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Afif Ben Amar

On an integral equation under Henstock–Kurzweil–Pettis integrability

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Abstract In this paper, we investigate the set of solutions for nonlinear Volterra type integral equations in Banach spaces in the weak sense and under Henstock–Kurzweil–Pettis integrability. Moreover, a fixed point result is presented for weakly sequentially continuous mappings defined on the function space $C(K, X)$, where K is compact Hausdorff and X is a Banach space. The main condition is expressed in terms of axiomatic measure of weak noncompactness.

Mathematics Subject Classification 47H10 · 28B05 · 45D05

الملخص

نبحث في هذه الورقة مجموعة الحل لمعادلات تكاملية غير خطية من نوع فولتيرا في فضاءات باناخ بالمعنى الضعيف في ظل قابلية تكامل هنستوك – كورتسوايل – بيتيس. بالإضافة إلى ذلك، يتم عرض نتيجة نقطة ثابتة للرواسم متوالية الاتصال المعرفة على فضاء الدوال $C(K,X)$، حيث K فضاء هاوسدورف متراص و X فضاء باناخ. يتم التعبير عن الشرط الرئيس بدلالة مقياس بَدَهي لعدم التراص الضعيف.

1 Introduction

The differential, integral and integro-differential problems in Banach spaces have been widely studied by many authors. Recently, for problems involving highly oscillating functions, many authors have examined the existence of solutions under Henstock–Kurzweil–Pettis integrability [1,7–9,19,20,24–28].

In this paper, motivated by these examinations we focus on the existence of solutions in the weak sense for the nonlinear Volterra type integral equation in Banach spaces

$$x(t) = h(t) + \int_0^t G(t,s)f(s, x(s), \int_0^s k(s,\tau)x(\tau)\mathrm{d}\tau)\mathrm{d}s, \tag{1.1}$$

involving the Henstock–Kurzweil–Pettis integral. The main tools used in our study are associated with the techniques of measure of weak noncompactness, properties and convergence theorems mainly of Vitali type for Henstock–Kurzweil–Pettis integrals based on the notion of equi-integrability (see [11]). By using these tools, we are able to prove not only the existence of solutions of the considered integral equation, but also we can obtain a topological structure of the set of these solutions.

In the next section, we give some preliminary facts and present a fixed point result for function spaces. Our ideas were motivated originally by a theorem of Dobrakov [13] which has the interest to characterize weakly convergent sequences in $C(K, X)$ with weakly convergent sequences in X without equicontinuity

A. Ben Amar (✉)
Department of Mathematics, Faculty of Sciences, Sfax University, LA 1171, 3000 Sfax, Tunisia
E-mail: afif.benamar@ipeis.rnu.tn

conditions. In the last section, we use our fixed point result and the techniques of the theory of measure of weak noncompactness presented in Section 2 to establish existence principle for the nonlinear Volterra integral equation (1.1) under Henstock–Kurzweil–Pettis integrability. By imposing some conditions expressed in terms of the measure of weak noncompactness on f and k, we define an operator over the Banach space of continuous functions from a compact interval to a Banach space, whose fixed points are solutions of (1.1).

2 Preliminaries

Let X be a Banach space with the norm $||.||$ and let K be a compact and Hausdorff space. In what follows, we denote by $C(K, X)$ the Banach space of all continuous functions from K to X, endowed with the sup-norm $||.||_\infty$ defined by $||x||_\infty = \sup\{\|x(t)\|, t \in K\}$ for each $x \in C(K, X)$.

Definition 2.1 Let X be a Banach space and C a lattice with a least element, which is denoted by 0. By a measure of weak noncompactness on X, we mean a function Φ defined on the set of all bounded subsets of X with value in C satisfying:

(1) $\Phi(\overline{\text{conv}}(\Omega)) = \Phi(\Omega)$, for all bounded subsets $\Omega \subseteq X$, where $\overline{\text{conv}}$ denotes the closed convex hull of Ω,
(2) for any bounded subsets Ω_1, Ω_2 of X we have

$$\Omega_1 \subseteq \Omega_2 \Longrightarrow \Phi(\Omega_1) \le \Phi(\Omega_2),$$

(3) $\Phi(\Omega \cup \{a\}) = \Phi(\Omega)$ for all $a \in X$, Ω bounded set of X,
(4) If $\Phi(\Omega) = 0$, then Ω is relatively weakly compact in X.

The above notion is a generalization of the well-known De Blasi measure of weak noncompactness β (see [10]) defined on each bounded set Ω of X by

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \text{ there exists a weakly compact set } D \text{ such that } \Omega \subseteq D + B_\varepsilon(\theta)\}.$$

Note for all bounded subsets $\Omega, \Omega_1, \Omega_2$ of X,

(5) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$,
(6) $\beta(\lambda\Omega) = \lambda\beta(\Omega)$ for all $\lambda > 0$,
(7) $\beta(\Omega_1 + \Omega_2) \le \beta(\Omega_1) + \beta(\Omega_2)$.
(8) $\beta\left(\bigcup_{|\lambda|\le h} \lambda\Omega\right) = h\beta(\Omega)$.

Note that β is the counterpart for the weak topology of the classical Hausdorff measure of noncompactness. For more examples and properties of measures of weak noncompactness, we refer the reader to [2,4,5,21,22].

Definition 2.2 A function $f : X_1 \longrightarrow X_2$, where X_1 and X_2 are Banach spaces, is said to be weakly–weakly sequentially continuous if for each weakly convergent $(x_n)_n \subset X_1$ with $x_n \rightharpoonup x$, we have $fx_n \rightharpoonup fx$. Here, $\rightharpoonup$ denotes weak convergence.

The following fixed point result due to Arino et al. [3] will be used throughout this section.

Theorem 2.3 *Let X be a metrizable locally convex linear topological space and let C be a weakly compact convex subset of X. Then, any weakly sequentially continuous map $F : C \longrightarrow C$ has a fixed point.*

To characterize weak convergence in $C(K, X)$, we use Dobrakov's theorem

Theorem 2.4 ([13, Theorem 9]) *Let K be a compact Hausdorff space and X a Banach space. Let (x_n) be a bounded sequence in $C(K, X)$, and $x \in C(K, X)$. Then, (x_n) is weakly convergent to x if and only if $(x_n(t))$ is weakly convergent to $x(t)$ for each $t \in K$.*

Now, we state the following Ambrosetti's type lemma which will be useful in the sequel.

Lemma 2.5 *Let $V \subseteq C(K, X)$ be a family of strongly equicontinuous functions. Then*
(a)

$$\beta(V) = \sup_{t\in K} \beta(V(t)) = \beta(V(K))$$

where $V(t) = \{x(t) : x \in V\}$ and $V(K) = \bigcup_{t\in K}\{x(t) : x \in V\}$.

(b) *The function $t \longmapsto \beta(V(t))$ is continuous.*

The next fixed point result has the advantage to omit any equicontinuity conditions.

Theorem 2.6 *Let K be a compact Hausdorff space, X is a Banach space with Q a non-empty closed convex subset of $C(K, X)$ and Φ is a measure of weak noncompactness on X. Suppose $F : Q \longrightarrow Q$ satisfying:*

(i) *If $\{x_n\} \subset Q$ is a sequence with $x_n(t) \rightharpoonup x(t)$ for each $t \in K$, then $Fx_n(t) \rightharpoonup Fx(t)$ for each $t \in K$.*
(ii) *$F(Q)$ is bounded and F is Φ-condensing (i.e., $\Phi(F(Y)) < \Phi(Y)$ for all bounded subsets $Y \subset Q$ such that $\Phi(Y) \neq 0$).*

Then, the set of fixed points of F is non-empty and weakly compact in $C(K, X)$.

Proof Let $\mathcal{S}$ be the set of fixed points of F in Q. We claim that $\mathcal{S}$ is non-empty. Indeed, let $x_0 \in F(Q)$ and $\mathcal{G}$ the family of all closed bounded convex subsets D of $C(K, X)$ such that $x_0 \in D$ and $F(D) \subset D$. Obviously $\mathcal{G}$ is non-empty, since $\overline{\text{conv}}(F(Q)) \in \mathcal{G}$ (the closed convex hull of $F(Q)$ in $C(K, X)$). We denote $H = \bigcap_{D \in \mathcal{G}} D$. Then, H is closed and convex, and $x_0 \in H$. If $x \in H$, then $F(x) \in D$ for all $D \in \mathcal{G}$ and hence $F(H) \subset H$. Therefore, we have $H \in \mathcal{G}$. We claim that H is a weakly compact subset of $C(K, X)$. Denote $H_* = \overline{\text{conv}}(F(H) \cup \{x_0\})$. We have $H_* \subset H$, which implies that $F(H_*) \subset F(H) \subset H_*$. Therefore, $H_* \in \mathcal{G}$, $H \subset H_*$. Hence $H = H_*$. Clearly, H is bounded and if $\Phi(H) \neq 0$, we obtain

$$\Phi(H) = \Phi(\overline{\text{conv}}(F(H) \cup \{x_0\}) = \Phi(\text{conv}(F(H)) \cup \{x_0\}) = \Phi(F(H)) < \Phi(H),$$

which is a contradiction, so $\Phi(H) = 0$. Since, H is a weakly closed subset of $C(K, X)$ (notice a convex subset of a Banach space is closed iff it is weakly closed), then H is a weakly compact subset of $C(K, X)$. We claim that F is weakly sequentially continuous. Indeed, let $\{x_n\}$ be a sequence in H such that $x_n \rightharpoonup x$ in $C(K, X)$. By Theorem 2.4, we have $x_n(t) \rightharpoonup x(t)$ for each $t \in K$. Thus by hypothesis (i), $Fx_n(t) \rightharpoonup Fx(t)$ for each $t \in K$, and with the same argument we obtain $Fx_n \rightharpoonup Fx$ in $C(K, X)$. So, F is weakly sequentially continuous. It follows using Theorem 2.3 that $F : H \longrightarrow H$ has a fixed point and so $\mathcal{S} \neq \emptyset$. Because $\mathcal{S} \subset F(Q)$, $F(\mathcal{S}) = \mathcal{S}$ and F is Φ-condensing, we have $\Phi(\mathcal{S}) = 0$ and so $\mathcal{S}$ is a relatively weakly compact subset of $C(K, X)$. Also, by the sequentially weak continuity of F, the set $\mathcal{S}$ is weakly sequentially closed. Let $x \in Q$, be weakly adherent to $\mathcal{S}$. Since $\overline{\mathcal{S}^w}$, the weak closure of $\mathcal{S}$ in $C(K, X)$, is weakly compact, by the Eberlein–Šmulian theorem [14, Theorem 8.12.4, p. 549], there exists a sequence $\{x_n\} \subset \mathcal{S}$ such that $x_n \rightharpoonup x$, so $x \in \mathcal{S}$. Hence $\mathcal{S}$ is a weakly closed subset of Q. Therefore, $\mathcal{S}$ is weakly compact in $C(K, X)$. □

Remark 2.7 Theorem 2.6 is a special case of Theorem 12 in [16], namely where the Banach space is $C(K, X)$.

3 Main result

Let $I = [0, 1]$ and X be a real Banach space. In this section, we investigate topological structure of the set of solutions in weak sense of following nonlinear Volterra type integral equation (1.1), $x \in C(I, X)$ and involving the Henstock–Kurzweil–Pettis integral [7,8].

First, we introduce the concept of Henstock–Kurzweil–Pettis integrability and give some related facts which are useful in the sequel.

Definition 3.1 ([18]) A $\mathcal{K}$-partition of $[0, T]$ is a finite collection $\mathcal{P} = \{([c_k, d_k], t_k) : 1 \le k \le n\}$ such that $\{[c_k, d_k] : 1 \le k \le n\}$ is a nonoverlapping family of subintervals of $[0, T]$ covering $[a, b]$ and $t_k \in [c_k, d_k]$ for $k = 1, 2, \ldots, n$. A gauge on $[0, T]$ is a function $\delta : [0, T] \longrightarrow (0, \infty)$. A $\mathcal{K}$-partition $\mathcal{P} = \{([c_k, d_k], t_k) : 1 \le k \le n\}$ is δ-fine if $[c_k, d_k] \subseteq (t_k - \delta(t_k), t_k + \delta(t_k))$ for $k = 1, 2, \ldots, n$. A function $f : [0, T] \longrightarrow X$ is said to be Henstock–Kurzweil-integrable, or simply HK-integrable, on $[0, T]$ if there exists $w \in X$ with the following property: for each $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon(.)$ defined on $[0, T]$ such that

$$\left\| \sum_{k=1}^{n} f(t_k)(d_k - c_k) - w \right\| < \varepsilon$$

for any δ_ε-fine $\mathcal{K}$-partition $\mathcal{P} = \{([c_k, d_k], t_k) : 1 \le k \le n\}$ of $[0, T]$. We set $w = (\text{HK})\int_0^T f(s)\mathrm{d}s$.

Remark 3.2 This definition includes the generalized Riemann integral defined by Gordon [17]. In a special case, when δ is a constant function, we get the Riemann integral.

We will also use the following equi-integrability notion, specific to the HK integrability that allows to obtain a Vitali-type convergence result.

Definition 3.3 A family $\mathcal{F}$ of HK-integrable functions defined on $[0, T]$ is said to be HK-equi-integrable if there exists $w \in X$ with the following property: for each $\varepsilon > 0$, there exists a gauge $\delta_\varepsilon(.)$ defined on $[0, T]$ such that

$$\left\| \sum_{k=1}^{n} f(t_k)(d_k - c_k) - w \right\| < \varepsilon$$

for any δ_ε-fine $\mathcal{K}$-partition $\mathcal{P} = \{([c_k, d_k], t_k) : 1 \le k \le n\}$ of $[0, T]$ and every $f \in \mathcal{F}$.

The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Henstock–Kurzweil integrability produces the Henstock–Kurzweil–Pettis integral (for the definition of Pettis integral, see [12]).

Definition 3.4 ([8]) A function $f : [0, T] \longrightarrow X$ is said to be Kurzweil–Henstock–Pettis integrable, or simply HKP-integrable, on $[0, T]$ if there exists a function $g : [0, T] \longrightarrow X$ with the following properties:

(i) $\forall x^* \in X^*$, $x^* f$ is Henstock–Kurzweil integrable on $[0, T]$.
(ii) $\forall t \in [0, T], \forall x^* \in X^*, x^* g(t) = \text{(HK)} \int_0^t x^* f(s)\mathrm{d}s$.

This function g will be called a primitive of f and by $g(T) = \int_0^T f(t)\mathrm{d}t$ we will denote the Henstock–Kurzweil–Pettis integral of f on the interval $[0, T]$.

Remark 3.5 (i) Any HK-integrable function is HKP-integrable. The converse is not true (see an example in [15]). Thus, the family of all Kurzweil–Henstock–Pettis integrable functions is larger than the family of all Kurzweil–Henstock integrable ones.
(ii) Since each Lebesgue integrable function is HK-integrable, we find that any Pettis integrable function is HKP-integrable. The converse is not true (see also [15]).

For $b > 0$ we denote by $B_b = \{y \in X \text{ such that } \|y\| \le b\}$, $D_b = \{z \in C(I, X) \text{ such that } \|z\| \le b\}$, and integrals are taken in the sense of (HKP) integrals. The closed unit ball of the dual X^* is denoted by $B(X^*)$.

Theorem 3.6 *Let $f : I \times X^2 \longrightarrow X$, $h : I \longrightarrow X$ and $G, k : I \times I \longrightarrow \mathbb{R}$ satisfy the following conditions:*

(1) *h is weakly continuous on I.*
(2) *For each $t \in I$, $G(t, .)$, is continuous, $G(t, .)$ and $k(t, .) \in BV(I, \mathbb{R})$. Also, the applications $t \longmapsto G(t, .)$ and $t \longmapsto k(t, .)$ are $\|.\|_{BV}$-continuous. (Here, $BV(I, \mathbb{R})$ represents the space of real bounded variation functions with its classical norm $\|.\|_{BV}$).*
(3) *f is weakly–weakly sequentially continuous (for each convergent sequence $\{t_n\} \subset [0, 1]$ and for all weakly convergent sequences $\{x_n\}, \{y_n\} \subset X$, the sequence $\{f(t_n, x_n, y_n)\}$ is weakly convergent in X) such that for all $r > 0$ and $\varepsilon > 0$, there exists $\delta_{\varepsilon,r} > 0$ such that*

$$\left\| \int_\tau^t f\left(s, x(s), \int_0^s k(s, \eta)x(\eta)\mathrm{d}\eta\right)\mathrm{d}s \right\| < \varepsilon, \quad \forall\, |t - \tau| < \delta_{\varepsilon,r}, \forall x \in D_r. \tag{3.1}$$

(4) *There exists $b_0 > \max(1, \sup_{t\in I} \|k(t, .)\|_{BV}) \sup_{t\in I} \|h(t)\|$ such that for every weakly convergent $(x_n)_n \subset D_{b_0}$, the set*

$$\left\{ x^* f\left(., x_n(.), \int_0^{(.)} k(., \tau)x_n(\tau)d\tau\right), n \in \mathbb{N}, x^* \in B(X^*) \right\} \tag{3.2}$$

is HK-equi-integrable.
(5) *There exists $b_1 > \sup_{t\in I} \|h(t)\|$, and a positive constant α such that*

$$\beta(k(J, J)V(J)) \le \alpha\beta(V(J)) \quad \text{for each closed interval } J \subset I, \text{ for every } V \subset D_{b_1}, \tag{3.3}$$

where $k(J, J)V(J) = \{k(t, s)x(r), t, s, r \in J, x \in V\}$.
(6) *There exists a nonnegative function $L(., .)$ such that*
(a) *for each closed subinterval J of I and bounded subsets Y, Z of X,*

$$\beta(f[J \times Y \times Z]) \le \sup\{L(t, max(\beta(Y), \beta(Z))), t \in J\}, \tag{3.4}$$

where $f[J \times Y \times Z]) = \{f(t, y, z) : t \in J, (y, z) \in Y \times Z\}$.

(b) *The function* $s \longmapsto L(s,r)$ *is continuous for each* $r \in [0, +\infty[$, *and*

$$\sup_{t\in I}\left\{(\mathrm{HK})\int_0^t |G(t,s)|\, L(s,r)\mathrm{d}s\right\} < r \tag{3.5}$$

for all $r > 0$.

Then, there exists an interval $J = [0,a] \subset I$ *such that the set of solutions of* (1.1) *defined on* J *is non-empty and weakly compact in the space* $C(J,X)$.

Proof To simplify, we denote $Tx(t) = \int_0^t k(t,s)x(s)\mathrm{d}s$ $b'_1 = \sup_{t\in I}\|k(t,.)\|_{BV}$ and $c = \sup_{t\in I}||h(t)||$. Let $c < b_2 < min(b_0, \frac{b_0}{b'_1})$. For $x \in D_{b_2}$ and $x^* \in B(X^*)$ we have

$$\begin{aligned}\left|x^*(Tx(s))\right| &= \left|(\mathrm{HK})\int_0^s x^*(k(s,\tau)x(\tau))\mathrm{d}\tau\right| \\ &\le \sup_{t\in I}\|k(t,.)\|_{BV}\int_0^1 \left|x^*x(\tau)\right|\mathrm{d}\tau \le b'_1 b_2 \le b_0.\end{aligned}$$

From here

$$\sup\left\{\left|x^*Tx(s)\right|, x^* \in B(X^*)\right\} \le b_0,$$

so $Tx \in D_{b_0}$. We notice that for $x \in C(I,X)$, the function $f(.,x(.), \int_0^{(.)} k(.,\tau)\, x(\tau)\mathrm{d}\tau)$ is HKP-integrable on [0, 1] (assumption (*4*)). Now let $\varepsilon > 0$. By assumption (3), there exists $\delta_{\varepsilon,b_2} > 0$ such that for $|t_2 - t_1| \le \delta$ we have for all $x \in D_{b_2}$,

$$\left\|\int_{t_1}^{t_2} f(s,x(s),Tx(s))\mathrm{d}s\right\| < \varepsilon. \tag{3.6}$$

Now, Put $d = \sup_{t\in I}\|G(t,.)\|_{BV}$ and $0 < \mu < \frac{b_2-c}{d}$. By the previous analysis, there exists $a \le 1$ with $\alpha a < 1$ such that

$$\sup_{t\in[0,a]}\left\|\int_0^t f(s,x(s),Tx(s))\mathrm{d}s\right\| < \mu$$

for any $x \in C(I,X)$, satisfying $||x|| \le b_2$. Put $J = [0,a]$, denote by $C(J,X)$ the space of continuous functions $J \longrightarrow X$, endowed with the topology of uniform convergence, and by $\widetilde{B}$ the set of all continuous functions $J \longrightarrow B_{b_2}$. We shall consider $\widetilde{B}$ as a topological subspace of $C(J,X)$. It is clear that the set $\widetilde{B}$ is convex and closed. Define the operator F by

$$F_x(t) = h(t) + \int_0^t G(t,s)f(s,x(s), \int_0^s k(s,\tau)x(\tau)\mathrm{d}\tau)\mathrm{d}s,\ \ x \in C(I,X).$$

First notice that for $x \in C(I,X)$, the family

$$\left\{x^*f(.,x(.), \int_0^{(.)} k(.,\tau)x(\tau)\mathrm{d}\tau), x^* \in B(X^*)\right\}$$

is HK-equi-integrable [see (3.2)]. Since for $t \in I$ the function $s \longmapsto G(t,s)$ is of bounded variation then by [24, Lemma 25] and assumption (*4*), the function

$$G(t,.)f(.,x(.), \int_0^{(.)} k(.,\tau)x(\tau)\mathrm{d}\tau)$$

is HKP-integrable on $[0,t]$ and thus the operator F makes sense.

We assert that $F : \widetilde{B} \longrightarrow \widetilde{B}$ is weakly–weakly sequentially continuous.

1. Let us firstly prove that the values of F are in $\widetilde{B}$. For any $x^* \in B(X^*)$, for any $x \in \widetilde{B}$

$$\begin{aligned}\left|x^*F_x(t)\right| &= \left|x^*h(t) + x^*\int_0^t G(t,s)f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\leq \left|x^*h(t)\right| + \left|(\mathrm{HK})\int_0^t G(t,s)x^*f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\leq c + \|G(t,.)\|_{BV}\left|(\mathrm{HK})\int_0^t x^*f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\leq c + d\left\|\int_0^t f(s,x(s),Tx(s))\mathrm{d}s\right\| \\ &\leq c + d\sup_{t\in[0,a]}\left\|\int_0^t f(s,x(s),Tx(s))\mathrm{d}s\right\| \\ &\leq c + d\mu \leq b_2.\end{aligned}$$

From here

$$\sup\left\{\left|x^*F_x(t)\right|, x^* \in B(X^*)\right\} \leq b_2.$$

So, $F_x(t) \in B_{b_2}$.

2. Next, we will prove that $F(\widetilde{B})$ is a strongly equicontinuous subset. Let $0 \leq t_1 < t_2 \leq a$ and $x \in \widetilde{B}$. We suppose without loss of generality that $F_x(t_1) \neq F_x(t_2)$. By the Hahn–Banach theorem, there exists $x^* \in X^*$, such that $\|x^*\| = 1$ and

$$\begin{aligned}\|F_x(t_2) - F_x(t_1)\| &= \left|x^*(F_x(t_2) - F_x(t_1))\right| \\ &\leq \left|x^*(h(t_1)) - x^*(h(t_2))\right| \\ &\quad + \left|(\mathrm{HK})\int_0^{t_1}(G(t_2,s) - G(t_1,s))x^*f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\quad + \left|(\mathrm{HK})\int_{t_1}^{t_2} G(t_2,s)x^*f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\leq \left|x^*(h(t_2) - h(t_1))\right| \\ &\quad + \|G(t_2,.) - G(t_1,.)\|_{BV}\left|(\mathrm{HK})\int_0^{t_1} x^*f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\quad + \sup_{\zeta\in I}\|G(\zeta,.)\|_{BV}\left|\int_{t_1}^{t_2} x^*f(s,x(s),Tx(s))\mathrm{d}s\right| \\ &\leq \left|x^*(h(t_2)) - x^*(h(t_1))\right| \\ &\quad + \|G(t_2,.) - G(t_1,.)\|_{BV}\sup_{\upsilon\in J}\left\|\int_0^{\upsilon} f(s,x(s),Tx(s))\mathrm{d}s\right\| \\ &\quad + d\left\|\int_{t_1}^{t_2} f(s,x(s),Tx(s))\mathrm{d}s\right\|.\end{aligned}$$

So, the result follows from hypotheses (1), (2) and inequality (3.6).

3. Now we will show that F is weakly–weakly sequentially continuous. Let $(x_n(.))_n$ a weakly convergent sequence to x in $\widetilde{B}$. Then by Theorem 2.4, $x_n(t) \rightharpoonup x(t)$ for each $t \in [0,a]$. Let $s \in [0,a]$ and $x^* \in X^*$. We have

$$\left|x^*k(s,\tau)x_n(\tau)\right| \leq b_2\|x^*\|\|k(s,.)\|_\infty,$$

for all $\tau \in [0,s]$. So,

$$\lim_{n\to\infty}(\mathrm{HK})\int_0^s x^*k(s,\tau)x_n(\tau)\mathrm{d}\tau = (\mathrm{HK})\int_0^s x^*k(s,\tau)x(\tau)\mathrm{d}\tau.$$

Then

$$\lim_{n\to\infty} x^* \int_0^s k(s,\tau)x_n(\tau)\mathrm{d}\tau = x^* \int_0^s k(s,\tau)x(\tau)\mathrm{d}\tau,$$

and so $Tx_n(s) \rightharpoonup Tx(s)$. Therefore, the operator T is weakly–weakly sequentially continuous on $\widetilde{B}$. Moreover, because f is weakly–weakly sequentially continuous, so

$$f(s, x_n(s), Tx_n(s)) \rightharpoonup f(s, x(s), Tx(s)),$$

for each $s \in [0, a]$. Now, for each $t \in [0, a]$, applying Theorem 5 in [11] and Lemma 25 in [24] to the sequence $(G(t, .)f(., x_n(.), Tx_n(.))_n$, we find that the function $G(t, .)f(., x(.), Tx(.))$ is HKP-integrable on $[0, t]$ and

$$\int_0^t G(t,s)f(s, x_n(s), Tx_n(s))\mathrm{d}s \rightharpoonup \int_0^t G(t,s)f(s, x(s), Tx(s))\mathrm{d}s.$$

Whence, $F_{x_n}(t) \rightharpoonup F_x(t)$ for each $t \in [0, a]$ and by Theorem 2.4 the operator F is weakly–weakly sequentially continuous. Next, we consider $Q = \overline{\text{conv}} F(\widetilde{B})$. Because $F(\widetilde{B})$ is bounded and strongly equicontinuous, so Q is a weakly closed bounded and strongly equicontinuous subset of $\widetilde{B}$. Clearly $F(Q) \subset Q$. We claim that $F : Q \longrightarrow Q$ is β-condensing. Indeed, let V be a subset of Q such $\beta(V) \neq 0$, $V(t) = \{x(t), x \in V\}$ and $F(V)(t) = \{F_x(t), x \in V\}$. Because V is bounded and strongly equicontinuous, we have by Lemma 2.5(a) that $\sup_{t\in J} \beta(V(t)) = \beta(V) = \beta(V(J))$. For fixed $t \in J$, we divide the interval $[0, t]$ into n parts: $0 = t_0 < t_1 < \cdots < t_n = t$ and put $T_i = [t_{i-1}, t_i]$. By Henstock–Kurzweil–Pettis integral mean value theorem [8], we obtain

$$\begin{aligned}
F_x(t) &= h(t) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} G(t,s)f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\in h(t) + \sum_{i=1}^n (t_i - t_{i-1})\overline{\text{conv}}\left\{G(t,s)f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\mathrm{d}\tau\right), s \in T_i, x \in V\right\} \\
&\in h(t) + \sum_{i=1}^n (t_i - t_{i-1})\overline{\text{conv}}\{G(t,s)f(T_i, V(T_i), \overline{\text{conv}}k([0,s]\times[0,s])V([0,s])), s \in T_i\}.
\end{aligned}$$

Using the properties of the measure of weak noncompactness, we have

$$\begin{aligned}
&\beta(F(V)(t)) \\
&\leq \sum_{i=1}^n (t_i - t_{i-1})\beta(\overline{\text{conv}}\{G(t,s)f(T_i, V(T_i), \overline{\text{conv}}(k([0,s][0,s])V([0,s])), s \in T_i\} \\
&\leq \sum_{i=1}^n (t_i - t_{i-1})\beta\left(\overline{\text{conv}}\left\{G(t,s)f(T_i, V(T_i), \overline{\text{conv}}\left\{\underset{\omega\in[0,t]}{\cup} \omega((k([0,s][0,s])V([0,s])))\right\}, s \in T_i\right\}\right. \\
&\leq \sum_{i=1}^n (t_i - t_{i-1})\beta\left(\overline{\text{conv}}\left\{G(t,s)f(T_i, V(T_i), \overline{\text{conv}}\left\{\underset{\omega\in[0,t]}{\cup} \omega((k([0,t][0,t])V([0,t])))\right\}, s \in T_i\right\}\right. \\
&\leq \sum_{i=1}^n (t_i - t_{i-1}) \sup_{s\in T_i} |G(t,s)|\, \beta\left(\overline{\text{conv}}\left\{f(T_i, V(T_i), \overline{\text{conv}}\left\{\underset{|\omega|\in[0,a]}{\cup} \omega((k(J,J)V(J)))\right\}, s \in T_i\right\}\right. \\
&\leq \sum_{i=1}^n (t_i - t_{i-1}) \sup_{s\in T_i} |G(t,s)| \sup_{\tau\in T_i} L\left(\tau, \max(\beta(V(J)), \beta\left(\left\{\overline{\text{conv}}\left\{\underset{|\omega|\in[0,a]}{\cup} \omega((k(J,J)V(J)))\right\}\right)\right.\right. \\
&\leq \sum_{i=1}^n (t_i - t_{i-1})\, |G(t,\tau_i)|\, L\left(s_i, \max(\beta(V(J)), \beta\left(\left\{\overline{\text{conv}}\left\{\underset{|\omega|\in[0,a]}{\cup} \omega((k(J,J)V(J)))\right\}\right),\right.\right.
\end{aligned}$$

here for each i, $\tau_i, s_i \in T_i$ are numbers such that

$$|K(t,\tau_i)| = \sup_{s\in T_i}|G(t,s)| \text{ and } L(s_i, \max(\beta(V(J)), \beta\Big(\Big\{\overline{\text{conv}}\Big\{\bigcup_{|\omega|\in[0,a]}\omega((k(J,J)V(J)))\Big\}\Big)$$
$$= \sup_{s\in T_i} L(s, \max(\beta(V(J)), \beta\Big(\Big\{\overline{\text{conv}}\Big\{\bigcup_{|\omega|\in[0,a]}\omega((k(J,J)V(J)))\Big\}\Big)\Big).$$

Next, by property $(viii)$ of β and (3.3), we have

$$\beta\Big(\Big\{\overline{\text{conv}}\Big\{\bigcup_{|\omega|\in[0,a]}\omega((k(J,J)V(J)))\Big\}\Big)\Big) = a\beta(k(J,J)V(J)) \le \alpha a\beta(V(J)) < \beta(V(J)),$$

so

$$\beta(F(V)(t)) \le \sum_{i=1}^{n}(t_i - t_{i-1})\,|G(t,\tau_i)|\,L(s_i, \beta(V)).$$

From the continuity of the functions $s \longmapsto G(t,s)$ and $s \longmapsto L(s,\beta(V))$ on $[0,t]$, it follows that there exists $\eta > 0$ such that

$$|G(t,\tau)L(q,\beta(V)) - G(t,s)L(s,\beta(V))| < \varepsilon, \tag{3.7}$$

if $|\tau - s| < \eta$, $|q - s| < \eta$, $q, s, \tau \in [0,t]$. So, taking t_i in the manner that $|t_i - t_{i-1}| < \eta$ and by (3.7) we infer that

$$\begin{aligned}\beta(F(V)(t)) &\le \sum_{i=1}^{n}(t_i - t_{i-1})\Big((\text{HK})\int_{t_{i-1}}^{t_i}|G(t,\tau_i)L(s_i,\beta(V)) - G(t,s)L(s,\beta(V))|\mathrm{d}s\Big)\\ &\quad+\sum_{i=1}^{n}(t_i - t_{i-1})\Big((\text{HK})\int_{t_{i-1}}^{t_i}|G(t,s)|\,L(s,\beta(V))\mathrm{d}s\Big)\\ &\le \varepsilon t + (\text{HK})\int_0^t |G(t,s)|\,L(s,\beta(V))\mathrm{d}s\\ &\le \varepsilon t + \sup\Big\{(\text{HK})\int_0^{t'}|G(t,s)|\,L(s,\beta(V)),\, t' \in J\Big\}.\end{aligned}$$

As the last inequality is satisfied for every $\varepsilon > 0$, we get

$$\beta(F(V)(t)) \le \sup\Big\{(\text{HK})\int_0^{t'}|G(t,s)|\,L(s,\beta(V))\mathrm{d}s,\, t' \in J\Big\}.$$

Applying again Lemma 2.5(a) for the bounded strongly equicontinuous subset $F(V)$, we obtain that $\beta(F(V)) = \sup_{t\in J}\{F(V)(t)\}$. Accordingly, by (3.5)

$$\beta(F(V)) \le \sup\Big\{(\text{HK})\int_0^{t'}|G(t,s)|\,L(s,\beta(V))\mathrm{d}s,\, t' \in J\Big\} < \beta(V),$$

so, F is β-condensing. Now, applying Theorem 2.6, we infer that F has a fixed pint in $Q \subset \widetilde{B}$. Therefore, $\mathcal{S}$ the fixed point set of F in $\widetilde{B}$ is non-empty. Finally, we have $F(\mathcal{S}) = \mathcal{S} \subset Q$, so $\beta(F(\mathcal{S})) = 0$ and hence $\mathcal{S}$ is relatively weakly compact subset of $\widetilde{B}$. Because the operator F is weakly–weakly sequentially continuous, so $\mathcal{S}$ is weakly sequentially closed. The use of Eberlein–Šmulian theorem [14, Theorem 8.12.4, p. 549], proves that $\mathcal{S}$ is weakly compact. This achieves the proof. □

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