Arab. J. Math. (2014) 3:437–447
DOI 10.1007/s40065-014-0108-3

Arabian Journal of Mathematics

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Direct sums of trace maps and self-adjoint extensions

Received: 4 February 2014 / Accepted: 2 June 2014 / Published online: 25 June 2014

Abstract We give a simple criterion so that a countable infinite direct sum of trace (evaluation) maps is a trace map. An application to the theory of self-adjoint extensions of direct sums of symmetric operators is provided; this gives an alternative approach to results recently obtained by Malamud–Neidhardt and Kostenko–Malamud using regularized direct sums of boundary triplets.

Mathematics Subject Classification 47B25 · 47B38

الملخص

نعطي معياراً بسيطاً لتحديد فيما إذا كان جمع مباشر نهائي قابل للعد لرواسم أثر (تقييم) هو راسم أثر. نعطي أيضاً تطبيقاً في نظرية التمديدات ذاتية التقارن لمجاميع مباشرة لمؤثرات متناظرة؛ يعطي هذا مقاربة بديلة لنتائج حصل عليها مؤخراً محمود – نيدهارت وَ كوستنكو – محمود باستخدام مجاميع مباشرة لثلاثيات حدية تم تنظيمها.

1 Introduction

We begin with a simple example. Let $\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}$ be the Laplace–Beltrami operator on the two-dimensional cylinder $\mathbb{M}_0 := \mathbb{R}_+ \times \mathbb{T}$ with respect to the flat Riemannian metric $g_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Its minimal realization with domain $C_c^\infty(\mathbb{M}_0)$ is symmetric and negative as a linear operator in the Hilbert space $L^2(\mathbb{M}_0) = L^2(\mathbb{R}_+) \otimes L^2(\mathbb{T})$. We denote its Friedrichs' self-adjoint extension by Δ_0^D; it corresponds to imposing Dirichlet boundary conditions at the boundary $\mathbb{T}$, i.e., $\mathscr{D}(\Delta_0^D) = \{u \in H^2(\mathbb{M}_0) : \lim_{x\downarrow 0} u(x,\theta) = 0\}$. Here $H^2(\mathbb{M}_0)$ is the usual Sobolev–Hilbert space of order two. Let us denote by $H^s(\mathbb{T})$ the (fractional) Sobolev–Hilbert space of square-integrable functions f on the 1-dimensional torus $\mathbb{T}$ such that $\sum_{k\in\mathbb{Z}} |k|^{2s} |\hat{f}_k|^2 < +\infty$, where $\hat{f}_k$ is the usual Fourier coefficient $\hat{f}_k := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ik\theta} f(\theta)\, d\theta$. Then $\gamma_0 : \mathscr{D}(\Delta_0^D) \to H^{\frac{1}{2}}(\mathbb{T})$, the unique continuous linear map which on regular functions acts by

$$\gamma_0 u(\theta) = \lim_{x\downarrow 0} \frac{\partial u}{\partial x}(x,\theta),$$

is a concrete example of what we call an *abstract trace map* (see the next section), i.e., γ_0 is continuous (w.r.t. graph norm), surjective and its kernel is dense in $L^2(\mathbb{M}_0)$. By partial Fourier transform with respect to the angular variable one gets

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$$L^2(\mathbb{M}_0) = \bigoplus_{k\in\mathbb{Z}} L^2(\mathbb{R}_+), \quad \Delta_0^D = \bigoplus_{k\in\mathbb{Z}} d_k^2,$$

where

$$d_k^2 : \mathscr{D}(d_k^2) \subset L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \quad d_k^2 f := f'' - k^2 f,$$

$$\mathscr{D}(d_k^2) = \mathscr{D}_0 := \{f \in L^2(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}}_+) : f'' \in L^2(\mathbb{R}_+),\ f(0) = 0\}.$$

On $\mathscr{D}_0$ one can define the trace map

$$\hat{\gamma}_0 : \mathscr{D}_0 \to \mathbb{C}, \quad \hat{\gamma}_0 f := f'(0),$$

which is bounded, surjective and with a kernel dense in $L^2(\mathbb{R}_+)$. Moreover $\hat{\gamma}_0$ is bounded uniformly in $k \in \mathbb{Z}$ w.r.t. the graph norm of d_k^2, and so the infinite direct sum

$$\bigoplus_{k\in\mathbb{Z}} \hat{\gamma}_0 : \mathscr{D}(\bigoplus_{k\in\mathbb{Z}} d_k^2) \to \ell^2(\mathbb{Z}). \tag{1.1}$$

is a well defined bounded operator. Since γ_0 corresponds to $\bigoplus_{k\in\mathbb{Z}} \hat{\gamma}_0$ by partial Fourier transform, (1.1) does not define a trace map since it is not surjective: its range space is the strict subspace of $\ell^2(\mathbb{Z})$ defined by

$$h^{\frac{1}{2}}(\mathbb{Z}) := \left\{ \{s_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{k\in\mathbb{Z}} |k|\, |s_k|^2 < +\infty \right\} \simeq H^{\frac{1}{2}}(\mathbb{T}).$$

This simple example shows that an infinite direct sum of trace maps can fail to be a trace map: the direct sum of the range spaces can be different from the range space of the sum.

In Sect. 2 we provide a simple criterion which selects the right range space in order that the direct sum of trace maps is a trace map. Such a simple criterion uses a hypothesis involving the boundedness of operator-valued sequences obtained composing the trace maps with their right inverses [see (2.1)]. Such a hypothesis seems a very strong one (indeed that allows an easy proof), however, we show that always there exist right inverses such that (2.1) holds true (see Lemma 2.3).

In Sect. 3 we give an application to self-adjoint extensions of direct sums of symmetric operators and provide a couple of examples. We obtain that the methods here presented permit to obtain results equivalent to the ones recently obtained in [8] and [7] using regularized boundary triplets (see Remark 3.5).

In Example 1 we determine the trace space for the evaluation map $f \mapsto \{f'(x_n)\}_{n\in\mathbb{N}}$ acting on functions $f \in H^2(\mathbb{R}\backslash X) \cap H_0^1(\mathbb{R}\backslash X)$ where $X = \{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $x_n < x_{n+1}$. In this case Theorem 2.1 easily implies that the range space is a weighted ℓ^2-space with weight $w_n = d_n^{-1}$, where $d_n := x_{n+1} - x_n$. By Theorem 3.2 such a trace map can be used to define one-dimensional Schrödingier operators with δ and δ' interaction supported on the discrete set X, thus providing a construction alternative to the one presented in [7].

In Example 2 we show that our criterion easily gives the correct trace space $H^{\frac{1}{2}}(\mathbb{T})$ for the example provided at the beginning. Then we point out that the same criterion allows to prove that $H^s(\mathbb{T})$, $s = \frac{1}{2} - \frac{\alpha}{1+\alpha}$, is (isomorphic to) the defect space of $\Delta_\alpha^{\min}$, $-1 < \alpha < 1$, the minimal realization of the Laplace–Beltrami operator $\Delta_\alpha := \frac{\partial^2}{\partial x^2} - \frac{\alpha}{x}\frac{\partial}{\partial x} + x^{2\alpha}\frac{\partial^2}{\partial \theta^2}$ corresponding to the degenerate/singular Riemannian metric $g_\alpha(x,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2\alpha} \end{pmatrix}$. We refer to the papers [3] and [4] for the almost-Riemannian geometric considerations leading to the study of Δ_α and to [12] for the classification of all self-adjoint extensions of $\Delta_\alpha^{\min}$.

2 Direct sums of abstract trace maps

Let $\mathscr{H}_k$, $k \in \mathbb{Z}$, be a sequence of Hilbert spaces, with scalar product $\langle\cdot,\cdot\rangle_k$ and corresponding norm $\|\cdot\|_k$. On each $\mathscr{H}_k$ we consider a self-adjoint operator

$$A_k : \mathscr{D}(A_k) \subset \mathscr{H}_k \to \mathscr{H}_k$$

and we denote by $\mathscr{H}_{(k)}$ the Hilbert space consisting of $\mathscr{D}(A_k)$ equipped with a scalar product $\langle\cdot,\cdot\rangle_{(k)}$ giving rise to a norm $\|\cdot\|_{(k)}$ equivalent to the graph one.

Let $\mathfrak{h}_k$, $k \in \mathbb{Z}$, be a sequence of auxiliary Hilbert spaces with scalar product $[\cdot, \cdot]_k$ and corresponding norm $|\cdot|_k$.

Let

$$\tau_k : \mathscr{H}_{(k)} \to \mathfrak{h}_k, \quad k \in \mathbb{Z},$$

be a sequence of *abstract trace maps*, i.e., τ_k is a linear, continuous and surjective map such that its kernel $\mathscr{K}(\tau_k)$ is dense in $\mathscr{H}_k$. Since τ_k is continuous and surjective there exists a linear continuous right inverse

$$\iota_k : \mathfrak{h}_k \to \mathscr{H}_{(k)}, \quad \tau_k \iota_k = 1$$

(see e.g. [2, Proposition 1, Section 6, Chapter 4]). Since τ_k is surjective, ι_k is injective and so we can define a new scalar product on $\mathfrak{h}_k$ by

$$[\phi_k, \psi_k]_{(k)} := [\iota_k^* \iota_k \phi_k, \psi_k]_k \equiv \langle \iota_k \phi_k, \iota_k \psi_k \rangle_{(k)}.$$

It is immediate to check that $\mathfrak{h}_k$ is complete w.r.t. the norm

$$|\phi_k|_{(k)} := \|\iota_k \phi_k\|_{(k)} \equiv |(\iota_k^* \iota_k)^{1/2} \phi_k|_k.$$

Let us denote by $\mathfrak{h}_{(k)}$ the Hilbert space given by $\mathfrak{h}_k$ equipped with the scalar product $[\cdot, \cdot]_{(k)}$. We define

$$\mathscr{H} := \bigoplus_{k \in \mathbb{Z}} \mathscr{H}_k, \qquad \mathscr{H}_\circ := \bigoplus_{k \in \mathbb{Z}} \mathscr{H}_{(k)},$$

$$\mathfrak{h} := \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_k, \qquad \mathfrak{h}_\circ := \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_{(k)}$$

with corresponding norms $\|\cdot\|$, $\|\cdot\|_\circ$, $|\cdot|$, $|\cdot|_\circ$.

We denote by $|\!|\!|\cdot|\!|\!|$ the operator norm of bounded linear operators.

Theorem 2.1 *Let ι_k be a linear continuous right inverse of τ_k and suppose that*

$$\sup_{k \in \mathbb{Z}} |\!|\!|\iota_k \tau_k|\!|\!| < +\infty. \tag{2.1}$$

Then the linear map

$$\tau : \mathscr{H}_\circ \to \mathfrak{h}_\circ, \quad \tau\Big(\bigoplus_{k \in \mathbb{Z}} v_k\Big) := \bigoplus_{k \in \mathbb{Z}} (\tau_k v_k)$$

is an abstract trace map, i.e., is continuous, surjective and its kernel $\mathscr{K}(\tau)$ is dense in $\mathscr{H}$.

Proof (continuity) Let $v = \bigoplus_{k \in \mathbb{Z}} v_k \in \mathscr{H}_\circ$. Then

$$\begin{aligned} |\tau v|_\circ^2 &= \sum_{k \in \mathbb{Z}} \|\iota_k \tau_k v_k\|_{(k)}^2 \le \Big(\sup_{k \in \mathbb{Z}} |\!|\!|\iota_k \tau_k|\!|\!|\Big)^2 \sum_{k \in \mathbb{Z}} \|v_k\|_{(k)}^2 \\ &= \Big(\sup_{k \in \mathbb{Z}} |\!|\!|\iota_k \tau_k|\!|\!|\Big)^2 \|v\|_\circ^2. \end{aligned}$$

(surjectivity) Given $\phi = \bigoplus_{k \in \mathbb{Z}} \phi_k \in \mathfrak{h}_\circ$, let us define $v := \bigoplus_{k \in \mathbb{Z}} v_k$ by $v_k = \iota_k \phi_k \in \mathscr{H}_{(k)}$. Then $v \in \mathscr{H}_\circ$ by

$$\sum_{k \in \mathbb{Z}} \|v_k\|_{(k)}^2 = \sum_{k \in \mathbb{Z}} \|\iota_k \phi_k\|_{(k)}^2 = \sum_{k \in \mathbb{Z}} |\phi_k|_{(k)}^2 = |\phi|_\circ^2.$$

(density) Given $v := \bigoplus_{k \in \mathbb{Z}} v_k \in \mathscr{H}$ and $\epsilon > 0$, let $N_\epsilon \ge 0$ such that $\sum_{|k| > N_\epsilon} \|v_k\|_k^2 \le \epsilon/2$. Since $\mathscr{K}(\tau_k)$ is dense in $\mathscr{H}_k$, there exist $v_{k,\epsilon} \in \mathscr{K}(\tau_k)$ such that $\|v_k - v_{k,\epsilon}\|_k^2 \le 2^{-|k|}(\epsilon/6)$. Define $v_\epsilon := \bigoplus_{|k| \le N_\epsilon} v_{k,\epsilon}$. Then $v_\epsilon \in \mathscr{K}(\tau)$ and

$$\|v - v_\epsilon\|^2 \le \sum_{|k| \le N_\epsilon} \|v_k - v_{k,\epsilon}\|_k^2 + \frac{\epsilon}{2} \le \frac{\epsilon}{6} \sum_{k \in \mathbb{Z}} 2^{-|k|} + \frac{\epsilon}{2} = \epsilon.$$

□

Remark 2.2 Notice that Theorem 2.1 holds true for any sequence of Hilbert spaces $\mathscr{H}_{(k)}$, $k \in \mathbb{N}$, such that each $\mathscr{H}_{(k)}$ is densely embedded in $\mathscr{H}_k$. However, our hypotheses $\mathscr{H}_{(k)} = \mathscr{D}(A_k)$ permits to show that it is always possible to find right inverses ι_k such that hypothesis (2.1) is satisfied (see Lemma 2.3 below).

For any $z \in \rho(A_k)$, let us define the following bounded linear operators:

$$R_k(z) : \mathscr{H}_k \to \mathscr{H}_{(k)}, \quad R_k(z) := (-A_k + z)^{-1},$$

$$G_k(z) : \mathfrak{h}_k \to \mathscr{H}_k, \quad G_k(z) := (\tau_k R_k(\bar{z}))^*.$$

By resolvent identity one has

$$G_k(w) - G_k(z) =(z - w)R_k(w)G_k(z) = (z - w)R_k(z)G_k(w). \tag{2.2}$$

Now let us take $z = \pm i$ in the above definitions and pose

$$R_k := (-A_k + i)^{-1}, \quad G_k := G_k(-i), \quad G_k^+ := G_k(i),$$

$$\Gamma_k(z) := \tau_k\left(\frac{G_k + G_k^+}{2} - G_k(z)\right).$$

Then $z \mapsto \Gamma_k(z)$ is a Weyl function (equivalently a Krein's Q-function), i.e., it satisfied the identities

$$\Gamma_k(z) - \Gamma_k(w) = (z - w)G_k(\bar{w})^* G_k(z)$$

and

$$\Gamma_k(z)^* = \Gamma_k(\bar{z}).$$

Therefore, the set

$$Z_k := \{z \in \rho(A_k) : 0 \in \rho(\Gamma_k(z))\}.$$

is not void: $\mathbb{C}\backslash\mathbb{R} \subseteq Z_k$ (see e.g. [11, Theorem 2.1]).

Posing

$$\Gamma_k := \Gamma_k(-i),$$

one has the identities

$$G_k^+ - G_k = 2i R_k G_k, \tag{2.3}$$

$$G_k^* G_k = -i\, \Gamma_k \tag{2.4}$$

and so

$$\iota_k : \mathfrak{h}_k \to \mathscr{H}_{(k)}, \quad \iota_k := i R_k G_k \Gamma_k^{-1} = R_k G_k (G_k^* G_k)^{-1}. \tag{2.5}$$

is a linear bounded right inverse of τ_k. Moreover, since $R_k : \mathscr{H}_k \to \mathscr{H}_{(k)}$ is unitary w.r.t. the scalar product

$$\langle u_k, v_k\rangle_{(k)} := \langle(-A + i)u_k, (-A + i)v_k\rangle_k,$$

one has

$$\iota_k^* \iota_k = (G_k^* G_k)^{-1}. \tag{2.6}$$

Lemma 2.3 *Let ι_k be defined as in (2.5). Then $\|\iota_k \tau_k\| = 1$.*

Proof By (2.5) one has

$$\|\iota_k \tau_k v_k\|_{(k)} = \|G_k(G_k^* G_k)^{-1} G_k^*(-A_k + i)v_k\|_k.$$

Since the range of G_k is closed one has the decomposition $\mathscr{H}_k = \mathscr{R}(G_k) \oplus \mathscr{K}(G_k^*)$ and so $(-A_k + i)v_k = G_k\phi_k \oplus w_k$. Therefore

$$\|\iota_k \tau_k v_k\|_{(k)} = \|G_k\phi_k\|_k \le \|(-A_k + i)v_k\|_k = \|v_k\|_{(k)}.$$

If $v_k = R_k G_k \phi_k$ then $\|\iota_k \tau_k v_k\|_{(k)} = \|v_k\|_{(k)}$. □

Remark 2.4 In the case there exists $\lambda \in \cap_{k\in\mathbb{Z}} \rho(A_k) \cap \mathbb{R}$ the previous reasonings have the following variant. By (2.2) there follows

$$\begin{aligned} &G_k(-i)^* G_k(-i) \\ &\quad = G_k(\lambda)^*(1 + (\lambda - i)R_k(i))(1 + (\lambda + i)R_k(-i))G_k(\lambda). \end{aligned}$$

and so

$$\begin{aligned} |G_k(-i)^* G_k(-i)\phi_k|_k &\le \|1 + (\lambda - i)R_k(i)\|^2 |G_k(\lambda)^* G_k(\lambda)\phi_k|_k \\ &\le \left(1 + \sqrt{1 + \lambda^2}\right)^2 |G_k(\lambda)^* G_k(\lambda)\phi_k|_k. \end{aligned}$$

Since $G_k(-i)^* G_k(-i)$ is injective by (2.4), this shows that $G_k(\lambda)^* G_k(\lambda)$ is injective. Since it is self-adjoint and its range is closed [since the range of $G_k(\lambda)$ is closed], $G_k(\lambda)^* G_k(\lambda)$ is a continuous bijection. Then

$$\iota_k := R_k G_k (G_k^* G_k)^{-1},$$

is a bounded right inverse of τ_k, where in this case we used the notation

$$R_k := (-A_k + \lambda)^{-1}, \quad G_k := G_k(\lambda).$$

Moreover, using the scalar product

$$\langle u_k, v_k\rangle_{(k)} := \langle(-A_k + \lambda)u_k, (-A_k + \lambda)v_k\rangle_k,$$

one gets

$$\iota_k^* \iota_k = (G_k^* G_k)^{-1}.$$

and, proceeding as in the proof of Lemma 2.3,

$$\|\iota_k \tau_k\| = 1.$$

Theorem 2.1 has the following alternative version where one can still use the original trace space $\mathfrak{h}$ as long as one regularizes the traces τ_k:

Theorem 2.5 *Let us define* $r_k := (G_k^* G_k)^{1/2}$ *and*

$$\tilde\tau_k : \mathscr{H}_{(k)} \to \mathfrak{h}_k, \quad \tilde\tau_k := r_k^{-1}\tau_k.$$

Then the linear map

$$\tilde\tau : \mathscr{H}_\circ \to \mathfrak{h}, \quad \tilde\tau\Big(\bigoplus_{k\in\mathbb{Z}} v_k\Big) := \bigoplus_{k\in\mathbb{Z}} (\tilde\tau_k v_k)$$

is continuous, surjective and its kernel $\mathscr{K}(\tilde\tau) = \mathscr{K}(\tau)$ *is dense in* $\mathscr{H}$.

Proof The proof is the same as in Theorem 2.1. It suffices to notice that $\tilde\iota_k := \iota_k r_k$ is the right inverse of $\tilde\tau_k$ and that

$$(\tilde\iota_k)^* \tilde\iota_k = r_k \iota_k^* \iota_k r_k = r_k (G_k^* G_k)^{-1} r_k = 1.$$

□

Remark 2.6 Notice that in this section $\mathbb{Z}$ can be replaced by any other countable set N and that we can replace $[\cdot,\cdot]_{(k)}$ by a scalar product inducing an equivalent norm. Moreover, given a finite subset $F \subset N$, we can replace $\mathfrak{h}_{(k)}$ by $\mathfrak{h}_k$ for any $k \in F$.

3 Applications and examples

Let $S_k, k \in \mathbb{Z}$, be the sequence of symmetric operators defined by $S_k := A_k|\mathscr{K}(\tau_k)$, where A_k and τ_k are defined as in the previous section. Then $S := \bigoplus_{k\in\mathbb{Z}} S_k$ is a symmetric operator and $S = A|\mathscr{K}(\tau)$, where $A := \bigoplus_{k\in\mathbb{Z}} A_k$ and $\tau := \bigoplus_{k\in\mathbb{Z}} \tau_k$ is defined as in Theorem 2.1. Here τ_k is considered as a map on $\mathscr{H}_{(k)}$ to $\mathfrak{h}_{(k)}$, so that when calculating the adjoint $G_{(k)}(z)$ of $\tau_k(R_k(\bar{z}))$ one gets

$$G_{(k)}(z) := G_k(z)\iota_k^*\iota_k.$$

Next Lemma shows that the direct sums $\bigoplus_{k\in\mathbb{Z}} G_{(k)}(z)$ appearing in Theorem 3.2 below are well-defined bounded operators:

Lemma 3.1

$$\forall z \in \bigcap_{k\in\mathbb{Z}} \rho(A_k), \quad \sup_{k\in\mathbb{Z}} \|G_{(k)}(z)\| < +\infty.$$

Proof By (2.2) one has, posing $G_{(k)} := G_{(k)}(-i)$,

$$\|G_{(k)}(z)\| \le \|1 - (i+z)R_k\| \, \|G_{(k)}\| \le (2+|z|)\, \|G_{(k)}\|.$$

By (2.6),

$$\|G_{(k)}\phi_k\|_k = \|G_k\iota_k^*\iota_k\phi_k\|_k = \langle \iota_k^*\iota_k\phi_k, G_k^*G_k\iota_k^*\iota_k\phi_k\rangle_k = |\phi_k|_{(k)}$$

and so

$$\|G_{(k)}\| = 1.$$

□

By Theorem 2.1 and by the results provided in [9, Theorem 2.2] and [11, Theorem 2.1] one gets the following

Theorem 3.2 *The set of self-adjoint extensions of S is parametrized by couples (Π, Θ), where Π is an orthogonal projection in $\mathfrak{h}_\circ = \bigoplus_{k\in\mathbb{Z}} \mathfrak{h}_{(k)}$ and Θ is a self-adjoint operator in the Hilbert space* Range(Π). *Denoting by $A_{\Pi,\Theta}$ the self-adjoint extension associated with (Π, Θ) one has*

$$A_{\Pi,\Theta}(\bigoplus_{k\in\mathbb{Z}} v_k) = \bigoplus_{k\in\mathbb{Z}} \left(A_k v_k^\circ + \left(Re(z_\circ)G_{(k)}^\circ + i\, Im(z_\circ)G_{(k)}^\diamond\right)\phi_k\right),$$

$$\mathscr{D}(A_{\Pi,\Theta}) = \left\{ \bigoplus_{k\in\mathbb{Z}} v_k \in \mathscr{H} : v_k = v_k^\circ + G_{(k)}^\circ\phi_k,\ \bigoplus_{k\in\mathbb{Z}} v_k^\circ \in \bigoplus_{k\in\mathbb{Z}} \mathscr{D}(A_k), \right.$$
$$\left. \bigoplus_{k\in\mathbb{Z}} \phi_k \in \mathscr{D}(\Theta),\ \Pi(\bigoplus_{k\in\mathbb{Z}} \tau_k v_k^\circ) = \Theta(\bigoplus_{k\in\mathbb{Z}} \phi_k) \right\}.$$

Moreover, for any $z \in (\cap_{k\in\mathbb{Z}} \rho(A_k)) \cap \rho(A_{\Pi,\Theta})$,

$$(-A_{\Pi,\Theta} + z)^{-1} = \bigoplus_{k\in\mathbb{Z}} (-A_k + z)^{-1}$$
$$+ \bigoplus_{k\in\mathbb{Z}} G_{(k)}(z)\Pi\left(\Theta + \Pi \bigoplus_{k\in\mathbb{Z}} \tau_k(G_{(k)}^\circ - G_{(k)}(z))\Pi\right)^{-1}\Pi \bigoplus_{k\in\mathbb{Z}} G_{(k)}^*(z).$$

Here

$$G_{(k)}^\circ := \frac{1}{2}(G_{(k)}(z_\circ) + G_{(k)}(\bar{z}_\circ)), \quad G_{(k)}^\diamond := \frac{1}{2}(G_{(k)}(z_\circ) - G_{(k)}(\bar{z}_\circ))$$

and $z_\circ \in \cap_{k\in\mathbb{Z}} \rho(A_k)$.

Remark 3.3 By the definition of $\mathscr{D}(A_{\Pi,\Theta})$ one has that $A_{\Pi,\Theta}$ is a direct sum if and only if both Π and Θ are direct sums.

In the case $z_\circ \in \mathbb{R}$ one has $G^\diamond_{(k)} = 0$ and

$$\tau_k(G^\circ_{(k)} - G_{(k)}(z)) = z\, G_k(z_\circ)^* G_k(z)\, \iota_k^* \iota_k = z\, G_k(z)^* G_k(z_\circ)\, \iota_k^* \iota_k$$

In the case $0 \in \cap_{k\in\mathbb{Z}}\, \rho(A_k)$ one can take $z_\circ = 0$, so that

$$A_{\Pi,\Theta}(\oplus_{k\in\mathbb{Z}} v_k) = \oplus_{k\in\mathbb{Z}} A_k v_k^\circ.$$

Remark 3.4 Theorem 3.2 has an alternative version in the case one uses the trace map furnished by Lemma 2.5. In this case the extension parameter (Π, Θ) is such that Π is an orthogonal projection in $\mathfrak{h} = \oplus_{k\in\mathbb{Z}} \mathfrak{h}_k$ and Θ is a self-adjoint operator in the Hilbert space associated with Π. The statement of Theorem 3.2 remains unchanged replacing τ_k with $\tilde{\tau}_k$ and $G_{(k)}(z)$ with

$$\tilde{G}_k(z) := G_{(k)}(z) r_k = G_k(z) r_k^{-1} = (\tilde{\tau}_k R_k(\bar{z}))^*.$$

Also in this case the norm of $\tilde{G}_{(k)}(z) : \mathscr{H}_k \to \mathfrak{h}_k$ is bounded uniformly in $k \in \mathbb{Z}$ for any $z \in \cap_{k\in\mathbb{Z}}\, \rho(A_k)$.

Remark 3.5 By [10] and [11, Section 4], Theorem 3.2 and Lemma 2.5 provide results equivalent to the ones that can be obtained using Boundary Triplet Theory. Let us for simplicity take $z_\circ = i$. Then (see [10, Theorem 3.1])

$$\mathscr{D}(S_k^*) = \{v_k \in \mathscr{H} : v_k = v_k^\circ + G_k^\circ \phi_k,\ v_k^\circ \in \mathscr{D}(A_k),\ \phi_k \in \mathfrak{h}_k\},$$

$$S_k^* : \mathscr{D}(S_k^*) \subseteq \mathscr{H}_k \to \mathscr{H}_k, \quad S_k^* v_k := A_k v_k + R_k G_k \phi_k,$$

and the triple $\{\mathfrak{h}_k, \beta_{k,0}, \beta_{k,1}\}$, where

$$\beta_{k,0} : \mathscr{D}(S_k^*) \to \mathfrak{h}_k, \quad \beta_{k,0} v_k := \tau_k v_k^\circ,$$

$$\beta_{k,1} : \mathscr{D}(S_k^*) \to \mathfrak{h}_k, \quad \beta_{k,1} v_k := \phi_k,$$

is a boundary triple for S_k^*, i.e., $\beta_{k,1}$ and $\beta_{k,2}$ are surjective and satisfy the Green-type identity

$$\langle S_k^* u_k, v_k\rangle_k - \langle u_k, S_k^* v_k\rangle_k = [\beta_{1,k} u_k, \beta_{k,0} v_k]_k - [\beta_{k,0} u_k, \beta_{k,1} v_k]_k. \tag{3.1}$$

Moreover the Weyl function of S_k is

$$M_k(z) = \tau_k \left(\frac{G_k + G_k^+}{2} - G_k(z) \right)$$

(see [10, Theorem 3.1]). By (3.1) there follows that $\{\mathfrak{h}_k, r_k \beta_{k,1}, r_k^{-1} \beta_{k,2}\}$, where r_k is defined in Lemma 2.5, is a boundary triple for S_k^* as well with Weyl function $r_k^{-1} M_k(z)\, r_k^{-1}$.

By Lemma 2.5 and [10, Theorem 1.6] one gets

$$\begin{aligned}\mathscr{D}(S^*) &= \left\{ \oplus_{k\in\mathbb{Z}} v_k : v_k = v_k^\circ + \tilde{G}_k^\circ \phi_k,\ \oplus_{k\in\mathbb{Z}} v_k^\circ \in \oplus_{k\in\mathbb{Z}} \mathscr{D}(A_k),\ \oplus_{k\in\mathbb{Z}} \phi_k \in \mathfrak{h} \right\} \\ &\equiv \left\{ \oplus_{k\in\mathbb{Z}} v_k : v_k = v_k^\circ + G_k^\circ \psi_k,\ \oplus_{k\in\mathbb{Z}} v_k^\circ \in \oplus_{k\in\mathbb{Z}} \mathscr{D}(A_k),\ \oplus_{k\in\mathbb{Z}} r_k \psi_k \in \mathfrak{h} \right\}\end{aligned}$$

and the triple $\{\mathfrak{h}, \tilde{\beta}_0, \tilde{\beta}_1\}$ is a boundary triple for $S^* = \oplus_{k\in\mathbb{Z}} S_k^*$ with Weyl function $\oplus_{k\in\mathbb{Z}} (r_k^{-1} M_k(z)\, r_k^{-1})$, where

$$\tilde{\beta}_0 : \mathscr{D}(S^*) \to \mathfrak{h}, \quad \tilde{\beta}_0(\oplus_{k\in\mathbb{Z}} v_k) := \oplus_{k\in\mathbb{Z}} (r_k^{-1} \beta_{0,k})(\oplus_{k\in\mathbb{Z}} v_k) \equiv \oplus_{k\in\mathbb{Z}} (r_k^{-1} \tau_k v_k^\circ),$$

$$\tilde{\beta}_1 : \mathscr{D}(S^*) \to \mathfrak{h}, \quad \tilde{\beta}_1(\oplus_{k\in\mathbb{Z}} v_k) := \oplus_{k\in\mathbb{Z}} (r_k \beta_{1,k})(\oplus_{k\in\mathbb{Z}} v_k) \equiv \oplus_{k\in\mathbb{Z}} (r_k \psi_k) \equiv \oplus_{k\in\mathbb{Z}} \phi_k.$$

Let us notice that the norm of $r_k^{-1} \tau_k \equiv \tilde{\tau}_k : \mathscr{H}_{(k)} \to \mathfrak{h}_k$ is bounded uniformly in $k \in \mathbb{Z}$ by Lemma 2.5, and so $\tilde{\beta}_0$ is well defined. $\tilde{\beta}_1$ is well defined as well by the definition of $\mathscr{D}(S^*)$.

In conclusion this provides results on direct sums of regularized boundary triplets of the kind recently obtained in [8, Section 5], [7, Section 3], [5, Section 2].

Example 3.6 Given $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $x_n < x_{n+1}$, let $\mathscr{H}_n := L^2(I_n)$, $n \geq 0$, where $I_0 = (-\infty, x_1]$ and $I_n = [x_n, x_{n+1}]$, $n \in \mathbb{N}$. Let

$$A_n : \mathscr{D}(A_n) \subset L^2(I_n) \to L^2(I_n), \quad A_n u = u'', \quad n \geq 0,$$

$$\mathscr{D}(A_0) := \{u \in L^2(I_0) \cap C^1(I_0) : u'' \in L^2(I_0),\ u(x_1) = 0\},$$

$$\mathscr{D}(A_n) := \{u \in C^1(I_n) : u'' \in L^2(I_n),\ u(x_n) = u(x_{n+1}) = 0\}, \quad n \in \mathbb{N},$$

$$\tau_0 : \mathscr{H}_{(0)} \to \mathbb{C}, \quad \tau_0 u := -u'(x_1).$$

$$\tau_n : \mathscr{H}_{(n)} \to \mathbb{C}^2, \quad \tau_n u := (u'(x_n), -u'(x_{n+1})) \quad n \in \mathbb{N}.$$

For any $n \geq 0$, the map τ_n is continuous, surjective and has a kernel dense in $L^2(I_n)$.

By Remark 2.6 we can suppose $n \neq 0$ and, since $0 \in \cap_{n>0}\, \rho(A_n)$, we can use the results provided in Remark 2.4 with $\lambda = 0$.

The kernel of $(-A_n)^{-1}$, $n > 0$, is given by

$$\begin{aligned} &K_n(x, y)\\ &\quad = \frac{1}{d_n}\,((x_{n+1} - x)(y - x_n)\theta(x - y) + (x - x_n)(x_{n+1} - y)\theta(y - x)),\end{aligned}$$

where θ denotes Heaviside's function and $d_n := x_{n+1} - x_n$. Therefore

$$(G_n\xi)(x) = \frac{1}{d_n}\,(\xi_1(x_{n+1} - x) + \xi_2(x - x_n)), \quad \xi \equiv (\xi_1, \xi_2),$$

$$G_n^* u \equiv \frac{1}{d_n}\left(\int_{x_n}^{x_{n+1}} (x_{n+1} - x)u(x)\,\mathrm{d}x,\ \int_{x_n}^{x_{n+1}} (x - x_{n+1})u(x)\,\mathrm{d}x\right)$$

and so by straightforward calculations one gets that $G_n^* G_n : \mathbb{C}^2 \to \mathbb{C}^2$ corresponds to the positive-definite matrix

$$G_n^* G_n \equiv d_n \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}.$$

In conclusion on $\mathfrak{h}_{(n)} = \mathbb{C}^2$ we can put the equivalent scalar product

$$[\xi, \zeta]_{(n)} := \frac{\xi \cdot \zeta}{d_n}.$$

Hence, by Theorem 2.1, denoting by $\ell^2_d(\mathbb{N})$ the weighted ℓ^2-space

$$\ell^2_d(\mathbb{N}) := \left\{\{s_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} \frac{|s_n|^2}{d_n} < +\infty\right\},$$

one gets that

$$\tau := \tau_0 \oplus (\bigoplus_{n\in\mathbb{N}} \tau_n) : \mathscr{H}_0 \oplus (\bigoplus_{n\in\mathbb{N}} \mathscr{H}_{(n)}) \to \mathbb{C} \oplus \ell^2_d(\mathbb{N}) \oplus \ell^2_d(\mathbb{N}) \tag{3.2}$$

is continuous, surjective and has a kernel dense in $L^2(-\infty, x_\infty)$, $x_\infty := \sup_{n\in\mathbb{N}} x_n$. Notice that $\ell^2_d(\mathbb{N}) = \ell^2(\mathbb{N})$ if and only if

$$0 < d_* := \inf_{n\in\mathbb{N}} d_n \leq d^* := \sup_{n\in\mathbb{N}} d_n < +\infty.$$

Using Theorem 3.2 with trace map τ defined in (3.2), one gets the same kind of self-adjoint extensions given in [7] (the case in which $0 < d_* \le d^* < +\infty$ has been studied in [6]). Such extensions describe one-dimensional Schrödinger operators in $L^2(-\infty, x_\infty)$ with δ and δ' interactions supported on the discrete set $X = \{x_n\}_{n\in\mathbb{N}}$. These operators have been studied in [1, Chapters III.2 and III.3], when $0 < d_* \le d^* < +\infty$ and $x_\infty = +\infty$, and in [7] when $d^* < +\infty$. Analogous considerations, with A_n given by the one-dimensional Dirac operator with Dirichlet boundary conditions on the interval I_n, lead to self-adjoint extension describing one-dimensional Dirac operators with δ and δ' interactions on the discrete set $X = \{x_n\}_{n\in\mathbb{N}}$ (see [1, Appendix J], for the case X in which is a finite set and [5] for the general case).

Example 3.7 At first let us check that applying Thereom 2.1 to the example given in the introduction one gets the right trace space $\mathfrak{h}_\circ = h^{\frac{1}{2}}(\mathbb{Z})$. Hence here $A_k = d_0^2 - k^2$. By Remark 2.6 we can suppose $k \ne 0$ and, since $0 \in \cap_{k\in\mathbb{Z}\setminus\{0\}} \rho(A_k)$, we can use the results provided in Remark 2.4 with $\lambda = 0$. Since the kernel of $(-d_0^2 + z^2)^{-1}$, $\mathrm{Re}(z) > 0$, is given by

$$K(z; x, y) = \frac{\mathrm{e}^{-z\,|x-y|} - \mathrm{e}^{-z\,(x+y)}}{2z},$$

one easily gets

$$G_k^* \equiv \hat\gamma_0(-d_0^2 + k^2)^{-1} : L^2(\mathbb{R}_+) \to \mathbb{C}, \quad G_k^* f = \int_0^\infty \mathrm{e}^{-|k|\,x} f(x)\,\mathrm{d}x$$

and so $G_k^* G_k : \mathbb{C} \to \mathbb{C}$ is given by the multiplication by the real number

$$G_k^* G_k \equiv \int_0^\infty \mathrm{e}^{-2|k|\,x}\,\mathrm{d}x = \frac{1}{2\,|k|}.$$

Therefore $\mathfrak{h}_{(k)} = \mathbb{C}$ is equipped with the scalar product

$$[\xi, \zeta]_{(k)} := |k|\,\xi \cdot \zeta$$

and so

$$\mathfrak{h}_\circ = \mathbb{C} \oplus \Bigg(\bigoplus_{k\in\mathbb{Z}\setminus\{0\}} \mathfrak{h}_{(k)}\Bigg) = \Bigg\{\{s_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{k\in\mathbb{Z}} |k|\,|s_k|^2 < +\infty\Bigg\}.$$

Using Theorem 3.2 with trace map

$$\gamma_0 = \bigoplus_{k\in\mathbb{Z}} \hat\gamma_0 : \bigoplus_{k\in\mathbb{Z}} \mathscr{D}_0 \to h^{\frac{1}{2}}(\mathbb{Z}),$$

then one can determine all self-adjoint extensions of the minimal Laplacian on $\mathbb{M}_0$.

Such an example can be generalized in the following way: let $\mathbb{M}_\alpha$ be $\mathbb{R}_+ \times \mathbb{T}$ endowed with the singular/degenerate Riemannian metric

$$g_\alpha(x, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2\alpha} \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

The Riemannian volume form corresponding to g_α is $\mathrm{d}\omega = x^{-\alpha}\mathrm{d}x\mathrm{d}\theta$ and so we denote by $L^2(\mathbb{M}_\alpha)$ be the Hilbert space

$$L^2(\mathbb{M}_\alpha) := \Bigg\{u : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{C} : \int_0^{2\pi}\int_0^{+\infty} |u(x,\theta)|^2\, x^{-\alpha}\mathrm{d}x\mathrm{d}\theta < +\infty\Bigg\}.$$

In [4] it is shown that the minimal realization

$$\Delta_\alpha^{\min} : C_c^\infty(\mathbb{M}_\alpha) \subset L^2(\mathbb{M}_\alpha) \to L^2(\mathbb{M}_\alpha)$$

of the Laplace–Beltrami operator

$$\Delta_\alpha := \frac{\partial^2}{\partial x^2} - \frac{\alpha}{x}\frac{\partial}{\partial x} + x^{2\alpha}\frac{\partial^2}{\partial \theta^2} \tag{3.3}$$

corresponding to g_α is essentially self-adjoint whenever $\alpha \notin (-3, 1)$, has deficiency indices (1, 1) whenever $\alpha \in (-3, -1]$ and has infinite deficiency indices whenever $\alpha \in (-1, 1)$. Therefore, in order to determine and then study all self-adjoint realizations of $\Delta_\alpha^{\min}$, $-1 < \alpha < 1$, by Theorem 3.2 one needs to characterize the range space of the trace map

$$\gamma_\alpha u(\theta) := \lim_{x\downarrow 0} x^{-\alpha}\frac{\partial u}{\partial x}(x,\theta)$$

acting on function in the domain of the Friedrichs extensions Δ_α^D (corresponding to Dirichlet boundary conditions at $\mathbb{T}$) of $\Delta_\alpha^{\min}$ (see [12]). Let us sketch here a proof in the case $0 < \alpha < 1$, referring to [12] for more details and for the (more involved but still using Theorem 2.1) proof that holds in the case $-1 < \alpha < 1$.

By partial Fourier transform one gets

$$L^2(\mathbb{M}_\alpha) = \bigoplus_{k\in\mathbb{Z}} L^2_w(\mathbb{R}_+), \quad \Delta_\alpha^D = \bigoplus_{k\in\mathbb{Z}} (d_\alpha^2 - k^2 q_\alpha),$$

where $L^2_w(\mathbb{R}_+)$ is the weighted L^2 space

$$L^2_w(\mathbb{R}_+) := \left\{ f : \mathbb{R}_+ \to \mathbb{C} : \int_0^\infty |f(x)|^2 x^{-\alpha}\mathrm{d}x < +\infty \right\},$$

and

$$(d_\alpha^2 - k^2 q_\alpha) : \mathscr{D}_{\alpha,k} \subset L^2_w(\mathbb{R}_+) \to L^2_w(\mathbb{R}_+),$$

$$d_\alpha^2 f(x) := f''(x) - \frac{\alpha}{x} f'(x), \quad q_\alpha(x) = x^{2\alpha},$$

$$\mathscr{D}_{\alpha,k} := \{ f \in L^2_w(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}}_+) : (d_\alpha^2 - k^2 q_\alpha) \in L^2_w(\mathbb{R}_+),\ f(0) = 0 \}.$$

By Remark 2.6 we can suppose $k \neq 0$ and, since $0 \in \cap_{k\in\mathbb{Z}\setminus\{0\}} \rho(A_k)$, $A_k = d_\alpha^2 - k^2 q_\alpha$, whenever $0 < \alpha < 1$, we can use the results provided in Remark 2.4 with $\lambda = 0$. Since $f_\xi \equiv G_k \xi$ solves the boundary value problem

$$\begin{cases} f_\xi''(x) - \frac{\alpha}{x} f_\xi'(x) - k^2 x^{2\alpha} f_\xi = 0 \\ f_\xi(0) = \xi, \end{cases}$$

one gets

$$(G_k\xi)(x) = \xi \exp\left(-\frac{|k| x^{\alpha+1}}{\alpha+1} \right).$$

Therefore $G_k^* G_k : \mathbb{C} \to \mathbb{C}$ is given by the multiplication by the real number

$$G_k^* G_k \equiv \int_0^\infty \mathrm{e}^{-2\frac{|k|x^{\alpha+1}}{\alpha+1}} x^{-\alpha}\mathrm{d}x = |k|^{\frac{\alpha-1}{\alpha+1}} \int_0^\infty \mathrm{e}^{-2\frac{x^{\alpha+1}}{\alpha+1}} x^{-\alpha}\mathrm{d}x$$

and so $\mathfrak{h}_{(k)} = \mathbb{C}$ is equipped with the scalar product

$$[\xi, \zeta]_{(k)} := |k|^{\frac{1-\alpha}{1+\alpha}} \xi \cdot \zeta.$$

Thus by Theorem 2.1 the range space of γ_α (i.e., the defect space of $\Delta_\alpha^{\min}$) is given by the fractional Hilbert–Sobolev space

$$H^s(\mathbb{T}) \simeq h^s(\mathbb{Z}) := \left\{ \{s_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{k\in\mathbb{Z}} |k|^{2s} |s_k|^2 < +\infty \right\},$$

where $s = \frac{1}{2} - \frac{\alpha}{1+\alpha}$.

Acknowledgments I thank Ugo Boscain and Dario Prandi for the stimulating discussions which inspired this work.

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