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Studying monoids is not enough to study multiplicative properties of rings: an elementary approach

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Abstract The aim of these notes is to indicate, using very simple examples, that not all results in ring theory can be derived from monoids and that there are results that deeply depend on the interplay between "+" and "·".

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الملخص

الهدف من هذه الملاحظات هو الإشارة، باستخدام أمثلة بسيطة جدا، إلى أنه لا يمكن اشتقاق جميع النتائج في نظرية الحلقات من نظرية المونوئيدات، وأن هناك نتائج تعتمد بشكل عميق على التفاعل بين "+" وَ "·".

In a commutative unitary ring $(R, +, \cdot)$, with $0 \neq 1$, there are two binary operations, denoted by "+", addition, and by "·", multiplication, at work. So, in particular, we can consider R as an algebraic system closed under one of the binary operations, that is $(R, +)$ or $(R, \cdot)$. In both cases, R is at least a monoid. This state of affairs could lead one to think that perhaps it is sufficient to study monoids to get a handle on ring theory. The aim of these notes is to indicate, using elementary and easily accessible results available in literature, that not all results in ring theory can be derived separately from semigroups or monoids and that there are results that deeply depend on the interplay between "+" and "·". Following the custom, we choose to use the simplest examples that can be developed with a minimum of jargon to establish our thesis. We shall restrict our attention to integral domains and to results of multiplicative nature, as that is our area of interest. Indeed, we plan to show that there are results on integral domains that cannot be established for monoids, one way or another. Of course, to show that monoids cannot prove all the results on rings we have to have an idea of what monoids can do. For this, we start the paper with a review of the monoids. Our coverage of this topic will be tool specific, in that we shall concentrate more on what we need to establish our thesis. For this the best source we find in the field is Franz Halter-Koch's book [10] where, between the lines, further evidence to the aim of the present paper can be found.

Before we start the review, we wish to point out that the material presented here is a sort of fact-presenting mission and it is not our intention to belittle monoids which have their own uses and play active roles in various contexts. Moreover, on the other side, it is known that monoid theory can produce results of ring-theoretical relevance which cannot be derived from the ring structure alone. For example, in a Noetherian domain, the monoid of elements coprime to the conductor forms a Krull monoid, whose arithmetic properties cannot be detected by purely ring-theoretical methods [10, Chapter 22].

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1 A basic review of monoids

Recall that, following Halter-Koch's terminology [10, page 5], *a monoid* is a multiplicative semigroup M with (a uniquely determined) *identity element*, denoted by 1, and (a uniquely determined) *zero element*, denoted by 0, with $1 \neq 0$, satisfying $0m = 0$, for each $m \in M$. An *element* $x \in M$ is called *cancellative* if $xa = xb$ implies $a = b$, for $a, b \in M$. A cancellative monoid M is a monoid such that every element in $M^\bullet := M \setminus \{0\}$ is cancellative. As usual, let $a, b \in M$; we say that *a divides b* in M (in symbols, $a \mid b$), if $ax = b$ for some $x \in M$. Set $M^\times := \{x \in M \mid x \text{ is invertible in } M\}$. Clearly, $M^\bullet$ is a group if and only if $M^\bullet = M^\times$; in this situation the monoid M is called a *groupoid.* By *quotient groupoid* of a cancellative monoid M, we mean a uniquely determined (up to isomorphisms) groupoid $Q = Q(M)$ such that $M \subseteq Q$ and $Q = \{y^{-1}x \mid x \in M, y \in M^\bullet\}$ [10, page 38].

For each $x \in M$, we call $xM := \{xy \mid y \in M\}$ the *principal ideal generated by* x in M. A nonempty subset X of M is called an *ideal of* M, if $XM := \bigcup\{\{xM\} \mid x \in X\}$ is contained in (in fact, equal to) X.

If M is a cancellative monoid and Q is its quotient groupoid, then a nonempty subset $\mathcal{X} \subset Q$ is an *M-fractional ideal* if $c\mathcal{X} \subseteq M$ is an ideal of M, for some $c \in M^\bullet$. For each M-fractional ideal $\mathcal{X}$ of M, we call *v-ideal associated with* $\mathcal{X}$, the M-fractional ideal:

$$\mathcal{X}^v := \bigcap\{zM \mid zM \supseteq \mathcal{X}, z \in Q\},$$

and the *t-ideal associated with* $\mathcal{X}$, the M-fractional ideal:

$$\mathcal{X}^t := \bigcup\{(FM)^v \mid F \text{ nonempty finite subset of } \mathcal{X}\}.$$

A *t-ideal* (respectively, *v-ideal*) of a cancellative monoid M is an ideal $X \subseteq M$ such that $X = X^t$ (respectively, $X = X^v$). A *prime t-ideal* (respectively, *prime v-ideal*) of M is a t-ideal (respectively, v-ideal) $X \subsetneq M$ such that if $a, b \in M$, $ab \in X$, and $a \notin X$, then $b \in X$. A *maximal t-ideal* (respectively, *maximal v-ideal*) of M is a maximal element in the set of all proper t-ideals (respectively, v-ideals) of M, ordered by the set inclusion. It is well-known that each t-ideal of M is always contained in a maximal t-ideal of M which is a prime ideal of M [10, Theorem page 57 and Theorem page 58]. Denote by $\text{Max}^t(M)$ the set of all maximal t-ideal of M. It is clear that an ideal H in M is proper if and only if $H \cap M^\times = \emptyset$. Since a principal proper ideal in M is a t-ideal, clearly $M^\times = M \setminus (\bigcup\{Q \mid Q \in \text{Max}^t(M)\})$.

Note that, as an integral domain is, in particular, a monoid under multiplication, the various notions of v-ideals and t-ideals transfer to integral domains without much change.

Restricting to cancellative monoids, Halter-Koch in [10, Chapter 10, page 110] defines GCD monoids.

Given a (nonempty) finite subset B of M, an element $d \in M$ is called *greatest common divisor of B (in M)* if (1) $d \mid b$ for all $b \in B$ and (2) if $c \in M$, such that $c \mid b$ for all $b \in B$ then $c \mid d$. This is pretty much the same as the definition of a GCD of a finite set in an integral domain.

We denote by $\mathtt{GCD}(B)$ the set of all *greatest common divisors of* B. Obviously, $\mathtt{GCD}(B)$ may be empty, but if $d \in \mathtt{GCD}(B)$ then all the associates of d belong to $\mathtt{GCD}(B)$ and, more precisely, $\mathtt{GCD}(B) = dM^\times$, where *$M^\times$ denotes the set of all the invertible elements (or, units) of M.*

Halter-Koch defines a GCD-*monoid* as a cancellative monoid M such that for every finite (nonempty) subset B of M, $\mathtt{GCD}(B) \neq \emptyset$.

It is straightforward that M is a GCD-monoid if and only if $\mathtt{GCD}(a, b) \neq \emptyset$, for all $a, b \in M$.

On a monoid M the relation "$\simeq$" defined by $a \simeq b$ if $aM = bM$ is a congruence relation on M and $M_{\text{red}} := M/\simeq$ is called *the reduced monoid associated with* M. When M is cancellative, $a \simeq b$ if and only if $aM^\times = bM^\times$. A *monoid* is called *reduced* if M is canonically isomorphic to M_{red}. When M is cancellative, this happens if and only if $M^\times = \{1\}$. It is easy to see that M is a GCD-monoid if and only if M_{red} is a GCD-monoid and, when M is reduced GCD-monoid, $\mathtt{GCD}(a, b)$ consists of just one element, for all $a, b \in M$. In this situation, Halter-Koch denotes $\mathtt{GCD}(a, b)$ by $a \wedge b$ and calls it `inf` of a and b. In [10, Proposition, page 110], he shows that in a reduced GCD-monoid $x\,\mathtt{GCD}(a, b) = \mathtt{GCD}(xa, xb)$ or, equivalently, $x(a \wedge b) = xa \wedge xb$ for all $x, a, b \in M$ (that is, multiplication distributes over $\wedge$).

Obviously, an integral domain D is a GCD-domain if D (under multiplication) is a GCD-monoid, and conversely.

Bosbach in 1991, introduced an important class of lattice ordered semigroups called by Áhn et al. in ‘[2] Bézout monoids.

A *Bézout monoid* is a commutative monoid S with 0 such that under the natural partial order defined by:

$$a \leq b \in S \;:\Leftrightarrow\; bS \subseteq aS,$$

S is a distributive lattice, multiplication is distributive over both meets and joins, and for any $x, y \in S$, if $d = x \wedge y$ and $\mathrm{d}x_1 = x$, then there is an element $y_1 \in S$ with $\mathrm{d}y_1 = y$ and $x_1 \wedge y_1 = 1$.

Lemma 1.1 (a) *Let M be a reduced $\mathtt{GCD}$-monoid and $a, b \in M$. If $\mathtt{GCD}(a, b) = d$, then there exist $a_1, b_1 \in M$ such that $\mathtt{GCD}(a_1, b_1) = 1$, $a = a_1 d$ and $b = b_1 d$.*
(b) *A reduced $\mathtt{GCD}$-monoid is a Bézout monoid (after Áhn–Márki–Vámos).*

Proof (a) Let M be a reduced $\mathtt{GCD}$-monoid and let $a, b \in M$, then $\mathtt{GCD}(a, b) = a \wedge b = d$ for a unique $d \in M$. On the other hand, since $d \mid a$ and $d \mid b$ in M, we can write $a = a_1 d$ and $b = b_1 d$, for some $a_1, b_1 \in M$. Now as $d = a_1 d \wedge b_1 d = (a_1 \wedge b_1)d$ and M is cancellative, we get $1 = a_1 \wedge b_1 = \mathtt{GCD}(a_1, b_1)$.
(b) Let M be a reduced $\mathtt{GCD}$-monoid and let $a, b, d, a_1, b_1 \in M$ be as above. If for some x_1 we have $a = x_1 d$, then by cancellation $x_1 = a_1$ and since $b = b_1 d$ then necessarily $a_1 \wedge b_1 = \mathtt{GCD}(a_1, b_1) = 1$. □

2 First exhibit: units and maximal *t*-ideals

Given a $\mathtt{GCD}$-monoid M with group of units $M^{\times}$, the reduced associated ($\mathtt{GCD}$-) monoid $M_1 := M/M^{\times}$ has group of units $M_1^{\times} = \{1\}$. This indicates that the multiplicative structure of a $\mathtt{GCD}$-monoid does not have any control over the size of its group of units.

With this introduction, we start by stating a simple result for $\mathtt{GCD}$-domains, that apparently cannot be deduced using the multiplicative structure only. In other words, the theory of $\mathtt{GCD}$-monoids does not seem sufficient to show the following:

Proposition 2.1 *Every valuation domain that is not a field has infinitely many units.*

Recall that, the following conditions for a ring R and an ideal I of R are equivalent to each other:

(i) for any $x \in I$ the element $1 + x$ is invertible in R, that is, $1 + I \subseteq R^{\times}$;
(ii) an element $x \in I$ is invertible in R if and only if $x \bmod I$ is invertible in R/I;
(iii) I is contained in the Jacobson radical of R.

Therefore, Proposition 2.1 is a particular case of the fact that an integral domain D with nonzero Jacobson ideal J has at least countably many units (let $0 \neq x \in J$, then $1 + x^n$ is a unit of D, for all $n \geq 0$). In other words, we have:

Lemma 2.2 *If an integral domain (not a field) has finitely many units then its Jacobson ideal is zero.*

The next result generalizes Proposition 2.1.

Proposition 2.3 *If an integral domain, which is not a field, has only a finite number of maximal t-ideals then it has infinitely many units.*

The proof depends upon the fact that each nonzero nonunit element of an integral domain D (which is not a field) is contained in some (maximal) t-ideal and so if $\mathrm{Max}^t(D) := \{Q_1, Q_2, \ldots, Q_r\}$, for some $r \geq 1$, the set of units of D coincides with $D \backslash (Q_1 \cup Q_2 \cup \cdots \cup Q_r)$ which implies that each maximal ideal M of D is contained in $Q_1 \cup Q_2 \cup \cdots \cup Q_r$ and so $\mathrm{Max}^t(D) = \mathrm{Max}(D)$. In this situation, the Jacobson radical is obviously nonzero. (See also [14, Proposition 3.5]).

Remark 2.4 (a) Note that the Prime Avoidance Lemma is also a result of ring theory that cannot be derived only from the monoidal structure. The standard proof of the Prime Avoidance Lemma for rings depends on the interplay of both operations [9, Proposition 4.9]. We give now an example showing that an analogue of this lemma does not hold for monoids. Let D be a PID with exactly two maximal ideals P and Q. Consider D as a (multiplicative) monoid, then $I := P \cup Q$ is an ideal of the monoid D, since it coincides with $D \backslash D^{\times}$, i.e., with the set of the non invertible elements of the monoid D. Obviously, $I \subseteq P \cup Q$, $I \nsubseteq P$ and $I \nsubseteq Q$. Finally, note that I is not an ideal of the ring D.

(b) We can point out that the previous observation can be used to establish that a Krull domain with a finite number of height one primes is a PID while for a Krull monoid with finitely many prime divisors this is very far for being true [10, Chapter 22].

The point of interest in Proposition 2.3 too is that, *if an integral domain D (which is not a field) has only a finite number of units then D has an infinite number of maximal t-ideals.*

The converse does not hold in general, i.e., an integral domain D having an infinite number of maximal t-ideals does not mean that D has a finite number of units, nor that the Jacobson radical of D is zero. For instance, in $D := \mathbb{Z} + X\mathbb{Q}[\![X]\!]$ every maximal ideal is principal and hence is a t-ideal and so D has an infinite number of maximal t-ideals and $\mathrm{Max}^t(D) = \mathrm{Max}(D)$, yet this ring has infinitely many units and the Jacobson radical of D is $X\mathbb{Q}[\![X]\!]$.

One may remark that in the proof of Proposition 2.1, as in the proof of Proposition 2.3, we had to use the additive structure, in addition to the multiplicative structure. Now noting that most of the ideal theoretic notions known for integral domains have been translated to the language of monoids, we leave the following as a problem.

Problem 2.5 Prove or disprove: It can be shown using monoid theoretic techniques that if an integral domain D, that is not a field, has finitely many units then D has infinitely many maximal t-ideals.

With reference to the previous problem, let p be a prime integer and let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal (p). Clearly, $\mathbb{Z}_{(p)}$ is a GCD-domain and so $M := \mathbb{Z}_{(p)}$, considered only with the multiplicative structure is a GCD-monoid. If we consider the reduced monoid associated to M, i.e., $M_{\mathrm{red}} = \mathbb{Z}_{(p)}/\mathbb{Z}_{(p)}^{\times}$, by the previous observations this is a reduced GCD-monoid, and so $M_{\mathrm{red}}^{\times} = \{1\}$. Moreover, M_{red} has a unique maximal (t-) ideal, i.e., the principal ideal $p\mathbb{Z}_{(p)}/\mathbb{Z}_{(p)}^{\times}$. Therefore M_{red} is a GCD-monoid having a unique unit and a unique maximal t-ideal.

3 Exhibit 2: units and cardinality of the set of mutually (v-)coprime elements

We investigate now the case of GCD-monoids to show similar results. We extend the setting to include some more notions considered in recent years. Recall that an integral domain D is *an almost*-GCD-*domain* (for short, AGCD-*domain*) if for all $x, y \in D$ there is a positive integer $n := n(x, y)$ such that $x^n D \cap y^n D$ is principal (or, equivalently, $(x^n, y^n)^v$ is principal, since $(D : (a, b)) = (ab)^{-1}(aD \cap bD)$ for a, b nonzero elements in D, [13, Lemma 1.1]). These domains generalize GCD-domains, since an AGCD-domain is a GCD-domain if and only if $n(x, y) = 1$ for all $x, y \in D$. However, $\mathbb{F}_2[X^2, X^3]$ is a (non integrally closed) AGCD-domain that, in particular, is not a GCD-domain [3, Theorem 3.3]. On the other hand, it is well-known that an integrally closed domain D is an AGCD-domain if and only if D is a PvMD and $\mathrm{Cl}(D)$ is torsion (here $\mathrm{Cl}(D)$ represents a sort of generalization of the divisor class group, see [12, Theorems 3.9 and 5.6]). In particular, Krull domains with torsion divisor class groups are AGCD-domains; these domains were called *almost factorial domains* in Fossum [7, page 33] and were previously introduced and studied by Storch in [11] as *Fastfactorielle Ringe*.

In [1] an integral domain D was called an *almost Bézout domain* if for all $x, y \in D$ there is a positive integer $n := n(x, y)$ such that the ideal (x^n, y^n) is principal. An almost Bézout domain is an AGCD-domain and obviously it generalizes a Bézout domain. It was shown in [1, Corollary 5.4] that an AGCD-domain D is almost Bézout if and only if every maximal ideal of D is a t-ideal.

Generally two elements x, y of an integral domain D are called v-coprime if $xD \cap yD = xyD$ (or, equivalently, if $(x, y)^v = D$).

Theorem 3.1 *Let D be an AGCD-domain in which there can at most be finite sets of mutually v-coprime nonunits. Then D has infinitely many units.*

Proof The property "there can at most be finite sets of mutually v-coprime nonunits" can be translated into "there is no infinite sequence of mutually v-coprime nonunits". So, by [5, Proposition 2.1] D is an almost Bézout semilocal domain and this forces D to have finitely many maximal t-ideals and consequently a nonzero Jacobson radical, which forces D to have an infinity of units (Proposition 2.3). □

Corollary 3.2 *Let D be a GCD-domain in which there can at most be finite sets of mutually coprime nonunits. Then D has infinitely many units.*

Now note that both of Theorem 3.1 and Corollary 3.2 need both the addition and multiplication for their proofs and so do their contrapositives namely: *if D is an* AGCD- *(or a* GCD-*) domain with only a finite number of units then D contains infinitely many mutually v-coprime (or, coprime) nonunits.*

This seems to be a sort of analogue of Euclid's Theorem about infinitude of primes in the ring of integers. A similar result can be proven in much more generality. However, we restrict it here to the case of GCD-domains.

On the other hand, a monoid can always be reduced to have a single unit without changing the cardinality of sets of mutually (v-)coprime elements and the multiplicative structure of the monoid. Thus, we conclude that in a monoid the relationship between units and sets of mutually (v-)coprime elements is arbitrary and so the above material may be our exhibit 2.

4 Exhibit 3: it is hard to get from monoids to rings

At this stage, we can say that multiplicative monoids are a tool that can be used to settle questions about divisibility, or questions related to divisibility, by disregarding units and often the zero. Most of multiplicative ideal theory in ring theory does just that. But as the monoids are more general, proving results for monoids does not mean proving results about rings, except in some limited cases. The "limited cases" are essentially linked with the Krull–Jaffard–Ohm Theorem that says that given a lattice ordered abelian group G one can construct a Bézout domain with a group of divisibility isomorphic with G [9, Theorem 18.6].

But, there are well-known examples of directed partially ordered groups (not lattice ordered) that are not groups of divisibility (see e.g. [8, Example 4.7 on page 110, due to P. Jaffard]). Directed partially ordered groups play a central role in the present theory because the group of divisibility of an integral domain is a directed partially ordered group G [9, page 174] and the positive cone $G_+ := \{x \in G \mid x \geq 1\}$ of a partially ordered group determines a reduced monoid.

Now just to hammer home the point we present the part of this exhibit that has been in the literature for some time, in the form of the following example. Just to keep the reader interested, we want to show in the following example that the notion of product of ideals in a monoid is inadequate to describe the product of ideals in an integral domain.

Example 4.1 A class of directed partially ordered abelian groups which lies very close to the lattice ordered groups was systematically studied by L. Fuchs in 1965. A directed partially ordered abelian group G is a *Riesz group* [6, Theorem 2.2] if for all $x, y, z \in G_+$, $x \leq yz$ implies that $x = rs$ where $r \leq y$ and $s \leq z$.

P.M. Cohn [4] called *Schreier domain* an integrally closed integral domain D such that for all $x, y, z \in D\backslash\{0\}$, $x \mid yz$ implies that $x = rs$ where $r, s \in D$ are such that $r \mid y$ and $s \mid z$.

We denote by $G(D)$ the group of divisibility of an integral domain D with quotient field K , i.e., $G(D) := \{xD \mid x \in K\backslash\{0\}\}$, having as identity element D, ordered by setting $xD \leq yD$ if $xD \supseteq yD$, hence $G(D)_+ = \{xD \mid x \in D\backslash\{0\}\}$.

From this one can conclude that an integrally closed domain is a Schreier domain if and only if its group of divisibility is a Riesz group.

Now, if we drop off the integrally closed condition, we get what was termed as a *pre-Schreier domain* in [13]. So, a domain D is pre-Schreier if and only if $G(D)$ is a Riesz group.

Let Γ be a partially ordered group, let $\xi_1, \xi_2, \ldots, \xi_n \in \Gamma$, we set $U(\xi_1, \xi_2, \ldots, \xi_n) := \{\gamma \in \Gamma \mid \gamma \geq \xi_i, \text{ for } 1 \leq i \leq n\}$. So, in particular, if $x_1D, x_2D, \ldots, x_nD$ belong to $G(D)$ (with $x_1, x_2, \ldots, x_n \in K\backslash\{0\}$), then $U(x_1D, x_2D, \ldots, x_nD) = \{tD \mid tD \geq x_iD, \text{ for } 1 \leq i \leq n\} = \{tD \mid x_i \mid t, \text{ for } 1 \leq i \leq n\}$ (with $t \in K\backslash\{0\}$). Thus $U(x_1D, x_2D, \ldots, x_nD) \cup \{0\} = \bigcap_{1\leq i\leq n} x_iD$.

Now, by [6, Theorem 2.2 (3)] , $G(D)$ is a Riesz group if and only if, for all $a_1D, a_2D, \ldots, a_mD; b_1D, b_2D, \ldots, b_nD \in G(D)$, we have

$$U(a_1D, a_2D, \ldots, a_mD)U(b_1D, b_2D, \ldots, b_nD) = U(a_1b_1D, a_1b_2D, \ldots, a_ib_jD, \ldots, a_mb_nD).$$

This translates, in ring theoretic terms, to

$$(*) \quad \left(\bigcap_{1\leq i\leq m} a_iD\right)\left(\bigcap_{1\leq j\leq n} b_jD\right) = \bigcap_{1\leq i\leq m,\, 1\leq j\leq n} a_ib_jD.$$

In [13, pages 1905-6], where the above observation was made, an integral domain D was said to have *the $*$-property* if for for all $a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n \in K\backslash\{0\}$ the equality $(*)$ holds. It is easy to see that in the definition of the $*$-property we can restrict $a_i, b_j \in D\backslash\{0\}$. Moreover, it is not hard to see that the $*$-property is a local property (i.e., an integral domain D verifies the $*$-property if and only if D_P verifies the $*$-property for each $P \in \operatorname{Spec}(D)$) [13, Theorem 2.1]. It was shown in [13, Corollary 1.7] that a pre-Schreier domain has the ∗-property, yet a domain with the ∗-property may not be a pre-Schreier domain. In fact, since a valuation domain is a GCD-domain and a GCD-domain is a pre-Schreier domain and so it has the ∗-property, then a Prüfer domain has the ∗-property. However, a Prüfer domain which is a Schreier domain is a Bézout domain by [4, Theorem 2.8], so a Prüfer non-Bézout domain is an example of an integrally closed domain with the ∗-property which is not a (pre-)Schreier domain. This discrepancy was blamed on involvement of addition in the definition of products of ideals in an integral domain.

Example 4.1 provides an interesting exhibit in that we start with a pre-Schreier domain D, get to the multiplicative monoid of D as $G(D)_+$, the positive cone of the group of divisibility of D. Now, $G(D)_+$ is characterized by a property that involves product of ideals. On translation back the characterizing property of the multiplicative monoid gives rise to another integral domain!

From the previous considerations, it appears that monoids are an efficient tool but they come in handy for integral domains only when an interaction between addition and multiplication does not play a significant role.

Remark 4.2 Finally, we have restricted our attention to produce easy to see examples, otherwise there are more general results available. For instance, let us mention that if D is a pre-Schreier domain that does not allow an infinite sequence of mutually coprime nonunits (in a pre-Schreier domain v-coprime is the same as coprime) then $\text{Max}(D)$ is finite and so D must have an infinity of units. In fact, similar results can be proved for a lot more general domains. But we do not include the proofs here as treating those domains would entail a lot more jargon and in the end they will not contribute more to our arguments.

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References

1. Anderson, D.D.; Zafrullah, M.: Almost Bezout domains. J. Algebra **142**, 285–309 (1991)
2. Ánh, P.N.; Márki, L.; Vámos, P.: Divisibility theory in commutative rings: Bézout monoids. Trans. Am. Math. Soc. **364**, 3967–3992 (2012)
3. Chang, G.W.: Almost splitting sets in integral domains. J. Pure Appl. Algebra **197**, 279–292 (2005)
4. Cohn, P.: Bézout rings and their subrings. Proc. Camb. Philos. Soc. **64**, 251–264 (1968)
5. Dumitrescu, T.; Lequain, Y.; Mott, J.; Zafrullah, M.: Almost GCD domains of finite t-character. J. Algebra **245**, 161–181 (2001)
6. Fuchs, L.: Riesz groups. Ann. Sc. Norm. Sup. Pisa, Cl. Scienze, Ser. 3, **19**, 1–34 (1965)
7. Fossum, R.: The Divisor Class Group of a Krull Domain. Ergebnisse der Mathematik und ihrer Grenzgebiete Band, vol. 74. Springer, Berlin (1973)
8. Fuchs, L.; Salce, L.: Modules Over Non-Noetherian Domains. Mathematical Surveys and Monographs, vol. 84. American Mathematical Society, Providence (2001)
9. Gilmer, R.: Multiplicative Ideal Theory. M. Dekker, New York (1972)
10. Halter-Koch, F.: Ideal Systems: An Introduction to Multiplicative Ideal Theory. M. Dekker, New York (1998)
11. Storch, U.: Fastfaktorielle Ringe, vol. 36. Schriftenreihe Math. Inst. Univ., Münster (1967)
12. Zafrullah, M.: A general theory of almost factoriality. Manuscr. Math. **51**, 29–62 (1985)
13. Zafrullah, M.: On a property of pre-Schreier domains. Commun. Algebra **15**, 1895–1920 (1987)
14. Zafrullah, M.: What v-coprimality can do for you. In: Brewer, J.W., Glaz, S., Heinzer, W., Olberding, B. (eds.) Multiplicative Ideal Theory in Commutative Algebra, pp. 387–404. Springer, New York (2006)