases when the percentage of censoring becomes small (i.e.,  $m$  is large).
3. The method using highest posterior density performs better than the asymptotic normality-based procedure in giving closer coverage probability to the nominal value.

**Table 6** Expected test time  $E[X_m]$  for type-II progressive censoring with binomial removals for  $\theta = 2$

$n$	$m$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
6	6	1.7285	1.7285	1.7285	1.7285	1.7285	1.7285	1.7285	1.7285	1.7285
	5	1.4870	1.5436	1.5850	1.6142	1.6341	1.6472	1.6553	1.6603	1.6632
	4	1.2592	1.3092	1.3617	1.4154	1.4659	1.5086	1.5407	1.5618	1.5742
10	3	1.0678	1.1067	1.1485	1.1945	1.2451	1.2987	1.3516	1.3988	1.4354
	10	1.8698	1.8698	1.8698	1.8698	1.8698	1.8698	1.8698	1.8698	1.8698
	9	1.7149	1.7800	1.8112	1.8255	1.8319	1.8348	1.8363	1.8372	1.8378
15	8	1.5611	1.6691	1.7333	1.7674	1.7843	1.7925	1.7968	1.7992	1.8008
	7	1.9374	1.5178	1.6179	1.6836	1.7202	1.7388	1.7483	1.7536	1.7570
	6	1.2609	1.3457	1.4200	1.4824	1.5359	1.5851	1.6317	1.6721	1.6987
	5	1.1370	1.2286	1.3290	1.4244	1.5030	1.5600	1.5975	1.6204	1.6342
	15	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754
	14	1.6289	1.7832	1.8904	1.9365	1.9512	1.9550	1.9558	1.9562	1.9564
	13	1.3870	1.5967	1.7800	1.8759	1.9152	1.9291	1.9336	1.9350	1.9356
	12	1.2989	1.5442	1.7553	1.8596	1.8969	1.9075	1.9105	1.9118	1.9127
	11	1.2767	1.5032	1.7114	1.8216	1.8648	1.8790	1.8837	1.8859	1.8873
15	10	1.1895	1.2416	1.4048	1.5688	1.6956	1.7797	1.8273	1.8499	1.8581
	9	1.1087	1.1662	1.3440	1.5137	1.6350	1.7164	1.7723	1.8080	1.8245
	8	0.7774	0.8479	1.1056	1.3891	1.5942	1.7064	1.7578	1.7795	1.7880
	7	0.6990	0.7406	0.9559	1.2306	1.4674	1.6217	1.7002	1.7316	1.7430

**Table 7** Expected test time  $E[X_m]$  for type-II progressive censoring with binomial removals for  $\theta = 3$

$n$	$m$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
6	6	1.8414	1.8414	1.8414	1.8414	1.8414	1.8414	1.8414	1.8414	1.8414
	5	1.6128	1.6664	1.7055	1.7332	1.7520	1.7644	1.7721	1.7768	1.7796
	4	1.3967	1.4418	1.4903	1.5412	1.5899	1.6316	1.6630	1.6836	1.6954
10	3	1.2179	1.2538	1.2921	1.3345	1.3817	1.4326	1.4835	1.5291	1.5643
	10	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754	1.9754
	9	1.8283	1.8901	1.9198	1.9333	1.9394	1.9422	1.9436	1.9445	1.9450
15	8	1.6826	1.7849	1.8458	1.8782	1.8942	1.9021	1.9061	1.9084	1.9099
	7	1.5260	1.6391	1.7350	1.7982	1.8333	1.8511	1.8601	1.8652	1.8683
	6	1.3982	1.4737	1.5396	1.5963	1.6469	1.6956	1.7431	1.7850	1.8126
	5	1.2832	1.3690	1.4633	1.5533	1.6277	1.6818	1.7173	1.7390	1.7521
	15	2.0761	2.0761	2.0761	2.0761	2.0761	2.0761	2.0761	2.0761	2.0761
	14	1.7197	1.8774	1.9890	2.0372	2.0526	2.0564	2.0573	2.0576	2.0579
	13	1.4741	1.6856	1.8758	1.9760	2.0170	2.0315	2.0361	2.0375	2.0381
	12	1.3849	1.6350	1.8533	1.9616	2.0003	2.0112	2.0142	2.0154	2.0163
	11	1.3838	1.5997	1.8119	1.9251	1.9696	1.9840	1.9886	1.9907	1.9921
15	10	1.2752	1.3294	1.4933	1.6622	1.7944	1.8824	1.9322	1.9558	1.9643
	9	1.2490	1.2610	1.4362	1.6094	1.7347	1.8194	1.8778	1.9151	1.9323
	8	0.8360	0.9230	1.1885	1.4831	1.6969	1.8137	1.8671	1.8892	1.8978
	7	0.8176	0.8190	1.0380	1.3229	1.5695	1.7303	1.8118	1.8439	1.8550

4. Another values of the parameter  $\theta$  were considered and the results were not reported here because those cases exhibited a similar pattern to the case considered here. But in general, we note that the MSE of all estimates of the parameters  $\theta$  and  $p$  is reduced when the removal probability  $p$  is large.

*Example 7.2* In this example, we compute numerically the expected test time  $E[X_m]$  for different values of  $n, m, p$  and  $\theta$  under progressive type-II censoring with binomial removals. We consider the following values :  $n = 6, 10, 15; \theta = 1, 2, 3$  and  $p = 0.1(0.1)0.9$ . The cases of  $m = n$  correspond to the complete sample plan. The results are displayed in Tables 5, 6 and 7.

From Tables 5, 6 and 7, it is observed that the expected termination time for type-II progressive censoring sample is getting close to that of the complete sample when  $m$  is increasing. For fixed  $m$ , the expected experiment time of type-II progressive with binomial removals decreases when the sample size  $n$  increasing. Also, with respect to binomial removals, it is clear that for fixed  $n$  and  $m$ , the REET getting closer to one faster for increasing  $p$ . All results are due to the fact that a high removal probability implies a large number of dropouts. Thus, the removal probability  $p$  plays a major factor in the time required to complete the experiment. In all cases, a large number of test units  $n$  would shorten the experiment time of the test when the underlying scheme is type-II progressive with binomial removals.

## 8 Conclusions

The purpose of this paper is to develop a maximum likelihood estimation and Bayesian estimation of the Burr type-X distribution when data are collected under type-II progressive censoring with binomial removals. We investigate both point and interval estimations of the parameters and the expected time to complete the test. The results show that the MSE of different estimators of the parameter  $\theta$  is decreasing when the removal probability  $p$  increasing, on the other hand, the corresponding time required to complete the test increases significantly. We also computed the expected termination time for type-II progressive censoring with binomial removals. Illustrative Example and simulation study were conducted to examine the performance of the MLE and the Bayes estimators. Finally, we discussed some numerical results concerning the expected test time. In summary, results of the numerical examples demonstrate that when data are collected under type-II progressive censoring with binomial removals, the test time is most influenced by the removal probability  $p$ . From Tables 5, 6, 7, users can decide the censoring number in their life test under the consideration of expected termination point. Generally, the results would provide important information to experimenters in planning a life test.

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